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THE MECHANICS OF DELAMINATION IN FIBER-REINFORCED COMPOSITE MATERIALS

Part I - Stress Singularities and Solution Structure

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THE MECHANICS OF DELAMINATION IN FIBER-REINFORCED COMPOSITE MATERIALS: Part I - Stress Singularities and Solution Structure

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ABSTRACT

The fundamental mechanics of delamination in fiber composite laminates is studied. Mathematical formulation of the problem is based on recently developed laminate anisotropic elasticity theory and interlaminar fracture mechanics concepts. Stress singularities and complete solution structures associated with general composite delaminations are determined. For a fully open delamination with traction-free surfaces, oscillatory stress singularities always appear, leading to physically inadmissible field solutions. A refined model is introduced by considering a partially closed delamination with crack surfaces in finite-length contact. Stress singularities associated with a partially closed delamination having frictional crack-surface contact are determined, and are found to be different from the inverse square-root one of the frictionless-contact case. In the case of a delamination with very small area of crack closure, a simplified model having a square-root stress singularity is employed by taking the limit of the partially closed delamination. The possible presence of logarithmic-type stress singularity is examined; no logarithmic singularity of any kind is found in the composite delamination problem. Numerical examples of dominant stress singularities are shown for delaminations having crack-tip closure with different frictional coefficients between general θ_1 and θ_2 graphite-epoxy composites.

1. INTRODUCTION

Delamination has been a problem of significant concern in the reliable design and analysis of advanced fiber composite laminates. Separation of composite laminae, caused by high local interlaminar stress and low strength along the ply interface, can result in destruction of load transfer, reduction of stiffness, and loss of structural integrity, leading to final structural and functional failure. From the mechanics point of view, delamination involves initiation and growth of macroscopic cracks between dissimilar, strongly anisotropic solids. A rigorous mathematical study of delamination is recognized to be difficult, especially in a finite-dimensional fiber composite laminate. The complexities include the inherent crack-tip singularity, the effect of anisotropy of each constituent fiber lamina, and the abrupt change of stiffness or ply orientation through the laminate thickness direction. In addition, the three-dimensional state of stress and deformation associated with the composite delamination always gives rise to a combined opening (mode I), in-plane shearing (mode II), and out-of-plane tearing mode (mode III) fracture, which render the problem mathematically intractable in many cases. The mechanics of delamination in fiber composite laminates is, therefore, not only of significant academic interest but of practical importance. In this paper, the first of two articles in a row, the fundamental nature of stress singularities and associated field solutions for a delamination in a fiber composite laminate are investigated.

Owing to the aforementioned complexities, studies on an interface crack between dissimilar anisotropic materials have been limited. Gotoh [1] appears to be the first to examine the two-dimensional problem of partial debonding between dissimilar anisotropic plates under a plane stress condition. Clements [2] has used Stroh's approach [3] to study the problem of an

interface crack between two generally anisotropic half-spaces. Willis [4] has also conducted a two-dimensional stress analysis of a crack on the plane interface of two bonded dissimilar half-spaces. The analysis has been combined with the usual local form of Griffith's virtual work argument to give a failure criterion, involving a stress concentration vector and specific surface energy of the bonded interface. All of the asymptotic solutions obtained in [1,2,4] have an oscillatory displacement field that material interpenetration on either side of the crack surface is predicted. Similar to those found in the solutions for an interface crack between dissimilar isotropic materials [5-9], these physically unreasonable results have led to the argument of solution inadmissibility for the crack problem in dissimilar anisotropic media. To correct the unsatisfactory feature of oscillatory stress singularity, Wang and Choi [10] have recently reconsidered the problem of an interface crack between dissimilar, strongly anisotropic fiber-composite half-spaces by introducing a partially closed interface crack model, in which the crack is not completely open and that its surfaces are in frictionless contact near the tip. The formulation leads to a singular integral equation, which is solved numerically. Numerical results from this refined model [10] exhibit an inverse square-root stress singularity and, therefore, physically meaningful fracture mechanics parameters can be defined consistently with those in fracture problems of homogeneous materials [11-13] and in the model given by Comninou [14,15] for an interface crack between two isotropic media. Moreover, significantly global crack closure has been found [16] for an interface crack between dissimilar anisotropic elastic half-spaces subjected to mixed-mode loading--a situation that is generally experienced by a delamination in finite dimensional fiber composite laminates.

In this paper, we employ Lekhnitskii's complex-variable stress potentials [17] in conjunction with an eigenfunction expansion method to examine the mechanics and the mathematical solution structure for a delamination with frictional crack-tip closure in a composite laminate. Based on the general solution structure determined, an advanced numerical method using singular finite elements is then developed to study the detailed delamination behavior in finite dimensional fiber composite laminates with any arbitrary combinations of lamination parameters, geometric variables, and crack dimensions. Owing to space limitation, the numerical method and the detailed composite delamination behavior are reported in an accompanying article [18].

In the next section, the problem definition and basic assumptions are stated. Basic laminate anisotropic elasticity equations and formulation of the composite delamination problem are introduced in Section 3. General solution structures for asymptotic stress and displacement fields are obtained. Stress singularities associated with an open interlaminar crack and with a partially closed delamination tip with frictional crack-surface contact are determined respectively in Section 4. Influences of frictional coefficients on delamination stress singularities are examined. A simplified model for a delamination with a very small area of crack-tip closure is also introduced. The possibility of existence of additional singularities in logarithmic forms in homogeneous and particular solutions is investigated. Results are presented for delaminations with different local crack-surface traction boundary conditions in composite laminates containing various fiber orientations. The eigenvalues and associated stress singularities obtained in this study provide the most fundamental information on complete solution structures of delamination stress and deformation fields, and establish a basis for formulation of the singular finite elements used in the next paper

[18] to study the detailed delamination behavior in finite dimensional composites with general lamination and geometric variables.

2. STATEMENT OF THE PROBLEM AND ASSUMPTIONS

The problem considered here is a composite laminate (Fig. 1) composed of unidirectional fiber-reinforced plies of uniform thicknesses, h_1, h_2, h_3, ..., h_n. The composite has a finite dimension with a width equal to 2b. For simplicity and without loss of generality, we restrict our attention to the cases of symmetric composite laminates with fiber orientations $[\theta_1/\theta_2/\theta_3/\ldots\theta_3/\theta_2/\theta_1]$. Ply thicknesses are also symmetric with respect to the x-z plane, i.e., for each ply above the x-z plane ($y > 0$), there exists an identical ply with the same ply thickness below the x-z plane ($y < 0$). Delamination with a length a is assumed to occur in the form of an interface crack between dissimilar, strongly anisotropic fiber-reinforced composite laminae with fiber orientations θ_m and θ_{m+1}.

The composite laminate is assumed to be subjected to tractions acting in planes normal to the z-axis and distributed uniformly along the z-axis without variation. In the case that the finite dimensional composite laminate has a finite length, axial loads and moments are assumed to act on the ends of the composite body. The composite laminate is further assumed to be sufficiently long that in the region away from the ends, end effects are negligible by virtue of the Saint Venant principle. Consequently, the components of stresses in the laminate are independent of the z-axis. The special case in which all components of stresses and displacements in the composite are independent of the z is well-known as the generalized plane deformation problem [17].

The objectives of this study are to: (1) establish a mathematical basis for the mechanics of delamination based on laminate elasticity theories and interlaminar fracture mechanics concepts; (2) determine stress singularities and associated solution structures for composite delaminations with different

local crack-tip deformation configurations; (3) obtain asymptotic stress and deformation fields governing the fundamental behavior of delamination; and (4) study the influences of various lamination and material variables such as fiber orientation and crack-surface frictional coefficients on the delamination stress singularities.

3. LAMINATE ELASTICITY BASIS FOR COMPOSITE DELAMINATION

The development of the mechanics of composite delamination is based on recently established theories of anisotropic laminate elasticity [19,20,21] and fracture mechanics concepts of interface cracks between dissimilar, strongly anisotropic composites [4,10,16,21]. In this section, governing partial differential equations for composite laminate elasticity problems are established first, based on Lekhnitskii's complex-variable stress potentials [17]. General solutions for the laminate elasticity problem with interlaminar cracks are introduced. Stress singularities associated with a composite delamination having homogeneous local boundary conditions are defined. Solution structures of asymptotic stress and displacement fields are constructed for a delamination between dissimilar general fiber composite laminae. Additional terms of logarithmic forms in the homogeneous and particular solutions for the composite delamination problem are examined.

