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AERONAUTICAL RESEARCH COMMITTEE
REPORTS AND MEMORANDA

Round Jets in a General Stream

By

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LONDON : HIS MAJESTY'S STATIONERY OFFICE
[redacted]

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COMMUNICATED BY THE DIRECTOR OF SCIENTIFIC RESEARCH
MINISTRY OF AIRCRAFT PRODUCTION

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Summary.—The flow in a round jet issuing from an orifice in the same direction as a general external stream, is investigated theoretically as an extension of the problem of a jet issuing into still air. The flow in the upstream part of the jet (Region *A* of Fig. 1), in which a core of fluid of uniform velocity exists, and the flow in the downstream or developed part of the jet (Region *B* of Fig. 1), are investigated separately; the solutions fitted together at the section at which the core disappears. The deviation of the outside stream due to inflow into the jet is also considered. Numerical solutions are derived for several values of the ratio of jet exit velocity to stream velocity.

CONTENTS

1. *Introduction.*—The problem of the flow in a round jet, when the air surrounding the jet is at rest, has been fully investigated[1,2,3]. The flow in and near a jet with an external stream parallel to the jet axis is considered in the present report, with special reference to the inflow induced outside the jet. The fluid is assumed to be incompressible but the application to jets of heated or compressible fluids is briefly considered in para. 7.

The jet is assumed to have a uniform velocity over its cross-section at the nozzle exit. Mixing with the outside stream occurs at the edge of the jet, and the diameter of the central core of fluid of uniform velocity decreases with distance downstream, the core eventually disappearing altogether. Downstream of the apex of the core the velocity on the jet axis begins to fall and finally a quasi-steady state is reached for which the velocity profiles at all sections are similar.

* R.A.E. Report No. Aero. 1904, January, 1944.

The presence of the external stream makes the problem more complicated than for a jet with the surrounding air at rest, and the method of solution has been simplified by specifying a shape for the velocity distribution across the jet at all normal sections. For the first part of the jet, in which a core of undisturbed fluid exists, the radius of the core and the external radius of the jet are determined from the equations of motion, and after the core has disappeared the velocity on the axis and the jet radius are determined in the same way. The two solutions are then joined by making the velocity and radius continuous at the section containing the apex of the core, and in this way a complete picture of the development of the jet is obtained.

The flow in the mixing region is assumed to be turbulent and the shear stress is determined by application of the momentum transfer theory. Reichardt's criticisms[3] of this theory do not apply to the present application, as it is not used to determine the shape of the velocity distribution curve; it would be possible to present the theory in Reichardt's form, but the momentum transfer theory has been retained on account of its greater familiarity.

2. *Notation.*—The following notation, illustrated in Fig. 1, will be used:—

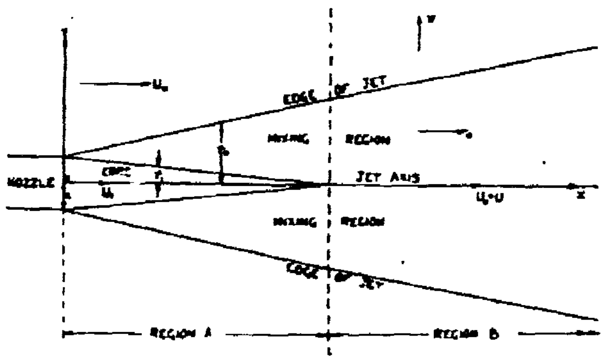

FIG. 1—Diagram illustrating Notation.

x	Distance from jet exit along axis of jet.
r	Distance from jet axis.
a	Radius of jet at exit.
r_0	Radius of outer boundary of jet.
r_1	Radius of inner boundary of mixing region, i.e. radius of core.
U_0	Stream velocity.
U_1	Jet exit velocity.
λ	U_0/U_1
U	Excess velocity on jet axis over stream velocity (after disappearance of core).
u	Velocity parallel to jet axis at point (x, r).
v	Velocity normal to jet axis at point (x, r).
ψ	Stokes's stream function for flow outside the jet.
M	Jet momentum/2π.
ρ	Stream density.
τ	Shear stress.
l	Mixing length.
c	Mixing length parameter.
θ_A θ_B R_A R_B	Defined in Fig. 12 (Appendix III).

* This requires that the radial inflow velocity into the mixing region from the core shall vanish and is probably not exactly true.

3. *Flow in Region Near Jet Exit.*—In this paragraph the flow in the part of the jet which contains a core of fluid of uniform velocity (Region A of Fig. 1) will be considered. At the jet exit, which is a circle of radius a, the jet has a uniform axial velocity U_1. It will be assumed that the pressure variations in and near the jet can be neglected and that the velocity throughout the central core is constant*.

The momentum equation for the flow across a circle of radius r, whose plane is normal to the jet axis and at a distance x from the exit, has the form (Ref. 4, p. 133) :—

$$\frac{\partial}{\partial x}\int_0^r \rho r u^2\, dr - u\frac{\partial}{\partial x}\int_0^r \rho u r\, dr = -\tau r, \qquad (1)$$

where u is the mean axial velocity at the point (x, r) and τ is the shear stress in the axial direction at (x, r) due to turbulent mixing.

The assumed axial velocity distribution in the mixing region is given by the equation

$$u = U_0 + \frac{(U_1 - U_0)}{2}\left[1 - \cos\pi\left(\frac{r_0 - r}{r_0 - r_1}\right)\right], \qquad (2)$$

and is shown in Fig. 2, where U_0 is the external stream velocity and r_0, r_1, are respectively the external and internal radii of the mixing region at the station x.

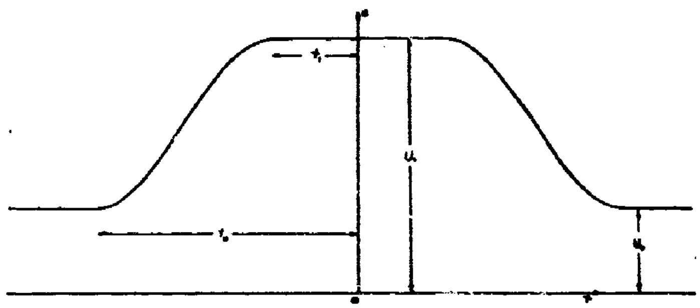

FIG. 2—Velocity Distribution across Jet in Region A. (Eqn. 2).

The usual formula for the shear stress, on the momentum transfer theory of turbulent flow (Ref. 4, p. 207), is

$$\tau = \rho l^2\left(\frac{\partial u}{\partial r}\right)^2, \qquad (3)$$

and this formula will be adopted. Here l stands for the mixing length, which will be assumed to be proportional to the width of the mixing region : we therefore put

$$l = c\,(r_0 - r_1), \qquad (4)$$

where c is a constant, characteristic of turbulent motion for the problem under consideration.

Hence
$$\tau = \rho c^2\,(r_0 - r_1)^2\left(\frac{\partial u}{\partial r}\right)^2 \qquad (5)$$

The quantities r_0 and r_1 will be determined by satisfying equation (1), with u and τ given by (2) and (5), at two positions. The first position is at the outer boundary of the jet $r = r_0$, where $\tau = 0$ and $u = U_0$, so that equation (1) becomes

$$\frac{\partial}{\partial x}\int_0^{r_0} \rho r\, u\,(u - U_0)\, dr = 0.$$

This equation is integrated and the constant of integration determined from the conditions at the jet exit. We obtain

$$\int_0^{r_0} \rho r u (u - U_0)\, dr = M = \frac{a^2}{2}\rho U_1 (U_1 - U_0), \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where a is the radius of the jet at the exit. The constant M is proportional to the jet thrust, which is equal to $2\pi M$ when the fluid is taken in from the stream; but if the fluid is supplied by other means the jet thrust is increased by the source thrust of the added fluid.

