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# Probability Distribution Associated With m on n Aerial Duels

by
Frank C. Reed
Systems Engineering and Synthesis Division
*Weapons Department*

DECEMBER 1976





Naval Weapons Center
CHINA LAKE, CALIFORNIA 93555

# Naval Weapons Center

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R. G. Freeman, III, RAdm., USN .............................................. Commander

G. L. Hollingsworth ............................................................ Technical Director

## FOREWORD

This interim technical report describes the mathematical derivation of a methodology for determining survival probabilities, exchange ratios, weapon usage, and other measures of effectiveness in air combat duels. Data inputs for the methodology can be from a variety of sources including digital, manned, and "real life" combat simulations. The methodology is capable of handling m on n type engagements.

This work was conducted under several tasks including: AirTask A03P-03P2/008B/3F32-311-000, Anti-Air Requirements Study, The Joint Short Range Air-to-Air Missile (JSRAAM) Project, and independent research funds.

The methodology described in this report is currently being programmed and will be available to continue evaluation of existing air combat data as well as being available for post analysis of the planned AIMVAL flight tests.

This report was reviewed for technical accuracy by Dr. E. A. Fay.

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(U) *Probability Distribution Associated With m on n Aerial Duels*, by Frank C. Reed. China Lake, Calif., Naval Weapons Center, December 1976. 52 pp. (NWC TP 5815, publication UNCLASSIFIED.)

(U) This report presents algorithms for determining the probability distributions of survivors and weapons expended in m on n simulated aerial combat. Both shoot-look-shoot and shoot-switch-shoot firing doctrines are considered. For shoot-look-shoot firing doctrines, a method of bounding numerical results due to the "ghost" effect in simulated combat is presented.

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## Section 1. INTRODUCTION

This report is written to derive survival probabilities, exchange ratios, weapons usage, and other measures of effectiveness in aerial duels based on simulated aerial combat. It is convenient to think of an m on n duel as taking place between two sides. Side I has m members and side II has n members. Simulated aerial combat geometry is generated by allowing all (m + n) participants to fly for the entire duration of the duel.

From the trajectories on these runs, each combatant is assigned firing opportunities consistent with missile flight characteristics, envelope considerations, an appropriate firing doctrine, and weapons loadout. The member of the side being fired at is assigned based on both geometry and previous firing history. Thus, an attempt is made to minimize overkill and spread kill potential uniformly over the history of the duel. The kill probability associated with the launch is based on relative geometries, single-shot kill probabilities and salvo size.

Unfortunately, an unavoidable difficulty is induced by the geometry. Whenever $m > 1$ or $n > 1$, it is possible for a combatant who would have died early in the engagement to continue to draw attention and fire from an adversary who would otherwise be free to engage viable targets. Thus, a pursuer has the chance of firing at "ghosts" or decoys, rather than directing his attention at real threats.

In addition to the geometry problem there is apparently some difficulty in making probability calculations correctly. Previous attempts to make the probability calculations using a recursive algorithm violate a critical independence assumption. A clear discussion of this fallacy in duel calculations is contained in a report by Lincicum.[1] It is a common error made by a number of contractors and government facilities. This report considers both aspects of these problems for general m on n duels. Section 2 provides a special algorithm which correctly obtains survival probabilities for the case $m = n = 1$. Weapons usage is also determined for this case.

[1] Naval Weapons Center. *Determinations of Kill Probabilities and Exchange Ratios for Multiple Firings by Two Combatants*, by L. Lincicum. China Lake, Calif., NWC, April 1973. (TN 3007-129, publication UNCLASSIFIED.)

Section 3 presents a different algorithm for the general m on n duel. The concept of $m/\alpha$ on $n/\beta$ duels is introduced as a method of handling the ghost problem in shoot-look-shoot aerial duels. The problem is then formulated as a Markov process with appropriate updating formulae at state space transition times. Survival probabilities and exchange ratio formulae are also presented.

Section 4 extends the approach discussed in Section 3 to include weapons usage. This extension is made at the cost of what appears to be an exceedingly large state space. In application, the actual size of the state space one must consider may be much smaller.

Section 5 discusses shoot-switch firing policies with updating formulae when the "ghost" problem is ignored. The problem of combining $m/\alpha$ on $n/\beta$ duels with a shoot-switch firing doctrine is discussed.

Appendixes A and B illustrate the updating formulae in Sections 3 and 4, respectively. The important observation in this example is that, even though the potential state space is quite large, only a small fraction of these states are ever used. Thus, a computer program which only introduces states as required may be quite efficient.

## Section 2. A SPECIAL ALGORITHM FOR 1 ON 1 DUELS

When $m = 1$ and $n = 1$, we are concerned with an important special case of m on n duels. The algorithm presented here is more efficient than the general algorithm of the next section and it is the preferred method of analysis of 1 on 1 duels. The "ghost" problem is not a consideration in 1 on 1 duels.

### 2.1 SURVIVAL PROBABILITIES FOR TWO DUELISTS

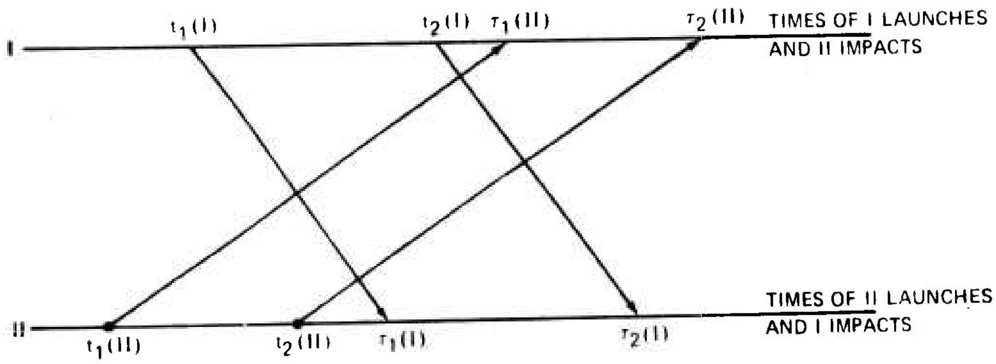
In this section, which is based on a report by Reed,[2] we describe an algorithm procedure for generating time-dependent survival functions for two duelists whom we call I and II. We allow time delay between launch and impact, so that the probability both duelists are killed may be greater than zero.

We assume from simulation records that I launches his jth missile at II at time $t_j(I)$, $j = 1,2,\cdots,r$, where $r \leq L_1$, the weapon load for I. We assume that the corresponding intercept times $\tau_j(I)$ are such that $\tau_j(I) \leq \tau_{j+1}(I)$ for all j and we allow $P_j(I)$, the kill probability of I's jth shot, to vary with j. Similarly, II launches his ith missile at I at times $t_i(II)$, $i = 1,2,\cdots,s \leq L_2$, II's weapon loadout. The ith impact of II's missiles occurs at $\tau_i(II)$ with $\tau_i(II) \leq \tau_{i+1}(II)$ for all i with probability of kill $P_i(II)$.

We define $P_{ij}$ as the probability that I is killed at time $\tau_i(II)$ and II is killed at time $\tau_j(I)$, for $j = 1,2,\cdots,r$; $i = 1,2,\cdots,s$. We define $P_{0j}$ as the probability II is killed at $\tau_j(I)$ and I survives the duel. Similarly, $P_{i0}$ is the probability I is killed at $\tau_i(II)$ and II survives the duel. We let $P_{00}$ be the probability that both I and II survive the duel.

The key to constructing a valid algorithm comes from considering a general diagram which describes the interactions between missile launches and subsequent impacts of the two duelists. As an example, consider the following figure:

[2] Naval Weapons Center. *Survival Probabilities for Two Duelists*, by F. C. Reed. China Lake, Calif., NWC, June 1972. (IDP 3381, publication UNCLASSIFIED.)




We say that the duel "ends" when one or the other duelist is killed. This does not mean that the first duelist killed is the only one killed. With this in mind, the first chance for the duel to "end" occurs with the first impact. In the figure above $\tau_1(I)$ is the earliest impact time.

The duel "ends" at $\tau_1(I)$ with probability $P_1(I)$. Given that the duel ends with II's death on this first impact, it is a simple probabilistic calculation to obtain $P_{i1}$ for $i = 1,2,\cdots,s$. One merely looks at the figure to see which launches II gets off before he is killed. In this example, II gets two shots off, so the appropriate conditional probabilities are

$$P_{11}|\text{II killed at } \tau_1(I) = P_1(II)$$

$$P_{21}|\text{II killed at } \tau_1(I) = (1 - P_1(II))\ P_2(II)$$

$$P_{i1}|\text{II killed at } \tau_1(I) = 0 \qquad\qquad 3 \le i \le s$$

$$P_{01}|\text{II killed at } \tau_1(I) = (1 - P_1(II))(1 - P_2(II))$$

The unconditional probabilities $P_{i1}$ for $0 \le i \le s$ are given by

$$P_{11} = P_1(I)\ P_1(II)$$

$$P_{21} = P_1(I)(1 - P_1(II))\ P_2(II)$$

$$P_{i1} = 0 \qquad i = 3,4,\cdots,s$$

$$P_{01} = P_1(I)(1 - P_1(II))(1 - P_2(II))$$

With probability $(1 - P_1(I))$ the duel does not end at $\tau_1(I)$. Thus, with probability $(1 - P_1(I))$ one gets a figure like that above with the vector $\overrightarrow{t_1(I)\tau_1(I)}$ eliminated. Proceeding in this way one generates all probabilities required, stopping when all vectors have been eliminated.

We now turn to an algorithm which formalizes these ideas. We let

$$t_{r+1}(I) = \tau_{r+1}(I) = t_{s+1}(II) = \tau_{s+1}(II) = T$$

where

$$T > \max\,[\tau_r(I),\ \tau_s(II)]$$

We also let

$$P_0(I) = P_0(II) = 0$$

For any $i_0, j_0$ we proceed as follows:

1. If $i_0 = s+1$ and $j_0 = r+1$, set

$$P_{00} = \prod_{i=0}^{s} (1 - P_i(II)) \prod_{j=0}^{r} (1 - P_j(I))$$

and end the algorithm; otherwise, go to 2.

2. If $\tau_{i_0}(II) \le \tau_{j_0}(I)$, go to 3; otherwise, go to 6.

3. Form

$$S(I) = \left| j : t_{j_0}(I) \le t_j(I) < \tau_{i_0}(II) \right|$$

If $S(I) = \phi$, go to 4; otherwise, go to 5.

4. Set

$$P_{i_0 0} = P_{i_0}(II) \prod_{i=0}^{i_0-1} (1 - P_i(II)) \prod_{j=0}^{j_0-1} (1 - P_j(I))$$

and go to 1 with $i_0$ replaced by $i_0+1$.

5. Set

$$P_{i_0 0} = P_{i_0}(II) \prod_{i=0}^{i_0-1} (1 - P_i(II)) \prod_{j=0}^{\max S(I)} (1 - P_j(I))$$

$$P_{i_0 j} = P_{i_0}(II)P_j(I) \prod_{i=0}^{i_0-1} (1 - P_i(II)) \prod_{k=0}^{j-1} (1 - P_k(I))$$

for $j \varepsilon S(I)$, and go to step 1 with $i_0$ replaced by $i_0+1$.

6. Form

$$S(II) = \left| i : t_{i_0}(II) \leq t_i(II) < \tau_{j_0}(I) \right|$$

If $S(II) = \phi$, go to 7; otherwise, go to 8.

7. Set

$$P_{0 j_0} = P_{j_0}(I) \prod_{i=0}^{i_0-1} (1 - P_i(II)) \prod_{j=0}^{j_0-1} (1 - P_j(I))$$

and go to step 1 with $j_0$ replaced by $j_0+1$.

8. Set

$$P_{0j_0} = P_{j_0}(I) \prod_{i=0}^{\max S(II)} (1 - P_i(II)) \prod_{j=0}^{j_0-1} (1 - P_j(I))$$

$$P_{ij_0} = P_{j_0}(I)P_i(II) \prod_{k=0}^{i-1} (1 - P_k(II)) \prod_{j=0}^{j_0-1} (1 - P_j(I))$$

for $i \varepsilon S(II)$ and go to step 1 with $j_0$ replaced by $j_0+1$.

The algorithm begins with $i_0 = j_0 = 1$.