3.1 Basic Equations

The fundamental mechanics of delamination in a fiber-reinforced composite laminate may be studied from the schematics illustrated in Fig. 1. The constitutive equations of each fiber-reinforced composite lamina with rectilinear anisotropy of a general form in the structural (x-y-z) coordinates are denoted by generalized Hooke's law in contracted notation as

$$\epsilon_i = S_{ij}\,\sigma_j \qquad (i,j = 1,2,3,\ldots,6), \tag{1}$$

where the repeated subscript indicates summation, and S_{ij} is a compliance tensor. The engineering strains ϵ_i in Eq. 1 are defined in a Cartesian coordinate system by

$$\varepsilon_1 = \varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_2 = \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \varepsilon_3 = \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\varepsilon_4 = \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \varepsilon_5 = \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \varepsilon_6 = \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}. \tag{2}$$

where u, v, and w are displacement components. The stresses σ_i are defined in an analogous manner in the Cartesian coordinate system. For a composite laminate in the aforementioned loading condition, mathematical formulation for this class of elastostatic problems can be made using the well-known Lekhnitskii complex-variable stress potentials [17], F(x,y) and Ψ(x,y), defined as

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x\, \partial y},$$

$$\tau_{yz} = -\frac{\partial \Psi}{\partial x}, \qquad \tau_{zx} = \frac{\partial \Psi}{\partial y}. \tag{3}$$

Following the same procedure in [17,19], we can easily obtain the following system of partial differential equations for each anisotropic composite lamina:

$$L_3 F + L_2 \Psi = -2A_4 + A_1 S_{34} - A_2 S_{35}, \tag{4a}$$

$$L_4 F + L_3 \Psi = 0, \tag{4b}$$

where L_2, L_3 and L_4 are differential operators of the second, third, and fourth orders which have the form:

$$L_2 = \hat{S}_{44} \frac{\partial^2}{\partial x^2} - 2\hat{S}_{45} \frac{\partial^2}{\partial x\, \partial y} + \hat{S}_{55} \frac{\partial^2}{\partial y^2},$$

$$L_3 = -\hat{S}_{24} \frac{\partial^3}{\partial x^3} + (\hat{S}_{25} + \hat{S}_{46}) \frac{\partial^3}{\partial x^2\, \partial y} - (\hat{S}_{14} + \hat{S}_{56}) \frac{\partial^3}{\partial x\, \partial y^2} + \hat{S}_{15} \frac{\partial^3}{\partial y^3},$$

$$L_4 = \hat{S}_{22} \frac{\partial^4}{\partial x^4} - 2\hat{S}_{26} \frac{\partial^4}{\partial x^3\, \partial y} + (2\hat{S}_{12} + \hat{S}_{66}) \frac{\partial^4}{\partial x^2\, \partial y^2} - 2\hat{S}_{16} \frac{\partial^4}{\partial x\, \partial y^3} + \hat{S}_{11} \frac{\partial^4}{\partial y^4}, \tag{5}$$

with

$$\tilde{S}_{ij} = S_{ij} - S_{i3}\, S_{3j}/S_{33} \qquad (i,j = 1,2,4,5,6). \tag{6}$$

The constants A_1 and A_2 in Eqs. 4(a) and 4(b) characterize the bending of the composite body in the x-z and y-z planes respectively, and A_4 is the relative angle of rotation about the z-axis.

Assume that external tractions on the lateral surface of the composite cross section are given as $\bar{T}_x$, $\bar{T}_y$ and $\bar{T}_z$. The boundary conditions on the lateral surface ∂B are as follows:

$$\begin{aligned} \sigma_x n_x + \tau_{xy} n_y &= \bar{T}_x, \\ \tau_{xy} n_x + \sigma_y n_y &= \bar{T}_y, \\ \tau_{xz} n_x + \tau_{yz} n_y &= \bar{T}_z, \end{aligned} \tag{7}$$

where n_i are directional cosines of the bounding surface ∂B. The conditions at the ends of the composite have the form:

$$\begin{aligned} &\iint_B \tau_{xz}\, dx\, dy = \iint_B \tau_{yz}\, dx\, dy = 0, \qquad \iint_B \sigma_z\, dx\, dy = P_z, \\ &\iint_B \sigma_z\, y\, dx\, dy = M_1, \qquad \iint_B \sigma_z\, x\, dx\, dy = M_2, \\ &\iint_B (\tau_{yz}\, x - \tau_{xz}\, y)\, dx\, dy = M_t, \end{aligned} \tag{8}$$

where the integrals are taken over the entire area of the cross section B, and P_z, M_1, M_2, and M_t are applied force, bending moments, and twisting moment at ends of the composite, respectively.

3.2 General Solutions

The general solutions for the governing differential equations have been shown [17] to have the form as

$$F = \sum_{k=1}^{6} F_k(Z_k) + F_o, \tag{9}$$

$$\Psi = \sum_{k=1}^{6} \eta_k F_k'(Z_k) + \Psi_o, \tag{10}$$

where the complex variables Z_k are defined as $Z_k = x + \mu_k y$; F_o and Ψ_o are particular solutions of the nonhomogeneous system; the prime (') denotes differentiation of the analytical functions F_k with respect to their arguments, and μ_k are roots of the following algebraic characteristic equation:

$$\ell_4(\mu)\ell_2(\mu) - \ell_3^2(\mu) = 0, \tag{11}$$

with

$$\ell_2(\mu) = \hat{S}_{55}\mu^2 - 2\hat{S}_{45}\mu + \hat{S}_{44}, \tag{12a}$$

$$\ell_3(\mu) = \hat{S}_{15}\mu^3 - (\hat{S}_{14} + \hat{S}_{56})\mu^2 + (\hat{S}_{25} + \hat{S}_{46})\mu - \hat{S}_{24}, \tag{12b}$$

$$\ell_4(\mu) = \hat{S}_{11}\mu^4 - 2\hat{S}_{16}\mu^3 + (2\hat{S}_{12} + \hat{S}_{66})\mu^2 - 2\hat{S}_{26}\mu + \hat{S}_{22}. \tag{12c}$$

The η_k in Eq. 10 are complex numbers equal to

$$\eta_k = -\ell_3(\mu_k)/\ell_2(\mu_k) = -\ell_4(\mu_k)/\ell_3(\mu_k). \tag{13}$$

We now choose the form of $F_k(Z_k)$ as

$$F_k(Z_k) = C_k Z_k^{\delta+2}/[(\delta+1)(\delta+2)], \tag{14}$$

where C_k and δ are arbitrary complex constants to be determined. Substituting Eq. 14 into Eqs. 3, 9 and 10, we obtain the homogeneous solutions for stress and displacement components in polar coordinates (r-ϕ-z) as follows:

$$\sigma_{\phi\phi}^{(h)} = \sum_{k=1}^{3} (C_k H_{1k} Z_k^{\delta} + C_{k+3} \overline{H}_{1k} \overline{Z}_k^{\delta}), \quad \tau_{\phi z}^{(h)} = \sum_{k=1}^{3} (C_k H_{2k} Z_k^{\delta} + C_{k+3} \overline{H}_{2k} \overline{Z}_k^{\delta}),$$

$$\tau_{\phi r}^{(h)} = \sum_{k=1}^{3} (C_k H_{3k} Z_k^{\delta} + C_{k+3} \overline{H}_{3k} \overline{Z}_k^{\delta}), \quad \sigma_{rr}^{(h)} = \sum_{k=1}^{3} (C_k H_{4k} Z_k^{\delta} + C_{k+3} \overline{H}_{4k} \overline{Z}_k^{\delta}), \tag{15}$$

$$\tau_{rz}^{(h)} = \sum_{k=1}^{3} (C_k H_{5k} Z_k^{\delta} + C_{k+3} \overline{H}_{5k} \overline{Z}_k^{\delta}),$$

$$u_r^{(h)} = \sum_{k=1}^{3} [C_k H_{6k} Z_k^{\delta+1}/(\delta+1) + C_{k+3} \overline{H}_{6k} \overline{Z}_k^{\delta+1}/(\delta+1)],$$

$$u_{\phi}^{(h)} = \sum_{k=1}^{3} [C_k H_{7k} Z_k^{\delta+1}/(\delta+1) + C_{k+3} \overline{H}_{7k} \overline{Z}_k^{\delta+1}/(\delta+1)], \tag{16}$$

$$u_z^{(h)} = \sum_{k=1}^{3} [C_k H_{8k} Z_k^{\delta+1}/(\delta+1) + C_{k+3} \overline{H}_{8k} \overline{Z}_k^{\delta+1}/(\delta+1)],$$

where

$$Z_k = r(e^{i\phi} + \lambda_k e^{-i\phi})/(1 + \lambda_k), \tag{17a}$$

$$\lambda_k = (1 + i\mu_k)/(1 - i\mu_k). \tag{17b}$$

The coefficients $H_{ik}(\phi)$ ($i = 1,2,3,\ldots,8$; $k = 1,2,3$) are known functions of ϕ, η_k, μ_k, and $\hat{S}_{ij}$, defined in Appendix 1.

