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Steady Flow in the Boundary Layer near the Surface of a Cylinder in a Stream

By L. HOWARTH
B.A., B.Sc.

JULY, 1934
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AERODYNAMICS SYMBOLS

1. GENERAL

m mass

t time

V resultant linear velocity

Ω resultant angular velocity

ρ density, σ relative density

ν kinematic coefficient of viscosity

R Reynolds number, $R = lV/\nu$ (where l is a suitable linear dimension), to be expressed as a numerical coefficient $\times 10^6$

Normal temperature and pressure for aeronautical work are 15° C. and 760 mm.

For air under these conditions $\begin{cases} \rho = 0.002378 \text{ slug/cu. ft.} \\ \nu = 1.59 \times 10^{-4} \text{ sq. ft./sec.} \end{cases}$

The slug is taken to be 32.2 lb.-mass.

α angle of incidence

ϵ angle of downwash

S area

c chord

s semi-span

A aspect ratio, $A = 4s^2/S$

L lift, with coefficient $k_L = L/S\rho V^2$

D drag, with coefficient $k_D = D/S\rho V^2$

γ gliding angle, $\tan \gamma = D/L$

L rolling moment, with coefficient $k_l = L/sS\rho V^2$

M pitching moment, with coefficient $k_m = M/cS\rho V^2$

N yawing moment, with coefficient $k_n = N/sS\rho V^2$

2. AIRSCREWS

n revolutions per second

D diameter

J V/nD

P power

T thrust, with coefficient $k_T = T/\rho n^2 D^4$

Q torque, with coefficient $k_Q = Q/\rho n^2 D^5$

η efficiency, $\eta = TV/P = J k_T / 2\pi k_Q$

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ON THE CALCULATION OF STEADY FLOW IN THE BOUNDARY LAYER NEAR THE SURFACE OF A CYLINDER IN A STREAM

By L. HOWARTH, B.A., B.Sc. Busk Student

Communicated by Dr. S. GOLDSTEIN

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17th July, 1934

Summary.—A critical survey of methods used by previous workers is given together with some extensions.

1. *Introduction*.—This paper has been written with a view to comparing the various methods, so far suggested, of boundary layer analysis for steady two-dimensional flow past an obstacle. Some of the methods, as given by their authors, can be reduced to more convenient forms for the solution of any given problem. This has been done where possible and conclusions drawn as to the usefulness of each method.

It will be seen that the method due to Blasius and Hiemenz[1] should be used whenever possible. This method gives the velocity at any point in the boundary layer as a power series in the distance from the forward stagnation point, the coefficients being functions of the distance measured normally from the obstacle. The coefficients can be expressed in a convenient non-dimensional form and tabulated once and for all. Unfortunately, owing to the amount of labour involved, only a limited number of these coefficients have been calculated by the present writer and, in general, the method cannot be used with confidence as far as the point of separation with this limited number of coefficients. A step by step method, theoretically due to Goldstein[2] and adapted by the present writer for purposes of calculation, has been given for continuing this solution. This method suffers from the drawback that the steps which can be taken are rather small. If the number of coefficients given in the solution of Blasius's method is not sufficient to give the solution as far as is required, an approximate method due to Kármán and Pohlhausen[3] is recommended either for a complete solution of the problem or for the continuation of Blasius's solution. This method appears to give the point of separation of the flow with as great an accuracy as experiment, though the full velocity distribution is given with doubtful accuracy in the neighbourhood of that point. Another method due to Bairstow and Green[4] may also be of value. This method has been used in the case of flow past a circle only, and even in this case, when the pressure distribution seems to be particularly

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simple, a fairly cumbersome step by step method of finding the skin friction is necessary. It is anticipated that with more complicated pressure distributions the work involved would be too great for the method to be of use. Given the skin friction the method will give the velocity distribution in the middle of the boundary layer with good accuracy, and it is suggested that it may be of value to use the skin friction given by the method of Kármán and Pohlhausen and then obtain the velocity distribution using the method of Bairstow and Green.

In certain circumstances, the method of Pohlhausen breaks down. Two alternative methods are suggested for use in these circumstances. The first of these is a modification of the original method of Pohlhausen due to Dryden[5] and is described below. The second is a modification of Falkner and Skan's[6] method due to the present writer.

The original method of Falkner and Skan and the modification are described below, but they both suffer from the drawback that they break down in the neighbourhood of the point of separation (see later). The modification is, however, of value in the circumstances indicated above.

The remaining methods due to Von Mises and Luckert[7] and Thom[8] are described but seem to be of theoretical interest only. The methods mentioned here and the original method of Falkner and Skan are not recommended for general use for one of two reasons. Either the labour involved in solving any given problem is too great, or the method is not capable of any wider application, and is more cumbersome, than one of the methods mentioned previously.

Where possible the results of the various methods are given first to enable the reader to judge whether a particular method is suitable in any given problem.

Several of the methods have been compared on the problem of flow past a circle. Blasius's method, with the coefficients as given, could be used to give the velocities to within 0·2 per cent. as far as the point of separation in this particular case. The point of separation is given as 82°, with a possible error of 2°. This method would take, at the outside, four hours to obtain a complete idea of the velocity distribution in the boundary layer as far as the point of separation. To obtain a greater accuracy Blasius's method should be stopped, say, at 60° and the method of continuation given in paragraph 5 used. Unfortunately, this continuation process has to be done taking very small steps of about 1° each, if the greater accuracy is to be retained, and four hours' work is required per step. This example is, however, an extreme case, since the velocity distribution given by Blasius's method is accurate enough as far as the point of separation for all practical purposes. Normally, this method of continuation will be applied when Blasius's method ceases to be of value. When this is so the maximum error permitted per step will be greater and the length of the step correspondingly greater.

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Kármán's approximate method would carry the solution as far as the point of separation in about six hours. It is not easy to give an estimate of possible error in this method.

It should be stressed that the example chosen is very favourable to Blasius's method, owing to the convenient expression for the velocity at the edge of the boundary layer. When this form is not so convenient, it will be necessary, in using the method, to continue the solution in order to obtain any idea of the point of separation and the velocity distribution in its neighbourhood. This continuation is laborious, as will be seen from the figures given, and for this reason Kármán's method is recommended in general.

If we denote by y distance measured normally to the surface considered, x distance measured along curves orthogonal to the normals (measured from the normal at the forward stagnation point), u and v the fluid velocity components in the directions x and y increasing, p the pressure, ϱ the density and ν the kinematic viscosity of the fluid, then the boundary layer equations are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \quad \ldots \quad (1.1)$$

$$0 = -\frac{1}{\varrho}\frac{\partial p}{\partial y} \quad \ldots \quad (1.2)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \ldots \quad (1.3)$$

In virtue of equation (1.2) the pressure gradient is independent of y, and so the term dp/dx can be replaced by the pressure gradient at the edge of the boundary layer. This is supposed known either from experiment or calculation. The latter method can, however, be used only when the points of separation are very near to the tail of the obstacle, so that the flow outside the boundary layer and wake is approximately the same as the perfect fluid flow round the obstacle, with the requisite circulation. For a symmetrical streamline obstacle the circulation is zero; for a non-symmetrical streamline obstacle with no sharp trailing edge the theoretical determination of the circulation is a problem on which the present writer is engaged.