Survival probabilities for the two duelists are easily obtained when the $P_{ij}$ have been obtained from the algorithm above. If $S_I(t)$ is the probability I survives through time t, we have

$$S_I(\tau_i(II)) = 1 - \sum_{k=1}^{i} \sum_{j=0}^{r} P_{kj}$$

and I's duel survival is

$$S_I = 1 - \sum_{k=1}^{s} \sum_{j=0}^{r} P_{kj} = \sum_{j=0}^{r} P_{0j}$$

Similarly, if $S_{II}(t)$ is II's probability of survival through t, we have

$$S_{II}(\tau_j(I)) = 1 - \sum_{i=0}^{s} \sum_{k=1}^{j} P_{ik}$$

and II's duel survival is

$$S_{II} = 1 - \sum_{i=0}^{s} \sum_{k=1}^{r} P_{ik} = \sum_{i=0}^{s} P_{i0}$$

It is important to notice that the above algorithm does not depend on a shoot-look-shoot firing doctrine. It only depends on the kth missile fired by I(II) not reaching II(I) before the k+rth missile fired by I(II) reaches II(I). We now turn to weapons usage.

## 2.2 WEAPONS USAGE

We define $X_1$ as the random variable which takes on value 1 if I is alive and 0 if I is dead. The random variable $X_2$ is defined similarly for II. The random variable $M_1$ takes on the integer values associated with I's use of weapons. The random variable $M_2$ is defined similarly for II. The joint distribution

$$P[X_1 = x_1, X_2 = x_2, M_1 = m_1, M_2 = m_2] \equiv P_{x_1x_2m_1m_2}$$

with

$$P_{11rs} = P_{00}$$

$$P_{10jL_2} = P_{0j} \qquad j = 1,2,\cdots,r$$

$$P_{01L_1i} = P_{i0} \qquad i = 1,2,\cdots,s$$

$$P_{00L_1L_2} = 1 - P_{00} - \sum_{j=1}^{r} P_{0j} - \sum_{i=1}^{s} P_{i0}$$

From $P_{x_1x_2m_1m_2}$ we may compute the following means, variances, and covariances:

$$E(X_1) = \sum_{j=0}^{r} P_{0j}$$

$$E(X_2) = \sum_{i=0}^{s} P_{i0}$$

$$Var(X_1) = \sum_{j=0}^{r} P_{0j} - \left(\sum_{j=0}^{r} P_{0j}\right)^2$$

$$Var(X_2) = \sum_{i=0}^{s} P_{i0} - \left(\sum_{i=0}^{s} P_{i0}\right)^2$$

$$Cov(X_1,X_2) = P_{00} - \left(\sum_{j=0}^{r} P_{0j}\right)\left(\sum_{i=0}^{s} P_{i0}\right)$$

$$E(M_1) = rP_{00} + \sum_{j=1}^{r} jP_{0j} + L_1\left(1 - \sum_{j=0}^{r} P_{0j}\right)$$

$$E(M_2) = sP_{00} + \sum_{i=1}^{s} iP_{i0} + L_2\left(1 - \sum_{i=0}^{s} P_{i0}\right)$$

$$Var(M_1) = r^2P_{00} + \sum_{j=1}^{r} j^2P_{0j} + L_1^2\left(1 - \sum_{j=0}^{r} P_{0j}\right) - E^2(M_1)$$

$$Var(M_2) = s^2P_{00} + \sum_{i=1}^{s} i^2P_{i0} + L_2^2\left(1 - \sum_{i=0}^{s} P_{i0}\right) - E^2(M_2)$$

$$\mathrm{Cov}(M_1M_2) = rsP_{00} + L_2 \sum_{j=1}^{r} jP_{0j} + L_1 \sum_{i=1}^{s} iP_{i0}$$

$$+ L_1L_2 \left(1 - P_{00} - \sum_{j=1}^{r} P_{0j} - \sum_{i=1}^{s} P_{i0}\right) - E(M_1)E(M_2)$$

$$\mathrm{Cov}(X_1M_1) = rP_{00} + \sum_{j=1}^{r} jP_{0j} - E(X_1)E(M_1)$$

$$\mathrm{Cov}(X_2M_2) = sP_{00} + \sum_{i=1}^{s} iP_{i0} - E(X_2)E(M_2)$$

$$\mathrm{Cov}(X_1M_2) = sP_{00} + L_2 \sum_{j=1}^{r} P_{0j} - E(X_1)E(M_2)$$

$$\mathrm{Cov}(X_2M_1) = rP_{00} + L_1 \sum_{i=1}^{s} P_{i0} - E(X_2)E(M_2)$$

# Section 3. SURVIVAL PROBABILITIES FOR m ON n SHOOT-LOOK-SHOOT DUELS

We now want to consider more general m on n shoot-look-shoot duels. For this report shoot-look-shoot implies each combatant may have at most one missile in flight at a time. The procedure will be to define a state space for which the Markov assumption holds and then provide the probabilistic updating required. First, we introduce the concept of $m/\alpha$ on $n/\beta$ duels as a means for handling the "ghost" problem.

## 3.1 PESSIMISTIC/OPTIMISTIC FIRING DOCTRINE AND $m/\alpha$ ON $n/\beta$ DUELS

From the simulated aerial engagement, missile simulation, and rules of assignment, one obtains a time history $t_1 \leq t_2 \cdots \leq t_N$ where each $t_k$ is either a missile launch or impact. For each $t_k$ the combatant firing the missile salvo, the intended victim, and missile salvo kill probability are all known. The victim is specified by the geometry of the engagement and does not account for the fact that he may not be alive. This means that whenever a combatant is being shot at for the ith time where $i \geq 2$ and kill probabilities are greater than zero, there is a nonzero probability that the shot is wasted. We think of such a constraint as a "pessimistic firing doctrine," since it is pessimistic from the point of view of the side doing the firing.

This geometric inaccuracy affects both sides and one would like to think the results have a balancing effect. Intuitively one would expect the effect to be more serious for the better side. Every time the better side kills an opponent it is replaced by a nonlethal decoy. These nonlethal decoys tend to increase the survivability of opponents of the better side. For 1 on 1 duels the "ghost" effect is not serious, for a nonlethal decoy does not degrade the survivability of the winner.

To compensate for the geometry inadequacy, we consider different target allocations than those implied by the geometry. Targets are assigned using an assumption that is easy to implement with the existing data base. One merely assumes that if, in the original geometry, the target is dead when a missile is launched at him, then this firing may be converted to a live target, if there is one. The ability to always convert is called an "optimistic firing doctrine" and intuitively it gives the pursuer more capability than he would have in a real duel.

We expand on this concept by defining an $m/\alpha$ on $n/\beta$ duel as an m on n duel in which side I has optimistic firings with probability $\alpha$, side II has optimistic firings with probability $\beta$, and pessimistic firings occur with the complements of these probabilities. If $E(m/\alpha, n/\beta)$ is the measure of effectiveness for side I associated with an $m/\alpha$ on $n/\beta$ duel and $E(m, n)$ is the unknown measure of effectiveness associated with an m on n duel with the geometry deficiency removed, then intuitively,

$$E(m/0, n/1) \leq E(m, n) \leq E(m/1, n/0)$$

This should allow one to bracket the payoff to side I with two duel calculations. Both $E(m/0, n/0)$ and $E(m/1, n/1)$ will lie in this interval and one wonders if there exists an $\alpha_0$ and $\beta_0$ so that $E(m/\alpha_0, n/\beta_0)$ best estimates $E(m, n)$ over a wide class of m on n duels.

## 3.2 A MARKOV CHAIN FORMULATION FOR HANDLING m/α ON n/β DUELS

We recall that the basic inputs to the problem are times $t_1 \leq t_2 \leq \cdots \leq t_N$ where each $t_k$ is either an impact or launch time. Moreover, for each $t_k$ the launcher, target, and kill probability are known. Our goal in this section is to define an appropriate state space so that the stochastic process defined on this state space at times $t_k$ satisfies the Markov assumption. Thus one may update probability distributions at time $t_k$ by knowing the distributions at time $t_{k-1}$ and the appropriate transition probabilities.

To formulate the problem as a Markov chain, one must be careful to construct a state space for which the Markov assumption holds. Once this is done the problem is merely one of properly updating the probability distributions of the state of the process at launch and impact times. We define

$$X_1 = (x_{11}, x_{12}, \cdots, x_{1m})$$

where $x_{1i} = 1$ if the ith member of side I is alive and $x_{1i} = 0$ if the ith member of side I is dead. The components of vector

$$X_2 = (x_{21}, \cdots, x_{2n})$$

are defined similarly for side II. We also have

$$T_1 = (t_{11}, t_{12}, \cdots, t_{1m})$$

where $t_{1i} = 0$ if the ith member of side I has no missile in flight and $t_{1i} = j$ if the missile of the ith member of side I is directed at the jth member of side II. In a similar way,

$$T_2 = (t_{21}, \cdots, t_{2n})$$

where $t_{2j} = 0, 1, \cdots, m$. The state space is determined by the vector $(X_1, X_2, T_1, T_2)$ and the associated Markov process has $2^{m+n}(n + 1)^m(m + 1)^n$ states. Clearly, if m and n are large, the number of states is large.

In order to reduce the size of the state space there is a real temptation to let $(X_1, X_2)$ describe the state space. If, however, the missile of element i of side I is to impact element j of side II at a transition time and if $x_{1i} = 0$, we do not know whether to allow the impact or not. Element i of side I may or may not have been alive at the time of the missile launch. The state space description described here is slightly more generous than actually required; however, this state space description allows us to assess the effect of the geometry deficiency described previously.

We are now ready to consider the updating of the probability distribution defined on the state space of the Markov process with the possibility of optimistic firings. It must be stressed that, with optimistic firings, targets other than those implied by the geometry may come under consideration. The state space we have constructed always allows us to decide who the target is, even if optimistic firings are considered.

We shall make use of the following notation:

| | |
|---|---|
| $L_r$ | An r-vector of 1's |
| $0_r$ | An r-vector of 0's |
| $\vec{\delta}_{r_i}$ | An r-vector of 0's with the exception of component i which is a 1 |
| $\delta_x$ | A scalar which is 0 if $x = 0$ and 1 if $x > 0$ |
| $P_{1i}(t_k)$ | Kill probability of salvo launched by member i of side I and impacting at $t_k$ |
| $P_{2j}(t_k)$ | Kill probability of salvo launched by member j of side II and impacting at $t_k$ |

For $s = 0, 1, \cdots, r-1$, we define

$$
\begin{aligned}
k_r(i, s) &= i + s && \text{if } i + s \leq r \\
&= i + s - r && \text{if } i + s > r
\end{aligned}
$$

With respect to side I we define $s_1{}^*$ to be the smallest integer s for which

$$x_{1k_m(i,\, s_1{}^*)} = 1$$

If

$$X_1 = 0_m$$

then

$$k_m(i, s_1^*) \equiv 0$$

With respect to side II, we define $s_2^*$ to be the smallest integer s for which

$$x_{2k_n}(j, s_2^*) = 1$$

If

$$X_2 = 0_n$$

then

$$k_n(j, s_2^*) \equiv 0$$

We define $P_{X_1X_2T_1T_2}(t)$ to be the probability that the stochastic process under consideration is in state $X_1, X_2, T_1, T_2$ at time t. Now for all $t_k$ we want to take $P_{X_1X_2T_1T_2}(t_{k-1})$, the probability distribution defined on the state space at $t_{k-1}$ and update it based on the Markov property to obtain $P_{X_1X_2T_1T_2}(t_k)$. We assume that at $t_0 < t_1$

$$P_{L_m L_n 0_m 0_n}(t_0) = 1$$

Now assume that at $t_k$ member i of side I shoots (according to the geometry) at member j of side II. Assuming $P_{X_1X_2T_1T_2}(t_{k-1})$ is known then the probability updating can be given as

$$P_{X_1X_2T_1T_2}(t_k) = \left(1 - \delta_{x_{1i}}\right)P_{X_1X_2T_1T_2}(t_{k-1})$$

$$+ \delta_{x_{1i}}\delta_{x_{2j}}P_{X_1X_2T_1-j\vec{\delta}_{m_i}T_2}(t_{k-1})$$

$$+ \delta_{x_{1i}}\left(1 - \delta_{x_{2j}}\right)(1 - \alpha)P_{X_1X_2T_1-j\vec{\delta}_{m_i}T_2}(t_{k-1})$$

$$+ \delta_{x_{1i}}\left(1 - \delta_{x_{2j}}\right)\alpha P_{X_1X_2T_1-k_n(j,s_2*)\vec{\delta}_{m_i}T_2}(t_{k-1})$$

This updating breaks down into particular cases as follows:

1. If at $t_{k-1}$, $x_{1i} = 0$

$$P_{X_1X_2T_1T_2}(t_k) = P_{X_1X_2T_1T_2}(t_{k-1})$$

2. If at $t_{k-1}$, $x_{1i} = 1$ and $x_{2j} = 1$

$$P_{X_1X_2T_1+j\vec{\delta}_{m_i}T_2}(t_k) = P_{X_1X_2T_1T_2}(t_{k-1})$$

3. If at $t_{k-1}$, $x_{1i} = 1$ and $x_{2j} = 0$

$$P_{X_1X_2T_1+j\vec{\delta}_{m_i}T_2}(t_k) = (1 - \alpha)P_{X_1X_2T_1T_2}(t_{k-1})$$

$$P_{X_1X_2T_1+k_n(j,s_2*)\vec{\delta}_{m_i}T_2}(t_k) = \alpha P_{X_1X_2T_1T_2}(t_{k-1})$$

A combination of these terms gives

$$\begin{aligned} P_{X_1X_2T_1T_2}(t_k) &= \left(1 - \delta_{x_{1i}}\right)P_{X_1X_2T_1T_2}(t_{k-1}) \\ &+ \delta_{x_{1i}}\left(1 - \alpha\left(1 - \delta_{x_{2j}}\right)\right)P_{X_1X_2T_1-j\vec{\delta}_{m_i}T_2}(t_{k-1}) \\ &+ \delta_{x_{1i}}\alpha\left(1 - \delta_{x_{2j}}\right)P_{X_1X_2T_1-k_n(j,s_2*)\vec{\delta}_{m_i}T_2}(t_{k-1}) \qquad (3.1) \end{aligned}$$

In a similar manner, if at $t_k$ member j of side II shoots (according to the geometry) at member i of side I, we have the following updating formula:

$$
\begin{aligned}
P_{X_1X_2T_1T_2}(t_k) &= \left(1 - \delta_{x_{2j}}\right) P_{X_1X_2T_1T_2}(t_{k-1}) \\
&+ \delta_{x_{2j}}\left(1 - \beta\left(1 - \delta_{x_{1i}}\right)\right) P_{X_1X_2T_1T_2 - i\vec{\delta}_{n_j}}(t_{k-1}) \\
&+ \delta_{x_{2j}} \beta\left(1 - \delta_{x_{1i}}\right) P_{X_1X_2T_1T_2 - k_m(j,s_1*)\vec{\delta}_{n_j}}(t_{k-1})
\end{aligned}
\tag{3.2}
$$

These launch formulae allow for the possibility of continued missile launchings by one side even after all members of the other side are killed. This does not affect survival results; however, it is not realistic in terms of missile usage. More complex updating formulae will be introduced for missile usage in Section 4.

Suppose that, from the geometry run, $t_k$ is a time point at which the missile of element i of side I is to impact some member of side II. The original geometry would have specified the member of side II receiving the impact. Previous impacts on this particular member of side II and the assumption of optimistic firings lead to the possibility that any member of side II may be the target. With this in mind, we consider the following updating:

$$
\begin{aligned}
P_{X_1X_2T_1T_2}(t_k) &= P_{X_1X_2T_1T_2}(t_{k-1}) \\
&+ \sum_{j=1}^{n}\left(1 - \delta_{x_{2j}}\right) P_{X_1X_2T_1+j\vec{\delta}_{m_i}T_2}(t_{k-1}) \\
&+ \sum_{j=1}^{n} \delta_{x_{2j}} (1 - P_{1i}(t_k)) P_{X_1X_2T_1+j\vec{\delta}_{m_i}T_2}(t_{k-1}) \\
&+ \sum_{j=1}^{n}\left(1 - \delta_{x_{2j}}\right) P_{1i}(t_k) P_{X_1X_2+\vec{\delta}_{n_j}T_1+j\vec{\delta}_{m_i}T_2}(t_{k-1})
\end{aligned}
$$

where $t_{1i}$, the ith component of $T_1$, is 0. The above updating formula may be condensed to yield

$$P_{X_1X_2T_1T_2}(t_k) = P_{X_1X_2T_1T_2}(t_{k-1})$$

$$+ \sum_{j=1}^{n} \left(1 - \delta_{x_{2j}} P_{1i}(t_k)\right) P_{X_1X_2T_1+j\vec{\delta}_{m_i}T_2}(t_{k-1})$$

$$+ \sum_{j=1}^{n} \left(1 - \delta_{x_{2j}}\right) P_{1i}(t_k) P_{X_1X_2+\vec{\delta}_{n_j}T_1+j\vec{\delta}_{m_i}T_2}(t_{k-1}) \qquad (3.3)$$

where $t_{1i} = 0$.

In a similar manner, if the missile fired by member j of side II is to impact side I at $t_k$, then

$$P_{X_1X_2T_1T_2}(t_k) = P_{X_1X_2T_1T_2}(t_{k-1})$$

$$+ \sum_{i=1}^{m} \left(1 - \delta_{x_{1i}} P_{2j}(t_k)\right) P_{X_1X_2T_1T_2+i\vec{\delta}_{n_j}}(t_{k-1})$$

$$+ \sum_{i=1}^{m} \left(1 - \delta_{x_{1i}}\right) P_{2j}(t_k) P_{X_1+\vec{\delta}_{m_i}X_2T_1T_2+i\vec{\delta}_{n_j}}(t_{k-1}) \qquad (3.4)$$

where $t_{2j} = 0$.

The updating for impacts is not nearly so straightforward as in launchings. Consider, for example, a 2 on 2 updating where member 1 of side I has launched at side II with an impact updating at $t_k$. Letting $T_1 = T_2 = 0_2$ at $t_k$ and assuming we want $P_{X_1(1,0)0_20_2}(t_k)$, we have

$$P_{X_1(1,0)0_20_2}(t_k) = P_{X_1(1,0)(0,0)(0,0)}(t_{k-1})$$

$$+ (1 - P_{11}(t_k))P_{X_1(1,0)(1,0)0_2}(t_{k-1})$$

$$+ P_{X_1(1,0)(2,0)0_2}(t_{k-1})$$

$$+ P_{11}(t_k)P_{X_1(1,1)(2,0)0_2}(t_{k-1})$$

A number of events may contribute to the calculation under consideration and all events must be accounted for properly.

If at $t_N$ the last possible impact occurs, we obtain $P_{X_1X_20_m0_n}(t_N)$ which we redefine as $P_{X_1X_2}$.

The joint probability of survival of member $i_1$, $i_2$, $\cdots$, $i_r$ of side I and $j_1$, $j_2$, $\cdots$, $j_s$ of side II is given by

$$P\left(x_{1i_1}, x_{1i_2}, \cdots x_{1i_r}, x_{2j_1}, \cdots x_{2j_s}\right) =$$

$$\sum_{x_{11}=0}^{1} \sum_{x_{12}=0}^{1} \cdots \sum_{x_{1m}=0}^{1} \sum_{x_{21}=0}^{1} \sum_{x_{22}=0}^{1} \cdots \sum_{x_{2n}=0}^{1} x_{1i_1} \cdots x_{1i_r} x_{21} \cdots x_{2j_s} P_{X_1X_2}$$

In particular, $P(x_{1i})$ is the probability of survival of the ith member of side I, and $P(x_{2j})$ is the probability of survival of the jth member of side II.

The expected number of survivors on side I is

$$E(X_1) = \sum_{x_{11}=0}^{1} \cdots \sum_{x_{2n}=0}^{1} (x_{11} + x_{12} + \cdots x_{1m}) P_{X_1X_2}$$

$$= \sum_{i=1}^{m} P(x_{1i})$$

The variance in the number of survivors on side I is

$$Var(X_1) = \sum_{x_{11}=0}^{1} \cdots \sum_{x_{2n}=0}^{1} (x_{11} + x_{12} + \cdots + x_{1m})^2 P_{X_1X_2} - E^2(X_1)$$

Similarly for side II,

$$E(X_2) = \sum_{j=1}^{n} P(x_{2j})$$

$$Var(X_2) = \sum_{x_{11}=0}^{1} \cdots \sum_{x_{2n}=0}^{1} (x_{21} + \cdots + x_{2n})^2 P_{X_1X_2} - E^2(X_2)$$

The covariance in number of survivors on sides I and II is

$$Cov(X_1X_2) = \sum_{x_{11}=0}^{1} \cdots \sum_{x_{2n}=0}^{1} (x_{11} + \cdots + x_{1m})(x_{21} + \cdots + x_{2n}) P_{X_1X_2} - E(X_1)E(X_2)$$

An important derived measure of effectiveness is side I's exchange ratio,

$$\theta = \frac{n - E(X_2)}{m - E(X_1)}$$

## Section 4. WEAPONS USAGE FOR m ON n SHOOT-LOOK-SHOOT DUELS

There is a straightforward way of expanding the state space to keep track of missile usage. One merely lets the state space be determined by the vector $(X_1,X_2,T_1,T_2,M_1,M_2)$ where $X_1,X_2,T_1$, and $T_2$ are defined in Section 3 and $M_1$ is an m-vector where $M_{1i}$, the ith component of $M_1$, is the number of missiles combatant i on side I has used. Similarly, $M_2$ is an n-vector associated with missile usage of side II.

If $L_1$ is the missile loadout for elements of side I and if $L_2$ is the missile loadout for elements of side II, then the state space has $2^{m+n}(n+1)^m(m+1)^n(L_1+1)^m(L_2+1)^n$ states. If, for example, m=n=2 and $L_1=L_2=4$ there are 810,000 states in the state space.

The numerical example in Appendix B suggests that the situation may not be as bad as one might expect. If one only introduces states as required, the actual number of states considered may not be excessive at all. If it should turn out that state space size starts to be a problem, then Monte Carlo procedures should be considered for looking at this problem for moderate m, n, and weapons loadout. The Monte Carlo approach to this problem will be considered in a later report.

### 4.1 UPDATING FORMULAE

For weapons usage formulae in this and the next section we have occasion to use the standard Kronecker delta,

$$\delta_{m_{1i}}^{L_1} = 1 \qquad \text{if } m_{1i} = L_1$$

$$= 0 \qquad \text{if } m_{1i} \neq L_1$$

Also,

$$\delta_{m_{2j}}^{L_2} = 1 \qquad \text{if } m_{2j} = L_2$$

$$= 0 \qquad \text{if } m_{2j} \neq L_2$$

The updating formulae are straightforward extensions of Equations 3.1, 3.2, 3.3, and 3.4 in Section 3. To obtain $P_{X_1X_2T_1T_2M_1M_2}(t)$ for any t, we assume

$$P_{L_m L_n 0_m 0_n 0_m 0_n}(t_0) = 1$$

where $t_0 < t_1$. Now assume $P_{X_1X_2T_1T_2M_1M_2}(t_{k-1})$ is known.

If, at $t_k$, member i of side I shoots, according to the geometry, at member j of side II, the updating formula (Equation 3.1) is modified as follows:

$$P_{X_1X_2T_1T_2M_1M_2}(t_k) = \left(1 - \delta_{x_{1i}}\right)P_{X_1X_2T_1T_2M_1M_2}(t_{k-1})$$

$$+ \delta_{x_{1i}}\left(1 - \alpha\left(1 - \delta_{x_{2j}}\right)\right)P_{X_1X_2T_1-j\vec{\delta}_{m_i}T_2M_1-\vec{\delta}_{m_i}M_2}(t_{k-1})$$

$$+ \delta_{x_{1i}}\alpha\left(1 - \delta_{x_{2j}}\right)P_{X_1X_2T_1-k_n(j,s_2^*)\vec{\delta}_{m_i}T_2M_1-\delta_{k_n(j,s_2^*)}\vec{\delta}_{m_i}M_2}(t_{k-1})$$

(4.1)

One may modify Equation 3.2 in a similar way to obtain a corresponding formula when member j of side II shoots at member i of side I.

If, at $t_k$, a missile of element i of side I is to impact some member of side II, we modify Equation 3.3 to obtain

$$P_{X_1X_2T_1T_2M_1M_2}(t_k) = P_{X_1X_2T_1T_2M_1M_2}(t_{k-1})$$

$$+ \sum_{j=1}^{n}\left(1 - \delta_{x_{2j}}P_{1i}(t_k)\right)P_{X_1X_2T_1+j\vec{\delta}_{m_i}T_2M_1M_2}(t_{k-1})$$

$$+ \sum_{r=0}^{L_2}\sum_{j=1}^{n}\delta_{m_{2j}}^{L_2}\left(1 - \delta_{x_{2j}}\right)P_{1i}(t_k)P_{X_1X_2+\vec{\delta}_{n_j}T_1+j\vec{\delta}_{m_i}T_2M_1M_2-(L_2-r)\vec{\delta}_{m_j}}(t_{k-1})$$

(4.2)

where $t_{1i} = 0$. A similar modification of Equation 3.4 is made when a missile of element j of side II is to impact some member of side I.

