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TRIGONOMETRIC SMOOTHING AND INTERPOLATION OF SAMPLED COMPLEX FUNCTIONS, VIA THE FFT

by
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ADMINISTRATIVE INFORMATION

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INTRODUCTION

In a recent memorandum [1], the use of the FFT for trigonometric interpolation of sampled complex functions was presented. Here we consider trigonometric smoothing of samples of a complex function, followed by interpolation of this smoothed function. Such a technique is useful for smoothing random data, for example. As a by-product, a short-cut for finding stationary points of a real function of complex variables is presented in the Appendix.

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PROBLEM STATEMENT

Suppose complex waveform $y(t)$ has been sampled at times $k\Delta t$, $0 \le k \le N-1$, giving values

$$\{y(k\Delta t)\}_0^{N-1} . \tag{1}$$

We wish to construct a smooth curve that approximates the sample values, according to a series of trigonometric functions. To be explicit, we assume that $y(t)$ is periodic*, of period $N\Delta t$, and we approximate it by low-order harmonics. That is, we choose a smoothed function of the form

$$\tilde{y}(t) = \sum_{m=-\frac{M}{2}}^{\frac{M}{2}} C_m \exp\left(i 2\pi \frac{m}{N\Delta t} t\right), \; M < N, \; M \text{ even}, \tag{2}$$

and choose the $M + 1$ complex constants $\{C_m\}$ such that the error between $\tilde{y}(t)$ and $y(t)$ is minimized. Our measure of error is

$$E = \sum_{k=0}^{N-1} \left| \tilde{y}(k\Delta t) - y(k\Delta t) \right|^2 . \tag{3}$$

Once the optimum coefficients $\{C_m\}$ have been obtained for minimum error, we wish to know (a) the sense in which smoothing is accomplished by (2), (b) how interpolated values of the smoothed function $\tilde{y}(t)$ can be obtained from the samples $\{y(k\Delta t)\}_0^{N-1}$, and (c) the minimum error in (3).

PROBLEM SOLUTION

The analytical details for the solution of the problem posed here are very similar to those given in [1]. Accordingly, reference is made to appropriate sections of [1] in order to avoid re-derivations here.

Substituting (2) in (3), we have for the error,

$$E = \sum_{k=0}^{N-1} \left| \sum_{m=-\frac{M}{2}}^{\frac{M}{2}} C_m \exp(i 2\pi m k/N) - y(k\Delta t) \right|^2 . \tag{4}$$

*See [1], Sample Modifications section.

Partially differentiating E with respect to c_j^* (See Appendix for this shortcut) and setting it equal to zero, we obtain

$$\sum_{m=-\frac{M}{2}}^{\frac{M}{2}} c_m \sum_{k=0}^{N-1} \exp\left(i2\pi(m-j)k/N\right) = \sum_{k=0}^{N-1} y(k\Delta t)\exp\left(-i2\pi jk/N\right), \quad -\frac{M}{2} \le j \le \frac{M}{2}. \tag{5}$$

We define the FFT of sequence $\{y(k\Delta t)\}_0^{N-1}$ as $\{Y_j\}_0^{N-1}$:

$$Y_j = \sum_{k=0}^{N-1} y(k\Delta t)\exp\left(-i2\pi kj/N\right), \quad 0 \le j \le N-1. \tag{6}$$

Then (5) gives for the optimum coefficients for minimum error,

$$c_j = \frac{1}{N}\begin{Bmatrix} Y_j, & 0 \le j \le \frac{M}{2} \\ Y_{j+N}, & -\frac{M}{2} \le j \le -1 \end{Bmatrix}. \tag{7}$$

Equation (7) indicates that the lowest frequency components of the FFT of the sampled waveform should be used for the smoothed function $\tilde{y}(t)$ in (2).