The second position at which equation (1) will be satisfied is half-way across the mixing region, where $r = \frac{r_0 + r_1}{2}$. At this position, from equation (2),

$$u = \frac{U_0 + U_1}{2}, \quad \frac{\partial u}{\partial r} = -\frac{\pi}{2}\left(\frac{U_1 - U_0}{r_0 - r_1}\right),$$

so that equation (5) gives

$$\frac{\tau}{\rho} = \frac{\pi^2}{4} c^2 (U_1 - U_0)^2.$$

With these expressions equation (1) becomes

$$\frac{\partial}{\partial x}\int^{(r_0+r_1)/2} \rho r u\left(u - \frac{U_0 + U_1}{2}\right) dr = -\frac{\pi^2}{8}\rho c^2 (U_1 - U_0)^2 (r_0 + r_1) \quad (7)$$

The next step is to substitute the form (2) for the velocity u into the momentum equations (6) and (7). The integrals occurring can be evaluated without difficulty and (6) and (7) reduce to

$$a_{11} r_1^2 + 2a_{10} r_1 r_0 + a_{00} r_0^2 = a^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$\frac{d}{dx}\left[A_{11} r_1^2 + 2A_{10} r_1 r_0 + A_{00} r_0^2\right] = B(r_1 + r_0), \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

where
$$a_{11} = 2\left(\frac{5-\lambda}{16} - \frac{1}{\pi^2}\right), \quad a_{10} = \frac{2}{\pi^2}, \quad a_{00} = 2\left(\frac{3+\lambda}{16} - \frac{1}{\pi^2}\right),$$

$$A_{11} = \frac{13 + 3\lambda}{16} - \frac{1+\lambda}{2\pi} - \frac{5 + 3\lambda}{4\pi^2}, \quad A_{10} = \frac{1-\lambda}{16} + \frac{5 + 3\lambda}{4\pi^2},$$

$$A_{00} = \frac{1-\lambda}{16} + \frac{1+\lambda}{2\pi} - \frac{5 + 3\lambda}{4\pi^2}, \quad B = -\frac{\pi^2}{2}(1-\lambda)c^2,$$

and λ stands for the velocity ratio U_0/U_1.

The development of the jet up to the disappearance of the core is determined by the equations (8) and (9) together with the conditions $r_0 = r_1 = a$ at the jet exit $x = 0$. Solutions have been derived by the method given in Appendix I for values of λ, the ratio of the stream velocity to the jet exit velocity, of 0, 0·125, 0·25, 0·5 and 0·75. The results are given in Tables 1–5 and in Fig. 3, which also shows the solution derived by Kuethe[3] for $\lambda = 0$ by a more elaborate method; the agreement with Kuethe's solution is fairly good and this gives confidence that the approximations introduced are reasonable.

4. *Flow in the Developed Jet.*—After the central core of uniform velocity has disappeared due to the inward spread of the turbulent mixing region, the velocity on the axis starts to fall and further downstream the velocity distribution across a section of the jet settles down to a steady shape. No attempt will be made to allow for the variation of the shape of the velocity distribution.

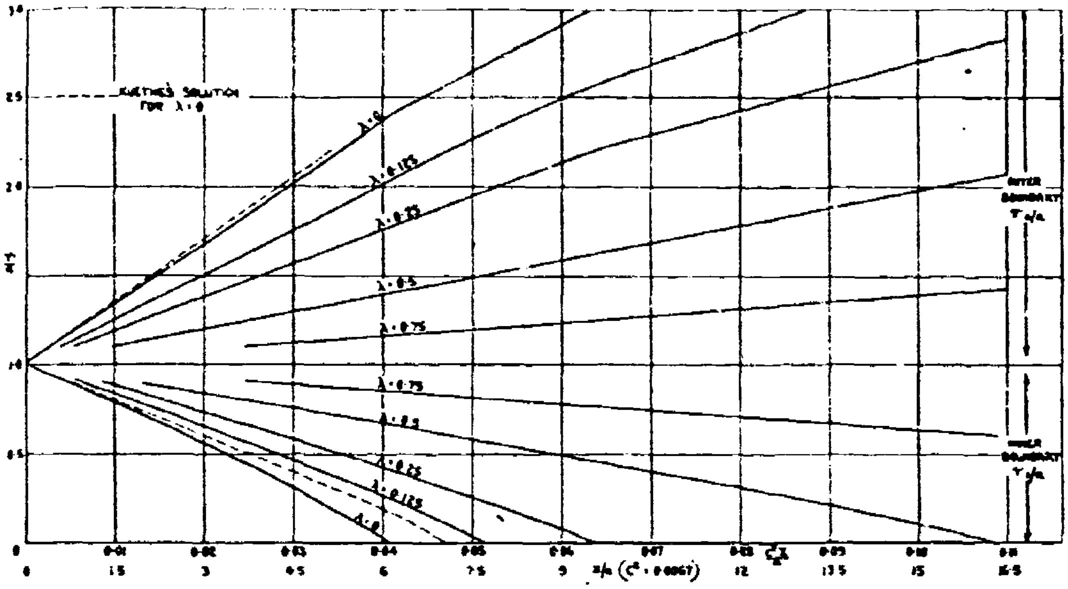

FIG. 3—Jet Boundaries.

It will be assumed that the axial velocity at all sections of the developed jet is given by the formula

$$u = U_0 + \frac{U}{2}\left(1 + \cos \pi \frac{r}{r_0}\right), \qquad \text{(10)}$$

where r_0 is the jet radius, and the velocity on the axis is $U_0 + U$, which falls with increase of distance downstream. The velocity distribution given by (10) is shown in Fig. 4. In addition, the alternative velocity distribution

$$u = U_0 + U\left[1 - \left(\frac{r}{r_0}\right)^{3/2}\right]^2. \qquad \text{(10a)}$$

also shown in Fig. 4, was considered. The latter form probably represents the correct shape better than the former at large distances from the exit, but the change was found to have little effect on the general development of the jet, and the calculations with it will not be discussed further.

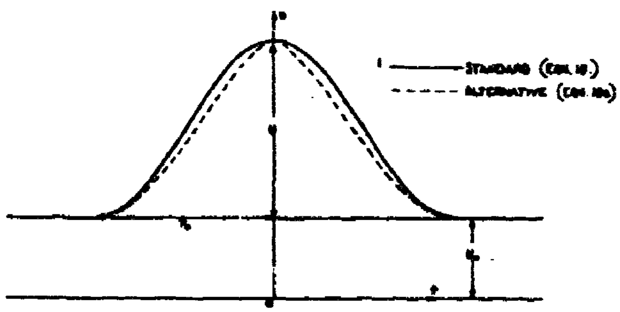

FIG. 4—Velocity Distributions across Jet in Region B.

The shear stress due to turbulent mixing is given as before by (3) but the mixing length l is now given by

$$l = c\, r_0, \qquad \text{(11)}$$

which is obtained from (4) by putting r_1 equal to zero.

The momentum equation (1) is still valid and equation (6) for the total momentum still holds in the form

$$\int_0^{r_0} \rho r\, u(u - U_0)\, dr = M, \quad \ldots \quad (12)$$

but with u now given by (10). In addition to satisfying this, we shall satisfy (1) at the half-radius $r = r_0/2$, at which, from (10),

$$u = U_0 + \frac{U}{2}, \quad \frac{\partial u}{\partial r} = \frac{\pi U}{2r_0},$$

and, from (3) and (11), $\dfrac{\tau}{\rho} = \dfrac{\pi^2}{4} c^2 U^2$.

Substituting these expressions in (1) gives

$$\frac{\partial}{\partial x}\int_0^{r_0/2} \rho r\, u^2\, dr - \left(U_0 + \frac{U}{2}\right)\frac{\partial}{\partial x}\int_0^{r_0/2} \rho r\, u\, dr = -\frac{\pi^2}{8}\rho c^2 U^2 r_0. \quad \ldots \quad (13)$$

Equations (12) and (13), with u given by (10), are sufficient to determine the variation of r_0, the jet radius, and U, the excess velocity on the axis, with distance downstream. Evaluation of the integrals involved leads to the equations

$$\rho\, U r_0^2 (0{\cdot}1486\, U_0 + 0{\cdot}0861\, U) = M, \quad \ldots \quad (14)$$

$$U \frac{dr_0}{dx}(0{\cdot}0578\, U_0 + 0{\cdot}0476\, U) + r_0 \frac{dU}{dx}(0{\cdot}0914\, U_0 + 0{\cdot}0933\, U) = -1{\cdot}234\, c^2 U^2. \quad (15)$$

Next, r_0 is eliminated from (14) and (15), and a differential equation obtained for U :—

$$\frac{1}{U}\frac{dU}{dx}\left[(0{\cdot}0914\, U_0 + 0{\cdot}0933\, U) - \frac{(0{\cdot}1486\, U_0 + 0{\cdot}1722\, U)(0{\cdot}0578\, U_0 + 0{\cdot}0476 U)}{2(0{\cdot}1486\, U_0 + 0{\cdot}0861\, U)}\right]$$

$$= -\frac{1{\cdot}234\, c^2 U^{3/2}}{(M/\rho)^{1/2}}(0{\cdot}1486\, U_0 + 0{\cdot}0861\, U)^{1/2}.$$

