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The Calculation of Aerodynamic Loading on Surfaces of any Shape

By
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AERODYNAMIC SYMBOLS

1. GENERAL

m	Mass
t	Time
V	Resultant linear velocity
Ω	Resultant angular velocity
ρ	Density, σ relative density
ν	Kinematic coefficient of viscosity
R	Reynolds number, $R = lV/\nu$ (where l is a suitable linear dimension)

Normal temperature and pressure for aeronautical work are 15° C and 760 mm.

For air under these conditions $\begin{cases} \rho = 0 \cdot 002378 \text{ slug/cu. ft.} \\ \nu = 1 \cdot 56 \times 10^{-4} \text{ ft.}^2\text{/sec.} \end{cases}$

The slug is taken to be 32·2 lb.-mass.

α	Angle of incidence
ϵ	Angle of downwash
S	Area
b	Span
c	Chord
A	Aspect ratio, $A = b^2/S$
L	Lift, with coefficient $C_L = L/\frac{1}{2}\rho V^2 S$
D	Drag, with coefficient $C_D = D/\frac{1}{2}\rho V^2 S$
γ	Gliding angle, $\tan \gamma = D/L$
L	Rolling moment, with coefficient $C_l = L/\frac{1}{2}\rho V^2 bS$
M	Pitching moment, with coefficient $C_m = M/\frac{1}{2}\rho V^2 cS$
N	Yawing moment, with coefficient $C_n = N/\frac{1}{2}\rho V^2 bS$

2. AIRSCREWS

n	Revolutions per second
D	Diameter
J	V/nD
P	Power
T	Thrust, with coefficient $k_T = T/\rho n^2 D^4$
Q	Torque, with coefficient $k_Q = Q/\rho n^2 D^5$
η	Efficiency, $\eta = TV/P = Jk_T/2\pi k_Q$

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Reports and Memoranda No. 1910
26*th August*, 1943

Summary.—The object of the report is to establish a routine method for the calculation of aerodynamic loads on wings of arbitrary shape. The method developed is based on potential theory and uses a general mathematical formula for continuous loading on a wing which is equivalent to a double Fourier series with unknown coefficients. In order to evaluate the unknown coefficients the continuous loading is split up into a regular pattern of horseshoe vortices, the strengths of which are proportional to the unknown coefficients and to standard factors which are given in a table. The total downwash at chosen pivotal points is obtained by summing the downwashes due to the individual vortices, a process which is simplified by the use of specially prepared tables of the properties of the horseshoe vortex. By equating the downwash to the slope of the wing at each pivotal point, simultaneous equations are obtained, the solution of which defines the unknown coefficients.

The first layout involves a total of 76 vortices over the wing, and a second layout, involving a total of 84, is shown to be of superior accuracy. The effect on the solution of the number of pivotal points is investigated and it is concluded that by a suitable choice, it is unnecessary to use a large number. Results for a rectangular wing at 0°, and an elliptic wing at 0° and 30° yaw are compared with those obtained by other workers and it appears that there may be errors in published results in at least one of these cases. Immediate development includes the application to the calculation of the characteristics of actual sweptback wings, including rotary derivatives, and future development includes also applications in wind tunnel design and technique.

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1. Recent design work on sweptback wings has drawn attention to the increasing need for a development of the simpler theory of aerodynamic loading which has served well in the past and will no doubt still be used for approximate calculations. Problems for which a more comprehensive theory is necessary include, in addition to the properties of sweptback wings, efficiency of wings, controls, wind tunnel interference, scale effect, design of wind tunnels, effect of airscrews and so on.

The present work was undertaken in order to reduce to a standard and easily understood routine the calculation of the loading distribution on a wing of arbitrary shape, initially to determine the simpler properties such as lift, induced drag, aerodynamic centre, effect of sweepback and twist, and with the immediate development in view of the calculation of rotary derivatives. Later developments will be directed towards the secondary characteristics such as effect of stalling and changes due to scale effect.

The work is based wholly on potential theory and, although the present work is confined to the simpler applications of this theory, the writer has no doubt that the effects of viscosity, often of considerable importance, can, for practical purposes, be represented by developments or modifications of potential theory. The work falls into two distinct categories (*a*) the purely mathematical problem of establishing solutions of known accuracy for certain assumed conditions, (*b*) the problem in applied mathematics of using these solutions to predict the physical properties of actual wings.

2. The present work is based on the theorem[1] that any continuous irrotational motion of an incompressible fluid, whether cyclic or not, can be represented by a distribution of vortices over the boundaries. The work will, as far as calculations are concerned, be limited for the present

to thin wing theory, in which any aerofoil is represented by a vortex sheet located on the surface which is the mean of the upper and lower surfaces. The effect of thickness is regarded as suitable for treatment either by modifications of potential theory or by correction factors.

The use of continuous loading in the spanwise direction was developed by Prandtl, Betz, Munk and others (1918–1919). Betz[2], in order to calculate the spanwise distribution of lift of a rectangular wing, uses the expression for the circulation (see Fig. 2), $\Gamma = \sqrt{1-\eta^2}\,(a_0 + a_1\eta + a_2\eta^2 + \ldots)$, and this was later expressed by Munk[3] and Glauert[4] in the more conventional Fourier series, with which it is identical with the exception, perhaps, of a difference in the mode of convergence. Each term can be identified with terms of the equivalent Fourier series; for instance $\sqrt{1-\eta^2} \equiv \sin\phi$, $\eta\sqrt{1-\eta^2} \equiv \frac{1}{2}\sin 2\phi$, and so on.

Continuous loading in the chordwise direction was developed by Birnbaum[5] in connection with the two-dimensional properties of wing sections; he used the form $\frac{k}{V} = a_0\sqrt{\frac{1+\xi}{1-\xi}} + \sqrt{1-\xi^2}\,(A_0 + A_1\xi + A_2\xi^2 + \ldots)$, where k is the vorticity loading per unit length. This forms the foundation for the thin wing theory developed by Munk[3] and Glauert[6], who use the corresponding Fourier form, $\frac{k}{V} = a_0 \cot\frac{\theta}{2} + \Sigma A_n \sin n\theta$.

The two systems were combined by Blenk[7] to give a formula for the continuous loading over a rectangular wing of finite aspect ratio, which can be expressed either in terms of the functions above, or as a double Fourier series. This formula is quite general within the limits of the assumptions involved, and, after generalisation for shape of wing, gives the following basic formula of the present work, the variables being defined in Fig. 1 :—

$$\begin{aligned}\frac{kc}{8sV\tan\alpha} = \sqrt{1-\eta^2}\Big[&\cot\frac{\theta}{2}(a_0 + b_0\eta + c_0\eta^2 + d_0\eta^3 + e_0\eta^4 + \ldots)\\ &+ \sin\theta\,(a_1 + b_1\eta + c_1\eta^2 + d_1\eta^3 + e_1\eta^4 + \ldots)\\ &+ \sin 2\theta\,(a_2 + b_2\eta + c_2\eta^2 + d_2\eta^3 + e_2\eta^4 + \ldots) + \ldots\Big] \qquad (1)\end{aligned}$$

In the formula α is the angle of incidence from zero lift. If we use the theorem that the effects of camber and twist are independent of incidence effects,[6, 8] the following additional form (also used by Blenk) represents conditions at zero lift :—

$$\begin{aligned}\frac{kc}{8sV} = \sqrt{1-\eta^2}\Big[&\cot\frac{\theta}{2}(a_0' + b_0'\eta + c_0'\eta^2 + \ldots) + \sin\theta\,(a_1' + b_1'\eta + c_1'\eta^2 + \ldots)\\ &+ \sin 2\theta\,(a_2' + b_2'\eta + c_2'\eta^2 + \ldots) + \ldots\Big] \qquad (2)\end{aligned}$$

The complete solution is the sum of the "loadings" given by these two forms. Relation (1) is used to calculate lift and moment derivatives and that part of the induced drag due to incidence, while relation (2), which is used with the condition that $C_L = 0$, is used for the calculation of moment at zero lift, angle of incidence for zero lift, and induced drag at zero lift.

Relations (1) and (2) may be written more concisely :—

$$\frac{kc}{8sV\tan\alpha} = \sqrt{1-\eta^2}\Big[F_0(\eta)\cot\frac{\theta}{2} + F_1(\eta)\sin\theta + F_2(\eta)\sin 2\theta\Big] \qquad (3)$$

$$\frac{kc}{8sV} = \sqrt{1-\eta^2}\Big[G_0(\eta)\cot\frac{\theta}{2} + G_1(\eta)\sin\theta + G_2(\eta)\sin 2\theta\Big] \qquad (4)$$

3. In order to consider how far the results given by (1) and (2) may be applicable to actual wings in a viscous fluid, the following list gives the assumptions which are involved in the use of (1) and (2) :—

(*a*) The fluid is incompressible.