If we denote by U the velocity in the main stream at the edge of the boundary layer, U and p are connected by Bernouilli's equation

$$\frac{p}{\varrho} + \tfrac{1}{2}U^2 = \text{const} \quad \ldots \quad (1.4)$$

which gives

$$\frac{1}{\varrho}\frac{dp}{dx} = -U\frac{dU}{dx} \quad \ldots \quad (1.5)$$

Further, (1.1) and (1.5) show that if, as y tends to infinity, u tends to a value independent of y and $\partial u/\partial y$ and $\partial^2 u/\partial y^2$ tend to zero then u tends to U.

We proceed to a detailed discussion of each method.

2. *Blasius's Method.—Summary.*—Two cases arise according as the obstacle is symmetrical about an axis in the direction of the flow at infinity, or not. We shall speak of them as the symmetrical and non-symmetrical cases respectively.

In both cases it is necessary to express the velocity in the main stream at the edge of the boundary layer as a power series or polynomial in x. In general, in order to obtain an accurate representation of this velocity, a power series will be required. Using this series, Blasius's method will give the accurate solution of the problem considered, provided a sufficient amount of time and labour is spent on it. Alternatively, it is often possible to express the velocity at the edge of the boundary layer as a polynomial with a small number of terms as far from the stagnation point as the solution is required. This considerably decreases the work attached and is advisable in all cases where it is possible.

The method expresses the velocity components at any point as power series in x whose coefficients are functions of y. These coefficients can be put in convenient forms and reduce any problem to simple arithmetic and the use of tables. Tables sufficient to determine the coefficients of the first three terms fully, and in special cases the fourth, in both the symmetrical and non-symmetrical cases, have been calculated by the present writer and are given below (pp. 51–54).

(i) *Symmetrical Case.*—The velocity distribution at the edge of the boundary layer can be expressed in the form

$$\mathrm{U} = u_1 x + u_3 x^3 + u_5 x^5 + \ldots . \qquad (2.1)$$

starting from the forward stagnation point.

Then, assuming that the stream function ψ can be expressed as a power series in x with coefficients that are functions of y, the values of the velocity components $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$ are given by

$$u = u_1 f_1' x + 4u_3 f_3' x^3 + 6\left(u_5 g_5' + \frac{u_3^2}{u_1} h_5'\right) x^5$$
$$+ 8\left(u_7 g_7' + \frac{u_3 u_5}{u_1} h_7' + \frac{u_3^3}{u_1^2} k_7'\right) x^7 + 10\left(u_9 g_9' + \frac{u_3 u_7}{u_1} h_9'\right.$$
$$\left. + \frac{u_5^2}{u_1} k_9' + \frac{u_3^2 u_5}{u_1^2} j_9' + \frac{u_3^4}{u_1^3} q_9'\right) x^9 + \ldots . \qquad (2.2)$$

$$-v = \sqrt{\frac{\nu}{u_1}}\Big[u_1 f_1 + 12u_3 f_3 x^2 + 30\Big(u_5 g_5 + \frac{u_3^2}{u_1}h_5\Big)x^4$$

$$+ 56\Big(u_7 g_7 + \frac{u_3 u_5}{u_1}h_7 + \frac{u_3^3}{u_1^2}k_7\Big)x^6 + 90\Big(u_9 g_9 + \frac{u_3 u_7}{u_1}h_9$$

$$+ \frac{u_5^2}{u_1}k_9 + \frac{u_3^2 u_5}{u_1^2}j_9 + \frac{u_3^4}{u_1^3}q_9\Big)x^8 + \dots\Big] \qquad (2.3)$$

$$\frac{\partial u}{\partial y} = \sqrt{\frac{u_1}{\nu}}\Big[u_1 f_1'' x + 4u_3 f_3'' x^3 + 6\Big(u_5 g_5'' + \frac{u_3^2}{u_1}h_5''\Big)x^5$$

$$+ 8\Big(u_7 g_7'' + \frac{u_3 u_5}{u_1}h_7'' + \frac{u_3^3}{u_1^2}k_7''\Big)x^7 + 10\Big(u_9 g_9'' + \frac{u_3 u_7}{u_1}h_9''$$

$$+ \frac{u_5^2}{u_1}k_9'' + \frac{u_3^2 u_5}{u_1^2}j_9'' + \frac{u_3^4}{u_1^3}q_9''\Big)x^9 + \dots\Big] \qquad (2.4)$$

The point of separation is given by $\left(\frac{\partial u}{\partial y}\right)_{,0} = 0$

$f_1, f_1', f_1'', f_3, f_3', f_3'', g_5, g_5', g_5'', h_5, h_5', h_5''$ and k_7, k_7', k_7'' are shown in Table 1 as functions of $\eta = y\left(\frac{u_1}{\nu}\right)^{\frac{1}{2}}$.

The remaining coefficients are defined by differential equations. Some of these are given below in equations (2.30) to (2.37).

We note the following special cases

(i) $u_3 = u_5 = u_7 = \dots\dots\dots = 0$

All the series terminate and give

$$u = u_1 f_1' x, \quad v = -\sqrt{\nu u_1} f_1, \quad \frac{\partial u}{\partial y} = \sqrt{\frac{u_1^3}{\nu}} f_1'' x \qquad (2.5)$$

As would be expected no separation occurs.

(ii) $u_5 = u_7 = \dots\dots\dots = 0$

The series do not terminate but give

$$u = u_1 f_1' x + 4u_3 f_3' x^3 + 6\frac{u_3^2}{u_1}h_5' x^5 + 8\frac{u_3^3}{u_1^2}k_7' x^7$$

$$+ 10\frac{u_3^4}{u_1^3}q_9' x^9 + \dots\dots\dots \qquad (2.6)$$

$$-v = \sqrt{\frac{\nu}{u_1}}\Big[u_1 f_1 + 12u_3 f_3 x^2 + 30\frac{u_3^2}{u_1}h_5 x^4 + 56\frac{u_3^3}{u_1^2}k_7 x^6$$

$$+ 90\frac{u_3^4}{u_1^3}q_9 x^8 + \dots\dots\dots\Big] \qquad (2.7)$$

$$\frac{\partial u}{\partial y} = \sqrt{\frac{u_1}{\nu}} \left[u_1 f_1'' x + 4u_3 f_3'' x^3 + 6 \frac{u_3^2}{u_1} h_5'' x^5 + 8 \frac{u_3^3}{u_1^2} k_7'' x^7 \right.$$

$$\left. + 10 \frac{u_3^4}{u_1^3} q_9'' x^9 + \ldots\ldots\ldots \right] \quad \ldots \quad \ldots \quad \ldots \qquad (2.8)$$

(ii) *Non-symmetrical Case.*—In this case the velocity at the edge of the boundary layer takes the form

$$U = u_1 x + u_2 x^2 + u_3 x^3 + \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (2.9)$$