If we put $U_0/U = z$, this equation becomes, on reduction,

$$\frac{dz}{dx}\cdot\frac{(0{\cdot}0626\, z + 0{\cdot}0458)}{(0{\cdot}1486\, z + 0{\cdot}0861)^{1/2}}\left[1 + \frac{0{\cdot}0012\, z}{(0{\cdot}1486\, z + 0{\cdot}0861)(0{\cdot}0626\, z + 0{\cdot}0458)}\right]$$

$$= \frac{1{\cdot}234\, c^2 U_0}{(M/\rho)^{1/2}}. \quad \ldots \quad (16)$$

The integration of this equation is greatly simplified if an approximation to the term in square brackets on the left-hand side is taken. This term is equal to unity for $z = 0$ and tends to unity as z tends to infinity, having a maximum value of $1{\cdot}05$ at $z = 0{\cdot}7$. A mean value of $1{\cdot}04$ has been adopted for it and, with this simplification, equation (16) becomes

$$\frac{dz}{dx}\cdot\frac{(z + 0{\cdot}732)}{(z + 0{\cdot}579)^{1/2}} = \frac{7{\cdot}31\, c^2 U_0}{(M/\rho)^{1/2}}, \quad \ldots \quad (17)$$

which integrates to

$$(z + 0{\cdot}579)^{1/2}(0{\cdot}667\, z + 0{\cdot}690) = \frac{7{\cdot}31\, c^2 U_0}{(M/\rho)^{1/2}}x + \text{const.} \quad \ldots \quad (18)$$

The next step is to link this solution with the solution with the central core, already derived in para. 3. We note first that, from (6),

$$\frac{(M/\rho)^{1/2}}{U_0} = \frac{1}{U_0}\left[\frac{U_1}{2}(U_1 - U_0)a^2\right]^{1/2} = \left(\frac{1-\lambda}{2\lambda^2}\right)^{1/2} a.$$

Also, from para. 3 and Fig. 3, the distance from the jet exit of the point of disappearance of the core is given by

$$\frac{c^2x}{a} = b,$$

where b is a number which depends upon the velocity ratio λ; at this point $z = \frac{U_0}{U} = \frac{U_0}{U_1 - U_0}$ $= \frac{\lambda}{1-\lambda}$. Substituting these values in (18) fixes the constant of integration, and this equation becomes

$$\left(\frac{U_0}{U} + 0\cdot579\right)^{1/2}\left(0\cdot667\frac{U_0}{U} + 0\cdot690\right) - \left(\frac{\lambda}{1-\lambda} + 0\cdot579\right)^{1/2}\left(\frac{0\cdot667\lambda}{1-\lambda} + 0\cdot690\right)$$

$$= 7\cdot31\left(\frac{2\lambda^2}{1-\lambda}\right)^{1/2}\left(\frac{c^2x}{a} - b\right). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

Equation (19) determines the variation of U, the excess velocity on the axis, with distance downstream; when U is known, the jet radius r_0 is given by (14). There is necessarily a smooth join in the jet radius and velocity at the junction of the two solutions because of the forms (2) and (10) adopted for the velocity distributions, which become the same when r_1 is put equal to zero in (2).

Calculations of the solutions of (19) and (14) have been made for the same values* of λ as before, viz., 0, 0·125, 0·25, and 0·5, up to c^2x/a equal to 0·4. These are given in Tables 2, 3, 4 and Figs. 5A, 5B and 6, which show the jet boundaries, the velocity on the axis, and the rate of spread of the surface on which the velocity is half-way between the stream velocity and the velocity on the axis at the same value of x. It is worth noting here that the outer boundary of the jet, which is difficult to define experimentally, may really be slightly wider than that calculated, because the actual velocity falls rather more slowly at the jet edge than the calculated distribution. For this reason the "half-velocity" line (Fig. 6), should be more reliable as an indication of the jet width.

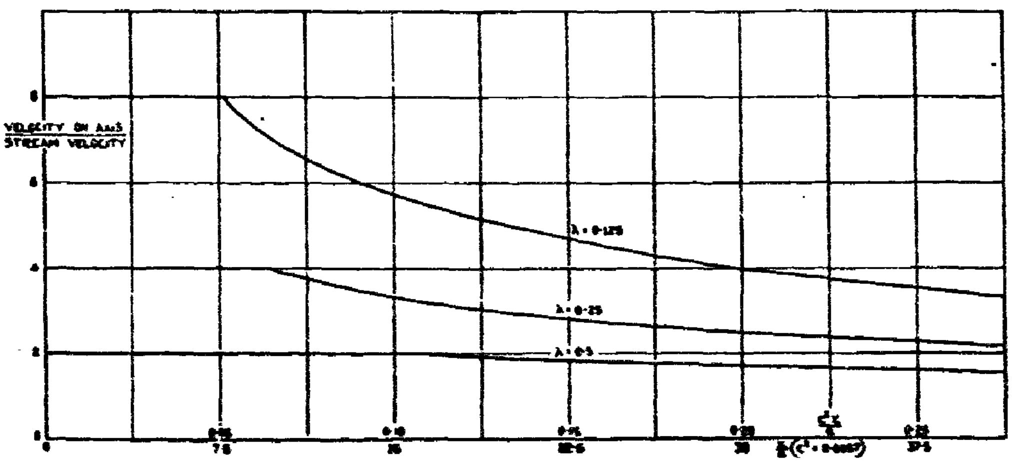

FIG. 5A—Velocity on Axis.

* $\lambda = 0\cdot75$ was omitted.

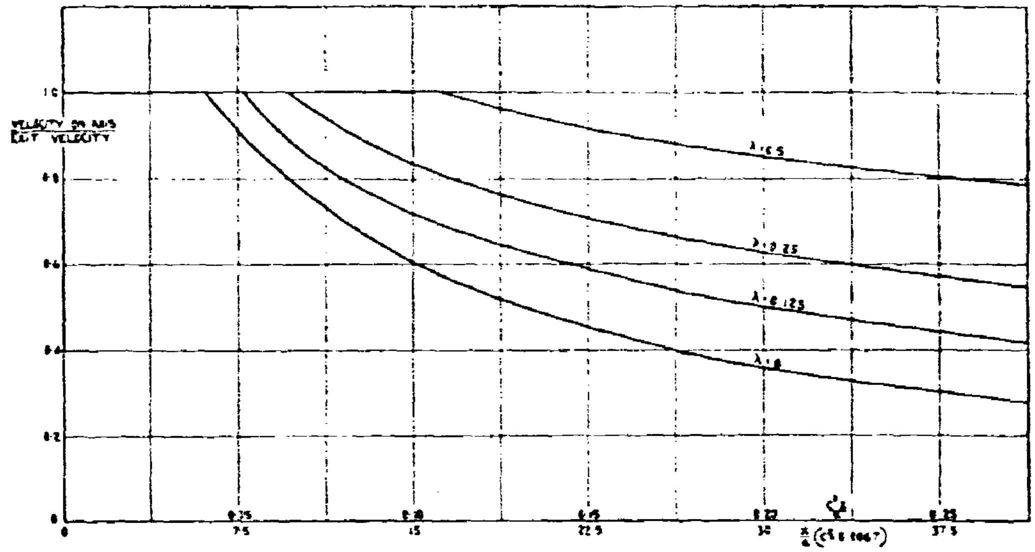

FIG. 5B—Velocity on Axis.

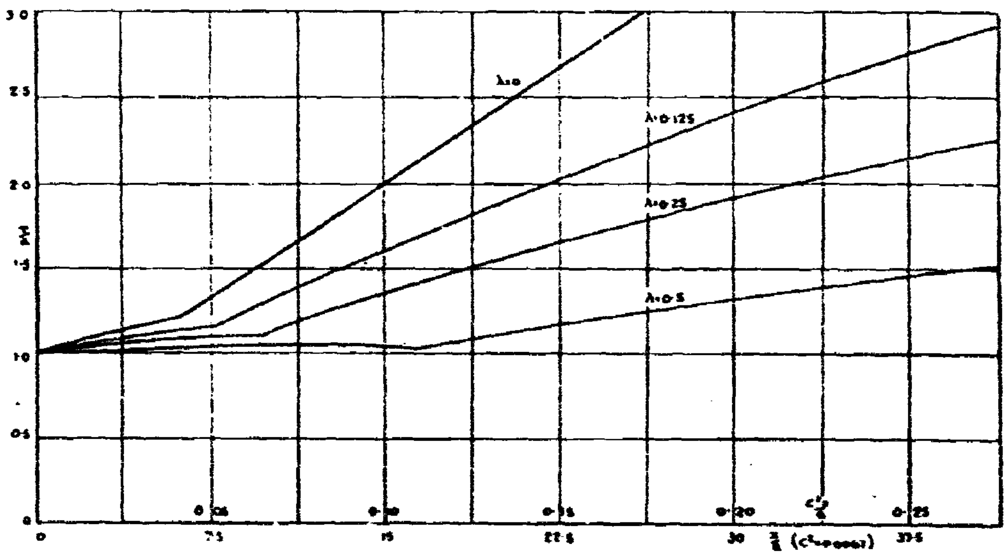

FIG. 6—Half-Velocity Lines.