(*b*) The flow is wholly potential.

(*c*) The wing is represented by a thin plate, the m[illegible]ium plane between the upper and lower surfaces.

(*d*) It is assumed that the wing tips are square or rounded off. Pointed tips would require a modified formula with the $\sqrt{1 - \eta^2}$ factor omitted.

(*e*) The application of theory, following Blenk, Glauert and others, in which the downwash ratio ω/v is equated to the local slope of the plate is equivalent to the assumption that the load is vanishingly small at all points. This condition cannot, in fact, be satisfied if camber and twist are present.

(*f*) The Kutta-Joukowski circulation giving the stagnation point at the trailing edge is assumed.

(*g*) Even if it is possible for the load generally to vanish everywhere at the same time, there is still a singularity at the leading edge arising from the cot $\theta/2$ term, which is not an adequate representation of the flow in that it gives the forward stagnation point at the leading edge. This singularity is discussed in Durand[8], and leads to a paradox regarding the resistance. The error is regarded as vanishing with the lift, and it is not known under what conditions it might be appreciable.

In spite of this formidable list it seems that much valuable work can be done with the bare theory before modifications are considered. Some corrections, e.g., the effect of the boundary layer on circulation and effects of partial stalling can. it is predicted, be treated quite easily by modifications, wholly potential, to the formulae 1 and 2; these will be introduced at a later stage of the work. Others can be effected by the use of simple factors obtained either theoretically or experimentally.

4. The most frequently used method for computing aerodynamic loading on wings is that which has reached its highest development in the Lotz[8] method, in which the loading represented by (1) is reduced to the first term in θ, i.e., cot $\theta/2$, the term which represents the vortex sheet of a flat aerofoil in two-dimensional motion. The load is taken as concentrated at 0·25 chord and the 0·25 chord line is assumed to be straight. The downwash due to the trailing vortices which spring from the 0·25 chord line can be readily calculated by Fourier analysis and, in effect, the solution is obtained by equating the downwash to the slope of the plate at selected points on the 0·25 chord line. This theory is notoriously inaccurate for small aspect ratios but it has not hitherto been realised that it is sufficiently inaccurate for conventional wings to make revision necessary of the methods used for computing section coefficients from results with a finite aspect ratio. This matter will be dealt with in §14.

The error is more serious when problems of control or effect of sweepback are in question. A modification of this method which consists in the calculation of downwash on the three-quarter chord line has been used by Weighardt[9] and Mutterperl[10]. The theory of thin aerofoils suggests that this method should be of superior accuracy. It is shown in Vol. II of Durand, p. 49, that if a thin aerofoil section is cambered parabolically or in the form of a circular arc, the effective angle of attack is the slope at the three-quarter chord line. Hence, as effective camber is always present in three-dimensional flow, the use of the single slope chordwise at 0·75 chord to define the incidence is more accurate than the use of the slope at 0·25 chord. It is hoped that this idea can be further developed at a later stage of the work when considering the most effective means of simplifying the calculations.

The effect of increasing the number of load lines in the chordwise direction while retaining continuity in the spanwise direction has been calculated by Weighardt[9] for a rectangular wing using 2 and 4 load lines

Continuous loading in both chordwise and spanwise direction has been dealt with by Blenk[7] for the rectangular plate, yawed and unyawed, and the arrow-shaped plate; by Kinner[11] for circular plates using the method of acceleration potential; by Krienes[12] for elliptic plates yawed and unyawed using the method of acceleration potential; and recently by W. P. Jones[13] as a side investigation in the calculation of derivatives for an oscillating wing. The position as regards some of these mathematical solutions is unsatisfactory, as they are not usually, in fact, complete mathematical solutions of the problem. Two examples are given :—Blenk gives the integrals for his problem, but in the analysis has firstly to evaluate these integrals by approximate methods involving series, and secondly to find the values of certain coefficients by the use of a limited number of pivotal points on the plate. The final solution is obtained only when these two processes have converged simultaneously. For the yawed elliptic aerofoil, Krienes gives no indication that his solution has reached convergence with respect to the number of pivotal points and there are indications of considerable error in his results.

Finally graphical methods of solving the continuous loading problems have been suggested and demonstrated by Cohen[14].

5. Having regard to the scope and object of the investigation, none of the work described in the preceding paragraph is of a sufficiently comprehensive nature to use as the general basis of the work. It is clear that it is difficult and specialised work to express in mathematical form even the integral relative to the simpler shapes of unyawed wings. When the investigation is extended to wings of arbitrary shape, yawed and with rotary motion, the mathematical expression of the downwash integral is so difficult as to be a practical impossibility. The proper function of the mathematician is to provide solutions of specified accuracy of some of the more simple problems which can be used as standards for the testing of easier approximate methods which offer a much wider field of utility by avoiding excessive mathematical rigidity

At the other extreme, graphical methods of solution have nothing to recommend them, as they fail to satisfy any of the essential conditions of a problem of this nature. Considering the possible uses and application of the work, the following conditions, which apply to the method which will be described below, are considered to be necessary :—

(*a*) The whole of the assumptions are contained in the original layout of the work. The number and disposition of the vortices to be used and the number and position of pivotal points are specified by the technical man on the basis of his previous experience. The remainder of the work is purely routine calculation which is suitable for the application of rigid checks for accuracy.

(*b*) The accuracy of a given result can be tested, frequently without undue labour, by revising the layout to the next higher approximation.

(*c*) Certain effects, such as effects of sweepback, derivatives with respect to yaw, and so on, can be calculated accurately with a comparatively simple layout, involving as they do only differences.

(*d*) Because of the rigid specification, the work can be repeated at any time to find the effect of modifications.

Graphical methods fail to satisfy the above conditions. For instance, it is not easy to specify a rigid layout for graphical methods; the work, if carried out by computers, could not be checked except by a complete recalculation, because of the difficulty of separating arithmetical errors from errors of judgment; the results could not be checked by proceeding to the next approximation; the rigid framework essential for the accurate calculation of derivatives, and effects of small variations, is lacking, and, finally, matters involving judgment may sometimes waste a considerable amount of time.

6. The present work is based on an idea which has been used frequently in other fields of research, that is, the replacement of a continuous loading by a patterned layout of isolated loac . It will not be disputed that, if the method of layout is sound, and the spacing is reduced indefinitel , the correct answer can be obtained. The important question is—can the layout be so arranged that good accuracy is obtained with a wide spacing of the loads, thus reducing the work of calculation, which involves the properties of the isolated loads, to a reasonable minimum? The present work aims to show and prove that this can be accomplished for the loading represented by vortex sheets.

Consider the distribution of vorticity given by relation (1). It is required to split this into a pattern of isolated vortices both chordwise and spanwise so that the coefficients a_0, b_0, etc., can be calculated for a specified wing. In the present work the chordwise loading is represented by four loads placed at 0·125, 0·375, 0·625 and 0·875 chord. The procedure for defining these loads is the same whatever the number of loads, and the choice of four was influenced by the circumstance that, having regard to possible developments, fewer than four would hardly be satisfactory and, in fact, may be inadequate for special problems. On the other hand, Prandtl[15] is satisfied that good accuracy for a flat wing can be obtained by the use of four load lines.

In the spanwise direction it was predicted that intervals of 0·1 semi-span would be satisfactory and later work has shown that these intervals, after slight modification by the addition of corrector vortices at each tip, are satisfactory. The maximum number of loads which have so far been used therefore total 84 for the complete wing.

7. The splitting up of the load in the chordwise direction is accomplished by the following process applied in turn to each term of (1). The pivotal points at which downwash will be equated to the slope of the plate are specified as the midpoints of the four chordwise loads, i.e., at the 0·25, 0·50 and 0·75 chord points. The fundamental condition which must be satisfied at these pivotal points, is that the downwash due to the isolated loads, in two-dimensional flow, shall be equal to that given by the continuous load. With the other condition that the sum of the isolated loads, which is in this case the circulation round the chord, is equal to the integral of the continuous load, the relation between isolated and continuous loads is specified exactly.