## 4.2 JOINT PROBABILITY DISTRIBUTIONS OF SURVIVORS AND MISSILES USED IN AN $m/\alpha$ ON $n/\beta$ DUEL

Finally at $t_N$, $T_1 = T_2 = 0$ and one obtains $P_{X_1X_2M_1M_2}$, the joint probability distribution of survivors and missile usage. Various statistical quantities of interest are

$$E(M_1) = \sum_{X_1X_2M_1M_2} (m_{11} + m_{12} + \cdots + m_{1m})P_{X_1X_2M_1M_2}$$

the expected number of missiles used by side I and

$$E(M_2) = \sum_{X_1X_2M_1M_2} (m_{21} + m_{22} + \cdots + m_{2n})P_{X_1X_2M_1M_2}$$

the expected number of missiles used by side II.

The corresponding variances are

$$\mathrm{Var}(M_1) = \sum_{X_1X_2M_1M_2} \left(\sum_{i=1}^{m} m_{1i}\right)^2 P_{X_1X_2M_1M_2} - E^2(M_1)$$

$$\mathrm{Var}(M_2) = \sum_{X_1X_2M_1M_2} \left(\sum_{j=1}^{n} m_{2j}\right)^2 P_{X_1X_2M_1M_2} - E^2(M_2)$$

It is convenient to set $K_1 = m$ and $K_2 = n$. Now various covariances may be computed.

$$\mathrm{Cov}(X_rM_s) = \sum_{X_1X_2M_1M_2} \left(\sum_{k=1}^{K_r} x_{rk}\right)\left(\sum_{k=1}^{K_s} m_{sk}\right)P_{X_1X_2M_1M_2} - E(X_r)E(M_s)$$

where $E(X_r)$ is given in Section 3 or may be computed directly as

$$E(X_r) = \sum_{X_1X_2M_1M_2} \left(\sum_{k=1}^{K_r} x_{rk}\right)P_{X_1X_2M_1M_2}$$

We also have

$$\mathrm{Cov}(M_1M_2) = \sum_{X_1X_2M_1M_2} \left(\sum_{i=1}^{m} m_{1i}\right)\left(\sum_{j=1}^{n} m_{2j}\right)P_{X_1X_2M_1M_2} - E(M_1)E(M_2)$$

# Section 5. SHOOT-SWITCH FIRING DOCTRINES

In this section we consider firing doctrines that allow a combatant to switch and fire at members of the opposing side while his missiles are in flight. The only restriction is that combatant i of one side cannot fire a missile at combatant j of the other side if he has a missile in flight at j. Providing $L_1 \geq n$, any member of side I can have as many as n missiles in flight at a time. Providing $L_2 \geq m$, any member of side II can have as many as m missiles in flight at a time.

The vectors $X_1$, $X_2$, $M_1$, and $M_2$ are defined as in previous sections. We replace vectors $T_1$ and $T_2$ by the m by n matrix, $\mathbf{T}_1$, and the n by m matrix, $\mathbf{T}_2$. The entry in the $\alpha$th row and $\beta$th column of matrix $\mathbf{T}_k$, $t_{k\alpha\beta}$, is 1 if combatant $\alpha$ on side k has a missile in flight at combatant $\beta$ on the other side. This entry is 0 otherwise.

With this notation the size of the state space is $2^{m+n+2mn}(L_1+1)^m (L_2+1)^n$. This is larger than the state space of Section 4 by a factor of

$$2^{2mn}/(n+1)^m(m+1)^n = \left(\frac{2^n}{n+1}\right)^m \left(\frac{2^m}{m+1}\right)^n \geq 1$$

with exponential growth in m and n.

For m=n=1, the ratio is 1; for m=n=2, the ratio is 3.16; and for m=n=3, the ratio is 64. Thus, the potential size of the state space can increase dramatically for shoot-switch firing doctrines. Appendixes A and B indicate that the number of states with non-zero probabilities may be relatively small for shoot-look-shoot firing doctrines. In fact, those numerical examples indicate that the total number of states that occur with non-zero probabilities in the entire probability updating is relatively small.

## 5.1 BOUNDS ON THE NUMBER OF NON-ZERO PROBABILITY STATES*

We assume that a duel consists of at most 2NS events, the NS launches and NS impacts for all participants. It is also assumed that, at each launch event, each state with a non-zero probability gives rise to exactly one state with a non-zero probability. Generally speaking, $\mathbf{T}_k$ and $M_k$ are updated one state for one depending on $X_1X_2\mathbf{T}_1\mathbf{T}_2M_1M_2$ at the launch time.

---

* This section is based on an analysis of the state space size problem by Dr. William Alltop of the Mathematical Services Branch, NWC.

Since the growth of non-zero probabilities can only occur at impact events, the nature of this growth can be determined by considering a single component i of $X_1$. Suppose i of side I is impacted $S_i$ times and all other components of $X_1$ and $X_2$ are 1. The vector $X_1$ with ith component equal to 1 (not killed) is a possible state on each impact. This same vector with ith component equal to 0 (killed) may occur for the first time on any of the $S_i$ impacts. In general, the impact number associated with the first time the zero occurs distinguishes different non-zero probability states. Consequently the vector $(X_1,X_2)$ with a 0 in the ith component of $X_1$ and 1's elsewhere is associated with $S_i$ non-zero probability states each having different entries in $T_1T_2M_1M_2$. That is, for $(X_1,X_2)$ as described there are $S_i$ different states $(X_1X_2T_1T_2M_1M_2)$ identifiable in the total state space.

More generally, if $S_j^*$ is the number of times participant j on side II is impacted and it can be assumed that the appropriate $t_{1ij}$ and $t_{2ji}$ are all 1 for all impacts, then the number of states with non-zero probabilities associated with each $(X_1,X_2)$ is

$$\prod_{i=1}^{m} S_i^{(1-\delta_{x_{1i}})} \prod_{j=1}^{n} S_j^{*(1-\delta_{x_{2j}})} .$$

A simple inductive argument shows that

$$\sum_{x_{11}=0}^{1} \cdots \sum_{x_{1m}=0}^{1} \sum_{x_{21}=0}^{1} \cdots \sum_{x_{2n}=0}^{1} \prod_{i=1}^{m} S_i^{(1-\delta_{x_{1i}})} \prod_{j=1}^{n} S_j^{*(1-\delta_{x_{2j}})} =$$

$$\prod_{i=1}^{m} (1 + S_i) \prod_{j=1}^{n} (1 + S_j^*)$$

When $t_{1ij}$ or $t_{2ji}$ is 0, the possibility for growth does not exist. Hence, if $|N_s|$ is the number of non-zero probability states at the final updating,

$$|N_s| \le \prod_{i=1}^{m} (1 + S_i) \prod_{j=1}^{n} (1 + S_j^*)$$

One may show that $|N_s|$ is maximized when impacts are spread as evenly as possible; hence,

$$|N_s| \le \left(1 + \frac{NS}{m+n}\right)^{m+n} .$$

If $L_1 = L_2 = 4$, $m = n = 2$, $NS = mL_1 + nL_2 = 16$,

$$|N_s| \le 625$$

considerably less than 2.5 million. We now turn to updating formulae for a number of cases.

## 5.2 SHOOT-SWITCH WITH $\alpha = 0$, $\beta = 0$

With $\alpha = \beta = 0$, every missile scheduled for firing is fired, if the attacker is alive. This occurs regardless of whether the target is alive or dead. In this case the duel consists of at most $mL_1 + nL_2$ launches and impacts. The only way a participant can end up with missiles is if he is not scheduled to fire all missiles and he survives the duel.

The bound in this case is

$$|N_s| \le \left(1 + \frac{mL_1}{n}\right)^n \left(1 + \frac{nL_2}{m}\right)^m$$

The comments about the $t_{1ij}$ or $t_{2ji}$ at impact apply here. Also, the fact that impact numbers will not be exactly $mL_1/n$ or $nL_2/m$ implies that this bound could be very conservative.

For this section we define $\delta_{rs}^{\alpha\beta}$ to be an r by s matrix of 0's with the exception that the element in the $\alpha$th row and $\beta$th column is a 1. Assuming that the geometry is such that no combatant is allowed to fire more missiles than his missile load, we arrive at updating formulae similar to those in Section 4 when $\alpha = \beta = 0$.

If, at $t_k$, i of side I shoots at j of side II, then

$$P_{X_1 X_2 \mathbf{T}_1 \mathbf{T}_2 M_1 M_2}(t_k) = \left(1 - \delta_{x_{1i}}\right) P_{X_1 X_2 \mathbf{T}_1 \mathbf{T}_2 M_1 M_2}(t_{k-1})$$

$$+ \delta_{x_{1i}} P_{X_1 X_2 \mathbf{T}_1 - \delta_{mn}^{ij} \mathbf{T}_2 M_1 - \vec{\delta}_{m_i} M_2}(t_{k-1}) \qquad (5.1)$$

If, at $t_k$, i of side I impacts j of side II,

$$P_{X_1X_2\mathbf{T}_1\mathbf{T}_2M_1M_2}(t_k) = P_{X_1X_2\mathbf{T}_1\mathbf{T}_2M_1M_2}(t_{k-1})$$

$$+ \left(1 - \delta_{x_{2j}} P_{1i}(t_k)\right) P_{X_1X_2\mathbf{T}_1+\delta^{ij}_{mn}\mathbf{T}_2M_1M_2}(t_{k-1})$$

$$+ \left(1 - \delta_{x_{2j}}\right)\delta^{L_2}_{m_{2j}} P_{1i}(t_k) \sum_{k=0}^{L_2} P_{X_1X_2+\vec{\delta}_j\mathbf{T}_1+\delta^{ij}_{mn}\mathbf{T}_2M_1M_2-(L_2-k)\vec{\delta}_j}(t_{k-1}) \tag{5.2}$$

where $t_{1ij}=0$.

Similar expressions exist when j of side II launches at i of side I and when i of side I incurs an impact of j of side II.

## 5.3 MODIFIED SHOOT-SWITCH, $\alpha = 0$, $\beta = 0$

By a modified shoot-switch firing doctrine with $\alpha = 0$, $\beta = 0$ we mean one that allows a combatant to save his missile if his opponent in the firing schedule is dead. This does not compensate for the ghost problem, but it does allow one to conserve missiles. As one might expect, the updating is more complex and the duel does not necessarily end with the $(mL_1 + nL_2)$th impact. In fact, with non-zero probability, each impact implied by the geometry will give rise to some new non-zero (perhaps very small) probability updating.

Letting $S_I$ be the total number of possible impacts on side I and $S_{II}$ be the total number of possible impacts on side II, the bound on the number of non-zero probability states is

$$|N_s| \le (1 + S_I/m)^m (1 + S_{II}/n)^n$$

If, for example, $S_I = S_{II} = 16$ and $m = n = 2$, then $|N_s| \le 6561$. However, $S_I = S_{II} = 40$ and $m = n = 2$ yield $|N_s| \le 194{,}481$. As the duel progresses, more and more $t_{1ij}$ and $t_{2ji}$ will be 0, reducing this bound considerably. One procedure is to set aside a fixed number of storage locations (say 10,000) and stop the updating when those locations are fully utilized or when the geometry indicates the duel has ended. In most applications the joint probability distribution of $(X_1, X_2, M_1, M_2)$ at such a stopping point will differ only slightly from that same distribution at the conclusion of the duel.

The updating formulae for i of I against j of II of the preceding section must be modified to reflect these considerations. If, at $t_k$, i of I launches at j of II, then

$$P_{X_1X_2T_1T_2M_1M_2}(t_k) = \left(1 - \delta_{x_{1i}} \delta_{x_{2j}} \left(1 - \delta_{m_{1i}}^{L_1}\right)\right) P_{X_1X_2T_1T_2M_1M_2}(t_{k-1})$$

$$+ \delta_{x_{1i}} \delta_{x_{2j}} \left(1 - \delta_{m_{1i}}^{L_1}\right) P_{X_1X_2T_1-\delta_{mn}^{ij}T_2M_1-\vec{\delta}_{m_i}M_2}(t_{k-1}) \qquad (5.3)$$

If, at $t_k$, i of I impacts j of II, then the updating formula is identical to Equation 5.2.