5. *Determination of the Parameter* c^2.—Up to this point the mixing length parameter c defined in equations (4) and (11) has been left undetermined. This has been done to keep the analysis as general as possible. But, before proceeding with the calculation of the inflow from outside the jet, it is convenient to select a value. We shall take $c^2 = 0 \cdot 0067$, this value being derived from experimental data on the spread of a jet issuing from a small orifice in still air; the analysis leading to this is given in Appendix II. This value is to be compared with Kuethe's value[2] of 0·00497 obtained from comparison between theory and experiment for region *A* of a jet issuing into still air.

After the calculations described below had been finished, the above discrepancy in the experimental data was reconsidered, and it is now thought possible that different values of c^2 apply in regions A and B. In practice c^2 may rise from about 0·005 in region A to 0·0067 in region B, over a transition region downstream of the core apex.

6. *Inflow Outside the Jet.*—The flow inside the jet is determined by the above analysis when c^2 has been specified. The mass flow across each section of the jet can be calculated, and is found to increase more rapidly than the increase provided by the natural inflow into the growing jet from the external stream: the jet thus induces an inflow towards itself. To do this, the pressure inside the jet must be lower than in the free stream and this appears to be in contradiction to the previous assumption of uniform pressure; it has, however, been shown by Tollmien[1] that this reduction in pressure is extremely small for a jet issuing into fluid at rest.

The next step is to calculate the inflow induced at all points near the jet. This is done in two stages; the first is based on the assumption that the inflow is purely radial, and the second stage is the correction of the first to allow for the interaction of the flow at neighbouring sections.

If v is the outward radial component of velocity at the point (x, r) and ψ is Stokes's stream function for the motion, we have

$$\psi = \int_0^r r\,u\,dr, \quad \frac{\partial \psi}{\partial x} = -\,r\,v \qquad \ldots \quad (20)$$

For region A we substitute for u from (2) into this integral and obtain, for $r > r_0$,

$$\psi = (U_1 - U_0)\left[\frac{r_1^2 + r_0^2}{4} - \frac{(r_1 - r_0)^2}{\pi^2}\right] + \frac{U_0 r^2}{2}$$

$$\frac{\partial \psi}{\partial x} = (U_1 - U_0)\frac{d}{dx}\left[\frac{r_1^2 + r_0^2}{4} - \frac{(r_1 - r_0)^2}{\pi^2}\right] \qquad \ldots \quad (21)$$

For region B we substitute from (10) and obtain, for $r > r_0$,

$$\psi = 0\cdot1486\; U r_0^2 + \frac{U_0 r^2}{2}$$

$$\frac{\partial \psi}{\partial x} = 0\cdot1486\,\frac{d}{dx}\,(U r_0^2) \qquad \ldots \quad (22)$$

This last formula can be reduced by use of (14) and (17) to

$$\frac{\partial \psi}{\partial x} = \left(\frac{M}{\rho}\right)^{1/2} \frac{4\cdot24\,c^2}{(z + 0\cdot732)\,(z + 0\cdot579)^{3/2}} \qquad \ldots \quad (23)$$

where, as before, z stands for U_0/U.

Equations (21) and (23) yield values of the inflow velocity

$$-\,v = \frac{1}{r}\,\frac{\partial \psi}{\partial x}$$

which, taken as they stand, correspond to the radial flow at each section of the jet being independent of the flow in other sections: this is approximately true very close to the boundary of the mixing region but is quite invalid at large distances from the jet axis. The actual flow outside the jet can be regarded as closely equivalent to that produced by a system of sinks along the jet axis, of strength sufficient to secure the inflow at the edge of the jet indicated by the above formulae. This system of sinks has a strength per unit length proportional to $\frac{\partial \psi}{\partial x}$, which is given by (21) and (23); numerical values of $\frac{1}{c^2 a}\frac{1}{U_1}\frac{\partial \psi}{\partial x}$ are given in Tables 2–4 and in Fig. 7, for the various jets considered.

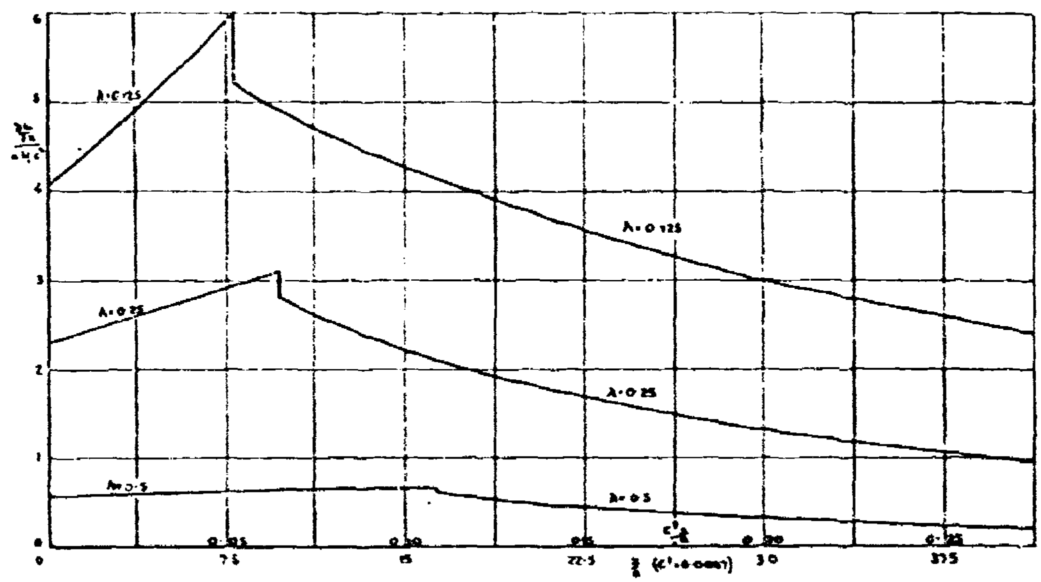

FIG. 7—Sink Strength along Axis of Jet.

It will be seen that there is a discontinuity in sink strength at the point of disappearance of the core; this is associated with the discontinuity of the rate of growth of the outer boundary of the jet there, and is a defect in the solution which, however, is only significant in the immediate neighbourhood of the jet edge at this particular section.

It is next required to find the inflow velocities* associated with the sink distributions shown in Fig. 7. This is a problem which is not amenable to analytical treatment and it has been solved by breaking up the jet axis into lengths over which the sink strength can be taken to be linear, and summing the contributions of each linear section at a large number of points. The details of this calculation are given in Appendix III.

Calculations were made for $c^2 = 0{\cdot}0067$ and for ratios of the stream velocity to the jet exit velocity $\lambda = 0{\cdot}125$, $0{\cdot}25$ and $0{\cdot}5$. The results are presented in Figs. 8–11, which give contours of the angular deviation (in degrees) of the outer stream towards the jet axis; for other velocity ratios the stream deviation at any point can be quickly determined by interpolation.

The results of similar inflow calculations made for $\lambda = 0{\cdot}25$, $c^2 = 0{\cdot}005$ are given in Fig. 11 for comparison with Fig. 9; the angle of inflow is slightly decreased by decrease of c^2.

7. *Application to Jets of Compressible or Heated Fluid.*—The solution of the corresponding problem for a jet of compressible or heated fluid is more complicated, as the energy equation would have to be satisfied as well as the momentum equation. It would therefore be convenient if an "equivalent" jet of incompressible fluid could be determined for which the inflow was the same as for a real jet of heated fluid.

Since the jet momentum M (equation (6) and (12)) is maintained for all sections, although the density, temperature and velocity of a hot jet will vary, it seems probable that an equivalent jet of incompressible fluid would be such as to have the same value of the exit momentum M, as defined by equation (6).

* Only the radial components are required. The axial components are negligible in comparison with the stream velocity.