Consider the first chordwise term V cot $\theta/2$. It can easily be shown that if $k = V \cot \theta/2$, ω V = $\frac{1}{2}$ at any point of the chord, and the integral of V cot $\theta/2$ along the chord is $\frac{1}{2}\pi Vc$. Hence if K_1. K_2, K_3 and K_4 be the four isolated loads

$$K_1 + K_2 + K_3 + K_4 = \tfrac{1}{2}\pi Vc.$$

The downwash factor at 0·25 chord due to K_1 at 0·125 chord is $\dfrac{\omega}{V} = \dfrac{8K_1}{2\pi Vc}$; that due to K_2 at 0·375 chord is $-\dfrac{8K_2}{2\pi Vc}$; and summing the total downwash and equating to the correct value, we obtain

$$8K_1 - 8K_2 - 2\cdot\dot{6}K_3 - 1\cdot 6K_4 = \pi Vc.$$

Similar relations for the 0·5 and 0·75 chord positions give

$$2\cdot\dot{6}K_1 + 8K_2 - 8K_3 - 2\cdot\dot{6}K_4 = \pi Vc$$

and

$$1\cdot 6K_1 + 2\cdot\dot{6}K_2 + 8K_3 - 8K_4 = \pi Vc.$$

The solution of this set of simultaneous equations gives the result that for $k = V \cot \theta/2$, the four isolated vortices are $0\cdot 2734Vc\pi$, $0\cdot 1172Vc\pi$, $0\cdot 0703Vc\pi$ and $0\cdot 0391Vc\pi$, summing to $0\cdot 5Vc\pi$.

A similar routine applied to sin θ and sin 2θ gives factors which are given in Table 2. The downwashes and integrals of vorticity relating to two-dimensional flow are given in Table 1.

If the plane in which the downwash is to be calculated is at a considerable distance from the horseshoe vortices, the set of four can be reduced without appreciable error to one at the centre of area. Table 2 gives this alternative representation—for instance, V cot $\theta/2$ is represented by $0{\cdot}5\pi Vc$ at 0·25 chord, V sin θ by $0{\cdot}25\pi Vc$ at 0·5 chord, and so on.

8. The splitting up of the loading in the spanwise direction is carried out by a rather different method. Along each of the four lines of concentrated load at 0·125, 0·375, 0·625 and 0·875 chord it is assumed that the vorticity loading and so the circulation remains constant for a set distance, then, after changing suddenly by the shedding of a trailing vortex, again remains constant for a similar distance, and so on. If the wing is divided into intervals of 0·1 semispan, this is equivalent to the use of the regular system of horseshoe vortices shown in Fig. 2 for layout 1. It was predicted that intervals of 0·1 would give good accuracy, and a side investigation suggested that the correct magnitudes of the vortices are the magnitudes of the continuous load at the points corresponding to the centres of the bound vortices, which define the location of the load. For example, consider the $\sqrt{1-\eta^2}$ term in (1). The appropriate strengths of the horseshoe vortices to represent this term are 1·0 on the median line or $\eta = 0$, $\sqrt{1-(0{\cdot}1)^2}$ or 0·9950 at $\eta = 0{\cdot}1$, 0·9798 at 0·2 and so on. Similarly the $\eta\sqrt{1-\eta^2}$ term is represented by 0 on the median line, $\pm 0{\cdot}0995$ at $\eta = \pm 0{\cdot}1$ and so on. All of these quantities vanish for $\eta = 1$, and the last vortex for this layout, termed layout 1, is at $\eta = 0{\cdot}9$. The factors for terms up to $\eta^{10}\sqrt{1-\eta^2}$ are given in Table 2.

Subsequent investigation showed that this method of representation was quite sound as long as the function representing the continuous load could be expressed over the interval concerned as a power series of the second degree. The form of the functions, for all of which the load at the tip vanishes as $\sqrt{1-\eta^2}$, shows that error will appear first at the tip. Integrations of one or two simple limiting cases, and comparison with a simple known solution, to be described below, suggested that the tip error could be corrected by the addition of an extra term near each tip for $\eta = \pm 0{\cdot}9625$, representing a vortex of width $\frac{1}{4}$ of the remaining vortices. These are termed corrector vortices and their strength is defined in exactly the same way as the other vortices. When used they convert the layout 1 shown in Fig. 2 with its 76-point loading, to the layout 2 with 84-point loading. The extra work involved in the use of layout 2 is small, and it is thought that the accuracy is at least equal to that which would be obtained from the next approximation with one half the interval in the spanwise direction. No work has yet been done on this higher approximation, which is held in reserve for future use.

The two layouts have a subsidiary distinction depending upon whether or not the reduction to 1-point loading in the chordwise direction is used. A description is given in Table 2.

9. A demonstration is now given of the exact relation between Table 2 and the relation (1). Suppose that the analysis is limited to a symmetrical wing at 0° yaw, which means that coefficients of odd powers of η are all zero, and that three terms chordwise and two terms spanwise are retained.

Then

$$\frac{kc}{8sV\tan\chi} = \sqrt{1-\eta^2}\left[\cot\frac{\theta}{2}\,(a_0 + c_0\eta^2) + \sin\theta\,(a_1 + c_1\eta^2) + \sin 2\theta\,(a_2 + c_2\eta^2)\right] \quad \ldots \quad (5)$$

For $\eta = 0$, the factor $\sqrt{1-\eta^2}$ is 1·0, while $\eta^2\sqrt{1-\eta^2} = 0$, and hence, using the factors for cot $\theta/2$, sin θ, and sin 2θ, the relative strength of the vortex at $\eta = 0$, 0·125 chord is $0{\cdot}2734a_0 + 0{\cdot}0488a_1 + 0{\cdot}0732a_2$; at $\eta = 0$, 0·375 chord is $0{\cdot}1172a_0 + 0{\cdot}0762a_1 + 0{\cdot}0381a_2$, and so on. Similarly for $\eta = 0{\cdot}1$, the strength of the vortex at $\eta = 0{\cdot}1$, 0·125 chord is

$$0{\cdot}2734 \times 0{\cdot}9950a_0 + 0{\cdot}0488 \times 0{\cdot}9950a_1 + 0{\cdot}0732 \times 0{\cdot}9950a_2$$
$$+\ 0{\cdot}2734 \times 0{\cdot}0099c_0 + 0{\cdot}0488 \times 0{\cdot}0099c_1 + 0{\cdot}0732 \times 0{\cdot}0099c_2$$

and so on.

All of the vortices are defined explicitly in terms of the unknown coefficients in (5), and the same applies however many coefficients occur in (5). The position and magnitude of the vortices being known, the downwash at any point can be calculated using the usual formula.

10. The work can be reduced to a minimum by tabulating the properties of the horseshoe vortex. This can be done simply because we are concerned only with downwashes on lines at regular distances, in terms of the vortex width, from the centre line of the vortex. The formulae are derived simply and are given in Glauert's book[6]. By the use of these formulae, downwash factors have been computed and printed on the National machine under the supervision of Dr. L. J. Comrie of Scientific Computing Service, Ltd. to five places of decimals, with first and second differences. The tables are computed for regular intervals of y^*, where $y^* = y/y_V$ (see Fig. 3), with $x^* = x/y_V$ as the variable. The tables give the value of a factor F, corresponding to x^* positive, and a complementary factor F′, corresponding to x^* negative, such that the downwash ratio ω/V is equal to $F \times \frac{K}{4\pi V y_V}$ where K is the strength of the vortex. These tables are not reproduced here but it is hoped that it will be possible later to circulate them after complete subtabulation. The writer has subtabulated to give correct answers to three places of decimals by the use of the first difference only, the use of second and higher differences not being recommended for inexperienced computers.

11. The solution of any problem involves the calculation of the downwash at a certain number of pivotal points by [illegible] the effect due to each individual vortex. The bare minimum number of points is equal to the number of unknowns in the relation (1). No final decision has yet been made as to the necessary number of points to give a specified accuracy. Evidence which will be given as each case is considered suggests that for a symmetrical wing without sweepback six points on the halfwing, those marked 1 to 6 in Fig. 2, are sufficient. By symmetry this is equivalent to the use of 12 points for the wing. For sweptback symmetrical wings it is probably necessary to use nine coefficients and nine points, those marked 1 to 9 in Fig. 2.

The calculated values of ω/V are equated to the slope of the plate, in this case $\tan \alpha$, at the point concerned, and the solution of the simultaneous equations gives the values of the coefficients in relation (1).