## 5.4 SHOOT-SWITCH, $\alpha \geq 0$, $\beta \geq 0$

For the general case when $\alpha \geq 0$ and $\beta \geq 0$, the problem gets out of hand. The reader should note that, if $t_k$ is a time when a missile of i of I is to impact j of II, $x_{2j} = 0$, $t_{1ij} = 0$, and $t_{1ij_k} = 1$ for some $j_k \neq j$, $k = 1,2,\cdots$, there is no way of knowing which one, if any, of these $j_k$ is the consequence of an optimistic firing when i was to fire at j. This presents a real problem on how to update on impact. It can apparently only be handled by expanding the state space to carry this type of information.

One possibility is to develop special updating algorithms for 2 on 1, 1 on 2, and 2 on 2 duels. Another possibility is to resort to Monte Carlo sampling procedures. More consideration will be given to this problem at a later time if the concepts of optimistic firings and shoot-switch policies both have merit within the framework of analyzing aerial combat data.

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## Appendix A

## A NUMERICAL EXAMPLE

To illustrate the use of the formulae in Section 3, we shall discuss in some detail the calculations associated with a sample 2 on 2 duel where each combatant fires, at most, two missiles. We shall use the following abbreviations:

| | |
|---|---|
| L | A launch event |
| H | An impact event |
| i of I | Member i of side I |
| j of II | Member j of side II |
| $\xrightarrow{L}$ | Launches at |
| $\xrightarrow{H}$ | Missile impacts |

Thus, i of I $\xrightarrow{H}$ j of II is read as member i of side I missile impacts member j of side II.

A typical time history from imaginary aerial combat data is as follows:

| Time | Event | Who against whom | Kill probability ($P_k$) |
|---|---|---|---|
| $t_1 = 5$ | L | 2 of I $\xrightarrow{L}$ 1 of II | 0.4 |
| $t_2 = 8$ | H | 2 of I $\xrightarrow{H}$ 1 of II | 0.4 |
| $t_3 = 10$ | L | 1 of II $\xrightarrow{L}$ 1 of I | 0.5 |
| $t_4 = 14$ | H | 1 of II $\xrightarrow{H}$ 1 of I | 0.5 |
| $t_5 = 14$ | L | 1 of I $\xrightarrow{L}$ 2 of II | 0.6 |
| $t_6 = 15$ | L | 2 of II $\xrightarrow{L}$ 1 of I | 0.3 |
| $t_7 = 16$ | H | 1 of I $\xrightarrow{H}$ 2 of II | 0.6 |
| $t_8 = 16$ | L | 1 of II $\xrightarrow{L}$ 2 of I | 0.8 |
| $t_9 = 19$ | H | 1 of II $\xrightarrow{H}$ 2 of I | 0.8 |
| $t_{10} = 20$ | H | 2 of II $\xrightarrow{H}$ 1 of I | 0.3 |
| $t_{11} = 20$ | L | 1 of I $\xrightarrow{L}$ 2 of II | 0.2 |

| Time | Event | Who against whom | Kill probability ($P_k$) |
|---|---|---|---|
| $t_{12} = 24$ | H | 1 of I $\xrightarrow{H}$ 2 of II | 0.2 |
| $t_{13} = 24$ | L | 2 of II $\xrightarrow{L}$ 2 of I | 0.7 |
| $t_{14} = 27$ | L | 2 of I $\xrightarrow{L}$ 2 of II | 0.6 |
| $t_{15} = 28$ | H | 2 of II $\xrightarrow{H}$ 2 of I | 0.7 |
| $t_{16} = 28$ | H | 2 of I $\xrightarrow{H}$ 2 of II | 0.6 |

To illustrate the updating formulae, we redefine each of the states with which a nonzero probability is associated sometime during the aerial combat. In fact, states are only defined as needed. The following key is required to understand the probability updating:

| State: Integer | State: $(X_1, X_2, T_1, T_2)$ |
|---|---|
| 1 | (1,1)(1,1)(0,0)(0,0) |
| 2 | (1,1)(1,1)(0,1)(0,0) |
| 3 | (1,1)(0,1)(0,0)(0,0) |
| 4 | (1,1)(1,1)(0,0)(1,0) |
| 5 | (0,1)(1,1)(0,0)(0,0) |
| 6 | (1,1)(1,1)(2,0)(0,0) |
| 7 | (1,1)(0,1)(2,0)(0,0) |
| 8 | (1,1)(1,1)(2,0)(0,1) |
| 9 | (0,1)(1,1)(0,0)(0,1) |
| 10 | (0,1)(1,1)(0,0)(0,2) |
| 11 | (1,1)(0,1)(2,0)(0,1) |
| 12 | (1,1)(1,1)(0,0)(0,1) |
| 13 | (1,1)(1,0)(0,0)(0,1) |
| 14 | (1,1)(0,1)(0,0)(0,1) |
| 15 | (1,1)(0,0)(0,0)(0,1) |
| 16 | (1,1)(1,1)(0,0)(2,1) |
| 17 | (1,1)(1,0)(0,0)(2,1) |
| 18 | (0,1)(1,1)(0,0)(2,1) |
| 19 | (0,1)(1,1)(0,0)(2,2) |
| 20 | (1,1)(0,1)(0,0)(0,1) |
| 21 | (1,1)(0,0)(0,0)(0,1) |
| 22 | (1,0)(1,1)(0,0)(0,1) |
| 23 | (1,0)(1,0)(0,0)(0,1) |
| 24 | (0,0)(1,1)(0,0)(0,1) |
| 25 | (0,0)(1,1)(0,0)(0,2) |
| 26 | (1,0)(1,1)(0,0)(0,0) |
| 27 | (0,0)(1,1)(0,0)(0,0) |
| 28 | (1,1)(1,0)(0,0)(0,0) |
| 29 | (0,1)(1,0)(0,0)(0,0) |
| 30 | (1,0)(1,0)(0,0)(0,0) |

| State: Integer | State: $(X_1, X_2, T_1, T_2)$ |
|---|---|
| 31 | (0,0)(1,0)(0,0)(0,0) |
| 32 | (0,1)(0,1)(0,0)(0,0) |
| 33 | (1,1)(0,0)(0,0)(0,0) |
| 34 | (0,1)(0,0)(0,0)(0,0) |
| 35 | (1,0)(1,1)(2,0)(0,0) |
| 36 | (1,1)(1,0)(2,0)(0,0) |
| 37 | (1,1)(1,0)(1,0)(0,0) |
| 38 | (1,0)(1,0)(2,0)(0,0) |
| 39 | (1,0)(1,0)(1,0)(0,0) |
| 40 | (1,1)(0,0)(2,0)(0,0) |
| 41 | (1,0)(0,0)(0,0)(0,0) |
| 42 | (1,1)(1,1)(0,0)(0,2) |
| 43 | (1,0)(1,1)(0,0)(0,2) |
| 44 | (1,1)(0,1)(0,0)(0,2) |
| 45 | (0,1)(0,1)(0,0)(0,2) |
| 46 | (1,1)(1,1)(0,2)(0,2) |
| 47 | (1,1)(1,0)(0,2)(0,0) |
| 48 | (1,1)(1,0)(0,1)(0,0) |
| 49 | (0,1)(1,1)(0,2)(0,2) |
| 50 | (0,1)(1,0)(0,2)(0,0) |
| 51 | (0,1)(1,0)(0,1)(0,0) |
| 52 | (1,1)(0,1)(0,2)(0,2) |
| 53 | (1,1)(0,0)(0,2)(0,0) |
| 54 | (0,1)(0,1)(0,2)(0,2) |
| 55 | (0,1)(0,0)(0,2)(0,0) |
| 56 | (1,1)(1,1)(0,2)(0,0) |
| 57 | (1,0)(1,1)(0,2)(0,0) |
| 58 | (0,1)(1,1)(0,2)(0,0) |
| 59 | (0,0)(1,1)(0,2)(0,0) |
| 60 | (1,1)(0,1)(0,2)(0,0) |
| 61 | (1,0)(0,1)(0,2)(0,0) |
| 62 | (0,1)(0,1)(0,2)(0,0) |
| 63 | (0,0)(0,1)(0,2)(0,0) |
| 64 | (1,0)(0,1)(0,0)(0,0) |
| 65 | (0,0)(0,1)(0,0)(0,0) |
| 66 | (0,0)(0,0)(0,0)(0,0) |

| Nonzero probability states | Updating formula | State probability |
|---|---|---|
| | Time: $t_0$; action: ... | |
| 1 | ... | 1 |
| | Time: $t_1$; action: 2 of I $\xrightarrow{L}$ 1 of II | |
| 2 | $P_2(t_1) = P_1(t_0)$ | 1 |
| | Time: $t_2$; action: 2 of I $\xrightarrow{H}$ II | |
| 1 | $P_1(t_2) = (1 - P_k(t_2))P_2(t_1)$ | 0.6 |
| 3 | $P_3(t_2) = P_k(t_2)P_2(t_1)$ | 0.4 |
| | Time: $t_3$; action: 1 of II $\xrightarrow{L}$ 1 of I | |
| 3 | $P_3(t_3) = P_3(t_2)$ | 0.4 |
| 4 | $P_4(t_3) = P_1(t_2)$ | 0.6 |
| | Time: $t_4$; action: 1 of II $\xrightarrow{H}$ I | |
| 1 | $P_1(t_4) = P_4(t_3)(1 - P_k(t_4))$ | 0.3 |
| 3 | $P_3(t_4) = P_3(t_3)$ | 0.4 |
| 5 | $P_5(t_4) = P_4(t_3)P_k(t_4)$ | 0.3 |
| | Time: $t_5$; action: 1 of I $\xrightarrow{L}$ 2 of II | |
| 5 | $P_5(t_5) = P_5(t_4)$ | 0.3 |
| 6 | $P_6(t_5) = P_1(t_4)$ | 0.3 |
| 7 | $P_7(t_5) = P_3(t_4)$ | 0.4 |
| | Time: $t_6$; action: 2 of II $\xrightarrow{L}$ 1 of I | |
| 8 | $P_8(t_6) = P_6(t_5)$ | 0.3 |
| 9 | $P_9(t_6) = (1 - \beta)P_5(t_5)$ | $0.3(1 - \beta)$ |
| 10 | $P_{10}(t_6) = \beta P_5(t_5)$ | $0.3\beta$ |
| 11 | $P_{11}(t_6) = P_7(t_5)$ | 0.4 |

| Nonzero probability states | Updating formula | State probability |
|---|---|---|
| Time: $t_7$; action: 1 of I $\xrightarrow{H}$ II | | |
| 9 | $P_9(t_7) = P_9(t_6)$ | $0.3(1 - \beta)$ |
| 10 | $P_{10}(t_7) = P_{10}(t_6)$ | $0.3\beta$ |
| 12 | $P_{12}(t_7) = (1 - P_k(t_7))P_8(t_6)$ | 0.12 |
| 13 | $P_{13}(t_7) = P_k(t_7)P_8(t_6)$ | 0.18 |
| 14 | $P_{14}(t_7) = (1 - P_k(t_7))P_{11}(t_6)$ | 0.16 |
| 15 | $P_{15}(t_7) = P_k(t_7)P_{11}(t_6)$ | 0.24 |
| Time: $t_8$; action: 1 of II $\xrightarrow{L}$ 2 of I | | |
| 16 | $P_{16}(t_8) = P_{12}(t_7)$ | 0.12 |
| 17 | $P_{17}(t_8) = P_{13}(t_7)$ | 0.18 |
| 18 | $P_{18}(t_8) = P_9(t_7)$ | $0.3(1 - \beta)$ |
| 19 | $P_{19}(t_8) = P_{10}(t_7)$ | $0.3\beta$ |
| 20 | $P_{20}(t_8) = P_{14}(t_7)$ | 0.16 |
| 21 | $P_{21}(t_8) = P_{15}(t_7)$ | 0.24 |
| Time: $t_9$; action: 1 of II $\xrightarrow{H}$ I | | |
| 9 | $P_9(t_9) = (1 - P_k(t_9))P_{18}(t_8)$ | $0.06(1 - \beta)$ |
| 10 | $P_{10}(t_9) = (1 - P_k(t_9))P_{19}(t_8)$ | $0.06\beta$ |
| 12 | $P_{12}(t_9) = (1 - P_k(t_9))P_{16}(t_8)$ | 0.024 |
| 13 | $P_{13}(t_9) = (1 - P_k(t_9))P_{17}(t_8)$ | 0.036 |
| 14 | $P_{14}(t_9) = P_{20}(t_8)$ | 0.16 |
| 21 | $P_{21}(t_9) = P_{21}(t_8)$ | 0.24 |
| 22 | $P_{22}(t_9) = P_k(t_9)P_{16}(t_8)$ | 0.096 |
| 23 | $P_{23}(t_9) = P_k(t_9)P_{17}(t_8)$ | 0.144 |
| 24 | $P_{24}(t_9) = P_k(t_9)P_{18}(t_8)$ | $0.24(1 - \beta)$ |
| 25 | $P_{25}(t_9) = P_k(t_9)P_{19}(t_8)$ | $0.24\beta$ |