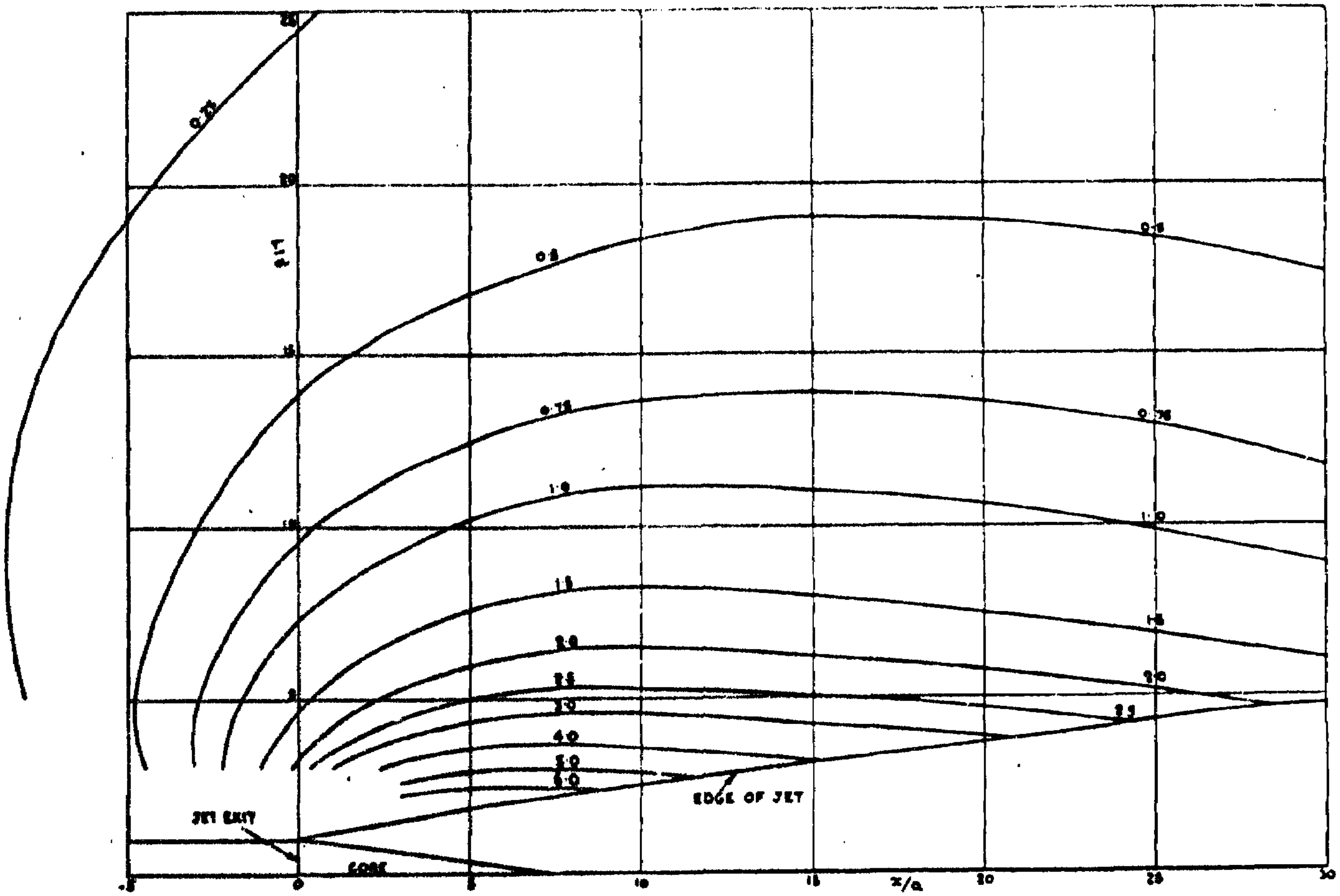

FIG. 8—Stream Deviation in Degrees. $\lambda = 0{\cdot}125$. $C^{1} = 0{\cdot}0087$.

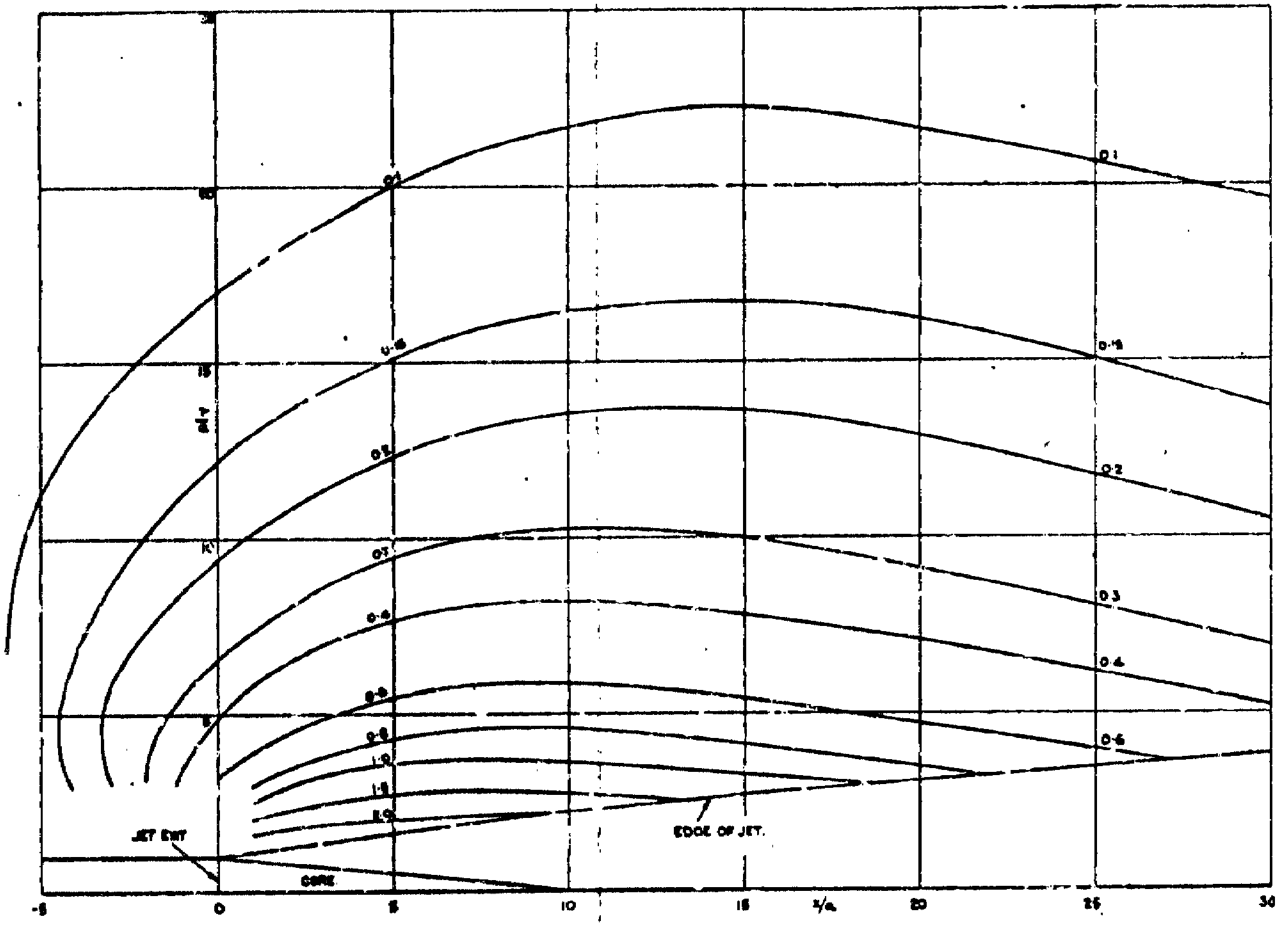

Fig. 9—Stream Deviation in Degrees. $\lambda = 0{\cdot}25$. $c^2 = 0{\cdot}0067$.