One important theorem, suggested originally by Dr. H. O. Hartley, assistant to Dr. Comrie, has been demonstrated by trial solutions. When using the bare minimum of pivotal points, they must agree in number in the two directions with the coefficients retained in the relation (1). For instance, if the coefficients a_0, c_0, a_1, c_1, a_2, c_2 are retained, three in the chordwise and two in the spanwise direction, the points 1, 2, 3 and 4, 5, 6 can be used. The points 1, 3, 4, 6, 7, 9 would probably give a false result unless used with a_0, c_0, e_0, a_1, c_1, e_1. It has not been considered advisable to place any pivotal point nearer the tip than 0·8 of the semispan.

12. The actual method of layout of the work with suitable checks for accuracy will vary depending on the machines and computing staff available. That devised by the writer at the laboratory differs from that used by Drs. Comrie and Hartley. As it may not be possible to show the complete layout for a wing, a demonstration is given of a simple problem, that is, the calculation by the present method of the loading on an elliptic wing with ratio of major to minor axis of 5 to 1, using the same assumptions as in the Glauert-Lotz method, i.e., load concentrated at 0·25 chord, the locus of which is a straight line. This case, for which the true analytical solution is given by the simple expression $\frac{dC_L}{d\alpha} = \frac{2\pi}{1 + \pi/10}$ or 4·781 forms a valuable test case for assessing the value of the present method and the effect of the corrector vortices.

In Table 3 the data conforms to the original layout 1, excluding the corrector vortices. The coefficients of odd powers of η vanish through symmetry; by the assumed conditions the coefficients of $\sin \theta$, $\sin 2\theta$, . . . are all zero and we retain four coefficients a_0, c_0, e_0 and g_0. The values of $\sqrt{1 - \eta^2}$, $\eta^2 \sqrt{1 - \eta^2}$, . . . from Table 2 are set out and denoted by A_1, A_2, A_3 and A_4. The four chosen pivotal points on the chord line are at 0·1, 0·4, 0·6 and 0·8 of the semispan. From the tables, the factors appropriate to the positions of each vortex with respect to each pivotal point are set down under the preceding values and denoted by B_1, B_2, B_3 and B_4. For this simple case, in which $x^* = 0$, the factor simplifies to the expression $\frac{1}{y^*+1} - \frac{1}{y^*-1}$. The sum of the A coefficients is denoted by ΣA, and the B coefficients by ΣB.

The sum of the products B_1A_1, B_1A_2, B_1A_3 and B_1A_4 for point 1, and similar products for points 2, 3 and 4 are computed and tabulated. The check for accuracy is that the sum should equal $\Sigma\ \Sigma A \times \Sigma B$, an error in the last figure being allowed on account of cumulative errors arising from the limited number of figures in the individual totals. For this case, including only the cot $\theta/2$ term concentrated at the centre of area, relation (1) becomes

$$\frac{\Gamma}{8sV\tan\alpha} = \sqrt{1-\eta^2}\,\frac{\pi}{2}\left[a_0 + c_0\eta^2 + e_0\eta^4 + g_0\eta^6\right].$$

Now

$$\frac{\omega}{V} = \frac{1}{4\pi V y_v}\,8sV\tan\alpha\,\frac{\pi}{2}\left[a_0\Sigma BA_1 + c_0\Sigma BA_2 + e_0\Sigma BA_3 + g_0\Sigma BA_3\right].$$

The element of lift is

$$8s\varrho V^2\tan\alpha\sqrt{1-\eta^2}\,\frac{\pi}{2}\,(a_0 + c_0\eta^2 + e_0\eta^4 + g_0\eta^6).$$

Alternatively, the element of lift is

$$2\pi\tfrac{1}{2}\varrho V^2 c\left[\tan\alpha - \frac{\omega}{V}\right].$$

Equating these

$$a_0\left[\frac{4s}{c}\sqrt{1-\eta^2} + 20\Sigma BA_1\right] + b_0\left[\frac{4s}{c}\,\eta\,\sqrt{1-\eta^2} + 20\Sigma BA_2\right] + \ldots = 1.$$

For the 5/1 ellipse, $c/s = 0{\cdot}4\sqrt{1-\eta^2}$, hence

$$a_0\left[1 + 2\Sigma BA_1\right] + b_0\left[\eta^2 + 2\Sigma BA_2\right] + c_0\left[\eta^4 + 2\Sigma BA_3\right] + d_0\left[\eta^6 + 2\Sigma BA_4\right] = 0{\cdot}1.$$

The resulting equations for the four points $\eta = 0{\cdot}1$, 0·4, 0·6 and 0·8 are given in the table. The solution gives (see Appendix I) $\frac{dC_L}{d\alpha} = 4{\cdot}746$, the exact solution being 4·781. A repetition of the solution with six pivotal points $\eta = 0{\cdot}1$, 0·3, 0·4, 0·6, 0·7, 0·8 gave 4·740, and a repetition using the corrector vortices at $\eta = \pm 0{\cdot}9625$, and using the four points $\eta = 0{\cdot}1$, 0·4, 0·6 and 0·8 gave $\frac{dC_L}{d\alpha} = 4{\cdot}778$. This result is taken by the writer as evidence that (1) no appreciable error is involved in the use of only four pivotal points (2) the addition of the corrector vortices is an effective means of obtaining a higher approximation.

13. The layout for a wing using distributed load does not differ in principle from that shown above. The factors A_1 to A_4 would be the same; an extra table derived from the plan of the wing and giving the relative positions of each vortex is necessary so that values of x^* and y^* applicable to any pivotal point can be computed and tabulated. The factors are then read from the tables and set down under the A coefficients, and when the full 4-point loading chordwise is adopted there will be four corresponding factors at each position along the span. The downwashes are computed in terms of sums of products and the coefficients a_0, etc., and are equated directly to the slope of the aerofoil at the point concerned. The solution of the simultaneous equations gives the values of the coefficients.

The solution of the properties at $C_L = 0$ is obtained by the use of relation (2), the equations being derived in precisely the same way as when finding $dC_L/d\alpha$. The unknown a_0' is eliminated by using the condition for no lift (see Appendix I) i.e., $16a_0' + 8a_1' + 4c_0' + 2c_1' + 2e_0' + e_1' = 0$, and in place of this the unknown α_0, the angle of incidence for no lift, is introduced. The downwash at any pivotal point is equated to α_0 plus the slope of the plate at that point. From this solution α_0 and C_{m0} are derived.

14. *Rectangular Wing, Aspect Ratio* 6 *to* 1.—The results of various calculations of the centre of pressure and lift derivative for steady motion are given in Table 4. The first point to be noted is the close agreement between the straight solution and the least squares solution computed for layout 1. This provides effective evidence that there is very little, if any, error involved in limiting the number of pivotal points to six. Another important point is the difference between the layouts 1 and 2, i.e., without and with the corrector vortices. The effect of the corrector vortices is to increase $dC_L/d\alpha$ by only about $2{\cdot}4\%$, and this is the order of correction which has been found in all cases which have been tried. It seems justifiable to assume that the answer given by layout 2 must be very nearly correct. The figure $\dfrac{dC_L}{d\alpha} = 4{\cdot}296$ is in close agreement with that obtained by W. P. Jones, i.e., $4{\cdot}303$, by a different method.

The values accepted as correct by the writer are $\dfrac{dC_L}{d\alpha} = 4{\cdot}30$, C.P. at $0{\cdot}239$ chord. The acceptance of these values involves a modification in the formulae for converting results for $A = 6$ to infinite aspect ratio. The new ratio of lift slopes will be $2\pi/4{\cdot}30$ instead of $2\pi/4{\cdot}53$ and there is an additional correction of $+0\ 011$ on the centre of pressure. The new value modifies the computed section values of $dC_L/d\alpha$ by about 5%.

In converting from $A = 6$ to $A = \infty$, it is always assumed that the values of C_{m0} and α_0 are unchanged. In Table 5 are given the corrections which should be applied to the N.A.C.A. series for various positions of maximum camber. The values for $A = 6$ have been computed by the method described in this paper using six pivotal points. The values corresponding to $A = \infty$ were computed by the thin wing theory described in Glauert, using the same three points of coincidence in the chordwise direction at $0{\cdot}25$, $0{\cdot}5$ and $0{\cdot}75$ chord. For the particular type of camber of the N.A.C.A. series, this may be too few to give the absolute values, and the differences only, which are corrections, are given. The corrections apply to a camber of 2%, and are proportional to the camber.