| Nonzero probability states | Updating formula | State probability |
|---|---|---|
| | Time: $t_{10}$; action: 2 of II $\xrightarrow{H}$ I | |
| 1 | $P_1(t_{10}) = (1 - P_k(t_{10}))P_{12}(t_9)$ | 0.0168 |
| 3 | $P_3(t_{10}) = (1 - P_k(t_{10}))P_{14}(t_9)$ | 0.112 |
| 5 | $P_5(t_{10}) = P_k(t_{10})P_{12}(t_9)$ $+ P_9(t_9) + (1-P_k(t_{10}))P_{10}(t_9)$ | 0.0672 - 0.018β |
| 26 | $P_{26}(t_{10}) = (1-P_k(t_{10}))P_{22}(t_9)$ | 0.0672 |
| 27 | $P_{27}(t_{10}) = P_k(t_{10})P_{22}(t_9)$ $+ P_{24}(t_9) + P_{25}(t_9)$ $+ P_k(t_{10})P_{10}(t_9)$ $= 0.3(0.096) + 0.24(1 - \beta)$ $+ 0.24\beta + 0.3(0.06)\beta$ | 0.2688 + 0.018β |
| 28 | $P_{28}(t_{10}) = (1 - P_k(t_{10}))P_{13}(t_9)$ | 0.0252 |
| 29 | $P_{29}(t_{10}) = P_k(t_{10})P_{13}(t_9)$ | 0.0108 |
| 30 | $P_{30}(t_{10}) = (1 - P_k(t_{10}))P_{23}(t_9)$ | 0.1008 |
| 31 | $P_{31}(t_{10}) = P_k(t_{10})P_{23}(t_9)$ | 0.0432 |
| 32 | $P_{32}(t_{10}) = P_k(t_{10})P_{14}(t_9)$ | 0.048 |
| 33 | $P_{33}(t_{10}) = (1 - P_k(t_{10}))P_{21}(t_9)$ | 0.168 |
| 34 | $P_{34}(t_{10}) = P_k(t_{10})P_{21}(t_9)$ | 0.072 |
| | Time: $t_{11}$; action: 1 of I $\xrightarrow{L}$ 2 of II | |
| 5 | $P_5(t_{11}) = P_5(t_{10})$ | 0.0672 - 0.018β |
| 6 | $P_6(t_{11}) = P_1(t_{10})$ | 0.0168 |
| 7 | $P_7(t_{11}) = P_3(t_{10})$ | 0.112 |
| 27 | $P_{27}(t_{11}) = P_{27}(t_{10})$ | 0.2688 + 0.018β |
| 29 | $P_{29}(t_{11}) = P_{29}(t_{10})$ | 0.0108 |
| 31 | $P_{31}(t_{11}) = P_{31}(t_{10})$ | 0.0432 |
| 32 | $P_{32}(t_{11}) = P_{32}(t_{10})$ | 0.048 |
| 33 | $P_{33}(t_{11}) = \alpha P_{33}(t_{10})$ | 0.168α |
| 34 | $P_{34}(t_{11}) = P_{34}(t_{10})$ | 0.072 |

| Nonzero probability states | Updating formula | State probability |
|---|---|---|
| | Time: $t_{11}$; action: 1 of I $\xrightarrow{L}$ 2 of II (contd.) | |
| 35 | $P_{35}(t_{11}) = P_{26}(t_{10})$ | 0.0672 |
| 36 | $P_{36}(t_{11}) = (1 - \alpha)P_{28}(t_{10})$ | $(1 - \alpha)(0.0252)$ |
| 37 | $P_{37}(t_{11}) = \alpha P_{28}(t_{10})$ | $\alpha(0.0252)$ |
| 38 | $P_{38}(t_{11}) = (1 - \alpha)P_{30}(t_{10})$ | $(1 - \alpha)(0.1008)$ |
| 39 | $P_{39}(t_{11}) = \alpha P_{30}(t_{10})$ | $\alpha(0.1008)$ |
| 40 | $P_{40}(t_{11}) = (1 - \alpha)P_{33}(t_{10})$ | $(1 - \alpha)(0.168)$ |
| | Time: $t_{12}$; action: 1 of I $\xrightarrow{H}$ II | |
| 1 | $P_1(t_{12}) = (1 - P_k(t_{12}))P_6(t_{11})$ | 0.01344 |
| 3 | $P_3(t_{12}) = (1 - P_k(t_{12}))P_7(t_{11})$ | 0.0896 |
| 5 | $P_5(t_{12}) = P_5(t_{11})$ | $0.0672 - 0.018\beta$ |
| 26 | $P_{26}(t_{12}) = (1 - P_k(t_{12}))P_{35}(t_{11})$ | 0.05376 |
| 27 | $P_{27}(t_{12}) = P_{27}(t_{11})$ | $0.2688 + 0.018\beta$ |
| 28 | $P_{28}(t_{12}) = P_6(t_{11})P_k(t_{12})$ $+ P_{36}(t_{11})$ $+ P_{37}(t_{11})(1 - P_k(t_{12}))$ | $0.02856 - 0.00504\alpha$ |
| 29 | $P_{29}(t_{12}) = P_{29}(t_{11})$ | 0.0108 |
| 30 | $P_{30}(t_{12}) = P_{35}(t_{11})P_k(t_{12})$ $+ P_{38}(t_{11})$ $+ P_{39}(t_{11})(1 - P_k(t_{12}))$ | $0.11424 - 0.02016\alpha$ |
| 31 | $P_{31}(t_{12}) = P_{31}(t_{11})$ | 0.0432 |
| 32 | $P_{32}(t_{12}) = P_{32}(t_{11})$ | 0.048 |
| 33 | $P_{33}(t_{12}) = P_k(t_{12})P_7(t_{11})$ $+ P_{33}(t_{11}) + P_{40}(t_{11})$ $+ P_k(t_{12})P_{37}(t_{11})$ | $0.1904 + 0.00504\alpha$ |
| 34 | $P_{34}(t_{12}) = P_{34}(t_{11})$ | 0.072 |
| 41 | $P_{41}(t_{12}) = P_k(t_{12})P_{39}(t_{11})$ | $0.02016\alpha$ |

| Nonzero probability states | Updating formula | State probability |
|---|---|---|
| | Time: $t_{13}$; action: 2 of II $\xrightarrow{L}$ 2 of I | |
| 10 | $P_{10}(t_{13}) = P_5(t_{12})$ | $0.0672 - 0.018\beta$ |
| 22 | $P_{22}(t_{13}) = \beta P_{26}(t_{12})$ | $\beta(0.05376)$ |
| 25 | $P_{25}(t_{13}) = (1 - \beta)P_{27}(t_{12})$ | $(1 - \beta)(0.2688 + 0.018\beta)$ |
| 27 | $P_{27}(t_{13}) = \beta P_{27}(t_{12})$ | $\beta(0.2688 + 0.018\beta)$ |
| 28 | $P_{28}(t_{13}) = P_{28}(t_{12})$ | $0.02856 - 0.00504\alpha$ |
| 29 | $P_{29}(t_{13}) = P_{29}(t_{12})$ | $0.0108$ |
| 30 | $P_{30}(t_{13}) = P_{30}(t_{12})$ | $0.11424 - 0.02016\alpha$ |
| 31 | $P_{31}(t_{13}) = P_{31}(t_{12})$ | $0.0432$ |
| 33 | $P_{33}(t_{13}) = P_{33}(t_{12})$ | $0.1904 + 0.00504\alpha$ |
| 34 | $P_{34}(t_{13}) = P_{34}(t_{12})$ | $0.072$ |
| 41 | $P_{41}(t_{13}) = P_{41}(t_{12})$ | $0.02016\alpha$ |
| 42 | $P_{42}(t_{13}) = P_1(t_{12})$ | $0.01344$ |
| 43 | $P_{43}(t_{13}) = (1 - \beta)P_{26}(t_{12})$ | $(1 - \beta)0.05376$ |
| 44 | $P_{44}(t_{13}) = P_3(t_{12})$ | $0.0896$ |
| 45 | $P_{45}(t_{13}) = P_{32}(t_{12})$ | $0.048$ |
| | Time: $t_{14}$; action: 2 of I $\xrightarrow{L}$ 2 of II | |
| 22 | $P_{22}(t_{14}) = P_{22}(t_{13})$ | $0.05376\beta$ |
| 25 | $P_{25}(t_{14}) = P_{25}(t_{13})$ | $(1 - \beta)(0.2688 + 0.018\beta)$ |
| 27 | $P_{27}(t_{14}) = P_{27}(t_{13})$ | $\beta(0.2688 + 0.018\beta)$ |
| 30 | $P_{30}(t_{14}) = P_{30}(t_{13})$ | $0.11424 - 0.02016\alpha$ |
| 31 | $P_{31}(t_{14}) = P_{31}(t_{13})$ | $0.0432$ |
| 33 | $P_{33}(t_{14}) = \alpha P_{33}(t_{13})$ | $\alpha(0.1904 + 0.00504\alpha)$ |
| 34 | $P_{34}(t_{14}) = \alpha P_{34}(t_{13})$ | $0.072\alpha$ |
| 41 | $P_{41}(t_{14}) = P_{41}(t_{13})$ | $0.02016\alpha$ |
| 43 | $P_{43}(t_{14}) = P_{43}(t_{13})$ | $(1 - \beta)(0.05376)$ |
| 46 | $P_{46}(t_{14}) = P_{42}(t_{13})$ | $0.01344$ |

| Nonzero probability states | Updating formula | State probability |
|---|---|---|
| | Time: $t_{14}$; action: 2 of I $\xrightarrow{L}$ 2 of II (contd.) | |
| 47 | $P_{47}(t_{14}) = (1 - \alpha)P_{28}(t_{13})$ | $(1 - \alpha)[0.02856 - 0.00504\alpha]$ |
| 48 | $P_{48}(t_{14}) = \alpha P_{28}(t_{13})$ | $\alpha[0.02856 - 0.00504\alpha]$ |
| 49 | $P_{49}(t_{14}) = P_{10}(t_{13})$ | $0.0672 - 0.018\beta$ |
| 50 | $P_{50}(t_{14}) = (1 - \alpha)P_{29}(t_{13})$ | $(1 - \alpha)(0.0108)$ |
| 51 | $P_{51}(t_{14}) = \alpha P_{29}(t_{13})$ | $\alpha(0.0108)$ |
| 52 | $P_{52}(t_{14}) = P_{44}(t_{13})$ | $0.0896$ |
| 53 | $P_{53}(t_{14}) = (1 - \alpha)P_{33}(t_{13})$ | $(1 - \alpha)[0.1904 + 0.00504\alpha]$ |
| 54 | $P_{54}(t_{14}) = P_{45}(t_{13})$ | $0.048$ |
| 55 | $P_{55}(t_{14}) = (1 - \alpha)P_{34}(t_{13})$ | $(1 - \alpha)(0.072)$ |
| | Time: $t_{15}$; action: 2 of II $\xrightarrow{H}$ I | |
| 26 | $P_{26}(t_{15}) = P_{43}(t_{14})$ $+ (1 - P_k(t_{15}))P_{22}(t_{14})$ | $0.05376 - 0.037632\beta$ |
| 27 | $P_{27}(t_{15}) = P_k(t_{15})P_{22}(t_{14})$ $+ P_{25}(t_{14}) + P_{27}(t_{14})$ | $0.2688 + 0.055632\beta$ |
| 30 | $P_{30}(t_{15}) = P_{30}(t_{14})$ | $0.11424 - 0.02016\alpha$ |
| 31 | $P_{31}(t_{15}) = P_{31}(t_{14})$ | $0.0432$ |
| 33 | $P_{33}(t_{15}) = P_{33}(t_{14})$ | $\alpha(0.1904 + 0.00504\alpha)$ |
| 34 | $P_{34}(t_{15}) = P_{34}(t_{14})$ | $0.072\alpha$ |
| 41 | $P_{41}(t_{15}) = P_{41}(t_{14})$ | $0.02016\alpha$ |
| 47 | $P_{47}(t_{15}) = P_{47}(t_{14})$ | $(1 - \alpha)[0.02856 - 0.00504\alpha]$ |
| 48 | $P_{48}(t_{15}) = P_{48}(t_{14})$ | $\alpha[0.02856 - 0.00504\alpha]$ |
| 50 | $P_{50}(t_{15}) = P_{50}(t_{14})$ | $(1 - \alpha)(0.0108)$ |
| 51 | $P_{51}(t_{15}) = P_{51}(t_{14})$ | $\alpha(0.0108)$ |
| 53 | $P_{53}(t_{15}) = P_{53}(t_{14})$ | $(1 - \alpha)[0.1904 + 0.00504\alpha]$ |
| 55 | $P_{55}(t_{15}) = P_{55}(t_{14})$ | $(1 - \alpha)(0.072)$ |