SMOOTHING FUNCTION

To find the smoothed function $\tilde{y}(t)$, we substitute (7) in (2) and obtain [1, Appendix A]

$$\tilde{y}(t) \doteq \sum_{k=0}^{N-1} y(k\Delta t)\, S_{NM}\left(\frac{t}{\Delta t} - k\right), \tag{8}$$

where smoothing function S_{NM} is

$$S_{NM}(x) = \frac{\sin[(M+1)\pi x/N]}{N\sin[\pi x/N]}. \tag{9}$$

Equations (8) and (9) indicate the sense in which smoothing is accomplished. The smoothing function S_{NM} has the properties:

(a) S_{NM} is real

(b) $S_{NM}(0) = \frac{M+1}{N}$

(c) S_{NM} is periodic, of period N

(d) $S_{NM}\left(k\frac{N}{M+1}\right) = 0$ if $k \neq 0, \pm(M-1), \pm 2(M-1), \ldots$

Properties (b) and (d) indicate that the smoothed function does not pass through the sample points (1), but rather performs a local weighted average over approximately N/(M +1) sample values. The smoothing function behaves as

$$S_{NM}(x) \cong \frac{\sin[(M+1)\pi x/N]}{\pi x} \quad \text{for} \quad \left|\frac{\pi x}{N}\right| \ll 1, \tag{10}$$

which is a scaled sinc-function.

INTERPOLATION OF THE SMOOTHED FUNCTION

Now we investigate actual computation of smoothed function $\tilde{y}(t)$. Explicitly, suppose we wish to evaluate $\tilde{y}(t)$ at times

$$0, \frac{N\Delta t}{L}, 2\frac{N\Delta t}{L}, 3\frac{N\Delta t}{L}, \ldots, \tag{11}$$

where L is an integer larger than M (and larger than N if desired); there are no other restrictions on L. Substituting (7) in (2), and letting $t = \ell\frac{N\Delta t}{L}$, we find [1, Appendix B]

$$\tilde{y}\left(\ell\frac{N\Delta t}{L}\right) = \frac{L}{N}\tilde{y}_\ell, \quad 0 \leq \ell \leq L-1, \tag{12}$$

where sequence $\{\tilde{y}_\ell\}_0^{L-1}$ is the inverse FFT of sequence $\{\tilde{Y}_n\}_0^{L-1}$, which is obtained from sequence $\{Y_j\}_0^{N-1}$ according to

$$\tilde{Y}_n = \begin{cases} Y_n, & 0 \leq n \leq \frac{M}{2} \\ 0, & \frac{M}{2}+1 \leq n \leq L-\frac{M}{2}-1 \\ Y_{N+n-L}, & L-\frac{M}{2} \leq n \leq L-1 \end{cases}. \tag{13}$$

That is, sequence $\{Y_j\}_0^{N-1}$ is split into three parts; the first part, of length $\frac{M}{2}+1$, is saved intact; the second part, of length N-M-1, is discarded; the third part, of length $\frac{M}{2}$, is shifted to the end of the $\tilde{Y}_n$ sequence.

We can therefore obtain values of the smoothed function as summarized below:

$$\begin{aligned}
&\{Y_j\}_0^{N-1} = \mathrm{FFT}\{y(k\,\Delta t)\}_0^{N-1} \\
&\{\tilde{Y}_n\}_0^{L-1} \text{ found from } \{Y_j\}_0^{N-1} \text{ according to (13)} \\
&\{\tilde{y}_\ell\}_0^{L-1} = \mathrm{IFFT}\{\tilde{Y}_n\}_0^{L-1} \\
&\tilde{y}\left(\ell \frac{N\,\Delta t}{L}\right) = \frac{L}{N}\tilde{y}_\ell\,,\ 0 \le \ell \le L-1.
\end{aligned} \tag{14}$$

MINIMUM ERROR

The minimum value of error E is found by substituting (7) into (4). After some manipulations and employment of (6), there follows

$$E_{min} = \frac{1}{N} \sum_{k=\frac{M}{2}+1}^{N-\frac{M}{2}-1} |Y_k|^2 . \tag{15}$$

That is, the minimum error is proportional to the discarded high-frequency components representing the original samples $\{y(k\,\Delta t)\}_0^{N-1}$. This is consistent with the observations under (7) that only the lowest frequency components are used for $\{c_j\}$, and also with (13) where frequency components $Y_{\frac{M}{2}+1}$ through $Y_{N-\frac{M}{2}-1}$ were discarded.

COMMENTS

The sample modifications and comments in Ref. 1, pp. 6-8, are also relevant here and should be reviewed before application of the formulas given above.