15. *Elliptic wing, major axis/minor axis* = 5 *to* 1.—The results of calculations on a wing of this plan form at 0° and 30° yaw in steady motion are given in Table 6. For 0° yaw, the aspect ratio is $20/\pi$ or $6{\cdot}37$, the Glauert value of $dC_L/d\alpha$ is $4{\cdot}78$, and the C.P. at $0{\cdot}288$ of the median chord. For 30° yaw, at which angle the span is reduced in the ratio $0{\cdot}872$ to 1, the aspect ratio is $4{\cdot}84$, the Glauert value of $dC_L/d\alpha$ is approximately $4{\cdot}45$ and the C.P. is approximately at $0{\cdot}288$ of the median chord.

Values obtained by Krienes using the acceleration potential method are $4{\cdot}55$ and $0{\cdot}283$ at 0° yaw, and $3{\cdot}26$ for $dC_L/d\alpha$ at 30° yaw. An unofficial examination of Krienes work is in hand by Dr. Hartley. The complete results are not yet available, but it seems that there is very little error if any in the result for 0° yaw.

The straight solution for 0° yaw was computed for layout 1, which gives $\dfrac{dC_L}{d\alpha} = 4{\cdot}49$. This would agree with Krienes' result if increased by $1{\cdot}3\%$: it will be seen from the results at 30° yaw that the addition of corrector vortices increases $dC_L/d\alpha$ by $1{\cdot}3\%$, hence it is deduced that the present method, using layout 2, would give complete agreement with Krienes result for 0° yaw.

Three solutions have been computed by Scientific Computing Service Ltd. for 30° yaw. The first two demonstrate that there is no appreciable error in limiting the number of pivotal points to 12 over the wing, and the third shows that the corrector vortices increase $dC_L/d\alpha$ by 1·3%. Hence, unless there is some hidden flaw in the present method, it seems that the value of $dC_L/d\alpha$ for the wing at 30° yaw cannot differ appreciably from 3·81. Any further discussion on Krienes' results is held over until the receipt of a report from Dr. Hartley.

16. Work is proceeding on calculations for sweptback wings, and, as far as can be seen, good agreement with wind tunnel tests will be obtained. These results will be given in a later paper, as the matter cannot be treated effectively until examination has been made of the present inadequate knowledge of section coefficients.

17. The immediate programme of work includes :—

(*a*) Revision of section coefficient calculations as described in §14.

(*b*) Calculation of lift, moment and induced drag for various shapes of sweptback wings using the bare theory.

(*c*) Modification to include effects due to loss of circulation and incipient stalling.

(*d*) Establishment of the proper routine for predicting actual wing properties from (*b*) and (*c*).

Work scheduled for the near future includes

(*e*) Calculation of rotary derivatives.

(*f*) Effect of flaps.

(*g*) Effect of airscrews.

(*h*) Effect of fins.

(*i*) Effect of body.

(*j*) Effect of controls.

The work under (*e*) and (*h*) will involve the computation of further tables relating to the horseshoe vortex. This can be carried out most effectively by Scientific Computing Service, Ltd. who have also expressed their willingness to undertake the subtabulation of the original tables so that interpolation will require only the use of first differences.

The writer wishes to express his indebtedness to Drs. Comrie and Hartley for helpful advice given during discussion of the work, and to state that the success of the investigation is in no small part due to having been able to hand over the more difficult computation problems to Scientific Computing Service Ltd. For the problems in asymmetry, the work involves, in the words of Dr. Comrie " that pitfall for the inexperienced, a large number of simultaneous equations which are not always well-conditioned ". If it is possible to hand over further work in the same way, the progress of the whole investigation—which may also be used in connection with wind tunnel interference and wind tunnel design—will be expedited.

The writer also wishes to express his thanks to Professor W. G. Bickley for helpful advice given during a discussion of the problem.

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APPENDIX I

Calculation of Lift Coefficient

The analysis is limited to three terms in the chordwise direction and five in the spanwise direction. If Γ be the total circulation around any chord c,

$$\Gamma = \int_{-c/2}^{+c/2} k\, dx.$$

Therefore, from (1)

$$\frac{\Gamma}{8sV\tan\alpha} = F_0\int_{-\frac{1}{2}}^{\frac{1}{2}} \cot\frac{\theta}{2}\, d\,\frac{x}{c} + F_1\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin\theta\, d\frac{x}{c} + F_2\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin 2\theta\, d\frac{x}{c}$$

But

$$\frac{x}{c} = \tfrac{1}{2}\cos\theta, \text{ and } d\left(\frac{x}{c}\right) = -\tfrac{1}{2}\sin\theta\, d\theta\cdot$$

Hence

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \cot\frac{\theta}{2}\, d\,\frac{x}{c} = -\tfrac{1}{2}\int_{\pi}^{0} \cot\frac{\theta}{2}\sin\theta\, d\theta = \frac{\pi}{2}\cdot$$

Similarly

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin\theta\, d\,\frac{x}{c} = \frac{\pi}{4} \text{ and} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin 2\theta\, d\,\frac{x}{c} = 0.$$

Therefore

$$\frac{\Gamma}{8sV\tan\alpha} = \sqrt{1-\eta^2}\left[\frac{\pi}{2}(a_0+\eta b_0+\eta^2 c_0+\eta^3 d_0+\eta^4 e_0)+\frac{\pi}{4}(a_1+\eta b_1+\eta^2 c_1+\eta^3 d_1+\eta^4 e_1)\right].$$

The element of lift on a chord is $\varrho V\Gamma dy$ or total lift is $\int_{-s}^{s}\varrho V\Gamma dy$. Hence

$$C_L = \frac{\int_{-s}^{s}\varrho V\Gamma dy}{\frac{1}{2}\varrho V^2 S} = \frac{2s}{VS}\int_{-1}^{1}\Gamma d\eta.$$

Evaluating the integrals

$$C_L = \frac{16s^2\pi\tan\alpha}{S}\left[\frac{\pi}{2}(\tfrac{1}{2}a_0+\tfrac{1}{4}a_1)+\frac{\pi}{8}(\tfrac{1}{2}c_0+\tfrac{1}{4}c_1)+\frac{\pi}{16}(\tfrac{1}{2}e_0+\tfrac{1}{4}e_1)\right]$$

$$C_L = \frac{1}{4}\frac{s^2\pi^2\tan\alpha}{S}\left[16a_0+8a_1+4c_0+2c_1+2e_0+e_1\right]$$

$$\frac{dC_L}{d\alpha} = \frac{1}{4}\frac{s^2\pi^2}{S}\left[16a_0+8a_1+4c_0+2c_1+2e_0+e_1\right].$$

These formulae are independent of the wing shape.

APPENDIX II

Calculation of Centre of Pressure and no Lift Moment Coefficient for Rectangular Wing at 0° Yaw (Symmetrical Loading)

The moment of a strip about the line $\theta = \frac{\pi}{2}$ is given by :—

$$\frac{dM}{8sV\tan\alpha} = \tfrac{1}{2}\varrho Vc\,dy\int_{-1}^{1}\left[F_0\cot\frac{\theta}{2}+F_1\sin\theta+F_2\sin 2\theta\right]\cos\theta\, d\,\frac{x}{c}$$

or $$\frac{S\,dC_m}{8s\tan\alpha} = \left[\tfrac{1}{4}\pi F_0+\tfrac{1}{8}\pi F_2\right]dy$$

$$\frac{dC_m}{d\alpha} = \frac{8s^2}{S}\left[\tfrac{1}{4}\pi\int_{-1}^{1}F_0\,d\eta+\tfrac{1}{8}\pi\int_{-1}^{1}F_2\,d\eta\right]$$

$$\frac{dC_m}{d\alpha} = \frac{1}{16}\cdot\frac{s^2\pi^2}{S}\left[16a_0+4c_0+2e_0+8a_2+2c_2+c_2\right].$$

The centre of pressure in terms of the chord c forward of the midpoint of the chord is given by dC_m/dC_L or

$$\frac{1}{4}\cdot\frac{16a_0+4c_0+2e_0+8a_2+2c_2+e_2}{16a_0+8a_1+4c_0+2c_1+2e_0+e_1}.$$

Similarly,

$$C_{m0} = \frac{1}{16}\frac{s^2\pi^2}{S}\left[16a_0'+4c_0'+2e_0'+8a_2'+2c_2'+e_2'\right]$$

or eliminating a_0' from the condition that $C_L = 0$

$$C_{m0} = \frac{1}{16}\frac{s_2\pi^2}{S}\left[-8a_1'+8a_2'-2c_1'+2c_2'-e_1'+e_2'\right].$$