Proceeding as before we find

$$u = u_1 f_1' x + 3u_2 f_2' x^2 + 4\left(u_3 g_3' + \frac{u_2^2}{u_1} h_3'\right) x^3$$

$$+ 5\left(u_4 g_4' + \frac{u_2 u_3}{u_1} h_4' + \frac{u_2^3}{u_1^2} k_4'\right) x^4 + 6\left(u_5 g_5' + \frac{u_2 u_4}{u_1} h_5'\right.$$

$$\left. + \frac{u_3^2}{u_1} k_5' + \frac{u_2^2 u_3}{u_1^2} j_5' + \frac{u_2^4}{u_1^3} q_5'\right) x^5 + \ldots \quad \ldots \qquad (2.10)$$

$$-v = \sqrt{\frac{\nu}{u_1}} \left[u_1 f_1 + 6u_2 f_2 x + 12\left(u_3 g_3 + \frac{u_2^2}{u_1} h_3\right) x^2 \right.$$

$$+ 20\left(u_4 g_4 + \frac{u_2 u_3}{u_1} h_4 + \frac{u_2^3}{u_1^2} k_4\right) x^3 + 30\left(u_5 g_5 + \frac{u_2 u_4}{u_1} h_5\right.$$

$$\left.\left. + \frac{u_3^2}{u_1} k_5 + \frac{u_2^2 u_3}{u_1^2} j_5 + \frac{u_2^4}{u_1^3} q_5\right) x^4 + \ldots \right] \quad \ldots \qquad (2.11)$$

$$\frac{\partial u}{\partial y} = \sqrt{\frac{u_1}{\nu}} \left[u_1 f_1'' x + 3u_2 f_2'' x^2 + 4\left(u_3 g_3'' + \frac{u_2^2}{u_1} h_3''\right) x^3 \right.$$

$$+ 5\left(u_4 g_4'' + \frac{u_2 u_3}{u_1} h_4'' + \frac{u_2^3}{u_1^2} k_4''\right) x^4 + 6\left(u_5 g_5'' + \frac{u_2 u_4}{u_1} h_5''\right.$$

$$\left.\left. + \frac{u_3^2}{u_1} k_5'' + \frac{u_2^2 u_3}{u_1^2} j_5'' + \frac{u_2^4}{u_1^3} q_5''\right) x^5 + \ldots \right] \quad \ldots \qquad (2.12)$$

The point of separation is given by $\left(\frac{\partial u}{\partial y}\right)_0 = 0$

$f_1, f_1', f_1'', f_2, f_2', f_2'', g_3, g_3', g_3'', h_3, h_3', h_3''$ and k_4, k_4', k_4'' are shown in Table 2 as functions of $\eta = y\left(\frac{u_1}{\nu}\right)^{\frac{1}{2}}$.

The remaining coefficients are defined by differential equations. Some of these are given below in equations (2.47) to (2.54).

The following are special cases of interest :—

(i) If $u_2 = u_3 = u_4 = \dots\dots\dots = 0$

the series terminate and the results are identical with those in the symmetrical case when $u_3 = u_5 = \dots\dots = 0$.

(ii) If $u_3 = u_4 = \dots\dots = 0$

$$u = u_1 f_1' x + 3u_2 f_2' x^2 + 4\frac{u_2^2}{u_1} h_3' x^3 + 5\frac{u_2^3}{u_1^2} k_4' x^4$$
$$+ 6\frac{u_2^4}{u_1^3} q_5' x^5 + \dots \qquad (2.13)$$

$$-v = \sqrt{\frac{\nu}{u_1}}\left[u_1 f_1 + 6u_2 f_2 x + 12\frac{u_2^2}{u_1} h_3 x^2 + 20\frac{u_2^3}{u_1^2} k_4 x^3\right.$$
$$\left. + 30\frac{u_2^4}{u_1^3} q_5 x^4 + \dots\right] \qquad (2.14)$$

$$\frac{\partial u}{\partial y} = \sqrt{\frac{u_1}{\nu}}\left[u_1 f_1'' x + 3u_2 f_2'' x^2 + 4\frac{u_2^2}{u_1} h_3'' x^3 + 5\frac{u_2^3}{u_1^2} k_4'' x^4\right.$$
$$\left. + 6\frac{u_2^4}{u_1^3} q_5'' x^5 + \dots\right] \qquad (2.15)$$

Details.—This method, originally due to Blasius, was elaborated by Hiemenz (*loc. cit.*). They were both concerned with the symmetrical case. Their calculations for this case have been extended, and the details for the non-symmetrical case have been worked out by the present writer.

(1) *Symmetrical Case.*—The velocity at the edge of the boundary layer is supposed known and expressed as a power series

$$U = u_1 x + u_3 x^3 + u_5 x^5 + \dots \qquad (2.16)$$

This series may, or may not, terminate.

Equation (1.3) implies the existence of a stream function defined by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \qquad (2.17)$$

Blasius's method of solution is to assume that ψ can be expressed in the form of a power series

$$\psi = F_1 x + F_3 x^3 + F_5 x^5 + \dots \qquad (2.18)$$

where the F's are functions of y.

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$$u = F_1' x + F_3' x^3 + F_5' x^5 + \dots \qquad (2.19)$$

$$-v = F_1 + 3F_3 x^2 + 5F_5 x^4 + \dots \qquad (2.20)$$

Substituting these values in equation (1.1) and equating coefficients of powers of x we find

$$F_1'^2 - F_1F_1'' = u_1^2 + \nu F_1''' \quad \ldots \quad (2.21)$$

$$4F_1'F_3' - 3F_1''F_3 - F_1F_3'' = 4u_1u_3 + \nu F_3''' \quad \ldots \quad (2.22)$$

$$6F_1'F_5' - 5F_1''F_5 - F_1F_5'' = 6u_1u_5 + 3u_3^2 - 3(F_3'^2 - F_3F_3'') + \nu F_5''' \quad \ldots \quad (2.23)$$

$$8F_1'F_7' - 7F_1''F_7 - F_1F_7'' = 8(u_1u_7 + u_3u_5) - (8F_3'F_5' - 3F_3F_5'' - 5F_3''F_5) + \nu F_7''' \quad \ldots \quad (2.24)$$

$$10F_1'F_9' - 9F_1''F_9 - F_1F_9'' = 10(u_1u_9 + u_3u_7) + 5u_5^2 - 5(F_5'^2 - F_5F_5'') - (10F_3'F_7' - 3F_3F_7'' - 7F_7F_3'') + \nu F_9''' \quad \ldots \quad (2.25)$$

and so on, dashes here denoting differentiations with respect to y.

Writing

$$\eta = \left(\frac{u_1}{\nu}\right)^{\frac{1}{2}} y, \quad F_1 = f_1 (u_1\nu)^{\frac{1}{2}}, \quad F_3 = 4f_3u_3\left(\frac{\nu}{u_1}\right)^{\frac{1}{2}},$$

$$F_5 = 6u_5\left(\frac{\nu}{u_1}\right)^{\frac{1}{2}}\left(g_5 + \frac{u_3^2}{u_1u_5}h_5\right),$$

$$F_7 = 8u_7\left(\frac{\nu}{u_1}\right)^{\frac{1}{2}}\left(g_7 + \frac{u_3u_5}{u_1u_7}h_7 + \frac{u_3^3}{u_1^2u_7}k_7\right),$$

$$F_9 = 10u_9\left(\frac{\nu}{u_1}\right)^{\frac{1}{2}}\left(g_9 + \frac{u_3u_7}{u_1u_9}h_9 + \frac{u_5^2}{u_1u_9}k_9 + \frac{u_3^2u_5}{u_1^2u_9}j_9 + \frac{u_3^4}{u_1^3u_9}q_9\right), \text{ etc.}$$