The complete laminate elasticity solutions for the composite mechanics problem can be written as

$$\sigma_i = \sigma_i^{(h)} + \sigma_i^{(p)}, \qquad (i = 1,2,3,4,5,6), \tag{18a}$$

$$u_j = u_j^{(h)} + u_j^{(p)}, \qquad (j = 1,2,3), \tag{18b}$$

where $\sigma_i^{(p)}$ and $u_j^{(p)}$ are particular solutions associated with the loading condition of each individual case studied. The expressions for $\sigma_z^{(h)}$ and $\sigma_z^{(p)}$ can be obtained as

$$\sigma_z^{(h)} = - S_{3j}\ \sigma_j^{(h)}/S_{33}, \tag{18c}$$

and

$$\sigma_z^{(p)} = (A_1 x + A_2 y + A_3) - S_{3j}\ \sigma_j^{(p)}/S_{33} \qquad (j = 1,2,4,5,6). \tag{18d}$$

3.3 Asymptotic Stress and Displacement Fields

Using Eqs. 15 and 16 and applying local homogeneous traction boundary conditions on crack surfaces ∂B_c (Fig. 2) and interface continuity (matching) conditions along ∂B_I, we obtain a system of twelve homogeneous linear equations in $C_k^{(\alpha)}(\alpha = m, m + 1)$, i.e.,

$$\underset{\sim}{D}\ \underset{\sim}{C} = \underset{\sim}{0}, \tag{19}$$

where $\underset{\sim}{D}(\delta)$ is a 12 x 12 coefficient matrix involving δ in a transcendental form, and $\underset{\sim}{C}$ is an unknown 12 x 1 column eigenvector. The nontrivial solution for $\underset{\sim}{C}$ requires that the determinant of the coefficient matrix vanishes, i.e.,

$$\|D(\delta)\| = 0. \tag{20}$$

This leads to a standard eigenvalue problem, and the δ can be determined from the transcendental characteristic equation. Standard numerical methods such as the Müller method [22] with the aid of a digital computer are needed for this purpose. The eigenvalues determined from Eq. 20 provide important information on the fundamental structure of stress and displacement solutions for the composite delamination problem. Furthermore, the eigenvalues δ_n which satisfy the following condition:

$$-1 < \mathrm{Re}[\delta_n] < 0 \tag{21}$$

characterize the fundamental nature of stress singularities and provide the asymptotic stress and deformation fields at the delamination tip. In the case

that local crack surface tractions are nonvanishing, for example, the crack closure problem, Eq. 19 needs to be modified. Delaminations having crack-tip deformation configurations with nonvanishing local traction boundary conditions are discussed in detail in the next section.

For a delamination problem in composite laminates with general lamination variables and fiber orientations, the algebraic multiplicity of the eigenvalues determined from Eq. 20 may give rise to additional terms of the logarithmic form $Z_k^{\delta_n}(\ln Z_k)^{\ell_m}$ in the homogeneous solution, as first suggested by Dempsey and Sinclair [23,24]. In this situation, the following terms may also be a part of the homogeneous solution in addition to Eqs. 15 and 16:

$$\sigma_i^{(h)} = \sum_{k=1}^{6} \frac{\partial^{\ell_m}}{\partial \delta_n^{\ell_m}}[C_k H_{ik} Z_k^{\delta_n}] \qquad (i = 1,2,4,5,6), \quad (22a)$$

$$u_j^{(h)} = \sum_{k=1}^{6} \frac{\partial^{\ell_m}}{\partial \delta_n^{\ell_m}}[C_k H_{(j+5)k} Z_k^{(\delta_n+1)}/(\delta_n+1)] \qquad (j = 1,2,3), \quad (22b)$$

where ℓ_m is the order of the logarithmic multiplier in the eigenfunction corresponding to eigenvalue δ_n, and is related to the property of the $\underset{\sim}{D}$ matrix by $\ell_m = M-(N-R)$, in which M is the algebraic multiplicity of the root δ_n, and N and R are the order and the rank of the D matrix, respectively. The presence of the logarithmic terms, Eqs. 22(a) and 22(b), in the homogeneous solution equires a nontrivial solution for C_k. Detailed discussion of the conditions for the existence of Eqs. 22(a) and 22(b) in the composite delamination problem can be found in [25].

In the construction of asymptotic solutions for delamination stress and displacement fields, the particular solution for the system of governing differential equations also contributes to the complete solution. It is apparent that the structure of the particular solution is related to the

applied loading and deformation of the delaminated composite. For the convenience of further developments, we consider here the case of a composite laminate with delaminations subjected to a uniform axial stretching, i.e., $\varepsilon_z = \varepsilon_o$. Under this circumstance, it has been shown [25] that the particular solution has a similar form as Eqs. 22(a) and 22(b),

$$\sigma_i^{(p)} = \sigma_{oi} + \left[\frac{\partial^{\ell_o}}{\partial\delta_n^{\ell_o}}\left(\sum_{k=1}^{6} C_k^{(p)} H_{ik} z_k^{\delta_n}\right)\right]_{\delta_n=0} \qquad (i = 1,2,4,5,6), \quad (23a)$$

$$u_j^{(p)} = u_{oj} + \left[\frac{\partial^{\ell_o}}{\partial\delta_n^{\ell_o}}\left(\sum_{k=1}^{6} C_k^{(p)} H_{(j+5)k} z_k^{\delta_{n+1}/(\delta_n+1)}\right)\right]_{\delta_n=0} \qquad (j = 1,2,3), \quad (23b)$$

where the components σ_{oi} and u_{oj} are known quantities determined by the remote loading condition. The ℓ_o in Eqs. 23(a) and 23(b) is the order of the logarithmic eigenfunction at $\delta_n = 0$ and is related to the multiplicity M_o of the root $\delta_n = 0$ and the rank and the order of the matrix $\underset{\sim}{D}$ by $\ell_o = M_o-(N-R)$. [Note that Eqs. 23(a) and 23(b) contain logarithmic terms of the forms $(\ln z_k)$, $(\ln z_k)^2$, ... $(\ln z_k)^{\ell_o}$.] The necessary and sufficient conditions for the existence of the particular solution, Eqs. 23(a) and 23(b), can be shown [25] as

$$\underset{\sim}{C}_h^{*(L)} \cdot \underset{\sim}{p}^* = 0 \qquad (24)$$

for every left eigenvector $\underset{\sim}{C}_h^{*(L)}$ of $\underset{\sim}{D}^*(0)$ defined in [25], where $\underset{\sim}{p}^*$ is a loading vector resulting from σ_{oi} and u_{oj}, and the dot ($\cdot$) denotes the inner product of the two column vectors. In the case that Eq. 24 does not hold, one needs to consider the logarithmic terms of a higher order through a higher-order differentiation $\partial^{(\ell_o+1)}/\partial\delta_n^{(\ell_o+1)}$ in Eqs. 23(a) and 23(b). A detailed discussion on this is given in Reference 25.

4. DELAMINATION STRESS SINGULARITIES IN COMPOSITE LAMINATES

Based on the general solution structures given in Eqs. 15, 16, 22 and 23, it is possible at this point to examine the detailed nature of stress singularity associated with a delamination in a fiber-reinforced composite laminate. Because of the local nature of the stress singularity, we focus our attention on the crack-tip region of a delamination between the mth and (m+1)th laminae with fiber orientations θ_m and θ_{m+1}, respectively. Both fully open and partially closed delaminations are considered. In the case of a delamination with an extremely small area of crack-tip closure, a simplified model by taking the limiting case of a partially closed crack is introduced.