APPENDIX II

Calculation of Centre of Pressure for Elliptic Wing at 0° Yaw

The moment of a strip about the line $\theta = \frac{\pi}{2}$ is given by

$$\frac{\delta M}{8sV \tan \alpha} = \tfrac{1}{2}\, \rho V c\, dy \int_{-1}^{1} \left[F_0 \cot \frac{\theta}{2} + F_1 \sin \theta + F_2 \sin 2\theta \right] \cos \theta \, d\, \frac{x}{c}\,.$$

Substituting $c = c_0 \sqrt{1 - \eta^2}$ and evaluating the integral,

$$\frac{\delta M}{8sV \tan \alpha} = \tfrac{1}{2}\, \rho V c_0 \sqrt{1 - \eta^2} \left[\tfrac{1}{4}\pi\, F_0 + \tfrac{1}{8}\pi\, F_2 \right] dy\,.$$

Therefore, for the complete wing

$$\frac{dM}{d\alpha} = 4\, s^2 \rho V^2 c_0 \pi \left[\tfrac{1}{4} \int_{-1}^{1} F_0 \sqrt{1 - \eta^2}\, d\eta + \tfrac{1}{8} \int_{-1}^{1} F_2 \sqrt{1 - \eta^2}\, d\eta \right]$$

$$= \frac{2}{105}\, \pi s^2 \rho V^2 c_0 \left[70a_0 + 14c_0 + 6e_0 + 35a_2 + 7c_2 + 3e_2 \right].$$

Also $dL/d\alpha$ where L is the lift is given by

$$\frac{dL}{d\alpha} = \frac{1}{8}\, \rho V^2 s^2 \pi^2 \left[16a_0 + 8a_1 + 4c_0 + 2c_1 + 2e_0 + e_1 \right].$$

Hence C.P. is at

$$\frac{16c_0}{105\pi} \left[\frac{70a_0 + 14c_0 + 6e_0 + 35a_2 + 7c_2 + 3e_2}{16a_0 + 8a_1 + 4e_0 + 2c_1 + 2e_0 + e_1} \right] \quad 4c_2$$

forward of the major axis, where c_0 is the minor axis.

TABLE 1

Table of Downwashes due to Continuous Chordwise Load, Two-Dimensional Flow.

If c be the chord, and x length in the chordwise direction with the origin at the midpoint of c :—

$$\frac{k}{V} = \cot \frac{\theta}{2}, \quad \frac{\omega}{V} = \tfrac{1}{2}, \qquad \int_{-c/2}^{+c/2} \cot \frac{\theta}{2}\, dx = \tfrac{1}{2}\pi c$$

$$\frac{k}{V} = \sin \theta, \quad \frac{\omega}{V} = -\tfrac{1}{2} \cos \theta, \quad \int_{-c/2}^{+c/2} \sin \theta\, dx = \tfrac{1}{4}\pi c$$

$$\frac{k}{V} = \sin n\theta, \quad \frac{\omega}{V} = -\tfrac{1}{2} \cos n\theta, \quad \int_{-c/2}^{+c/2} \sin n\theta\, dx = 0$$

Layout of Vortices to represent continuous loading on wing.

	Port	Location in terms of s and magnitude	Factors of horseshoe vortices spanwise.																	Starboard.	
η	-0.9625	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.9625
$\sqrt{1-\eta^2}$	0.2713	0.4359	0.6000	0.7141	0.8000	0.8660	0.9165	0.9539	0.9798	0.9950	1.0000	0.9950	0.9798	0.9539	0.9165	0.8660	0.8000	0.7141	0.6000	0.4359	0.2713
$\eta\sqrt{1-\eta^2}$	-0.2611	-0.3923	-0.4800	-0.4999	-0.4800	-0.4330	-0.3666	-0.2862	-0.1960	-0.0995	0	0.0995	0.1960	0.2862	0.3666	0.4330	0.4800	0.4999	0.4800	0.3923	0.2611
$\eta^2\sqrt{1-\eta^2}$	0.2513	0.3531	0.3840	0.3499	0.2880	0.2165	0.1466	0.0859	0.0392	0.0099	0	0.0099	0.0392	0.0859	0.1466	0.2165	0.2880	0.3499	0.3840	0.3531	0.2513
$\eta^3\sqrt{1-\eta^2}$	-0.2419	-0.3178	-0.3072	-0.2450	-0.1728	-0.1083	-0.0587	-0.0258	-0.0078	-0.0010	0	0.0010	0.0078	0.0258	0.0587	0.1083	0.1728	0.2450	0.3072	0.3178	0.2419
$\eta^4\sqrt{1-\eta^2}$	0.2328	0.2860	0.2458	0.1715	0.1037	0.0541	0.0235	0.0077	0.0016	0.0001	0	0.0001	0.0016	0.0077	0.0235	0.0541	0.1037	0.1715	0.2458	0.2860	0.2328
$\eta^6\sqrt{1-\eta^2}$	0.2157	0.2316	0.1573	0.0840	0.0373	0.0135	0.0038	0.0007	0.0001	0	0	0	0.0001	0.0007	0.0038	0.0135	0.0373	0.0840	0.1573	0.2316	0.2157
$\eta^8\sqrt{1-\eta^2}$	0.1998	0.1876	0.1007	0.0412	0.0134	0.0034	0.0006	0.0001	0	0	0	0	0	0.0001	0.0006	0.0034	0.0134	0.0412	0.1007	0.1876	0.1998
$\eta^{10}\sqrt{1-\eta^2}$	0.1851	0.1520	0.0644	0.0202	0.0048	0.0008	0.0001	0	0	0	0	0	0	0	0.0001	0.0008	0.0048	0.0202	0.0644	0.1520	0.1851

Location and magnitude Factors of horseshoe vortices chordwise.

4 Point			
Position on chord from L.E.	cot $\theta/2$	Sin θ	Sin 2θ
0.125	0.2785	0.0488	0.0753
0.375	0.1172	0.0762	0.0981
0.625	0.0703	0.0762	-0.0881
0.875	0.0[illegible]31	0.04[illegible]	-0.0742

1 Point			
Position on chord from L.E.	cot $\theta/2$	Sin θ	Sin 2θ
0.25	0.5	0	0
0.50	0	0.25	0

Layout 1 Spanwise vortices as shown, omitting those for $\eta = \pm 0.9625$. The width of all vortices is $2y_v = 0.1$ semispan s. Chordwise vortices as shown for the 4 point layout.

Layout 1A As for 1, but 4 point layout used for chordwise vortices up to and including $y_* = 8$, and 1 point layout for higher values of y_*.

Layout 2 Spanwise vortices complete as shown. The width of the corrector vortices at $\eta = \pm 0.9625$ is $2y_v = 0.025$ semispan s. Chordwise vortices as shown for the 4 point layout.

Layout 2A As for 2, but 4 point layout used for chordwise vortices up to and including $y_* = 8$, and 1 point layout for higher values of y_*.

Note:- Layout numbers are not altered by the addition of terms such as $\eta^n\sqrt{1-\eta^2}$ spanwise and sin $n\theta$ chordwise

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TABLE 3

Demonstration of method for simple case of elliptic wing with load concentrated at 0·25 chord.