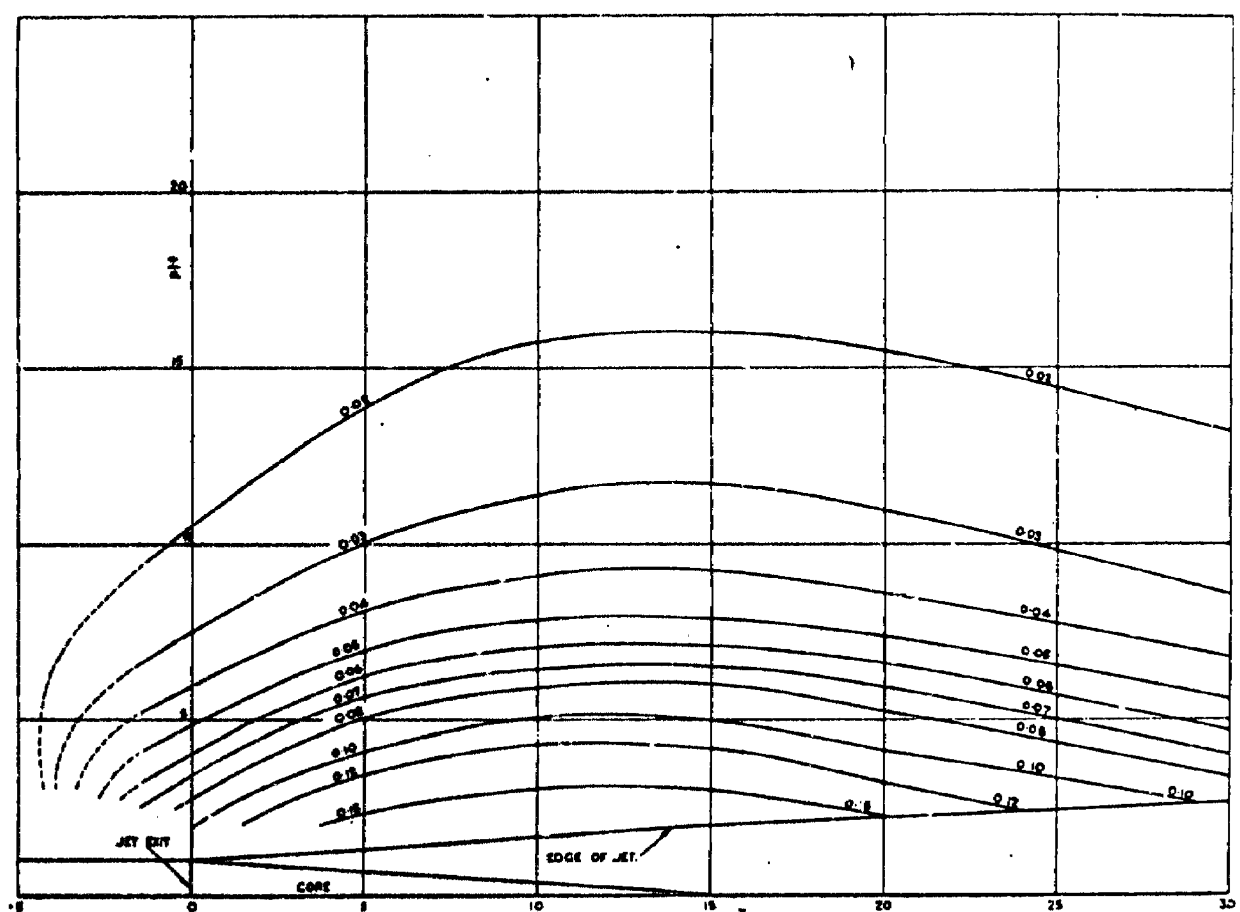

FIG. 10—Stream Deviation in Degrees. $\lambda = 0 \cdot 5$. $c^2 = 0 \cdot 0067$.

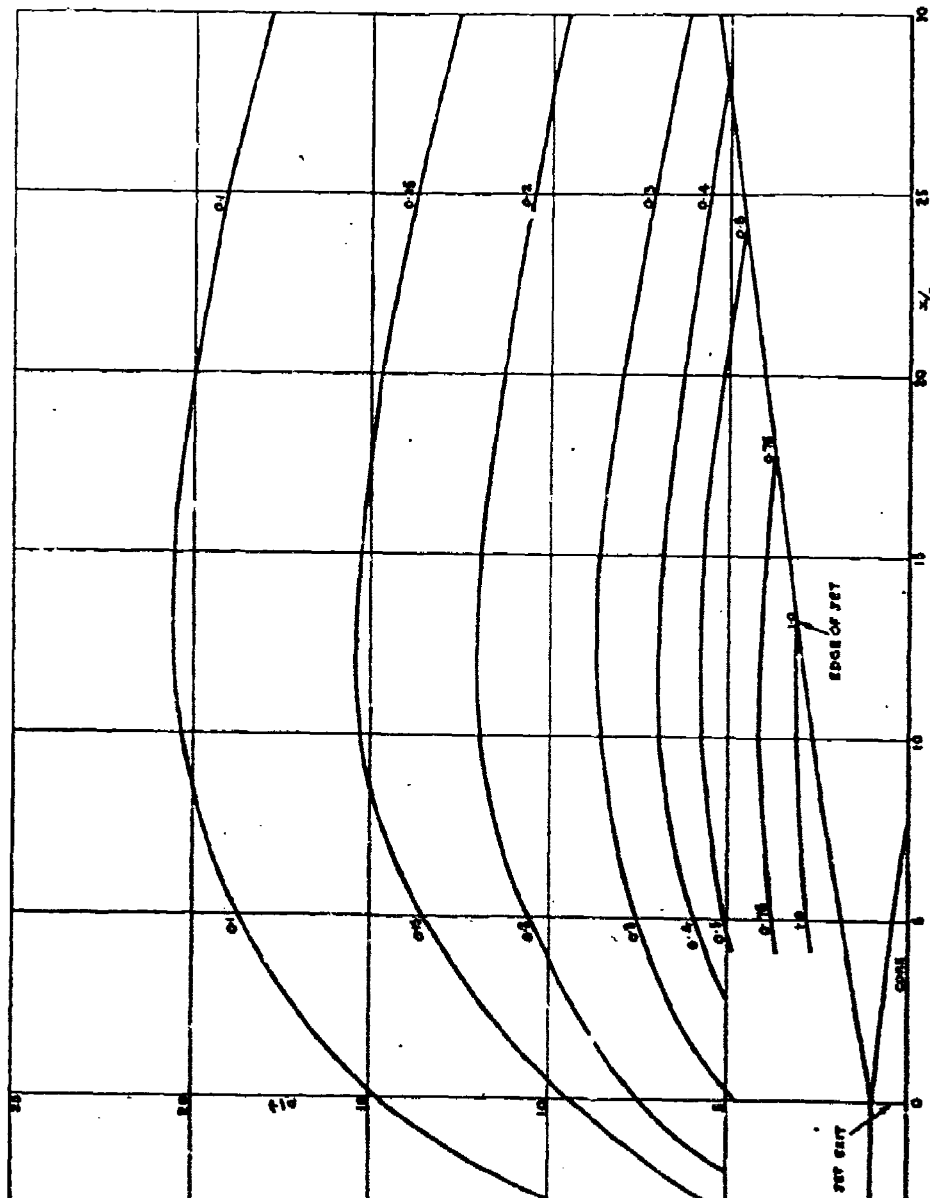

FIG. 11—Stream Deviation in Degrees. $\lambda = 0 \cdot 25$. $c^2 = 0 \cdot 005$.

REFERENCES

No.	Author	Title, etc.
1	Tollmien	*Z.A.M.M.*, Vol. 6 (1926), p. 468.
2	Kuethe	*Journal of Applied Mechanics*, Vol. 2 (1935), p. 87.
3	Reichardt	*Z.A.M.M.*, Vol. 21 (1941), p. 257. *V.D.I.*, Forschungsheft No. 414 (1942).
4	Goldstein (ed.)	Modern Developments in Fluid Dynamics. (Oxford, 1938.)
5	Milne-Thomson	Theoretical Hydrodynamics, p. 412. (McMillan and Co., Ltd.)

APPENDIX I

Solutions of Equations (8) *and* (9)

The equations are

$$a_{11} r_1^2 + 2a_{10} r_1 r_0 + a_{00} r_0^2 = a^2, \quad .. \quad .. \quad .. \qquad (8)$$

$$\frac{d}{dx}\left[A_{11} r_1^2 + 2A_{10} r_1 r_0 + A_{00} r_0^2\right] = B(r_1 + r_0). \quad .. \quad .. \qquad (9)$$

From (8) $$\frac{dr_1}{dr_0} = -\frac{a_{10}r_1 + a_{00}r_0}{a_{11}r_1 + a_{10}r_0},$$

and from (9) $$B(r_1 + r_0)\frac{dx}{dr_0} = 2(A_{11}r_1 + A_{10}r_0)\frac{dr_1}{dr_0} + 2(A_{10}r_1 + A_{00}r_0).$$

Elimination of $\frac{dr_1}{dr_0}$ gives

$$\frac{B}{2}\frac{dx}{dr_0} = \frac{r_1^2(a_{11}A_{10} - a_{10}A_{11}) + r_1 r_0(a_{11}A_{00} - a_{00}A_{11}) + r_0^2(a_{10}A_{00} - a_{00}A_{10})}{(a_{11}r_1 + a_{10}r_0)(r_1 + r_0)}$$

$$= \alpha_1 + \frac{\alpha_2 r_0}{a_{11}r_1 + a_{10}r_0} + \frac{\alpha_3 r_0}{r_1 + r_0}, \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (24)$$

where $$\alpha_1 = \frac{a_{11}A_{10} - a_{10}A_{11}}{a_{11}}, \quad \alpha_2 = \frac{(a_{11}A_{10} - a_{10}A_{11})(a_{11}a_{00} - a_{10}^2)}{a_{11}(a_{10} - a_{11})},$$

$$\alpha_3 = \frac{a_{11}(A_{10} - A_{00}) + a_{10}(A_{00} - A_{11}) + a_{00}(A_{11} - A_{10})}{(a_{10} - a_{11})}.$$

Also, from (8),

$$a_{11}r_1 + a_{10}r_0 = \left[a^2 a_{11} - r_0^2(a_{00}a_{11} - a_{10}^2)\right]^{1/2},$$

$$a_{11}(r_1 + r_0) = r_0(a_{11} - a_{10}) + \sqrt{a_{10}^2 r_0^2 + a_{11}(a^2 - a_{00}r_0^2)}$$

$$\doteq r_0(a_{11} - a_{10}) + \sqrt{-r_0^2(-a_{10}^2 + a_{11}a_{00}) + a^2 a_{11}}.$$