the equations become

$$f_1'^2 - f_1f_1'' = 1 + f_1''' \quad \ldots \quad (2.26)$$

$$4f_1'f_3' - 3f_1''f_3 - f_1f_3 = 1 + f_3''' \quad \ldots \quad (2.27)$$

$$6f_1'g_5' - 5f_1''g_5 - f_1g_5'' = 1 + g_5''' \quad \ldots \quad (2.28)$$

$$6f_1'h_5' - 5f_1''h_5 - f_1h_5'' = \tfrac{1}{2} + h_5''' - 8(f_3'^2 - f_3f_3'') \quad \ldots \quad (2.29)$$

$$8f_1'g_7' - 7f_1''g_7 - f_1g_7'' = 1 + g_7''' \quad \ldots \quad (2.30)$$

$$8f_1'h_7' - 7f_1''h_7 - f_1h_7'' = 1 + h_7''' - 3(8f_3'g_5' - 5f_3''g_5 - 3f_3g_5'') \quad \ldots \quad (2.31)$$

$$8f_1'k_7' - 7f_1''k_7 - f_1k_7'' = k_7''' - 3(8f_3'h_5' - 5f_3''h_5 - 3f_3h_5'') \quad \ldots \quad (2.32)$$

$$10f_1'g_9' - 9f_1''g_9 - f_1g_9'' = 1 + g_9''' \quad \ldots \quad (2.33)$$

$$10f_1'h_9' - 9f_1''h_9 - f_1h_9'' = 1 + h_9'''$$
$$- \frac{16}{5}(10f_3'g_7' - 3f_3g_7'' - 7f_3''g_7) \quad .. \quad .. \quad .. \quad (2.34)$$

$$10f_1'k_9' - 9f_1''k_9 - f_1k_9'' = \tfrac{1}{2} + k_9''' - 18(g_5'^2 - g_5g_5'') \quad .. \quad (2.35)$$

$$10f_1'j_9' - 9f_1''j_9 - f_1j_9'' = j_9''' - \frac{16}{5}(10f_3'h_7' - 3f_3h_7''$$
$$- 7f_3''h_7) - 18(2g_5'h_5' - g_5h_5'' - g_5''h_5) \quad .. \quad .. \quad (2.36)$$

$$10f_1'q_9' - 9f_1''q_9 - f_1q_9'' = q_9''' - \frac{16}{5}(10f_3'k_7' - 3f_3k_7''$$
$$- 7f_3''k_7) - 18(h_5'^2 - h_5h_5'') .. \quad .. \quad .. \quad .. \quad (2.37)$$

where dashes now denote differentiation with regard to η.

The boundary conditions satisfied are

$f_1 = f_1' = 0$ at $\eta = 0$ $\qquad f_1' = 1$ at $\eta = \infty$

$f_3 = f_3' = 0$ at $\eta = 0$ $\qquad f_3' = \frac{1}{4}$ at $\eta = \infty$

$g_5 = g_5' = 0$ at $\eta = 0$ $\qquad g_5' = \frac{1}{6}$ at $\eta = \infty$

$g_7 = g_7' = 0$ at $\eta = 0$ $\qquad g_7' = \frac{1}{8}$ at $\eta = \infty$

$g_9 = g_9' = 0$ at $\eta = 0$ $\qquad g_9' = \frac{1}{10}$ at $\eta = \infty$

$h_5 = h_5' = h_7 = h_7' = k_7 = k_7' = h_9 = h_9' = k_9 = k_9' = j_9$
$= j_9' = q_9 = q_9' = 0$ at $\eta = 0$

and

$h_5' = h_7' = k_7' = h_9' = k_9' = j_9' = q_9' = 0$ at $\eta = \infty$.

These equations are independent of the u's and can be solved once and for all and the solutions used in any problem. Blasius gave the transformations for reducing the equations for F_1 and F_3 to the forms given above. Hiemenz solved these reduced equations and remarked that it seemed to be impossible to find a transformation which would reduce, similarly, the remaining equations. That this is not the case will be seen from the above.

(2) *Non-symmetrical Case.*—This case is precisely analogous to the preceding one. We can write the velocity at the edge of the boundary layer, in this case, as the power series

$$U = u_1x + u_2x^2 + u_3x^3 + \ldots.$$

Assuming

$$\psi = F_1x + F_2x^2 + F_3x^3 + \ldots.$$

the differential equations for the F's are

$$F_1'^2 - F_1F_1'' = u_1^2 + \nu F_1''' \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.38)$$

$$3F_1'F_2' - 2F_1''F_2 - F_1F_2'' = 3u_1u_2 + \nu F_2''' \quad .. \quad .. \quad (2.39)$$

$$4F_1'F_3' - 3F_1''F_3 - F_1F_3'' - 2F_2F_2'' + 2F_2'^2$$
$$= 4u_1u_3 + 2u_2^2 + \nu F_3''' \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (2.40)$$

$$5F_1'F_4' - 4F_1''F_4 - F_1F_4'' - 2F_2F_3'' - 3F_2''F_3 + 5F_2'F_3'$$
$$= 5(u_1u_4 + u_2u_3) + \nu F_4''' \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (2.41)$$

$$6F_1'F_5' - 5F_1''F_5 - F_1F_5'' - 3F_3F_3'' + 3F_3'^2 - 4F_4F_2''$$
$$- 2F_4''F_2 + 6F_4'F_2' = 6(u_1u_5 + u_2u_4) + 3u_3^2 + \nu F_5''' \quad \ldots \qquad (2.42)$$

and so on, dashes denoting differentiations with regard to y. The boundary conditions which u and v satisfy are

(i) $u = v = 0$ at $y = 0$

(ii) $u = U, v = 0$ at $y = \infty$.

These imply

$F_1 = F_2 = F_3 \ldots = 0$ at $y = 0$
$F_1' = F_2' = F_3' \ldots. = 0$ at $y = 0$
$F_1' = u_1, F_2' = u_2, F_3' = u_3 \ldots.$ at $y = \infty$.

Write

$$\eta = y\left(\frac{u_1}{\nu}\right)^{\frac{1}{2}}, \quad F_1 = f_1\,(u_1\nu)^{\frac{1}{2}}, \quad F_2 = 3f_2u_2\left(\frac{\nu}{u_1}\right)^{\frac{1}{2}},$$

$$F_3 = 4u_3\left(\frac{\nu}{u_1}\right)^{\frac{1}{2}}\left[g_3 + \frac{u_2^2}{u_1u_3}h_3\right],$$

$$F_4 = 5u_4\left(\frac{\nu}{u_1}\right)^{\frac{1}{2}}\left[g_4 + \frac{u_2u_3}{u_1u_4}h_4 + \frac{u_2^3}{u_1^2u_4}k_4\right],$$

$$F_5 = 6u_5\left(\frac{\nu}{u_1}\right)^{\frac{1}{2}}\left[g_5 + \frac{u_2u_4}{u_1u_5}h_5 + \frac{u_3^2}{u_1u_5}k_5 + \frac{u_2^2u_3}{u_1^2u_5}j_5\right.$$
$$\left. + \frac{u_2^4}{u_1^3u_5}q_5\right] \text{etc.}$$