		Port									Position along Semispan								Starboard	
		0·9	0·8	0·7	0·6	0·5	0 4	0·3	0·2	0·1	0	0·1	0·2	0 3	0·4	0·5	0 6	0 7	0 8	0 9
$\sqrt{1-\eta^2}$	A_1	0·4359	0·6000	0·7141	0·8000	0·8660	0·9165	0·9539	0·9798	0·9950	1	0·9950	0·9798	0·9539	0·9165	0·8660	0 8000	0·7141	0·6000	0·4359
$\eta^2\sqrt{1-\eta^2}$	A_2	0 3531	0·3840	0·3499	0 2880	0·2165	0·1466	0·0859	0·0392	0·0099	0	0·0099	0·0392	0·0859	0·1466	0·2166	0·2880	0·3499	0·3840	0·3531
$\eta^4\sqrt{1-\eta^2}$	A_3	0 2860	0·2458	0·1715	0·1037	0·0541	0·0235	0·0077	0·0016	0·0001	0	0·0001	0·0016	0 ·0077	0·0235	0·0541	0·1037	0 1715	0·2458	0·2860
$\eta^6\sqrt{1-\eta^2}$	A_4	0·2316	0·1575	0·0840	0·0373	0·0135	0·0038	0·0007	0·0001	0	0	0	0·0001	0·0007	0 0038	0·0135	0 0373	0·0840	0·1575	0·2316
	ΣA	1·3066	1·3871	1·3195	1·2290	1·1501	1·0904	1 0482	1·0207	1·0050	1	1·0050	1·0207	1·0482	1·0904	1 1501	1·2290	1·3195	1·3871	1·3066
Point 1 0·1 Semispan	B_1	-0·0050	-0 0062	-0·0078	-0·0103	-0 0140	-0·0202	-0 0317	-0·0571	-0·1333	-0·6667	2·0000	-0 6667	0·1333	-0·0571	-0·0317	-0 0202	-0·0140	-0·0103	-0·0078
Point 2 0·4 Semispan	B_2	-0·0030	-0·0035	-0·0041	-0·0050	-0 006	-0·0078	-0·0103	-0·0140	-0 0202	-0·0317	-0·0571	-0·1333	-0·6667	2·0000	-0 6667	-0·1333	-0 0571	-0·0317	-0·0202
Point 3 0·6 Semispan	B_3	-0·0022	-0·0026	-0·0030	-0·0035	-0·0041	-0·0050	-0·0062	-0·0078	-0·0103	-0·0140	-0·0202	-0·0317	-0·0571	-0·1333	-0·6667	2·0000	-0·6667	-0·1333	-0·0571
Point 4 0·8 Semispan	B_4	-0 0017	-0·0020	-0·0022	-0·0026	-0·0030	-0·0035	-0·0041	-0 0060	-0·0062	-0·0078	-0 0103	-0 0140	-0·0202	-0·0317	-0·0571	-0·1333	-0·6667	2·0000	-0·6667
	ΣB	-0·0119	-0 0143	-0·0171	-0·0214	-0·0273	-0·0365	-0·0523	-0·0859	-0·1700	-0·7202	1·9124	-0·8457	-0·8773	1·7779	-1·4222	1·7132	-1·4046	1·8247	-0·7518

Point	ΣBA_1	ΣBA_2	ΣBA_3	ΣBA_4	Check
1	0 15823	-0·07256	-0·02116	-0·00963	Sum =
2	0·15882	-0·00127	-0 03517	-0·02321	1·05620
3	0·16048	0·09476	0·00173	-0·02655	$\Sigma\Sigma A \times \Sigma B$
4	0·17074	0·23687	0·16684	0·09726	=1·05619

Sum of Products.

$1{\cdot}31645\,a_o \; -0{\cdot}18512\,c_o \; -0{\cdot}04220\,e_o \; -0{\cdot}01925\,g_o = 0{\cdot}1$
$1{\cdot}31765\,a_o \; +0{\cdot}15745\,c_o \; -0{\cdot}04474\,e_o \; -0{\cdot}04233\,g_o = 0{\cdot}1$
$1{\cdot}32096\,a_o \; +0{\cdot}54950\,c_o \; +0{\cdot}013505\,e_o \; -0{\cdot}00640\,g_o = 0{\cdot}1$
$1{\cdot}34148\,a_o \; +1{\cdot}11373\,c_o \; +0{\cdot}74328\,e_o \; -0{\cdot}45667\,g_o = 0{\cdot}1$

Simultaneous Equations

$a_o = 0{\cdot}07587$
$c_o = -0{\cdot}00056$
$e_o = 0{\cdot}00049$
$g_o = -0{\cdot}00333$
$dC_L/d\alpha = 4{\cdot}746$

Solution

Summary

$\frac{dC_L}{d\alpha}$	Solution
4·781	Exact solution
4·746	Solution above, four points
4·740	Solution as above, but six points used, η = 0·1, 0·3, 0·4, 0·6, 0·7, 0·8.
4·778	Solution as above, four points, but with corrector vortices added at $\eta = \pm 0{\cdot}9625$

TABLE 4

Calculations on Rectangular Wing, Aspect Ratio 6, 0° Yaw :—Centre of Pressure and Lift Derivative

Operator	Method	Description		Coefficients			$dC_L/d\alpha$	C.P.
Blenk	Blenk	6 pivotal points on half wing at 0·067, 0·5 and 0·933 chord for $\eta = 0{\cdot}25$ and 0·75.	a_0 a_1 a_2	+0·0668 −0·0015 +0·0001	c_0 c_1 c_2	+0·0295 −0·0205 −0·0037	4·196	0·240
Falkner	Falkner	Layout 1A. Two places of decimals used in factors : 6 pivotal points on half wing at 0·25, 0·5 and 0·75 chord for $\eta = 0{\cdot}2$ and 0·8.	a_0 a_1 a_2	+0·0670 −0·0012 +0·0006	c_0 c_1 c_2	+0·0303 −0·0268 −0·0030	4·182	0·236
Scientific Computing Service Ltd.	Falkner	Layout 1. Four places of decimals used in factors : 6 pivotal points on half wing at 0·25, 0·5 and 0·75 chord for $\eta = 0{\cdot}2$ and 0·8.	a_0 a_1 a_2	+0·0670 −0·0015 +0·0001	c_0 c_1 c_2	+0·0322 −0·0277 −0·0051	4·195	0·237
Scientific Computing Service Ltd.	Falkner	Layout 1. Four places of decimals used in factors : 12 pivotal points on half wing at 0·25, 0·5 and 0·75 chord for $\eta = 0$, 0·2, 0·5 and 0·8. Least squares solution.	a_0 a_1 a_2	+0·0668 −0·0014 +0·0004	c_0 c_1 c_2	+0·0314 −0·0247 −0·0065	4·196	0·239
Falkner	Falkner	Layout 2A. Three places of decimals used in factors : 6 pivotal points on half wing at 0·25, 0·5 and 0·75 chord for $\eta = 0{\cdot}2$ and 0·8.	a_0 a_1 a_2	+0·0677 −0·0009 −0·0002	c_0	+0·0347 −0·0267 −0·0050	4·296	0·239
W. P. Jones	W. P. Jones	C.P. on all sections assumed to be at 0·250 chord.					4·303	0·250
Glauert		Fourier series. Single straight vortex filament.					4·53	0·250

TABLE 5

Calculated Corrections on C_{m0} and α_0 to be Applied to Cambered Rectangular Wings of the N.A.C.A. Series when Converting from Aspect Ratio 6 to ∞.

The corrections are proportional to the camber.

Camber, per cent.	Position of max. camber	Correction on C_{m0}	Correction on α_0 : degrees
2	0·2 chord	−0·0011	+0·01
2	0·3 chord	−0·0015	−0·01
2	0·4 chord	−0·0020	−0·01
2	0·5 chord	−0·0027	+0·04
2	0·6 chord	−0·0033	+0·08
2	0·7 chord	−0·0026	+0·06