| Nonzero probability states | Updating formula | State probability |
|---|---|---|
| | Time: $t_{15}$; action: 2 of II $\xrightarrow{H}$ I (contd.) | |
| 56 | $P_{56}(t_{15}) = (1 - P_k(t_{15}))P_{46}(t_{14})$ | 0.004032 |
| 57 | $P_{57}(t_{15}) = P_k(t_{15})P_{46}(t_{14})$ | 0.009408 |
| 58 | $P_{58}(t_{15}) = (1 - P_k(t_{15}))P_{49}(t_{14})$ | $0.02016 - 0.0054\beta$ |
| 59 | $P_{59}(t_{15}) = P_k(t_{15})P_{49}(t_{14})$ | $0.04704 - 0.0126\beta$ |
| 60 | $P_{60}(t_{15}) = (1 - P_k(t_{15}))P_{52}(t_{14})$ | 0.02688 |
| 61 | $P_{61}(t_{15}) = P_k(t_{15})P_{52}(t_{14})$ | 0.06272 |
| 62 | $P_{62}(t_{15}) = (1 - P_k(t_{15}))P_{54}(t_{14})$ | 0.0144 |
| 63 | $P_{63}(t_{15}) = P_k(t_{15})P_{54}(t_{14})$ | 0.0336 |
| | Time: $t_{16}$; action: 2 of I $\xrightarrow{H}$ II | |
| 1 | $P_1(t_{16}) = (1 - P_k(t_{16}))P_{56}(t_{15})$ | 0.0016128 |
| 3 | $P_3(t_{16}) = (1 - P_k(t_{16}))P_{60}(t_{15})$ | 0.010752 |
| 5 | $P_5(t_{16}) = (1 - P_k(t_{16}))P_{58}(t_{15})$ | $0.008064 - 0.00216\beta$ |
| 26 | $P_{26}(t_{16}) = (1 - P_k(t_{16}))P_{57}(t_{15}) + P_{26}(t_{15})$ | $0.0575232 - 0.037632\beta$ |
| 27 | $P_{27}(t_{16}) = (1 - P_k(t_{16}))P_{59}(t_{15}) + P_{27}(t_{15})$ | $0.287616 + 0.050592\beta$ |
| 28 | $P_{28}(t_{16}) = P_k(t_{16})P_{56}(t_{15}) + P_{47}(t_{15}) + P_{48}(t_{15})(1 - P_k(t_{16}))$ | $0.0309792 - 0.022176\alpha + 0.003024\alpha^2$ |
| 29 | $P_{29}(t_{16}) = P_k(t_{16})P_{58}(t_{15}) + P_{50}(t_{15}) + P_{51}(t_{15})(1 - P_k(t_{16}))$ | $0.022896 - 0.00324\beta - 0.00648\alpha$ |
| 30 | $P_{30}(t_{16}) = P_k(t_{16})P_{57}(t_{16}) + P_{30}(t_{15})$ | $0.1198848 - 0.02016\alpha$ |
| 31 | $P_{31}(t_{16}) = P_k(t_{16})P_{59}(t_{15}) + P_{31}(t_{15})$ | $0.071424 - 0.00756\beta$ |

| Nonzero probability states | Updating formula | State probability |
|---|---|---|
| | Time: $t_{16}$; action: 2 of I $\xrightarrow{H}$ II (contd.) | |
| 32 | $P_{32}(t_{16}) = (1 - P_k(t_{16}))P_{62}(t_{15})$ | 0.00576 |
| 33 | $P_{33}(t_{16}) = P_{33}(t_{15}) + P_{53}(t_{15}) + P_k(t_{16})[P_{48}(t_{15}) + P_{60}(t_{15})]$ | $0.2065280 + 0.022176\alpha - 0.003024\alpha^2$ |
| 34 | $P_{34}(t_{16}) = P_{34}(t_{15}) + P_{55}(t_{15}) + P_k(t_{16})[P_{51}(t_{15}) + P_{62}(t_{15})]$ | $0.08064 + 0.00648\alpha$ |
| 41 | $P_{41}(t_{16}) = P_k(t_{16})P_{61}(t_{15}) + P_{41}(t_{15})$ | $0.037632 + 0.02016\alpha$ |
| 64 | $P_{64}(t_{16}) = (1 - P_k(t_{16}))P_{61}(t_{15})$ | 0.025088 |
| 65 | $P_{65}(t_{16}) = (1 - P_k(t_{16}))P_{63}(t_{15})$ | 0.01344 |
| 66 | $P_{66}(t_{16}) = P_k(t_{16})P_{63}(t_{15})$ | 0.02016 |

The survivability of various participants in the duel is given as follows:

$$P(x_{11} = 1) = 0.49 - 0.037632\beta$$

$$P(x_{12} = 1) = 0.367232 - 0.0054\beta$$

$$P(x_{21} = 1) = 0.6 - 0.048816\alpha + 0.003024\alpha^2$$

$$P(x_{22} = 1) = 0.409856 + 0.0108\beta$$

Also side I's exchange ratio which depends on $\alpha$ and $\beta$ is given by

$$\theta(\alpha, \beta) = \frac{2 - (1.009856 - 0.048816\alpha + 0.003024\alpha^2 + 0.0108\beta)}{2 - (0.857232 - 0.043032\beta)}$$

$$= \frac{0.990144 + 0.048816\alpha - 0.003024\alpha^2 - 0.0108\beta}{1.142768 + 0.043032\beta}$$

where

$$0.8258931 = \theta(0,1) \le \theta \le \theta(1,0) = 0.9065147$$

In particular

$$\theta(0,0) = 0.8664436$$

$$\theta(1,1) = 0.86451$$

and

$$\frac{\theta(0,1) + \theta(1,0)}{2} = 0.8662039$$

Thus

$$\theta = 0.866 \pm 0.041$$

becomes a reasonable estimate for $\theta$ with a maximum percentage error less than 5%.

## Appendix B

## THE NUMERICAL EXAMPLE CONTINUED TO INCLUDE WEAPONS USAGE

In the previous example we introduced the states to handle the updating equations when $2^4(3)^2(3)^2 = 1,296$ states are possible. Assuming each participant in the duel of Appendix A has two missiles we have a total of $1,296 \times 3^2 \times 3^2 = 104,976$ possible states. We shall now give a summarized version of the updating formulae when weapons usage is considered and we shall see that the 66 states required to handle this problem for when weapon usage is not considered are increased only slightly.

We associate with $(M_1,M_2)$ the integer to the base 3 and for $(M_{11},M_{12})(M_{21},M_{22})$ let the corresponding state be $M_{11}3^3 + M_{12}3^2 + M_{21}3 + M_{22}$. Since each of the states we consider may be thought of as an ordered pair of integers where the first integer describes a state in Appendix A and the second integer is associated with a particular missile count and varies from 0 to 80, we let the state space be described by $(i,j)$ where $1 \le i \le 66$ and $0 \le j \le 80$. The missile usage integers required in the updating and the corresponding $(M_{11},M_{12})(M_{21},M_{22})$ are as follows:

| | | | |
|---|---|---|---|
| 0 | (0,0)(0,0) | 44 | (1,1)(2,2) |
| 9 | (0,1)(0,0) | 52 | (1,2)(2,1) |
| 12 | (0,1)(1,0) | 53 | (1,2)(2,2) |
| 15 | (0,1)(2,0) | 66 | (2,1)(1,0) |
| 39 | (1,1)(1,0) | 67 | (2,1)(1,1) |
| 40 | (1,1)(1,1) | 70 | (2,1)(2,1) |
| 41 | (1,1)(1,2) | 71 | (2,1)(2,2) |
| 42 | (1,1)(2,0) | 79 | (2,2)(2,1) |
| 43 | (1,1)(2,1) | 80 | (2,2)(2,2) |

Thus, 18 out of 81 possible missile usage descriptions are required.

The probability updating for weapons usage is modified as follows:

| Time | Old state | New states | New probabilities (if required) |
|---|---|---|---|
| $t_0$ | 1 | (1,0) | |
| $t_1$ | 2 | (2,9) | |
| $t_2$ | 1 | (1,9) | |
| | 3 | (3,15) | |
| $t_3$ | 3 | (3,15) | |
| | 4 | (4,12) | |
| $t_4$ | 1 | (1,12) | |
| | 3 | (3,15) | |
| | 5 | (5,66) | |
| $t_5$ | 5 | (5,66) | |
| | 6 | (6,39) | |
| | 7 | (7,42) | |
| $t_6$ | 8 | (8,40) | |
| | 9 | (9,67) | |
| | 10 | (10,67) | |
| | 11 | (11,43) | |
| $t_7$ | 9 | (9,67) | |
| | 10 | (10,67) | |
| | 12 | (12,40) | |
| | 13 | (13,41) | |
| | 14 | (14,43) | |
| | 15 | (15,44) | |
| $t_8$ | 16 | (16,43) | |
| | 17 | (17,44) | |
| | 18 | (18,70) | |
| | 19 | (19,70) | |
| | 20 | (20,43) | |
| | 21 | (21,44) | |
| $t_9$ | 9 | (9,70) | |
| | 10 | (10,70) | |
| | 12 | (12,43) | |
| | 13 | (13,44) | |

| Time | Old state | New states | New probabilities (if required) |
|---|---|---|---|
| $t_9$ | 14 | (14,43) | |
| | 21 | (21,44) | |
| | 22 | (22,52) | |
| | 23 | (23,53) | |
| | 24 | (24,79) | |
| | 25 | (25,79) | |
| $t_{10}$ | 1 | (1,43) | |
| | 3 | (3,43) | |
| | 5 | (5,70) | |
| | 26 | (26,52) | |
| | 27 | (27,79) | |
| | 28 | (28,44) | |
| | 29 | (29,71) | |
| | 30 | (30,53) | |
| | 31 | (31,80) | |
| | 32 | (32,70) | |
| | 33 | (33,44) | |
| | 34 | (34,71) | |
| $t_{11}$ | 5 | (5,70) | |
| | 6 | (6,70) | |
| | 7 | (7,70) | |
| | 27 | (27,79) | |
| | 29 | (29,71) | |
| | 31 | (31,80) | |
| | 32 | (32,70) | |
| | 33 | (33,44) | |
| | 34 | (34,71) | |
| | 35 | (35,79) | |
| | 36 | (36,71) | |
| | 37 | (37,71) | |
| | 38 | (38,80) | |
| | 39 | (39,80) | |
| | 40 | (40,71) | |
| $t_{12}$ | 1 | (1,70) | |
| | 3 | (3,70) | |
| | 5 | (5,70) | |
| | 26 | (26,79) | |
| | 27 | (27,79) | |
| | 28 | (28,71) | |
| | 29 | (29,71) | |
| | 30 | (30,80) | |

| Time | Old state | New states | New probabilities (if required) |
|---|---|---|---|
| | 31 | (31,80) | |
| | 32 | (32,70) | |
| | 33 | (33,71) | $P_k(P_{7,70} + P_{37,71}) + P_{40,71} = 0.1904 - 0.16296\alpha$ |
| | | (33,44) | $P_{(33,44)} = 0.168\alpha$ |
| | 34 | (34,71) | |
| | 41 | (41,80) | |
| $t_{13}$ | 10 | (10,71) | |
| | 22 | (22,80) | |
| | 25 | (25,80) | |
| | 27 | (27,79) | |
| | 28 | (28,71) | |
| | 29 | (29,71) | |
| | 30 | (30,80) | |
| | 31 | (31,80) | |
| | 33 | (33,71) | $0.1904 - 0.16296\alpha$ |
| | | (33,44) | $0.168\alpha$ |
| | 34 | (34,71) | |
| | 41 | (41,80) | |
| | 42 | (42,71) | |
| | 43 | (43,80) | |
| | 44 | (44,71) | |
| | 45 | (45,71) | |
| $t_{14}$ | 22 | (22,80) | |
| | 25 | (25,80) | |
| | 27 | (27,79) | |
| | 30 | (30,80) | |
| | 31 | (31,80) | |
| | 33 | (33,71) | $\alpha(0.1904 - 0.16296\alpha)$ |
| | | (33,44) | $0.168\alpha^2$ |
| | 34 | (34,71) | |
| | 41 | (41,80) | |
| | 43 | (43,80) | |
| | 46 | (46,80) | |
| | 47 | (47,80) | |
| | 48 | (48,80) | |
| | 49 | (49,80) | |
| | 50 | (50,80) | |
| | 51 | (51,80) | |
| | 52 | (52,80) | |