4.1 Delamination with Traction-Free (Fully Open) Surfaces

Assuming that the crack surfaces are fully open and that the interface ∂B_I between the plies is perfectly bonded along $r > 0$ as shown in Fig. 2, we can immediately introduce the local traction-free boundary conditions along the delamination surfaces $\phi = \pm\pi$,

$$\sigma_{\phi\phi}^{(m)}(r,\pi) = \tau_{r\phi}^{(m)}(r,\pi) = \tau_{\phi z}^{(m)}(r,\pi) = 0, \tag{25a}$$

$$\sigma_{\phi\phi}^{(m+1)}(r,-\pi) = \tau_{r\phi}^{(m+1)}(r,-\pi) = \tau_{\phi z}^{(m+1)}(r,-\pi) = 0. \tag{25b}$$

The continuity (or matching) conditions of interlaminar stresses and displacements along the ply interface $\phi = 0$ read as follows:

$$\{\sigma_{\phi\phi}^{(m)}(r,0),\tau_{\phi z}^{(m)}(r,0),\tau_{r\phi}^{(m)}(r,0)\} = \{\sigma_{\phi\phi}^{(m+1)}(r,0),\tau_{\phi z}^{(m+1)}(r,0),\tau_{r\phi}^{(m+1)}(r,0)\}, \tag{25c}$$

$$\{u_r^{(m)}(r,0),u_\phi^{(m)}(r,0),u_z^{(m)}(r,0)\} = \{u_r^{(m+1)}(r,0),u_\phi^{(m+1)}(r,0),u_z^{(m+1)}(r,0)\}. \tag{25d}$$

More explicitly, the local homogeneous boundary conditions, Eqs. 25(a) and 25(b), and continuity conditions, Eqs. 25(c) and 25(d), have the forms as

$$\sum_{k=1}^{3} \{C_k^{(m)} H_{ik}^{(m)}(\pi) [Q_k^{(m)}(\pi)]^{\delta} + C_{k+3}^{(m)} \overline{H_{ik}^{(m)}(\pi)} \overline{[Q_k^{(m)}(\pi)]^{\delta}}\} = 0, \tag{26a}$$

$$\sum_{k=1}^{3} \{C_k^{(m+1)} H_{ik}^{(m+1)}(-\pi) [Q_k^{(m+1)}(-\pi)]^{\delta} + C_{k+3}^{(m+1)} \overline{H_{ik}^{(m+1)}(-\pi)} \overline{[Q_k^{(m+1)}(-\pi)]^{\delta}}\} = 0, \tag{26b}$$

$$\sum_{k=1}^{3} \{[C_k^{(m)} \Gamma_{jk}^{(m)} + C_{k+3}^{(m)} \overline{\Gamma}_{jk}^{(m)}] - [C_k^{(m+1)} \Gamma_{jk}^{(m+1)} + C_{k+3}^{(m+1)} \overline{\Gamma}_{jk}^{(m+1)}]\} = 0,$$

$$(i = 1,2,3;\ j = 1,2,3,4,5,6), \tag{26c}$$

where

$$Q_k^{(\alpha)}(\phi) = (e^{i\phi} + \lambda_k^{(\alpha)} e^{-i\phi})/(1 + \lambda_k^{(\alpha)}), \tag{26d}$$

and

$$\Gamma_{1k}^{(\alpha)} = 1, \quad \Gamma_{2k}^{(\alpha)} = \eta_k^{(\alpha)}, \quad \Gamma_{3k}^{(\alpha)} = \mu_k^{(\alpha)}, \quad \Gamma_{4k}^{(\alpha)} = p_k^{(\alpha)}, \quad \Gamma_{5k}^{(\alpha)} = q_k^{(\alpha)}, \quad \Gamma_{6k}^{(\alpha)} = t_k^{(\alpha)}$$

$$(\alpha = m,\ m+1). \tag{26e}$$

Equations 26(a), 26(b) and 26(c) consist of a system of twelve homogeneous linear algebraic equations in $C_k^{(m)}$ and $C_k^{(m+1)}$. The existence of a nontrivial solution requires that Eq. 20 holds, leading to a standard eigenvalue problem. The solution for δ_n can be obtained easily from Eq. 20 and shown to have the form,

$$\delta_n = (n - 1/2) \pm i\gamma, \quad (n - 1/2), \quad n = 1, n, \tag{27}$$

where $n = 0,1,2,\ldots,\infty$; and γ is a constant related to material elastic properties, $S_{ij}^{(m)}$ and $S_{ij}^{(m+1)}$, of the adjacent mth and (m+1)th fiber-composite laminae. In general, the value of γ needs to be determined numerically from Eq. 20, which involves δ_n in a transcendental form. It is important to note that the eigenvalues of δ_n obtained from Eq. 27 give critically important information on the fundamental structure of stress and displacement solutions

for delaminated composite materials. We remark that the eigenvalues $(n-1/2) \pm i\gamma$ and $(n-1/2)$ are single roots and that all the integers, n, including zero, have an algebraic multiplicity of 3 in general. As mentioned in the preceding section, the values of δ_n which meet the condition Eq. 21 provide the exact strength (or order) of the inherently stress singularity for the asymptotic stress solution at the delamination crack tip. The possible presence of weaker singularities and related terms in logarithmic forms as discussed in Section 3.3, i.e., $(\ln Z_k)^{\ell_o}$ and/or $Z_k^{\delta_n}(\ln Z_k)^{\ell_m}$, in the homogeneous solution as well as in the particular solution will be discussed later.

For the purpose of illustration, consider a delamination located along the ply interface of θ/90° graphite-epoxy composite laminae (Fig. 2). The following material elastic constants* of high-modulus unidirectional graphite-epoxy are used in the computation:

$$\begin{aligned} E_L &= 20 \times 10^6 \text{ psi} \quad (137.9 \text{ GPa}), \\ E_T &= E_z = 2.1 \times 10^6 \text{ psi} \quad (14.48 \text{ GPa}), \\ G_{LT} &= G_{Tz} = G_{Lz} = 0.85 \times 10^6 \text{ psi} \quad (5.86 \text{ GPa}), \\ \nu_{LT} &= \nu_{Lz} = \nu_{Tz} = 0.21, \end{aligned} \tag{28}$$

*These ply elastic constants are used in the computation here only to illustrate the general nature of the current problem. (These constants are selected for historical reasons because they have been used in many previous studies of the mechanics of composite laminates [19,26,27].) Numerical results based on real material constants of the commonly employed T300/5208 graphite-epoxy with

$$\begin{aligned} &E_L = 19.5 \times 10^6 \text{ psi (134.45 GPa)}, \quad E_T = E_z = 1.48 \times 10^6 \text{ psi (10.2 GPa)}, \\ &G_{LT} = G_{Lz} = 0.8 \times 10^6 \text{ psi (5.52 GPa)}, \quad G_{Tz} = 0.49 \times 10^6 \text{ psi (3.38 GPa)}, \\ &\nu_{LT} = \nu_{Lz} = 0.3, \quad \nu_{Tz} = 0.49, \end{aligned} \tag{28a}$$

are also given in Tables 1 and 2 for comparison. The differences between the two cases are generally very small.

where the subscripts L, T and z denote the fiber, transverse and thickness directions of the composite lamina, respectively. The first three eigenvalues $\delta_i (i = 1,2,3)$ which satisfy the aforementioned constraint condition Eq. 21 are given in Table 1 to illustrate the exact strength of the stress singularity associated with the delamination. To demonstrate further the general characteristics of the stress singularities for delamination, results for an interlaminar crack between 30°/θ graphite-epoxy composites with the same ply properties are also shown (Table 2) for various fiber orientations θ's. From Tables 1 and 2, we observe that a fully opened delamination between dissimilar highly anisotropic laminae always possesses three distinct stress singularities, i.e., a pair of complex conjugates, $\delta_{1,2} = -\frac{1}{2} \pm i\gamma$, and a real constant, $\delta_3 = -0.5$. This situation is unique and apparently different from the cases of an interface crack between two dissimilar isotropic media or orthotropic solids in that the three distinct dominant stress singularities, δ_1, δ_2 and δ_3, always exist simultaneously in the present fiber-composite delamination problem. In the special cases when a delamination is located in the 90°/90° or 30°/30° composite system the classical inverse square-root singularity for crack-tip stresses is fully recovered as shown in the Tables, because the composite laminate becomes unidirectional. We note here that the imaginary part of δ_1 and δ_2, i.e., the value of γ, is generally very small as compared with the real part of δ_i in all cases studied.