The small letters then satisfy the following equations :—

$$f_1'^2 - f_1f_1'' = 1 + f_1''' \quad \ldots \quad \ldots \quad \ldots \qquad (2.43)$$

$$3f_1'f_2' - 2f_1''f_2 - f_1f_2'' = 1 + f_2''' \quad \ldots \quad \ldots \quad \ldots \qquad (2.44)$$

$$4f_1'g_3' - 3f_1''g_3 - f_1g_3'' = 1 + g_3''' \quad \ldots \quad \ldots \quad \ldots \qquad (2.45)$$

$$4f_1'h_3' - 3f_1''h_3 - f_1h_3'' = \frac{1}{2} + h_3''' - \frac{9}{2}(f_2'^2 - f_2f_2'') \ldots \qquad (2.46)$$

$$5f_1'g_4' - 4f_1''g_4 - f_1g_4'' = 1 + g_4''' \quad \ldots \quad \ldots \quad \ldots \qquad (2.47)$$

$$5f_1'h_4' - 4f_1''h_4 - f_1h_4'' = 1 + h_4''' - \frac{12}{5}(5f_2'g_3' - 3f_2''g_3$$
$$- 2f_2g_3'') \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (2.48)$$

$$5f_1'k_4' - 4f_1''k_4 - f_1k_4'' = k_4''' - \frac{12}{5}(5f_2'h_3' - 3f_2''h_3 - 2f_2h_3'') \quad \ldots \quad (2.49)$$

$$6f_1'g_5' - 5f_1''g_5 - f_1g_5'' = 1 + g_5''' \quad \ldots \quad (2.50)$$

$$6f_1'h_5' - 5f_1''h_5 - f_1h_5'' = 1 + h_5''' - \frac{5}{2}(6f_2'g_4' - 4f_2''g_4 - 2f_2g_4'') \quad \ldots \quad (2.51)$$

$$6f_1'k_5' - 5f_1''k_5 - f_1k_5'' = \tfrac{1}{2} + k_5''' - 8(g_3'^2 - g_3g_3'') \quad \ldots \quad (2.52)$$

$$6f_1'j_5' - 5f_1''j_5 - f_1j_5'' = j_5''' - \frac{5}{2}(6f_2'h_4' - 4f_2''h_4 - 2f_2h_4'') - 8(2g_3'h_3' - g_3h_3'' - g_3''h_3) \quad \ldots \quad (2.53)$$

$$6f_1'q_5' - 5f_1''q_5 - f_1q_5'' = q_5''' - \frac{5}{2}(6f_2'k_4' - 4f_2''k_4 - 2f_2k_4'') - 8(h_3'^2 - h_3h_3'') \quad \ldots \quad (2.54)$$

etc., and the boundary conditions

$$\begin{aligned} f_1 &= f_1' = f_2 = f_2' = g_3 = g_3' = h_3 = h_3' = g_4 = g_4' \\ &= h_4 = h_4' = k_4 = k_4' = g_5 = g_5' = h_5 = h_5' = k_5 \\ &= k_5' = j_5 = j_5' = q_5 = q_5' = 0 \text{ at } \eta = 0 \end{aligned}$$

$$\left.\begin{aligned} f_1' = 1, f_2' = \frac{1}{3}, g_3' = \frac{1}{4}, g_4' = \frac{1}{5}, g_5' = \frac{1}{6} \\ h_3' = h_4' = k_4' = h_5' = k_5' = j_5' = q_5' = 0 \end{aligned}\right\} \text{ at } \eta = \infty$$

where dashes denote differentiations with respect to η.

The Numerical Integration of the Equations.—The method of numerical integration used throughout was that due to Adams.

All the equations are of the third order with two boundary conditions given at $\eta = 0$ and one at $\eta = \infty$. The first equations of both the symmetrical and non-symmetrical cases are the same, have the same boundary conditions and are non-linear. The remaining equations are all linear.

For each of the linear equations two numerical integrations to infinity are required. Denoting by f the dependent variable in any particular example the boundary conditions are

$$f = f' = 0 \text{ at } \eta = 0, \quad f' = \text{const. at } \eta = \infty.$$

The method of solution is to assume a value of f'' at $\eta = 0$ and find the corresponding particular integral I. Using the same values of f'', f' and f at $\eta = 0$, a complementary function G can be found. Any solution of the equation may be written (λG + I) where λ is an arbitrary constant. Application of the condition at infinity gives the requisite value of λ.

It has not been proved that all the equations have solutions satisfying the boundary conditions given. Solutions were found to exist, however, in all the cases which have been attempted.

The non-linear equation is more difficult to solve. In this case, however, Blasius and Hiemenz had given approximate solutions, and it was found that two integrations to infinity were sufficient to give the correct values. In general, a number of integrations to infinity have to be made with different values of f'' at $\eta = 0$, and interpolation used to find an approximately correct value for f''. Two integrations in the neighbourhood of this value are generally sufficient to give the correct integral.

Hiemenz has given tables of f_1 and f_3 in the symmetrical case, f_1 to four decimal places and f_3 to three. The integrations were not carried out sufficiently far to enable him to give the fourth figure for f_1 correctly for the higher values of η, and there appears to be a misprint. These integrations have been carried out afresh; and g_5, h_5 and k_7 for the symmetrical case, together with f_1, f_2, g_3, h_3 and k_4 for the non-symmetrical case, have also been found by the present writer. In the symmetrical case f_1 is given to four figures; f_3, since it depends on f_1, to three figures; g_5 and h_5 to two figures; k_7, since it depends on h_5, could only be found very roughly, but the figures in Table 1 serve to give the order of magnitude of this term. Similar remarks apply to the accuracy of the coefficients in Table 2.

The Range of Usefulness of the Method.—The range of usefulness of the method depends, primarily, on the number of coefficients tabulated. We consider this range when the coefficients are tabulated as in Table 1 or 2 and take as an example the symmetrical case.

(*a*) If U is expressible in the form $(u_1 x)$ for a range of values of x, $0 \leq x \leq a_1$, the problem is entirely solved by using the tables of f_1 and the results given in (2.5).

(*b*) If U is expressible in the form $U = u_1 x + u_3 x^3$ for a range $0 \leq x \leq a_2$, the velocity distribution in the boundary layer is given by equation (2.6) above, and the point of separation, if it exists in this range, by equating the right-hand side of (2.8) with $\eta = 0$, to 0. The coefficients of x, x^3, x^5 and x^7 in these expressions are determined by the tables given.

Whether the velocity u is given sufficiently accurately within the range $0 \leq x \leq a_2$ depends on how rapidly the series converges, *i.e.*, on the magnitudes of u_3/u_1 and a_2. A rough idea of the error committed in using the first four terms only can be found by assuming, as seems very probable from the form of the equations, that k_7', k_7'' are everywhere greater than or equal to q_9' and q_9'' respectively.

If the error, as given by this inequality, is sufficiently small then the method can be used to give the velocity distribution as far as $x = a_2$.

If not, the series will only give the velocity as far as $x = \beta_2$ where β_2 is less than a_2 and is determined by u_3/u_1 and the accuracy required.

Similar remarks apply to the equation for the point of separation, if it lies within this range.