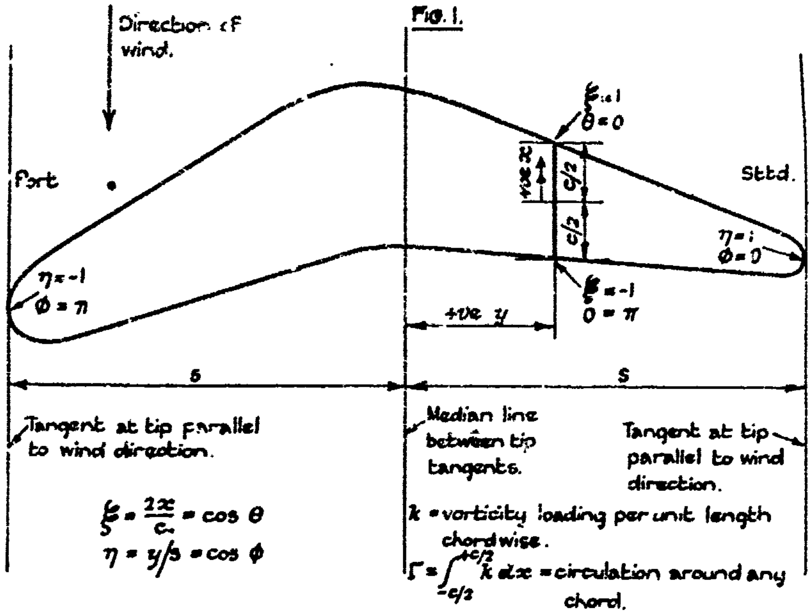
TABLE 6

Calculations on Elliptic Wing, Major/Minor Axis 5 to 1

Yaw	Aspect ratio	Operator	Method	Description	Coefficients				$\frac{dC_L}{d\alpha}$	C.P.
0°	6·37	Glauert ..		Single vortex filament					4·78	0·288
0°	6·37	Scientific Computing Service Ltd.	Falkner	Layout 1. Four places of decimals used in factors: 6 pivotal points on half wing at 0·25, 0·5 and 0·75 chord for η = 0·2 and 0·8.	a_0 a_1 a_2	+0·0739 −0·0034 0	c_0 c_1 c_2	+0·0015 −0·0093 −0·0016	4·49	0·280
0°	6·37	Krienes ..		Acceleration potential					4·55	0·283
30°	4·84	Glauert ..		Single vortex filament					4·45	
30°	4·84	Krienes ..		Acceleration potential					3·26	
30°	4·84	Scientific Computing Service Ltd.	Falkner	Layout 1. Four places of decimals used in factors: 12 pivotal points on wing at 0·25, 0·5, 0·75 chord for $\eta = \pm 0{\cdot}2$ and $\pm 0{\cdot}8$.	a_0 a_1 a_2 b_0 b_1 b_2	+0·0791 −0·0008 +0·0013 −0·0076 −0·0031 −0·0028	c_0 c_1 c_2 d_0 d_1 d_2	+0·0150 −0·0285 −0·0159 −0·0202 +0·0258 +0·0184	3·76	
30°	4·84	Scientific Computing Service Ltd.	Falkner	Layout 1. Four places of decimals used in factors: 21 pivotal points on wing at 0·25, 0·5 and 0·75 chord for $\eta = 0, \pm 0{\cdot}2, \pm 0{\cdot}5$ and $\pm 0{\cdot}8$. Least squares solution.	a_0 a_1 a_2 b_0 b_1 b_2	+0·0787 −0·0001 +0·0018 −0·0049 −0·0067 −0·0059	c_0 c_1 c_2 d_0 d_1 d_2	+0·0143 −0·0278 −0·0150 −0·0241 +0·0312 +0·0230	3·76	
30°	4·84	Scientific Computing Service Ltd.	Falkner	Layout 2. Four places of decimals used in factors: 12 pivotal points on wing at 0·25, 0·5 and 0·75 chord for $\eta = \pm 0{\cdot}2$ and $\pm 0{\cdot}8$.	a_0 a_1 a_2 b_0 b_1 b_2	+0·0793 −0·0008 +0·0013 −0·0073 −0·0035 −0·0027	c_0 c_1 c_2 d_0 d_1 d_2	+0·0164 −0·0269 −0·0613 −0·0204 +0·0254 +0·0191	3·81	

6997
Figs. 1 & 2.

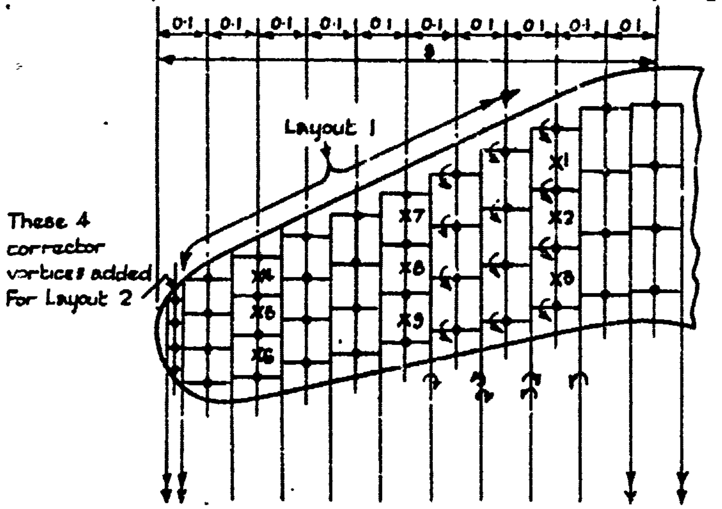
Fig. 1.



The y axis is perpendicular to the wind direction, with the origin on the median line.
For any chord c parallel to the wind direction, the x axis is parallel to the chord with the origin at the mid point of c.

General coordinates for wing of any shape.

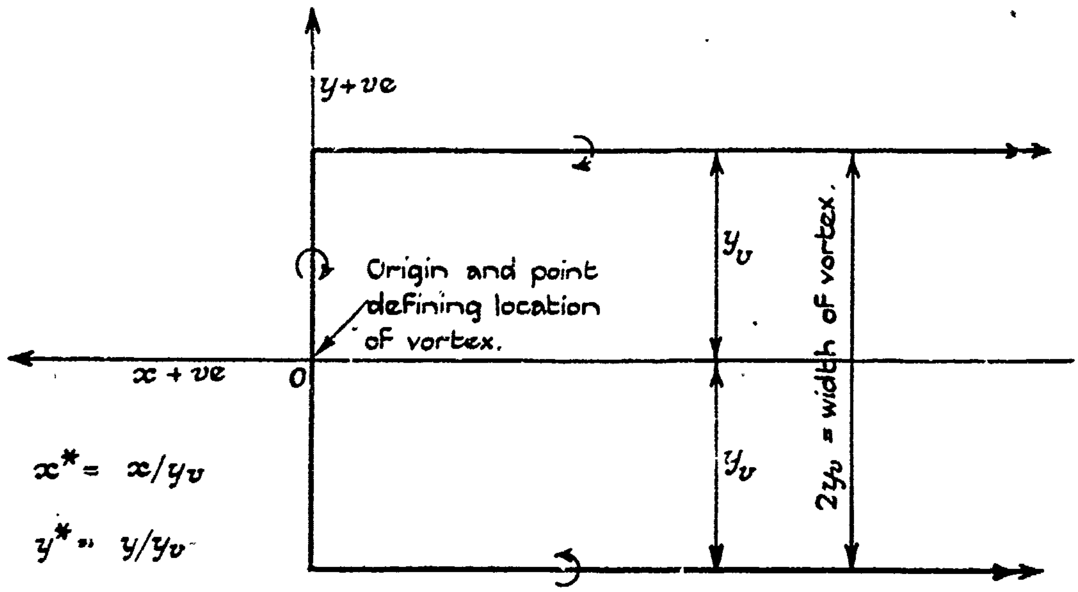
Fig. 2.



Pattern of horseshoe vortices representing continuos loading.

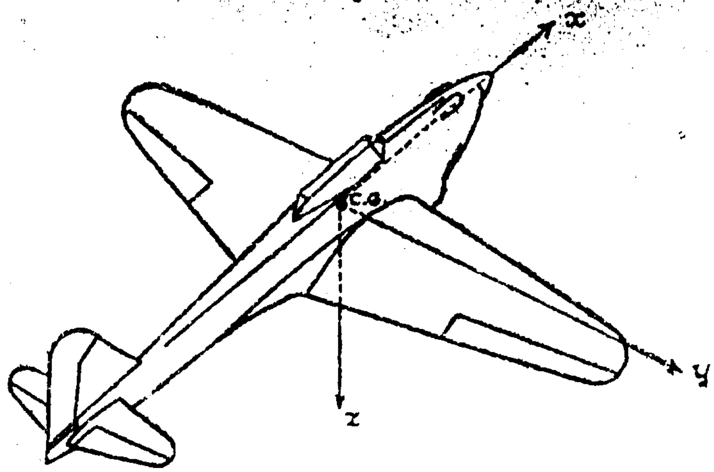
6997.

Fig. 3.



Dimensions of Horseshoe Vortex.

SYSTEM OF AXES



Axes	Symbol Designation Positive direction	x longitudinal forward	y lateral starboard	z normal downward
Force	Symbol	X	Y	Z
Moment	Symbol Designation	L rolling	M pitching	N yawing
Angle of Rotation	Symbol	ϕ	θ	ψ
Velocity	Linear Angular	u p	v q	w r
Moment of Inertia		A	B	C

Components of linear velocity and force are positive in the positive direction of the corresponding axis.

Components of angular velocity and moment are positive in the cyclic order y to z about the axis of x, z to x about the axis of y, and x to y about the axis of z.

The angular movement of a control surface (elevator or rudder) is governed by the same convention, the elevator angle being positive downwards and the rudder angle positive to port. The aileron angle is positive when the starboard aileron is down and the port aileron is up. A positive control angle normally gives rise to a negative moment about the corresponding axis.

The symbols for the control angles are :—

ξ aileron angle
η elevator angle
η_T tail setting angle
ζ rudder angle