Using these expressions integration of (24) gives

$$0\cdot 5\ Bx = \alpha_1 r_0 - \frac{\alpha_2\sqrt{a^2 a_{11} - r_0^2(a_{00}a_{11} - a_{10}^2)}}{a_{00}a_{11} - a_{10}^2}$$

$$+ \alpha_3 \int \frac{a_{11} r_0 dr_0}{r_0(a_{11} - a_{10}) + \sqrt{a^2 a_{11} - r_0^2(a_{11}a_{00} - a_{10}^2)}}. \qquad (25)$$

To evaluate the last integral let

$$I = \int \frac{a_{11} r_0 dr_0}{r_0(a_{11} - a_{10}) + \sqrt{a_{11}a^2 - r_0^2(a_{11}a_{00} - a_{10}^2)}}$$

$$= c_0 \int \frac{r_0 dr_0}{r_0 + \sqrt{c_1^2 - c_2 r_0^2}},$$

where $c_0 = \dfrac{a_{11}}{(a_{11} - a_{10})}, \quad c_1^2 = \dfrac{a_{11}a^2}{(a_{11} - a_{10})^2},$

and $c_2 = \dfrac{a_{11}a_{00} - a_{10}^2}{(a_{11} - a_{10})^2}.$

Then, rationalising the denominator,

$$I = c_0 \int \frac{r_0(r_0 - \sqrt{c_1^2 - c_2 r_0^2})dr_0}{r_0^2(1 + c_2) - c_1^2} = c_0(I_1 + I_2),$$

where $I_1 = \displaystyle\int \frac{r_0^2 dr_0}{r_0^2(1 + c_2) - c_1^2}, \quad I_2 = -\int \frac{r_0\sqrt{c_1^2 - c_2 r_0^2}\, dr_0}{r_0^2(1 + c_2) - c_1^2}$

We have $I_1 = \dfrac{1}{2(1 + c_2)} \displaystyle\int \left[2 + \frac{c_1}{r_0\sqrt{1 + c_2} - c_1} - \frac{c_1}{r_0\sqrt{1 + c_2} + c_1}\right] dr_0$

$$= \frac{1}{2(1 + c_2)}\left[2r_0 + \frac{c_1}{\sqrt{1 + c_2}}\log\left(\frac{r_0\sqrt{1 + c_2} - c_1}{r_0\sqrt{1 + c_2} + c_1}\right)\right]$$

To evaluate I_2, put $c_1^2 - c_2 r_0^2 = c_2 R_0^2$, so that

$$I_2 = -\sqrt{c_2}\int \frac{R_0^2 dR_0}{(1 + c_2)R_0^2 - c_1^2/c_2}$$

$$= \frac{-\sqrt{c_2}}{2(1 + c_2)}\left[2R_0 + \frac{c_1}{\sqrt{c_2(1 + c_2)}}\log\left(\frac{R_0\sqrt{1 + c_2} - c_1/\sqrt{c_2}}{R_0\sqrt{1 + c_2} + c_1/\sqrt{c_2}}\right)\right]$$

$$= \frac{-1}{2(1 + c_2)}\left[2\sqrt{c_1^2 - c_2 r_0^2} + \frac{c_1}{\sqrt{1 + c_2}}\log\left(\frac{\sqrt{(1 + c_2)(c_1^2 - c_2 r_0^2)} - c_1}{\sqrt{(1 + c_2)(c_1^2 - c_2 r_0^2)} + c_1}.\right)\right]$$

Hence,

$$I = \frac{c_0}{2(1 + c_2)}\left[2(r_0 - \sqrt{c_1^2 - c_2 r_0^2}) + \frac{c_1}{\sqrt{1 + c_2}} \log\left(\frac{r_0\sqrt{1 + c_2} - c_1}{r_0\sqrt{1 + c_2} + c_1} \cdot \frac{\sqrt{1 + c_2}(c_1^2 - c_2 r_0^2)^{1/2} + c_1}{\sqrt{1 + c_2}(c_1^2 - c_2 r_0^2)^{1/2} - c_1}\right)\right]$$

With this value of I, x is given as a function of r_0 by (25) and r_1 as a function of r_0 by (8).

APPENDIX II

Determination of c^2 from Measured Rate of Spread of a Jet in Still Air

Experimental data obtained from tests made at the R.A.E. on the rate of spread of a jet issuing from a small orifice in still air show that the jet spreads conically. The cone on which the velocity is equal to half the velocity on the jet axis was found to have a semi-angle of 5 degrees.

With the cosine velocity distribution adopted (equation (10)) this gives

$$r_0/2 = x \tan 5°, \quad \text{i.e. } r_0 = 0 \cdot 175x. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

For zero stream velocity (14) and (15) become

$$0 \cdot 0861 \, \rho U^2 r_0^2 = M,$$

$$0 \cdot 0476 \, U^2 \frac{dr_0}{dx} + 0 \cdot 0933 \, U r_0 \frac{dU}{dx} = - 1 \cdot 234 \, c^2 U^2 .$$

Elimination of U between these equations leads to

$$\frac{dr_0}{dx} = 27 \cdot 0 \, c^2,$$

and hence, from (26),

$$c^2 = 0 \cdot 0065.$$

The accuracy of the experimental data is not sufficient to distinguish between an angular spread of 5 deg. and 5·1 deg., and a value of $c^2 = 0 \cdot 0067$ has been adopted as the calculations were a little simpler with this value.

The approximate form assumed for the velocity distribution will give the jet radius rather less than the actual value, which is difficult to specify exactly.

APPENDIX III

Determination of Inflow Velocity from the Axial Sink Distribution

As stated in para. 6, the inflow velocities corresponding to the sink distributions of Fig. 7 were determined by breaking up the latter into sections, along each of which the sink strength could be treated as linear. We therefore require formulae to determine the inflow velocity for a linear source distribution of any length ; this in turn is easily determined if the velocity fields of (*a*) a uniform source distribution of unit strength between the points A $(-1,0)$ and B $(+1,0)$, and (*b*) a source distribution from A to B of strength proportional to the distance from the origin, can be calculated.

For case (*a*) the stream function is[5]

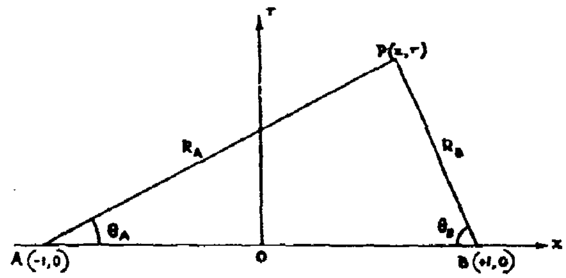

FIG. 12. Notation used in Appendix III.

$$\psi = \int_{-1}^{1} \frac{(x-\xi)\, d\xi}{[r^2 + (x-\xi)^2]^{1/2}} = \left[- \left\{ r^2 + (x-\xi)^2 \right\}^{1/2} \right]_{-1}^{1}$$

$$= \sqrt{r^2 + (x+1)^2} - \sqrt{r^2 + (x-1)^2}$$

and

$$\frac{\partial \psi}{\partial x} = \frac{(x+1)}{\sqrt{r^2 + (x+1)^2}} - \frac{(x-1)}{\sqrt{r^2 + (x-1)^2}} \, .$$

$$= \cos \theta_A + \cos \theta_B, \quad \ldots \quad (27)$$

where θ_A and θ_B are defined in Fig. 12.

For case (*b*)

$$\psi = \int_{-1}^{1} \frac{\xi(x-\xi)\, d\xi}{[r^2 + (x-\xi)^2]^{1/2}} = \int_{-1}^{1} \xi d\left[- \left\{ r^2 + (x-\xi)^2 \right\}^{1/2} \right]$$

$$= - \left[\sqrt{r^2 + (x-1)^2} + \sqrt{r^2 + (x+1)^2} \right]_{-1}^{1} + \int_{-1}^{1} \sqrt{r^2 + (x-\xi)^2} d\xi,$$

so that

$$\frac{\partial \psi}{\partial x} = - \frac{(x-1)}{\sqrt{r^2 + (x-1)^2}} - \frac{(x+1)}{\sqrt{r^2 + (x+1)^2}} + \sqrt{r^2 + (x+1)^2} - \sqrt{r^2 + (x-1)^2}$$

$$= (R_A - \cos \theta_A) - (R_B - \cos \theta_B) . \quad \ldots \quad (28)$$

The radial velocity v is equal to $-\frac{1}{r}\frac{\partial \psi}{\partial x}$ so that $\frac{\partial \psi}{\partial x}$ is the quantity which we require to determine. From (27) and (28) contours of equal values of $\partial \psi / \partial x$ for the two standard source distributions were drawn. With the aid of these the contribution of any set of linear source distributions to the value of $\partial \psi / \partial x$ could be determined in a straightforward manner by summation.