(c) If U is expressible in the form $U = u_1 x + u_3 x^3 + u_5 x^5$ for a range of $0 \lesssim x \lesssim a_3$ the velocity distribution in the boundary layer is given by

$$u = u_1 f_1' x + 4 u_3 f_3' x^3 + 6\left(u_6 g_5' + \frac{u_3^{\,2}}{u_1} h_5'\right) x^5$$

$$+ 8\left(\frac{u_3 u_5}{u_1} h_7' + \frac{u_3^{\,3}}{u_1^{\,2}} k_7'\right) x^7 + \ldots \quad \ldots \quad \ldots \qquad (2.55)$$

and the point of separation, if it exists in this range, is given by

$$0 = f_1'' x + 4 \frac{u_3}{u_1} f_3'' x^3 + 6\left(\frac{u^5}{u^1} g_5'' + \frac{u_3^{\,2}}{u_1^{\,2}} h_5''\right) x^5$$

$$+ 8\left(\frac{u_3 u_5}{u_1^{\,2}} h_7'' + \frac{u_3^{\,3}}{u_1^{\,3}} k_7''\right) x^7 + \ldots \quad \ldots \quad \ldots \qquad (2.56)$$

The coefficients of x, x^3 and x^5 in these expressions are determined by the tables given.

As in (b), the error can be estimated by assuming that h_5', h_5'' are everywhere greater than h_7', h_7'' respectively (k_7', k_7'' are given in Table 1).

(d) If, however, it is necessary to introduce terms in x^7 or higher powers to obtain an adequate representation of U in the range $0 \lesssim x \lesssim a_4$, it is evident that the tables given will not be sufficient to determine the velocity distribution in the entire range.

With the coefficients as given in Table 1 the most useful way of applying the method seems to be to express the velocity U in the form $(u_1 x + u_3 x^3)$ with u_3 less than u_1, for as large a range as possible, and then proceed as in (b). Alternatively, if the range covered by this means is too small it may possibly be increased by expressing the velocity U in the form $(u_1 x + u_3 x^3 + u_5 x^5)$ with u_3 less than u_1 and u_5 less than u_3, for as large a range as possible, and using (c) above.

Similar remarks apply to the non-symmetrical case.

If the range of the solution, as given by the above means, is not sufficient, one must either find some means of continuing this solution or else determine more of the coefficients required by the present method. Before the latter alternative could be attempted it would be necessary to recalculate the coefficients given to greater accuracies.

We proceed, in the next four paragraphs, to discuss methods of continuation. Paragraphs 3 and 4 may be used either to continue the solution of the present paragraph or to solve the problem independently. Paragraphs 5 and 6 are primarily methods of continuation since they are step by step processes.

The method of this paragraph is used (*see* paragraph 10) to determine the velocity distribution for the flow past a circle. Some of the velocity profiles are shown in Figs. 2, 3, 4, 5 and 6.

3. *Kármán-Pohlhausen's Method.—Summary.*—The differential equation given by Pohlhausen may be put in the form

$$\frac{dz}{dx} = \frac{f(\lambda)}{U} + z^2 U'' g(\lambda) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.1)$$

where dashes denote differentiation with respect to x, U is the velocity in the main stream at the edge of the boundary layer, z is equal to δ^2/ν, δ is the thickness of the boundary layer and λ is the non-dimensional quantity $U'z$. The functions $f(\lambda)$ and $g(\lambda)$ are independent of the particular problem under consideration and have been tabulated by the present writer in Table 3.*

In certain circumstances this method breaks down owing to λ becoming equal to 12. The corresponding value of $d\lambda/dx$ is infinite. It is found that, in general, λ then becomes imaginary for an interval of values of x. A modification due to Dryden may be used in these circumstances. A description of the modification is appended to this paragraph.

The method consists in solving equation (3.1). The particular integral which is required is given by

$$U'z = 7{\cdot}052 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

at $x = 0$.

The point of separation is given by

$$U'z = -12. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

The velocity distribution in the boundary layer can then be expressed in the form

$$\frac{u}{U} = F\left(\frac{y}{\delta}\right) + \lambda G\left(\frac{y}{\delta}\right). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

Again, $F(y/\delta)$ and $G(y/\delta)$ are definite functions independent of any particular problem and are shown graphically in Figs. A and B.

A neater mathematical expression can be found for equation (3.1) by putting it entirely in terms of λ and x, viz. :—

$$\frac{d\lambda}{dx} = \frac{U'}{U} f(\lambda) + \frac{U''}{U'} h(\lambda) \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.5)$$

* A quartic form for u is assumed, *see* pp. 17–19 below.

The particular integral required is given by $\lambda = 7\cdot052$ at $x = 0$. (That the differential equation given by Pohlhausen could be put in the form (3.1) or (3.5) was pointed out to me by Dr. Goldstein.) Unfortunately, difficulty arises at the point where $U' = 0$ (i.e. $\lambda = 0$) since this is a singular point of equation (3.5). An infinite number of integrals pass through this point and thus a straightforward application of a graphical or numerical method of integration is of little value, in this neighbourhood, owing to the difficulty in choosing the correct value of $d\lambda/dx$ at the singular point.

In some cases, for example for flow past a circle, this difficulty can be overcome by expanding λ as a power series in the neighbourhood of the singular point.

If we denote by x_0 the singular point, then we find

$$\lambda = \lambda_1(x - x_0) + \lambda_2 (x - x_0)^2 + \lambda_3 (x - x_0)^3 + \ldots .$$

where

$$\lambda_2 = 1\cdot5 \frac{u_3}{u_2} \lambda_1 + 0\cdot018 \lambda_1^2 + 68\cdot108 \frac{u_2}{u_0} .$$

$$\lambda_3 = 2\frac{u_4}{u_2}\lambda_1 - 1\cdot125 \frac{u_3^2}{u_2^2}\lambda_1 + 0\cdot0135 \frac{u_3}{u_2}\lambda_1^2 + 0\cdot0021 \lambda_1^3$$

$$+ 0\cdot75 \frac{u_3}{u_2}\lambda_2 + 0\cdot018 \lambda_1\lambda_2 + 51\cdot081 \frac{u_3}{u_0}$$

$$- 5\cdot150 \lambda_1 \frac{u_2}{u_0}, \qquad \ldots \quad (3.6)$$

and U is given by

$$U = u_0 + u_2 (x - x_0)^2 + u_3 (x - x_0)^3 . \qquad \ldots \quad (3.7)$$

λ_1 is then chosen so that the integral in the neighbourhood of the singular point joins on smoothly to the one found by the usual methods of integration starting from $\lambda = 7\cdot052$ at $x = 0$. This series for λ can then be used in the neighbourhood of the singular point and the usual methods of integration used again once the singular point has been passed.

The value of this series for λ depends entirely on the rapidity of its convergence; if the derivatives of U are large then it appears to be of little value.

Details.—Integrating equation (1.1) for y between the limits 0 and ∞ we have

$$\int_0^\infty \tfrac{1}{2}\frac{\partial}{\partial x}(u^2 - U^2)\, dy + \int_0^\infty v \frac{\partial u}{\partial y}\, dy = \nu \left[\frac{\partial u}{\partial y}\right]_0^\infty . \qquad \ldots \quad (3.8)$$

Considering now $\int_0^\infty v\,\frac{\partial u}{\partial y}$, integrating by parts and substituting for $\partial v/\partial y$ from equation (1.3), we find

$$\int_0^\infty v\,\frac{\partial u}{\partial y}\,dy = \int_0^\infty v\,\frac{\partial}{\partial y}\,(u - \mathrm{U})dy = \int_0^\infty (u - \mathrm{U})\,\frac{\partial u}{\partial x}\,dy \qquad (3.9)$$

since $[v\,(u - \mathrm{U})]$ vanishes at both top and bottom of the boundary layer. (The term U is introduced to ensure convergence of the integral.) Therefore (3.8) becomes

$$\int_0^\infty \frac{\partial}{\partial x}\,(u^2 - \tfrac{1}{2}\mathrm{U}^2)\,dy - \mathrm{U}\int_0^\infty \frac{\partial u}{\partial x}\,dy = \nu\left[\frac{\partial u}{\partial y}\right]_0^\infty \quad .. \quad (3.10)$$

The boundary conditions for u are

(i) $u = 0$ when $y = 0$

(ii) $u = \mathrm{U}$, $\quad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \ldots. = 0$ when $y = \infty$

(iii) $\nu\,\frac{\partial^2 u}{\partial y^2} = -\,\mathrm{U}\,\frac{d\mathrm{U}}{dx}$ when $y = 0$.