4.2 Delamination with Crack-Tip Closure

From Eq. 27 and from the results shown in Tables 1 and 2, it is clearly seen that the asymptotic delamination stress field in dissimilar anisotropic composites possesses the well-known oscillatory singularities. The associated displacement field also exhibits oscillatory characteristics, leading to controversial crack-surface interpenetration or overlapping, which is

physically inadmissible. Similar results have also been noted by several investigators in studying an interface crack between dissimilar isotropic materials. In recent studies, Wang and Choi [10,16] have shown that for a delamination between dissimilar, strongly anisotropic fiber composites with certain combinations of laminar elastic properties, ply orientations, and loading conditions, global crack surface closure may occur. Under these circumstances, interlaminar crack-surface contact or closure needs to be considered.

Consider the case that a delamination a is located between the mth and $(m+1)$th laminae and a portion of the crack surface, c, is closed as shown in Fig. 3. Frictional coefficients associated with $\tau_{\phi z}$ and $\tau_{r\phi}$ on the interface $\phi = 0$ are denoted by $f_{\phi z}$ and $f_{r\phi}$, respectively. An exact analytical complete elasticity solution for the delamination problem with crack closure is generally difficult to obtain because the unknown contact stress distributions along the crack-closure region need to be determined as a part of the final solution [10,14]. However, the local stress singularities, asymptotic field solutions, and associated characteristics can still be determined exactly by using the same procedure discussed in Section 4.1 but with some modifications. Referring to Fig. 3 for a partially closed delamination, instead of using Eqs. 25(a) and 25(b) we can introduce the local boundary conditions in the crack-closure region ($-c < r < 0$) as follows:

$$\sigma_{\phi\phi}^{(m)}(r,\pi) - \sigma_{\phi\phi}^{(m+1)}(r,-\pi) = 0, \quad \tau_{\phi z}^{(m)}(r,\pi) - \tau_{\phi z}^{(m+1)}(r,-\pi) = 0,$$

$$\tau_{r\phi}^{(m)}(r,\pi) - \tau_{r\phi}^{(m+1)}(r,-\pi) = 0, \quad u_{\phi}^{(m)}(r,\pi) - u_{\phi}^{(m+1)}(r,-\pi) = 0,$$

$$\tau_{\phi z}^{(m)}(r,\pi) = -f_{\phi z}\ \sigma_{\phi\phi}^{(m)}(r,\pi), \qquad (29)$$

$$(\alpha = m,\ m+1).$$

$$\tau_{r\phi}^{(m)}(r,\pi) = -f_{r\phi}\ \sigma_{\phi\phi}^{(m)}(r,\pi),$$

Along the ply interface $\phi = 0$, the same continuity conditions Eqs. 25(c) and 25(d) for $\sigma_i^{(\alpha)}$ and $u_j^{(\alpha)}$ ($\alpha = m,\ m+1$) are applicable.

Using Eqs. 25 and 29 and following the same procedure given in Section 4.1, we can immediately determine the eigenvalues δ_n for a composite delamination with crack-tip closure. The numerical example of a delamination located along the interface of θ/−θ graphite-epoxy composites is studied here first. Stress singularities associated with the partially closed delamination crack tip with different values of frictional coefficients $f_{r\phi}$ and $f_{\phi z}$ are shown in Table 3. The crack-tip stress singularity is found to be always −0.5 with an algebraic multiplicity of 2 (i.e., double roots $\delta_1 = \delta_2 = -0.5$) for the delamination having crack surfaces in frictionless contact (i.e., $f_{r\phi} = f_{\phi z} = 0$). In fact, the inverse square-root stress singularity, $\delta_1 = \delta_2 = -0.5$, is found for all delaminated θ/−θ fiber composites with frictionless crack-surface contact. In the cases of $f_{r\phi} \neq 0$ and/or $f_{\phi z} \neq 0$, stress singularities always possess an invariant constant $\delta_1 = -0.5$ (single root) as in the aforementioned frictionless contact case, and a δ_2 (with $\delta_2 \neq -0.5$, single root), which depends on values of the frictional coefficients (Tables 3 and 4). In Table 3, values of δ_2 for all delaminated θ/−θ fiber composites studied are observed to be slightly larger than −0.5, when $f_{\phi z} > 0$ is considered. That is, frictional contributions lead to a weaker delamination stress singularity δ_2 than that in a frictionless contact case and in a conventional homogeneous open crack case. We note here that for a delamination between θ/−θ graphite-epoxy composites, the stress singularities

are always independent of the value of the frictional coefficient $f_{r\phi}$, because of the symmetry (and antisymmetry) of components in elastic stiffness matrices of the θ and $-\theta$ plies and the decoupling of τ_{xy} from σ_y and τ_{yz} in the formulation. This phenomenon is clearly seen in Table 3, where δ_1 is always -0.5 and δ_2 differs from -0.5 gradually as the value of $f_{\phi z}$ increases. Note further that deviations of δ_2 in the frictional contact cases from the conventional square-root singularity are rather small for all $\theta/-\theta$ graphite-epoxy delamination problems.

Stress singularities are also determined for delaminations in more general cases of θ_1/θ_2 fiber composites with $\theta_1 \neq \theta_2$. For illustration, the results of a delamination between 30°/90° graphite-epoxy composites are presented in Table 4 for various values of $f_{r\phi}$ and $f_{\phi z}$. It is seen from the Table that the double roots $\delta_1 = \delta_2 = -0.5$ also appear for a delamination in θ_1/θ_2 composites with crack surfaces in frictionless contact ($f_{r\phi} = f_{\phi z} = 0$). However, in the cases of a delamination with crack surfaces in frictional contact, δ_2 is apparently influenced by the values of both $f_{r\phi}$ and $f_{\phi z}$. In Table 4, values of δ_2 for the partially closed delamination in 30°/90° graphite-epoxy composites with different $f_{r\phi}$ and $f_{\phi z}$ are observed to be smaller than the classical square-root stress singularity. Thus, δ_2 can be either greater or smaller than the conventional inverse square-root singularity, depending upon the values of $f_{r\phi}$ and $f_{\phi z}$ and fiber orientations of the composites. Owing to the complex algebraic structure of the transcendental characteristic equation, Eq. 20, it is generally not possible to predict in explicit form whether $\delta_2 > -0.5$ or $\delta_2 < -0.5$ for a delamination with crack surfaces in frictional contact without solving the transcendental equation numerically. We remark that in the case of a delamination with crack surfaces in frictionless contact between dissimilar anisotropic media, the

dominant stress singularity, $\delta = -0.5$, has also been determined independently by using a singular integral equation approach in [10,16]. Furthermore, a similar phenomenon of stress singularity $\delta > -0.5$ or $\delta < -0.5$ has been observed in studying the interface crack between dissimilar isotropic media with crack surfaces in frictional contact [28].

4.3 Delamination with a Very Small Area of Crack-Tip Closure

The delamination with open crack surfaces between dissimilar fiber composites has been shown mathematically in Section 4.1 to possess controversial oscillatory crack-tip stress and displacement fields. This abnormality is thought to be artifacts resulting from the method of approach by using eigenfunction expansion in the formulation and solution. As first pointed out by England [7], Malyshev et al. [6], and later by Wang and Choi [10], the region of oscillatory solutions for a delamination with open crack surfaces in a nominal tensile field is generally extremely small in comparison with the size of the interface crack and this very localized abnormality may not be significant in practical terms of linear fracture mechanics. In fact, using the partially closed crack model, Wang and Choi [10] have shown that a composite delamination in a tensile field has an extremely small crack-tip closure with $c/a \sim 0(10^{-6})$. A simplified model which disregards the small closure (or oscillatory) region and approximates the asymptotic field by an inverse square-root stress singularity is, therefore, proposed for this situation and shown to provide excellent results comparing with those determined by using a partially closed crack model [10].

Under certain loading conditions other than pure tension, however, an interlaminar crack may also possess a very small area of crack-tip closure, depending upon loading modes and material elastic properties of the dissimilar composite laminae [16]. A simplified solution for this case can be obtained

by taking the limit of the crack-closure length, i.e., $c \to 0$, in the results derived from a partially closed delamination. In particular, the delamination stress singularity can be taken directly from the partially closed crack model (with frictionless crack surface contact) as an inverse square-root one. Mathematically, this is equivalent to finding an analytical solution for a fully open crack by following the same formulation and procedure for a partially closed crack case with infinitesimal closure length, and the approximation introduced has the effect of smoothing the oscillatory singularity to an inverse square-root singularity for the composite delamination. Therefore, interlaminar stress intensity factors and strain energy release rates can be defined in a manner consistent with those for a homogeneous crack and for the refined model of an interface crack between dissimilar isotropic solids introduced by Comninou [14]. As will be shown later [18], this simplification leads to a very effective and efficient approach to the complex problem of delamination with a fully open crack tip or with a very small area of crack-tip closure, and provides meaningful information on the fundamental mechanics of delamination problems in composite laminates under general loading conditions.