TABLE 1

Stream Velocity Zero. $\lambda = 0$

Distance from jet exit	Radius of jet	Radius of core	Velocity on axis / Exit velocity	Half-velocity line
Region A $\frac{c^2x}{a}$	$\frac{r_0}{a}$	$\frac{r_1}{a}$		$\frac{r_0 + r_1}{2a}$
0	1	1	1·0	1
0·01	1·34	0·79	1·0	1·07
0·02	1·68	0·56	1·0	1·12
0·03	2·02	0·31	1·0	1·17
0·0408	2·41	0	1·0	1·21
Region B $\frac{c^2x}{a}$	$\frac{r_0}{a}$		$\frac{U}{U_1}$	$\frac{r_0}{2a}$
0·0408	2·41		1	1·21
0·06	2·92		0·83	1·46
0·08	3·46		0·70	1·73
0·10	4·00		0·60	2·00
0·12	4·54		0·53	2·27
0·14	5·08		0·48	2·54
0·16	5·62		0·43	2·81
0·18	6·16		0·39	3·08
0·20	6·70		0·36	3·35
0·22	7·24		0·33	3·62
0·24	7·78		0·31	3·89
0·26	8·32		0·29	4·16
0·28	8·86		0·27	4·43
0·30	9·40		0·255	4·70
0·32	9·94		0·24	4·97
0·34	10·48		0·23	5·24
0·36	11·02		0·22	5·51
0·38	11·56		0·21	5·78
0·40	12·10		0·20	6·05

TABLE 2

$$\frac{\textit{Jet Exit Velocity}}{\textit{Stream Velocity}} = 8. \quad \lambda = 0\cdot125.$$

Distance from jet exit	Radius of jet	Radius of core	Velocity on axis / Stream velocity	Half-velocity line	Axial sink strength
Region A $\frac{c^2x}{a}$	$\frac{r_0}{a}$	$\frac{r_1}{a}$		$\frac{r_0+r_1}{2a}$	$\frac{1}{aU_1c^2}\frac{\partial\psi}{\partial x}$
0	1·00	1·00	8·0	1·00	4·05
0·01	1·25	0·83	8·0	1·04	4·40
0·02	1·50	0·65	8·0	1·07	4·78
0·03	1·76	0·47	8·0	1·11	5·13
0·04	2·02	0·26	8·0	1·14	5·51
0·0514	2·31	0	8·0	1·16	6
Region B $\frac{c^2x}{a}$	$\frac{r_0}{a}$		$1+\frac{U}{U_0}$	$\frac{r_0}{2a}$	$\frac{1}{aU_1c^2}\frac{\partial\psi}{\partial x}$
0·0514	2·31		8·00	1·16	5·21
0·06	2·49		7·34	1·25	5·00
0·08	2·87		6·39	1·43	4·60
0·10	3·21		5·75	1·60	4·28
0·12	3·55		5·27	1·78	3·98
0·14	3·89		4·89	1·94	3·69
0·16	4·22		4·53	2·11	3·43
0·18	4·54		4·24	2·27	3·20
0·20	4·83		4·00	2·42	2·99
0·22	5·11		3·79	2·56	2·81
0·24	5·37		3·62	2·69	2·65
0·26	5·64		3·45	2·83	2·51
0·28	5·90		3·30	2·95	2·38
0·30	6·17		3·16	3·08	2·24
0·32	6·42		3·07	3·20	2·13
0·34	6·65		3·00	3·32	2·03
0·36	6·87		2·92	3·44	1·94
0·38	7·09		2·84	3·55	1·84
0·40	7·30		2·75	3·65	1·74

TABLE 3

$$\frac{\textit{Jet Exit Velocity}}{\textit{Stream Velocity}} = 4. \quad \lambda = 0{\cdot}25.$$

Distance from jet exit	Radius of jet	Radius of core	Velocity on axis / Stream velocity	Half-velocity line	Axial sink strength
Region A $\frac{c^2x}{a}$	$\frac{r_0}{a}$	$\frac{r_1}{a}$		$\frac{r_0 + r_1}{2a}$	$\frac{1}{aU_1c^2} \frac{\partial \psi}{\partial x}$
0	1	1	4·0	1	2·29
0·02	1·38	0·73	4·0	1·05	2·54
0·04	1·76	0·42	4·0	1·09	2·79
0·06	2·14	0·08	4·0	1·11	3·04
0·0639	2·21	0	4·0	1·11	3·09
Region B $\frac{c^2x}{a}$	$\frac{r_0}{a}$		$1 + \frac{U}{U_0}$	$\frac{r_0}{2a}$	$\frac{1}{aU_1c^2} \frac{\partial \psi}{\partial x}$
0·0639	2·21		4	1·11	2·80
0·08	2·43		3·67	1·22	2·51
0·10	2·70		3·34	1·35	2·22
0·12	2·96		3·10	1·48	1·98
0·14	3·19		2·85	1·60	1·76
0·16	3·41		2·73	1·70	1·59
0·18	3·61		2·60	1·80	1·44
0·20	3·82		2·49	1·91	1·31
0·22	4·01		2·38	2·01	1·20
0·24	4·20		2·30	2·10	1·10
0·26	4·38		2·23	2·19	1·02
0·28	4·55		2·16	2·28	0·94
0·30	4·72		2·10	2·36	0·88
0·32	4·87		2·05	2·44	0·82
0·34	5·03		2·00	2·52	0·76
0·36	5·18		1·97	2·59	0·71
0·38	5·32		1·93	2·66	0·67
0·40	5·45		1·90	2·73	0·63

TABLE 4

$$\frac{\textit{Jet Exit Velocity}}{\textit{Stream Velocity}} = 2. \quad \lambda = 0 \cdot 5.$$

Distance from jet exit	Radius of jet	Radius of core	$\frac{\text{Velocity on axis}}{\text{Stream velocity}}$	Half-velocity line	Axial sink strength
Region A $\frac{c^2x}{a}$	$\frac{r_0}{a}$	$\frac{r_1}{a}$		$\frac{r_0 + r_1}{2a}$	$\frac{1}{aU_1c^2}\frac{\partial\psi}{\partial x}$
0	1	1	2·0	1	0·566
0·02	1·20	0·84	2·0	1·02	0·588
0·04	1·40	0·67	2·0	1·04	0·605
0·06	1·60	0·49	2·0	1·05	0·622
0·08	1·79	0·31	2·0	1·05	0·635
0·10	1·98	0·10	2·0	1·04	0·649
0·1084	2·06	0	2·0	1·03	0·655
Region B $\frac{c^2x}{a}$	$\frac{r_0}{a}$		$1 - \frac{U}{U_0}$	$\frac{r_0}{2a}$	$\frac{1}{aU_1c^2}\frac{\partial\psi}{\partial x}$
0·1084	2·06		2	1·03	0·615
0·12	2·15		1·95	1·07	0·558
0·14	2·28		1·87	1·14	0·483
0·16	2·41		1·80	1·20	0·422
0·18	2·54		1·74	1·27	0·372
0·20	2·66		1·69	1·33	0·330
0·22	2·77		1·65	1·38	0·296
0·24	2·87		1·61	1·43	0·267
0·26	2·97		1·58	1·48	0·242
0·28	3·06		1·55	1·53	0·220
0·30	3·15		1·52	1·57	0·202
0·32	3·23		1·50	1·62	0·186
0·34	3·32		1·48	1·66	0·172
0·36	3·40		1·46	1·70	0·160
0·38	3·49		1·44	1·74	0·148
0·40	3·57		1·43	1·78	0·137

TABLE 5

$$\frac{\textit{Jet Exit Velocity}}{\textit{Stream Velocity}} = 1\cdot33. \quad \lambda = 0\cdot75.$$

Distance from jet exit	Radius of jet	Radius of core	Half-velocity line
Region A $\frac{c^2x}{a}$	$\frac{r_0}{a}$	$\frac{r_1}{a}$	$\frac{r_0 + r_1}{2a}$
0	1	1	1
0·02	1·08	0·93	1·00
0·04	1·16	0·85	1·00
0·06	1·24	0·78	1·01
0·08	1·32	0·70	1·01
0·10	1·40	0·63	1·02
0·12	1·48	0·54	1·01
0·14	1·56	0·45	1·01
0·16	1·64	0·37	1·01
0·18	1·71	0·28	1·00
0·20	1·79	0·19	0·99
0·22	1·87	0·10	0·98
0·2386	1·94	0	0·97

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