This condition is obtained by using equation (1.1) in conjunction with condition (i).

Further conditions can be found at the wall by repeatedly differentiating equation (1.1) and using condition (i),

$$\text{e.g. } \frac{\partial^3 u}{\partial y^3} = 0, \quad \nu\,\frac{\partial^4 u}{\partial y^4} = -\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial x\,\partial y}\right).$$

Now u tends asymptotically to U as y tends to infinity in the accurate solution of the boundary layer equations. This leads to the assumptions made in this method.

Assumption 1.—Assume that for any value of x a length δ exists such that for $y > \delta$, $(\mathrm{U} - u)$, v,

$$\int_\delta^\infty (\mathrm{U} - u)\,dy,\ \int_\delta^\infty \frac{\partial}{\partial x}\,(\mathrm{U} - u)\,dy \quad \text{and} \quad \int_\delta^\infty \frac{\partial}{\partial x}\,(\mathrm{U} - u)^2\,dy$$

may be neglected. Then equation (3.10) becomes

$$\int_0^\delta \frac{\partial}{\partial x}\,(u^2 - \tfrac{1}{2}\mathrm{U}^2)\,dy - \mathrm{U}\int_0^\delta \frac{\partial u}{\partial x}\,dy = \nu\left[\frac{\partial u}{\partial y}\right]_0^\delta \quad .. \quad (3.11)$$

$$\text{i.e. } \frac{d}{dx}\int_0^\delta (u^2 - \tfrac{1}{2}\mathrm{U}^2)\,dy - \mathrm{U}\,\frac{d}{dx}\int_0^\delta u\,dy$$

$$-\left[u^2 - \tfrac{1}{2}\mathrm{U}^2 - \mathrm{U}u\right]_\delta \cdot \delta' = \nu\left[\frac{\partial u}{\partial y}\right]_0^\delta \cdot \quad .. \quad .. \quad (3.12)$$

Assumption 2.—Assume that the boundary conditions which u satisfies when y is infinite are satisfied when $y = \delta$. Hence

$$[u]_\delta = \mathrm{U}, \quad \left[\frac{\partial u}{\partial y}\right]_\delta = 0.$$

Therefore equation (3.12) becomes

$$\tfrac{1}{2}\mathrm{U}^2\delta' + \frac{d}{dx}\int_0^\delta (u^2 - \tfrac{1}{2}\mathrm{U}^2)\,dy - \mathrm{U}\frac{d}{dx}\int_0^\delta u\,dy = \nu\left[\frac{\partial u}{\partial y}\right]^\delta \qquad (3.13)$$

with the same boundary conditions as in equation (3.10) except that the boundary conditions in (3.10) when y is infinite apply in this case when $y = \delta$.

Assumption 3.—We can satisfy five of the boundary conditions

$$\left.\begin{aligned} &u = \mathrm{U}, \frac{\partial u}{\partial y} = 0, \frac{\partial^2 u}{\partial y^2} = 0 \text{ at } y = \delta \\ &u = 0, \nu\frac{\partial^2 u}{\partial y^2} = -\mathrm{UU}' \text{ at } y = 0 \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3.14)$$

by assuming for u a quartic form in y, whose coefficients are functions of x, say,

$$u = a(x)\,y + b(x)\,y^2 + c(x)\,y^3 + d(x)\,y^4. \quad \ldots \quad \ldots \quad (3.15)$$

The boundary conditions are just sufficient to determine a, b, c and d. Then substituting this expression for u in equation (3.13) gives an ordinary differential equation for δ.

A better approximation might be expected by making u a quintic and satisfying the five boundary conditions of (3.14) and an additional one $\frac{\partial^3 u}{\partial y^3} = 0$ at $y = 0$. A sextic would be made to satisfy the six last named conditions and an additional one $\frac{\partial^3 u}{\partial y^3} = 0$ at $y = \delta$, and so on. Of course, in general, the velocity distribution in the boundary layer cannot be accurately expressed as a polynomial in y; to assume any such form is necessarily an approximation. Pohlhausen has, however, shown that a satisfactory approximation is given for the flow past a straight wall by assuming a quartic form for the velocity distribution, thus making the gradient and curvature of the velocity profile correct at the top and the curvature correct at the bottom of the boundary layer. Any further increase of accuracy by assuming quintic or higher polynomial forms for u does not seem to justify the added labour. It will be seen later, too, that for the quartic form a good agreement is obtained in the case of flow past a circle when the results are compared with those given by a more accurate method. In the latter problem the point of separation, as given by the present method, lies within the limits of error (2°) of the more accurate solution.

We shall, therefore, discuss the quartic form more fully as being generally sufficient to obtain a good approximation to the velocity distribution through the boundary layer and to the point of separation of the flow.

The Quartic Form

$$u = ay + by^2 + cy^3 + dy^4 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.16)$$

a, b, c and d being functions of x.

The boundary conditions are

$$u = \mathrm{U}, \quad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0 \text{ at } y = \delta \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.17)$$

$$\nu \frac{\partial^2 u}{\partial y^2} = -\mathrm{UU}', \quad u = 0 \text{ at } y = 0 \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.18)$$

These give

$$a = \mathrm{U}(12 + \lambda)/6\delta, \quad b = -\mathrm{UU}'/2\nu = -\mathrm{U}\lambda/2\delta^2,$$
$$c = -\mathrm{U}(4 - \lambda)/2\delta^3, \quad d = \mathrm{U}(6 - \lambda)/6\delta^4, \quad \ldots \quad \ldots \quad (3.19)$$

where λ is the non-dimensional quantity $\mathrm{U}'\delta^2/\nu$ and dashes denote differentiation with respect to x.

Making use of these values we find

$$\int_0^\delta u\,dy = \frac{\mathrm{U}\delta}{120}(84 + \lambda) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.20)$$

$$\int_0^\delta u^2 dy = \frac{\mathrm{U}^2\delta}{1260}\left(734 + \frac{71}{6}\lambda + \frac{5\lambda^2}{36}\right) \quad \ldots \quad \ldots \quad \ldots \quad (3.21)$$

Substituting these values in equation (3.15) gives

$$\mathrm{U}\delta\frac{d\delta}{dx} = \frac{-2\nu + \frac{116}{315}\mathrm{U}'\delta^2 - \frac{\delta^4}{7560\nu}\left[79\mathrm{U}'^2 + 8\mathrm{UU}''\right] - \frac{\mathrm{U}'\delta^6}{4536}(\mathrm{U}'^2 + \mathrm{UU}'')}{\left[-\frac{37}{315} + \frac{\mathrm{U}'\delta^2}{315\nu} + \frac{5}{9072}\frac{\mathrm{U}'^2\delta^4}{\nu^2}\right]}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.22)$$

Put $z = \delta^2/\nu$.