4.4 Logarithmic Stress Singularities

As mentioned in Section 3 that besides the power-type stress singularities given in Sections 4.1, 4.2 and 4.3 for various conditions in the crack-tip region, weak logarithmic-type singularities may also appear in the homogeneous and particular solutions for the delamination stress field. Since the particular solution is related to the remote loading applied to the delaminated composite, it has to be considered and constructed for each individual case. To study the possible presence of the logarithmic stress singularities in a delamination mechanics problem, we consider a symmetric

composite laminate subjected to uniform in-plane stretching with $\varepsilon_z = \varepsilon_o$ for simplicity and without loss of generality. Also, we restrict our attention at this point to delaminations located in the following three composite systems: $\theta/-\theta$, $\theta/0°$, and $\theta/90°$ graphite-epoxy composites. Based on the preceding theoretical developments and the conditions for the presence of the logarithmic terms given in [25], we address each individual case separately.

(1) Delamination between $\theta/-\theta$ composites ($\theta \neq 0°$ and $90°$)

Numerical calculations by using the ply elastic constants given in Eq. 28 provide the following:

$$N = 12, \quad R = 9, \quad M = 3 \qquad (\delta_n = \text{integer}), \tag{30a}$$

$$N = 12, \quad R = 11, \quad M = 1 \qquad (\delta_n \neq \text{integer}). \tag{30b}$$

Applying both Eq. 30(a) and Eq. 30(b) to the condition $\ell_m = M-(N-R)$, we obtain $\ell_m = 0$. Thus, logarithmic terms of the form $Z_k^{\delta_n}(\ln Z_k)^{\ell_m}$ with $\ell_m > 1$ do not appear in the homogeneous solution for this class of problems.

Also, carrying out the computations of constructing the left eigenvector $\underset{\sim}{C}_h^{*(L)}$ and the loading vector $\underset{\sim}{p}^*$, we find that, for all three sets of $\underset{\sim}{C}_h^{*(L)}$ at $\delta_n = 0$ in this case, Equation 24 is satisfied identically. Therefore, logarithmic terms of the form $(\ln Z_k)^{(\ell_o+1)}$ do not occur in the particular solution either.

(2) Delamination between $\theta/0°$ or $\theta/90°$ composites ($\theta \neq 0°$ and $90°$)

In these two cases, following the same procedure and computations as discussed in (1) but with minor modifications, we obtain similar results as those in the $\theta/-\theta$ case, i.e.,

$$N = 12, \quad R = 9, \quad \text{and} \quad M = 3, \qquad (\delta_n = \text{integer}) \tag{31a}$$

$$N = 12, \quad R = 11, \quad \text{and} \quad M = 1, \qquad (\delta_n \neq \text{integer}) \tag{31b}$$

and, also, $\ell_m = 0$. Moreover, Eq. 24 also holds for this problem. Thus, we conclude that no logarithmic singularities of any kind would appear in the asymptotic solutions for delamination in $\theta/90°$ and $\theta/0°$ composites; only power-type singularities $z_k^{\delta_n}$ occur in these problems.

We further remark that, in fact, it has been shown in [25] that no logarithmic terms of any kind would occur in the solutions for a general case of a delamination located between θ_1 and θ_2 fiber composites with θ_1 and θ_2 being any arbitrary fiber orientations.

5. SUMMARY AND CONCLUSIONS

The mechanics of delamination in fiber composite laminates has been studied. Formulation of the problem is based on Lekhnitskii's complex-variable stress potentials and basic relationships in laminate elasticity theory for anisotropic fiber composites. The eigenfunction expansion method used in this study appears to be a suitable approach to determine delamination stress singularities and fundamental structures of stress and deformation field solutions. Stress singularities for a delamination are found to be related to adjacent ply material properties and local traction boundary conditions. Numerical results for interlaminar cracks in commonly used graphite-epoxy composites with different fiber orientations and crack-tip conditions are shown to illustrate the basic nature of stress singularities and general solutions for the composite delamination problem. Based on the information obtained, the following conclusions may be reached:

(1) Assuming the delamination is fully open and free from surface traction, we find that delamination stress singularities always possess an oscillatory form by simultaneous presence of three distinct eigenvalues, $-1/2+i\gamma$, $-1/2-i\gamma$, and $-1/2$. The oscillatory stress singularities and field solutions for composite delamination are physically inadmissible because of interpenetration of crack-surface displacements.

(2) For a delamination with partially closed crack surfaces in frictionless contact, the present eigenfunction expansion approach always gives an eigenvalue $\delta = -1/2$ with an algebraic multiplicity of two (i.e., double roots), indicating the classical square-root stress singularity is recovered in the closed crack case.

(3) In the case of a delamination with crack surfaces in frictional contact, crack-tip stress singularities depend not only on material elastic constants and fiber orientations of adjacent plies but also on frictional coefficients $f_{r\phi}$ and $f_{\phi z}$ along the delamination surface.

(4) The crack-surface friction may lead to either a stronger or weaker stress singularity than the conventional inverse square-root one, depending upon fiber orientations of the adjacent plies. Present numerical results, for example, show that a weaker stress singularity, i.e., $0 > \delta > -1/2$, occurs for a delamination between any θ and $-\theta$ fiber composites, but a stronger singularity, i.e., $-1/2 > \delta > -1$, occurs for a delamination between 30° and 90° composites, if $f_{r\phi} > 0$ and $f_{\phi z} > 0$.

(5) In the situation that the delamination contains a very small area of crack-surface closure (e.g., $c < 10^{-6}a$), a simplified model with the crack-tip stress field having an inverse square-root stress singularity, as determined by finding the solution from the limiting case of a partially closed crack solution, is suggested and later used for solving the complete boundary value problem.

(6) Examining the multiplicity of eigenvalues and the rank and order of the coefficient matrix in the eigenfunction solution, we find that no logarithmic stress singularities of any kind would appear in the homogeneous and particular solutions for the composite delamination problem; only power-type singularities of the form $z_k^{\delta_n}$ could occur.

(7) After determining of all the eigenvalues for each individual delamination case, general solution structures for composite deformation and stress fields can be established immediately. Numerical methods such as the singular finite-element technique, which can incorporate exact delamination stress singularities in the element formulation, can be easily developed to

solve the complete boundary-value problem for delaminations in composite laminates with any arbitrary combinations of lamination, geometric, and crack variables. One of such methods employing displacement-based singular crack-tip elements is given in the associated paper [18].

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TABLE 1

DOMINANT STRESS SINGULARITIES* FOR DELAMINATION BETWEEN θ/90° GRAPHITE-EPOXY COMPOSITES

θ	δ_1	δ_2	δ_3
0°†	-0.5 + 0.051101i (-0.5 + 0.032924i)	-0.5 - 0.051101i (-0.5 - 0.032924i)	-0.5 (-0.5)
15°	-0.5 + 0.050349i (-0.5 + 0.032484i)	-0.5 - 0.050349i (-0.5 - 0.032484i)	-0.5 (-0.5)
30°	-0.5 + 0.045138i (-0.5 + 0.025764i)	-0.5 - 0.045138i (-0.5 - 0.025764i)	-0.5 (-0.5)
45°	-0.5 + 0.034504i (-0.5 + 0.015604i)	-0.5 - 0.034504i (-0.5 - 0.015604i)	-0.5 (-0.5)
60°	-0.5 + 0.021119i (-0.5 + 0.008067i)	-0.5 - 0.021119i (-0.5 - 0.008067i)	-0.5 (-0.5)
75°	-0.5 + 0.008899i (-0.5 + 0.004265i)	-0.5 - 0.008899i (-0.5 - 0.004265i)	-0.5 (-0.5)
90°†	-0.5 (-0.5)	-0.5 (-0.5)	-0.5 (-0.5)

*Values in the parentheses are for T300/5280 graphite-epoxy with laminar elastic constants given in Eq. 28(a).

†These eigenvalues are for 0°/90° and 90°/90° composites in a general loading condition. In the cases of 0°/90° and 90°/90° composites under uniform stretching $\varepsilon_z = \varepsilon_0$, $\delta_3 = -0.5$ does not appear because of $\tau_{xz} = \tau_{yz} = 0$ and being decoupled from other stress components.