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REFERENCES

[1]. A. H Nuttall, "Trigonometric Interpolation of Sampled Complex Functions, via the FFT", NUSC/NL Tech. Memo. No. 2020-217-70, 12 Nov. 1970.

APPENDIX. STATIONARY POINTS OF A REAL FUNCTION OF COMPLEX VARIABLES

Consider the real funtion H of complex variables $z_j = x_j + iy_j$, $1 \le j \le N$:

$$H(x_1, y_1, \ldots, x_N, y_N) = H\left(\frac{z_1+z_1^*}{2}, \frac{z_1-z_1^*}{i2}, \ldots, \frac{z_N+z_N^*}{2}, \frac{z_N-z_N^*}{i2}\right). \quad \text{(A1)}$$

Define the real functions

$$\left.\begin{aligned} \frac{\partial}{\partial x_j} H(x_1, y_1, \ldots, x_N, y_N) &= {}_jH(x_1, y_1, \ldots, x_N, y_N) \\ \frac{\partial}{\partial y_j} H(x_1, y_1, \ldots, x_N, y_N) &= H_j(x_1, y_1, \ldots, x_N, y_N) \end{aligned}\right\} 1 \le j \le N. \quad \text{(A2)}$$

The necessary conditions for a stationary point of H are

$$\left.\begin{aligned} {}_jH(x_1, y_1, \ldots, x_N, y_N) &= 0 \\ H_j(x_1, y_1, \ldots, x_N, y_N) &= 0 \end{aligned}\right\} 1 \le j \le N. \quad \text{(A3)}$$

Eq. (A3) constitutes 2N real equations in 2N real unknowns.

Now consider the complex function G obtained from (A1) by replacing z_j by the complex variable u_j, and z_j^* by the complex variable v_j :

$$G(u_1, v_1, \ldots, u_N, v_N) \equiv H\left(\frac{u_1+v_1}{2}, \frac{u_1-v_1}{i2}, \ldots, \frac{u_N+v_N}{2}, \frac{u_N-v_N}{i2}\right). \quad \text{(A4)}$$

Using notation similar to that defined in (A2), and <u>assuming G analytic</u>, we have

$$\begin{aligned} {}_jG(u_1, v_1, \ldots, u_N, v_N) &= {}_jH\left(\frac{u_1+v_1}{2}, \frac{u_1-v_1}{i2}, \ldots, \frac{u_N+v_N}{2}, \frac{u_N-v_N}{i2}\right)\frac{1}{2} \\ &+ H_j\left(\frac{u_1+v_1}{2}, \frac{u_1-v_1}{i2}, \ldots, \frac{u_N+v_N}{2}, \frac{u_N-v_N}{i2}\right)\frac{1}{i2}, \end{aligned} \quad \text{(A5)}$$

and

$$\begin{aligned} G_j(u_1, v_1, \ldots, u_N, v_N) &= {}_jH\left(\frac{u_1+v_1}{2}, \frac{u_1-v_1}{i2}, \ldots, \frac{u_N+v_N}{2}, \frac{u_N-v_N}{i2}\right)\frac{1}{2} \\ &+ H_j\left(\frac{u_1+v_1}{2}, \frac{u_1-v_1}{i2}, \ldots, \frac{u_N+v_N}{2}, \frac{u_N-v_N}{i2}\right)\left(-\frac{1}{i2}\right). \end{aligned} \quad \text{(A6)}$$

In particular,

$$G_j(z_1, z_1^*, \ldots, z_N, z_N^*) = \frac{1}{2}\ {}_jH\left(\frac{z_1+z_1^*}{2}, \frac{z_1-z_1^*}{i2}, \ldots, \frac{z_N+z_N^*}{2}, \frac{z_N-z_N^*}{i2}\right)$$
$$+ i\frac{1}{2}\ H_j\left(\frac{z_1+z_1^*}{2}, \frac{z_1-z_1^*}{i2}, \ldots, \frac{z_N+z_N^*}{2}, \frac{z_N-z_N^*}{i2}\right), \qquad \text{(A7)}$$

and

$${}_jG(z_1, z_1^*, \ldots, z_N, z_N^*) = G_j^*(z_1, z_1^*, \ldots, z_N, z_N^*), \qquad \text{(A8)}$$

since ${}_jH$ and H_j in (A7) take on real values. Therefore the necessary conditions in (A3) for a stationary point of H dictate that

$$G_j(z_1, z_1^*, \ldots, z_N, z_N^*) = 0+i0, \quad 1 \le j \le N. \qquad \text{(A9)}$$