| Time | Old state | New states | New probabilities (if required) |
|---|---|---|---|
| $t_{14}$ | 53 | (53,80) | $(1 - \alpha)P_{33,71} = (1 - \alpha)(0.1904 - 0.16276\alpha)$ |
| | | (53,53) | $(1 - \alpha)P_{33,44} = (1 - \alpha)0.168\alpha$ |
| | 54 | (54,80) | |
| | 55 | (55,80) | |
| $t_{15}$ | 26 | (26,80) | |
| | 27 | (27,80) | $P_kP_{22,80} + P_{25,80} = 0.2688 - 0.213168\beta - 0.018\beta^2$ |
| | | (27,79) | $P_{27,79} = 0.2688\beta + 0.018\beta^2$ |
| | 30 | (30,80) | |
| | 31 | (31,80) | |
| | 33 | (33,71) | $\alpha(0.1904 - 0.16296\alpha)$ |
| | | (33,44) | $0.168\alpha^2$ |
| | 34 | (34,71) | |
| | 41 | (41,80) | |
| | 47 | (47,80) | |
| | 48 | (48,80) | |
| | 50 | (50,80) | |
| | 51 | (51,80) | |
| | 53 | (53,80) | $(1 - \alpha)(0.1904 - 0.16296\alpha)$ |
| | | (53,53) | $(1 - \alpha)0.168\alpha$ |
| | 55 | (55,80) | |
| | 56 | (56,80) | |
| | 57 | (57,80) | |
| | 58 | (58,80) | |
| | 59 | (59,80) | |
| | 60 | (60,80) | |
| | 61 | (61,80) | |
| | 62 | (62,80) | |
| | 63 | (63,80) | |
| $t_{16}$ | 1 | (1,80) | |
| | 3 | (3,80) | |
| | 5 | (5,80) | |
| | 26 | (26,80) | |
| | 27 | (27,79) | $P_{27,79} = 0.2688\beta + 0.018\beta^2$ |
| | | (27,80) | $P_{27,80} + (1 - P_k)P_{59} = 0.287616 - 0.218208\beta - 0.018\beta^2$ |
| | 28 | (28,80) | |
| | 29 | (29,80) | |
| | 30 | (30,80) | |
| | 31 | (31,80) | |

| Time | Old state | New states | New probabilities (if required) |
|---|---|---|---|
| | 32 | (32,80) | |
| | 33 | (33,71) | $P_{33,71} = \alpha(0.1904 - 0.16296\alpha)$ |
| | | (33,44) | $P_{33,44} = 0.168\alpha^2$ |
| | | (33,80) | $P_k[P_{48} + P_{60}] + P_{(53,80)} = 0.206528 - 0.336224\alpha + 0.159936\alpha^2$ |
| | | (33,53) | $P_{53,53} = (1 - \alpha)0.168\alpha$ |
| | 34 | (34,71) | $P_{34,71} = 0.072\alpha$ |
| | | (34,80) | $P_{55} + P_k(P_{51} + P_{62}) = 0.08064 - 0.06552\alpha$ |
| | 41 | (41,80) | |
| | 64 | (64,80) | |
| | 65 | (65,80) | |
| | 66 | (66,80) | |

The joint distribution of survival and weapons usage, $P_{X_1X_2M_1M_2}$, is given below:

$$P_{(1,1)(1,1)(2,2)(2,2)} = 0.0016128$$

$$P_{(1,1)(0,1)(2,2)(2,2)} = 0.010752$$

$$P_{(0,1)(1,1)(2,2)(2,2)} = 0.008064 - 0.00216\beta$$

$$P_{(1,0)(1,1)(2,2)(2,2)} = 0.0575232 - 0.037632\beta$$

$$P_{(0,0)(1,1)(2,2)(2,1)} = 0.2688\beta + 0.018\beta^2$$

$$P_{(0,0)(1,1)(2,2)(2,2)} = 0.287616 - 0.218208\beta - 0.018\beta^2$$

$$P_{(1,1)(1,0)(2,2)(2,2)} = 0.0309792 - 0.022176\alpha + 0.003024\alpha^2$$

$$P_{(0,1)(1,0)(2,2)(2,2)} = 0.022896 - 0.00324\beta - 0.00648\alpha$$

$$P_{(1,0)(1,0)(2,2)(2,2)} = 0.1198848 - 0.02016\alpha$$

$$P_{(0,0)(1,0)(2,2)(2,2)} = 0.071424 - 0.00756\beta$$

$$P_{(0,1)(0,1)(2,2)(2,2)} = 0.00576$$

$$P_{(1,1)(0,0)(2,1)(2,2)} = 0.1904\alpha - 0.16296\alpha^2$$

$$P_{(1,1)(0,0)(1,1)(2,2)} = 0.168\alpha^2$$

$$P_{(1,1)(0,0)(1,2)(2,2)} = 0.168\alpha - 0.168\alpha^2$$

$$P_{(1,1)(0,0)(2,2)(2,2)} = 0.206528 - 0.336224\alpha + 0.159936\alpha^2$$

$$P_{(0,1)(0,0)(2,1)(2,2)} = 0.072\alpha$$

$$P_{(0,1)(0,0)(2,2)(2,2)} = 0.08064 - 0.06552\alpha$$

$$P_{(1,0)(0,0)(2,2)(2,2)} = 0.037632 + 0.02016\alpha$$

$$P_{(1,0)(0,1)(2,2)(2,2)} = 0.025088$$

$$P_{(0,0)(0,1)(2,2)(2,2)} = 0.01344$$

$$P_{(0,0)(0,0)(2,2)(2,2)} = 0.02016$$

From this distribution one obtains the following numerical results for $X_1, X_2, M_1$, and $M_2$:

$$\begin{aligned} E(X_1) &= 1(0.357488 - 0.043032\beta) \\ &\quad + 2(0.249872) \\ &= 0.857232 - 0.043032\beta \end{aligned}$$

$$\begin{aligned} E(X_2) &= 1(0.300224 - 0.048816\alpha - 0.0108\beta + 0.003024\alpha^2) \\ &\quad + 2(0.354816 + 0.0108\beta) \\ &= 1.009856 - 0.048816\alpha + 0.0108\beta + 0.003024\alpha^2 \end{aligned}$$

$$E(X_1^2) = 1.356976 - 0.043032\beta$$

$$Var(X_1) = 0.622129 + 0.030745\beta - 0.001852\beta^2$$

$$E(X_2^2) = 1.719488 - 0.048816\alpha + 0.0324\beta + 0.003024\alpha^2$$

$$\mathrm{Var}(X_2) = 0.699679 + 0.049778\alpha + 0.010587\beta$$
$$- 0.005467\alpha^2 + 0.001054\alpha\beta - 0.000009\alpha^4$$

$$E(X_1X_2) = 1(0.1736288 - 0.00324\beta - 0.02664\alpha)$$
$$+ 2(0.1073184 - 0.039792\beta - 0.022176\alpha + 0.003024\alpha^2)$$
$$+ 4(0.0016128)$$
$$= 0.3947168 - 0.082824\beta - 0.070992\alpha + 0.006048\alpha^2$$

$$\mathrm{Cov}(X_1X_2) = -0.470964 - 0.048626\beta - 0.029145\alpha$$
$$+ 0.003456\alpha^2 - 0.002101\alpha\beta + 0.000465\beta^2$$
$$+ 0.000130\beta\alpha^2$$

Clearly, $X_1$ and $X_2$ display a strong negative correlation, as one would expect.

For missile usage we have

$$E(M_1) = 2(0.168\alpha^2) + 3(0.4304\alpha - 0.33096\alpha^2)$$
$$+ 4(1 - 0.4304\alpha + 0.16296\alpha^2)$$
$$= 4 - 0.4304\alpha - 0.00504\alpha^2 \quad ,$$

a strictly decreasing function of $\alpha$.

$$E(M_2) = 3(0.2688\beta + 0.018\beta^2) + 4(1 - 0.2688\beta - 0.018\beta^2)$$
$$= 4 - 0.2688\beta - 0.018\beta^2$$

$$E(M_1^2) = 4(0.168\alpha^2) + 9(0.4304\alpha - 0.33096\alpha^2)$$
$$+ 16(1 - 0.4304\alpha + 0.16296\alpha^2)$$
$$= 16 - 3.0128\alpha + 0.30072\alpha^2$$

and

$$\begin{aligned}
\mathrm{Var}(M_1) &= 0.4304\alpha + 0.075156\alpha^2 + 0.004338\alpha^3 \\
&\quad - 0.000025\alpha^4
\end{aligned}$$

$$\begin{aligned}
E(M_2^2) &= 9(0.2688\beta + 0.018\beta^2) + 16(1 - 0.2688\beta - 0.018\beta^2) \\
&= 16 - 1.8816\beta - 0.126\beta^2
\end{aligned}$$

$$\mathrm{Var}(M_2) = 0.2688\beta - 0.054253\beta^2 - 0.009677\beta^3 - 0.000324\beta^4$$

$$\begin{aligned}
E(M_1M_2) &= 8(0.168\alpha^2) + 12(0.2688\beta + 0.018\beta^2 + 0.4304\alpha - 0.33096\alpha^2) \\
&\quad + 16(1 - (0.2688\beta + 0.018\beta^2 + 0.4304\alpha - 0.16296\alpha^2)) \\
&= 16 - 1.0752\beta - 0.072\beta^2 - 1.7216\alpha - 0.02016\alpha^2
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(M_1M_2) &= -0.115692\alpha\beta - 0.007747\alpha\beta^2 - 0.001355\alpha^2\beta \\
&\quad - 0.000091\alpha^2\beta^2
\end{aligned}$$

$$\begin{aligned}
E(X_1M_1) &= 3(0.072\alpha) + 4(0.357488 - 0.043032\beta - 0.072\alpha + 0.168\alpha^2) \\
&\quad + 6(0.3584\alpha - 0.33096\alpha^2) \\
&\quad + 8(0.249872 - 0.3584\alpha + 0.16296\alpha^2) \\
&= 3.428928 - 0.7888\alpha - 0.01008\alpha^2 - 0.172128\beta
\end{aligned}$$

$$\mathrm{Cov}(X_1M_1) = -0.419847\alpha - 0.00576\alpha^2 - 0.018521\alpha\beta - 0.000217\alpha^2\beta$$

In a similar manner, one may show that

$$\begin{aligned}
\mathrm{Cov}(X_2M_2) &= -0.266151\beta - 0.01492\beta^2 - 0.013122\alpha\beta \\
&\quad - 0.000878\alpha\beta^2 + 0.000194\beta^3 + 0.000054\alpha^2\beta^2
\end{aligned}$$

The resulting negative correlations are not surprising, since $M = 4$, its largest value, whenever $X = 0$, its smallest value.

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1 Center for Naval Analyses, University of Rochester, Arlington, Va.
1 Farnsworth Cannon, Inc., Mclean, Va. (Attn: Leroy Riggs)
1 Flight Systems, Inc., Newport Beach, Calif. (Attn: J. Guthrie)
1 Hughes Aircraft Company, Canoga Park, Calif. (Attn: D. Spencer, Bldg. 262)
1 Jet Propulsion Laboratory, CIT, Pasadena, Calif. (Technical Library)
1 McDonnell Douglas Corporation, St. Louis, Mo. (Attn: Steve Rayhawk)
1 Naval Warfare Research Center, Stanford Research Institute, Menlo Park, Calif.
1 Science Applications, Inc., Huntsville, Ala.
1 Science Applications, Inc., LaJolla, Calif.
1 Teledyne Brown Engineering, Fort Walton Beach, Florida (Attn: B. Scudamore)
1 Walter V. Sterling, Orange, Calif. (Attn: J. Smith)