TABLE 2

DOMINANT STRESS SINGULARITIES* FOR DELAMINATION BETWEEN 30°/θ GRAPHITE-EPOXY COMPOSITES

θ	δ_1	δ_2	δ_3
0°	-0.5 + 0.012451i (-0.5 + 0.009456i)	-0.5 - 0.012451i (-0.5 - 0.009456i)	-0.5 (-0.5)
15°	-0.5 + 0.010491i (-0.5 + 0.007920i)	-0.5 - 0.010491i (-0.5 - 0.007920i)	-0.5 (-0.5)
30°	-0.5 (-0.5)	-0.5 (-0.5)	-0.5 (-0.5)
45°	-0.5 + 0.015968i (-0.5 + 0.011608i)	-0.5 - 0.015968i (-0.5 - 0.011608i)	-0.5 (-0.5)
60°	-0.5 + 0.030943i (-0.5 + 0.020441i)	-0.5 - 0.030943i (-0.5 - 0.020441i)	-0.5 (-0.5)
75°	-0.5 + 0.041030i (-0.5 + 0.024664i)	-0.5 - 0.041030i (-0.5 - 0.024664i)	-0.5 (-0.5)
90°	-0.5 + 0.045138i (-0.5 + 0.024764i)	-0.5 - 0.045138i (-0.5 - 0.025764i)	-0.5 (-0.5)

*Values in the parentheses are for T300/5280 graphite-epoxy with laminar elastic properties given in Eq. 28(a).

TABLE 3

DOMINANT STRESS SINGULARITIES* FOR DELAMINATION
WITH CRACK-TIP CLOSURE IN θ/-θ GRAPHITE-EPOXY COMPOSITES

$f_{r\theta}$ \ θ	15°	30°	45°	60°	75°
0.0	-0.5 -0.5	-0.5 -0.5	-0.5 -0.5	-0.5 -0.5	-0.5 -0.5
0.2	-0.5 -0.4994	-0.5 -0.4962	-0.5 -0.4940	-0.5 -0.4941	-0.5 -0.4964
0.4	-0.5 -0.4977	-0.5 -0.4923	-0.5 -0.4881	-0.5 -0.4881	-0.5 -0.4928
0.6	-0.5 -0.4965	-0.5 -0.4884	-0.5 -0.4821	-0.5 -0.4822	-0.5 -0.4892
0.8	-0.5 -0.4954	-0.5 -0.4846	-0.5 -0.4762	-0.5 -0.4763	-0.5 -0.4855

*δ_1 and δ_2 are found to be independent of the value of $f_{r\theta}$.

TABLE 4

DOMINANT STRESS SINGULARITIES FOR DELAMINATION
WITH CRACK-TIP CLOSURE IN 30°/90° GRAPHITE/EPOXY COMPOSITES

λ_{rz} \ $\lambda_{r\theta}$	0.0	0.1	0.3	0.5	0.7
0.0	-0.5 -0.5	-0.5 -0.5010	-0.5 -0.5031	-0.5 -0.5051	-0.5 -0.5072
0.1	-0.5 -0.5036	-0.5 -0.5046	-0.5 -0.5067	-0.5 -0.5087	-0.5 -0.5108
0.3	-0.5 -0.5108	-0.5 -0.5118	-0.5 -0.5138	-0.5 -0.5159	-0.5 -0.5179
0.5	-0.5 -0.5179	-0.5 -0.5189	-0.5 -0.5210	-0.5 -0.5230	-0.5 -0.5251
0.7	-0.5 -0.5251	-0.5 -0.5261	-0.5 -0.5281	-0.5 -0.5302	-0.5 -0.5322

7. LIST OF FIGURE CAPTIONS

Fig. 1 Delaminations in a $[\theta_1/\theta_2/\theta_3/.../\theta_3/\theta_2/\theta_1]$ Fiber-Reinforced Composite Laminate.

Fig. 2 Coordinates and Geometry of a Delamination with Open Crack Surfaces between θ_m and θ_{m+1} Plies.

Fig. 3 Coordinates and Geometry of a Delamination with Finite Length of Crack-Surface Closure between θ_m and θ_{m+1} Plies.

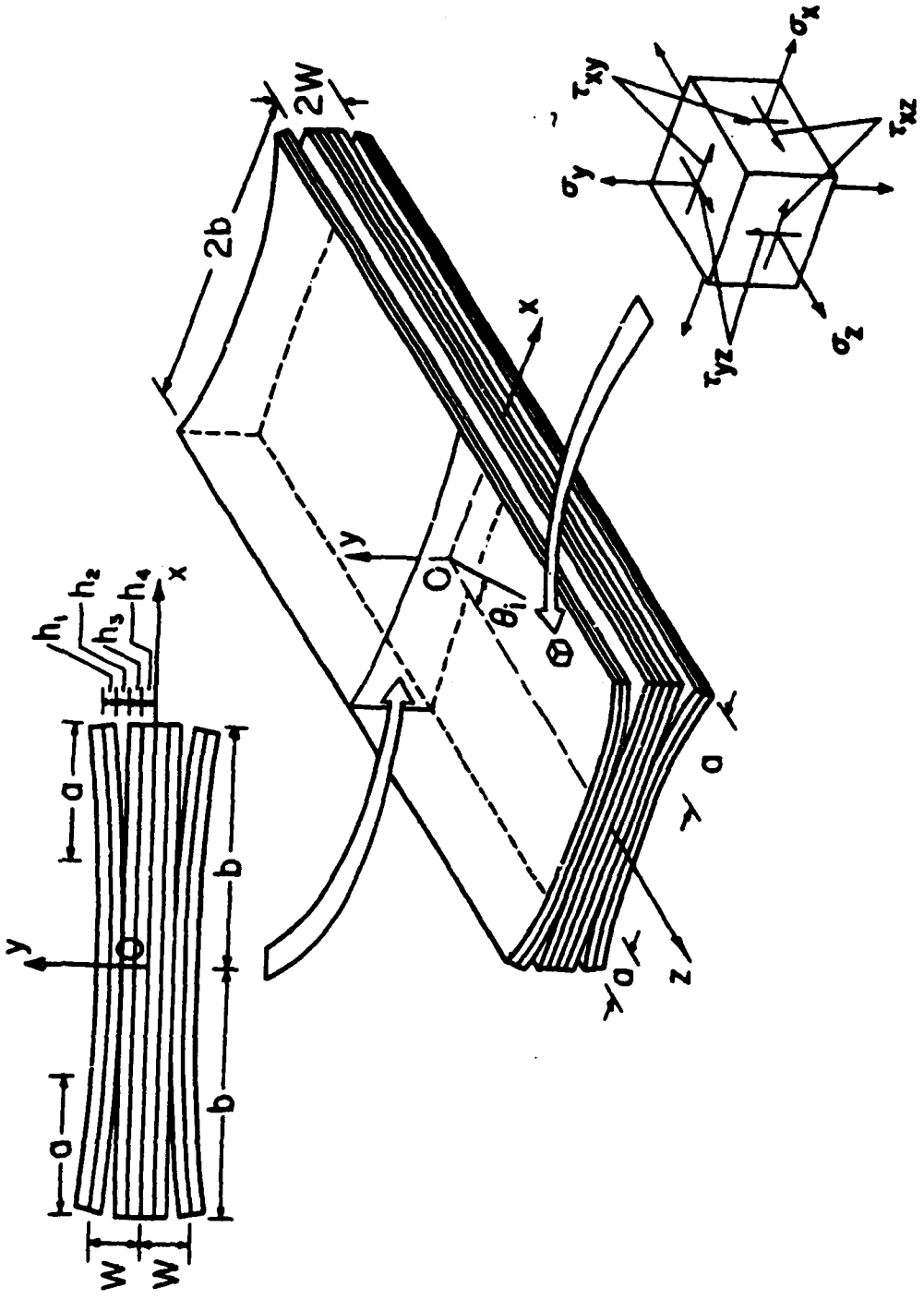

Fig. 1 Delaminations in a $[\theta_1/\theta_2/\theta_3/\ldots/\theta_3/\theta_2/\theta_1]$ Fiber-Reinforced Composite Laminate.