(The conditions (A3) also dictate that

$${}_jG(z_1, z_1^*, \ldots, z_N, z_N^*) = 0+i0, \quad 1 \le j \le N; \qquad \text{(A10)}$$

however (A10) follows from (A8) and (A9) and furnishes no new conditions.) Eq. (A9) constitutes N complex equations in N complex unknowns, and are often of simpler form than (A3).

A rigorous interpretation of (A9) is

$$\frac{\partial}{\partial v_j} H\left(\frac{u_1+v_1}{2}, \frac{u_1-v_1}{i2}, \ldots, \frac{u_N+v_N}{2}, \frac{u_N-v_N}{i2}\right)\Bigg|_{\substack{u_1=z_1\\ v_1=z_1^*\\ \vdots \\ u_N=z_N\\ v_N=z_N^*}} = 0+i0, \quad 1 \le j \le N. \qquad \text{(A11)}$$

However, the following imprecise interpretation is easier to use:

$$\frac{\partial}{\partial z_j^*} H\left(\frac{z_1+z_1^*}{2}, \frac{z_1-z_1^*}{i2}, \ldots, \frac{z_N+z_N^*}{2}, \frac{z_N-z_N^*}{i2}\right) = 0+i0, \quad 1 \le j \le N, \qquad \text{(A12)}$$

where z_j, $1 \le j \le N$, are considered fixed during the differentiation with respect to z_j^*. (A12) is the main result of this appendix.

<u>Example:</u>

$$H\left(\frac{z_1+z_1^*}{2}, \frac{z_1-z_1^*}{i2}, \ldots, \frac{z_N+z_N^*}{2}, \frac{z_N-z_N^*}{i2}\right) = \frac{\left|\sum_{n=1}^{N} z_n\right|^2}{\sum_{k=1}^{N}\sum_{q=1}^{N} z_k M_{k-q} z_q^*} \equiv \frac{\text{Num}}{\text{Den}}. \qquad \text{(A13)}$$

Matrix $[M_{k-j}]$ is Hermitian. Notice that a common complex scaling of every z does not change the value of H.

$$\frac{\partial}{\partial z_j^*} H\left(\frac{z_1+z_1^*}{2}, \frac{z_1-z_1^*}{i2}, \ldots, \frac{z_N+z_N^*}{2}, \frac{z_N-z_N^*}{i2}\right) = \frac{\mathrm{Den}\left(\sum_{n=1}^{N} z_n\right) - \mathrm{Num} \sum_{k=1}^{N} z_k M_{k-j}}{\mathrm{Den}^2} \quad \text{(A14)}$$

From (A12), we obtain

$$\sum_{k=1}^{N} z_k M_{k-j} = \frac{\mathrm{Den}}{\mathrm{Num}} \sum_{n=1}^{N} z_n , \quad 1 \le j \le N. \quad \text{(A15)}$$

Since complex scaling is irrelevant, and the right-hand side of (A15) is a constant, set

$$\sum_{k=1}^{N} z_k M_{k-j} = 1, \quad 1 \le j \le N. \quad \text{(A16)}$$

Then

$$\max\left\{\frac{\mathrm{Num}}{\mathrm{Den}}\right\} = \frac{\left|\sum_{n=1}^{N} z_n\right|^2}{\sum_{q=1}^{N} z_q^*} = \sum_{n=1}^{N} z_n , \quad \text{(A17)}$$

where $\{z_n\}_1^N$ follow from (A16). Alternately,

$$\max\left\{\frac{\mathrm{Num}}{\mathrm{Den}}\right\} = \sum_{j,k=1}^{N} m_{jk} , \quad \text{(A18)}$$

where m_{jk} is the j, k-th element of the inverse of matrix $[M_{k-j}]$.