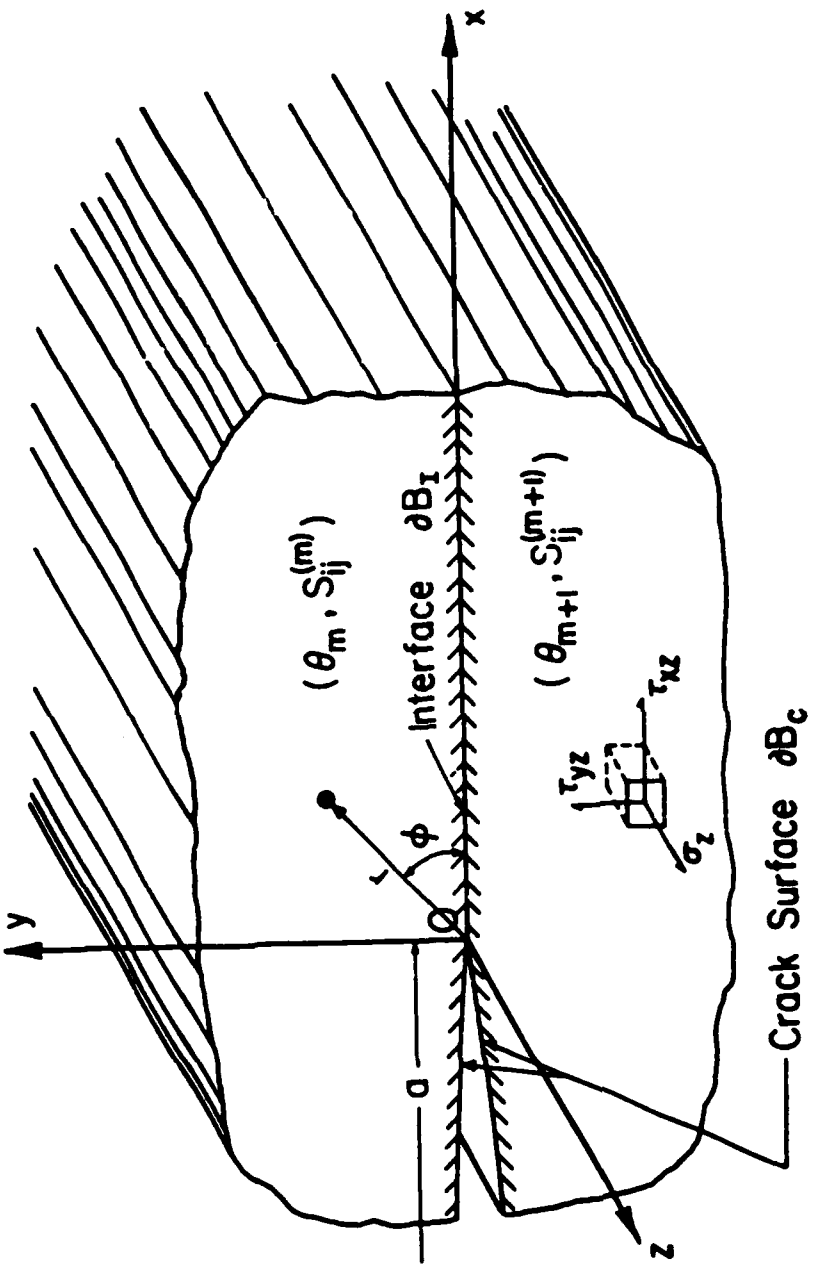

Fig. 2 Coordinates and Geometry of a Delamination with Open Crack Surfaces between θ_m and θ_{m+1} Plies.

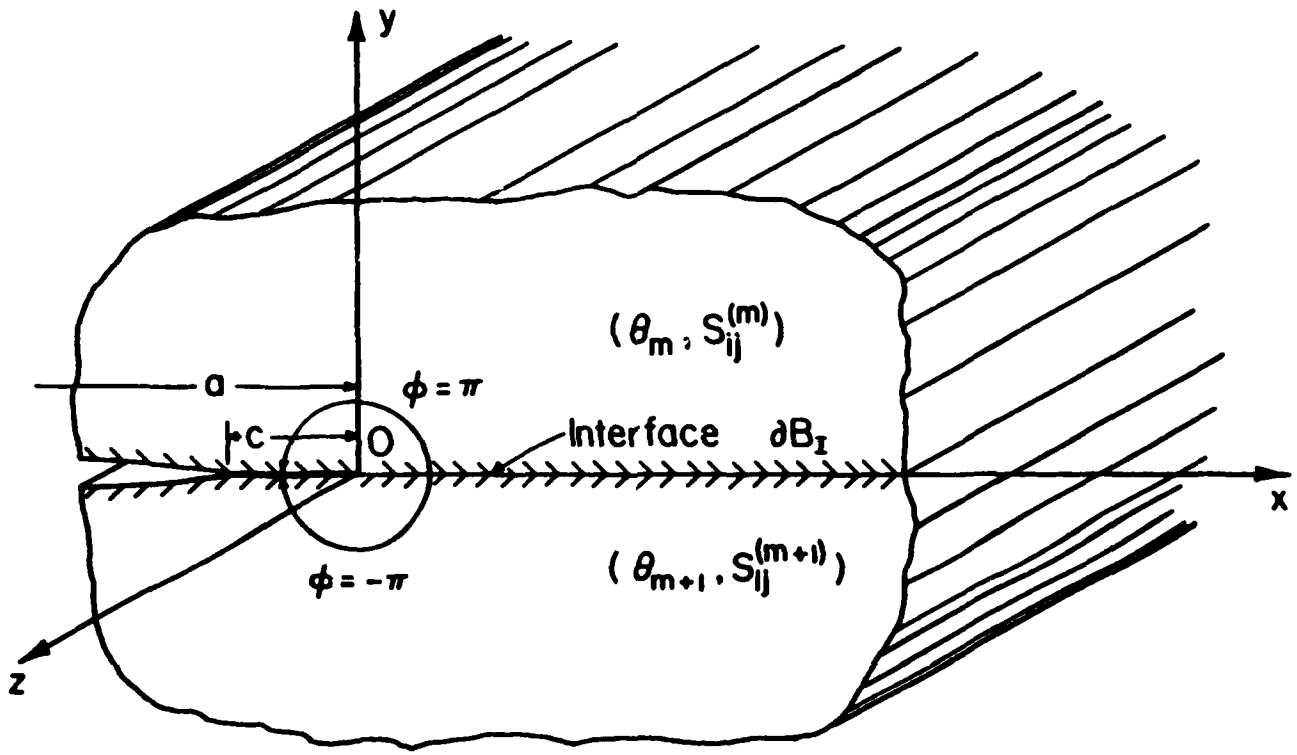

Fig. 3 Coordinates and Geometry of a Delamination with Finite Length of Crack-Surface Closure between θ_m and θ_{m+1} Plies.

APPENDIX 1

Expressions for coefficients $H_{1k}(\phi)$ in Eqs. 15 and 16 are as follows:

$$H_{1k} = (\mu_k \sin\phi + \cos\phi)^2, \qquad H_{2k} = -\eta_k(\mu_k \sin\phi + \cos\phi),$$

$$H_{3k} = -(\mu_k \sin\phi + \cos\phi)(\mu_k \cos\phi - \sin\phi), \qquad H_{4k} = (\mu_k \cos\phi - \sin\phi)^2, \quad \text{(A-1)}$$

$$H_{5k} = \eta_k(\mu_k \cos\phi - \sin\phi), \qquad H_{6k} = p_k \cos\phi + q_k \sin\phi,$$

$$H_{7k} = -p_k \sin\phi + q_k \cos\phi, \qquad H_{8k} = t_k,$$

where μ_k and η_k are defined in Eqs. 11, 12 and 13, and p_k, q_k and t_k are complex constants related to laminar elastic constants $\hat{S}_{ij}$ by

$$p_k = \hat{S}_{11}\mu_k^2 + \hat{S}_{12} - \hat{S}_{14}\eta_k + \hat{S}_{15}\eta_k\mu_k - \hat{S}_{16}\mu_k,$$

$$q_k = \hat{S}_{12}\mu_k + \hat{S}_{22}/\mu_k - \hat{S}_{24}\eta_k/\mu_k + \hat{S}_{25}\eta_k - \hat{S}_{26}, \quad \text{(A-2)}$$

$$t_k = \hat{S}_{14}\mu_k + \hat{S}_{24}/\mu_k - \hat{S}_{44}\eta_k/\mu_k + \hat{S}_{45}\eta_k - \hat{S}_{46}.$$