Then

$$\frac{dz}{dx} = \frac{0{\cdot}8[-9072 + 1670{\cdot}4\mathrm{U}'z - (47{\cdot}4\mathrm{U}'^2 + 4{\cdot}8\mathrm{UU}'')z^2 - \mathrm{U}'(\mathrm{U}'^2 + \mathrm{UU}'')z^3]}{\mathrm{U}[-213{\cdot}12 + 5{\cdot}76\mathrm{U}'z + \mathrm{U}'^2 z^2]}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.23)$$

U, U′ and U″ being given, this is a differential equation for z. There is a single infinity of integrals of this equation and the only difficulty remaining is to choose the correct one.

At the origin (the forward stagnation point) the velocity U vanishes and with it the denominator of equation (3.23). It can be shown that unless the numerator of the right-hand side of (3.23) also vanishes no integral exists having a finite value at the origin.

Thus, when $x = 0$

$$U'^3z^3 + 47\cdot4U'^2z^2 - 1670\cdot4U'z + 9072 = 0 \quad \ldots \quad (3.24)$$

This defines z at the origin as the root of a cubic equation. It is interesting to note that this cubic form is not an artificiality introduced by assuming a quartic form for the velocity distribution, but seems to appear if cubic, quintic or sextic forms are assumed for the velocity distribution.

Now $\lambda = U'z$ and equation (3.24) is a cubic in λ.

Thus, when $x = 0$, $\lambda = 7\cdot052, 17\cdot75$ or -70.

We have, now, to decide which of these values to choose. Since the particular integral required is defined by a condition at the forward stagnation point, it is necessary to consider

(i) the integral from the forward stagnation point over the upper surface of the body;

(ii) the integral from the forward stagnation point over the lower surface of the body.

For both these cases the positive direction of the axis of x is chosen in the direction of flow in the neighbourhood of the origin. Thus, U′ is positive at the origin in both cases. This immediately disposes of the negative value, since z is always positive.

To choose between the positive values is more difficult and involves a consideration of the point of separation as well as the stagnation point. Separation occurs when $\lambda = -12$. If we denote by $U\theta(\lambda)$ the denominator of the right-hand side of (3.23), we see that $\theta(\lambda) \equiv (\lambda - 12)(\lambda + 17\cdot76)$.

Moreover, the required integral for λ varies continuously between its value at the forward stagnation point and its value at the point of separation, as x varies from zero to its value at the point of separation. If, now, $\lambda = 17\cdot75$ when $x = 0$, $\theta(\lambda)$ vanishes at some point between the forward stagnation point and the point of separation, and the corresponding value of $\frac{d\lambda}{dx}$ is infinite. The infinite value of $\frac{d\lambda}{dx}$ implies an infinite value for the velocity v indicating the breakdown of the approximate method. Hence, if $\lambda = 17\cdot75$ at $x = 0$, the method always breaks down before the point of separation is reached.

If, however, $\lambda = 7\cdot052$ at $x = 0$, this difficulty is no longer necessarily present as $\theta(\lambda)$ does not vanish for any value of λ between $7\cdot052$ and -12, and this initial value for λ must be the required one. Hence the particular integral of (3.23) required is the one defined by

$$U'z = 7\cdot052 \text{ at } x = 0\,.$$

This value was used by Pohlhausen (*loc. cit.*) without mention of the other two.

Even when $\lambda = 7\cdot052$ initially, λ may take the value 12 during the course of the integration of (3.23). As above, this must be taken as an indication that this approximate method has broken down.

A modification of Pohlhausen's method, due to Dryden, is of value when the ordinary method of Pohlhausen breaks down. A summary of this modification is given at the end of this paragraph.

For purposes of calculation (3.23) can be written in the form (3.1) where $f(\lambda)$ and $g(\lambda)$ are given by

$$\left.\begin{aligned} f(\lambda) &= \frac{7257\cdot6 - 1336\cdot32\lambda + 37\cdot92\lambda^2 + 0\cdot8\lambda^3}{213\cdot12 - 5\cdot76\lambda - \lambda^2} \\ g(\lambda) &= \frac{(3\cdot84 + 0\cdot8\lambda)}{213\cdot12 - 5\cdot76\lambda - \lambda^2} \end{aligned}\right\} \quad \ldots \quad (3.25)$$

and are tabulated in Table 3.

The Solution of the Differential Equation.—To use Table 3 conveniently for the graphical solution of the differential equation requires a method rather different from the usual one.

For any value, λ_0, of λ used in Table 3, the curves $U'z = \lambda_0$ are drawn in the (z, x) plane. The value of dz/dx at any point of one of these curves can be obtained from the differential equation and Table 3. Beyond this point the solution follows the lines of the usual method.

To determine the value of dz/dx at $x = 0$ from (3.1) we notice that

$$\lim_{x \to 0} \frac{f(\lambda)}{U} = \left[\frac{f'(\lambda)}{U'}\lambda'\right]_{x=0} = \left[\frac{f'(\lambda)}{U'}(U''z + U'z')\right]_{x=0} \quad \ldots \quad (3.26)$$

The value of dz/dx is then found by algebra.

We find $(dz/dx)_0 = -5\cdot391\,\dfrac{U''}{U'^2}$. $\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad$ (3.27)

The corresponding value of

$(d\lambda/dx)_0$ is $+\,1\cdot661\,\dfrac{U''}{U'}$ $\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad$ (3.28)

The alternative form (3.5) and its drawbacks are discussed in the summary.

The Velocity Distribution.—From equation (3.19) we see that

$$\frac{u}{U} = \frac{(12+\lambda)}{6}\frac{y}{\delta} - \frac{\lambda}{2}\left(\frac{y}{\delta}\right)^2 - \frac{(4-\lambda)}{2}\left(\frac{y}{\delta}\right)^3 + \frac{(6-\lambda)}{6}\left(\frac{y}{\delta}\right)^4 . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.29)$$

This leads to equation (3.4) where

$$\left.\begin{aligned} F(y/\delta) &= 2(y/\delta) - 2(y/\delta)^3 + (y/\delta)^4 \\ G(y/\delta) &= \frac{1}{6}(y/\delta) - \tfrac{1}{2}(y/\delta)^2 + \tfrac{1}{2}(y/\delta)^3 - \frac{1}{6}(y/\delta)^4 \end{aligned}\right\} \quad \cdots \quad (3.30)$$

and are shown graphically in Figs. A and B.

The quantity δ which is determined by this method is not physically very significant. The skin friction is, however, given by

$$\mu\left(\frac{\partial u}{\partial y}\right)_0 = \frac{\mu(12+\lambda)U}{6\delta}.$$

Hence the method may be considered as primarily concerned with determining the skin-friction. Dr. Goldstein suggests that should it be required to use this method to determine the skin-friction without starting from the forward stagnation point—for example, it may be desired to finish off quickly the solution given by the method of the last paragraph when that method is not sufficient to reach the point of separation—the correct value of δ to use for starting the integration is given by making the skin-friction correct at the starting point. Many more examples will have to be discussed before it can be seen whether this method of procedure gives a better result than that given by using the method of the present paragraph throughout. On the face of it, it appears very probable that it should.

Summary of Dryden's Modification.—This method may be used when the ordinary method collapses owing to λ becoming equal to 12. The method makes use of equation (3.18). A quintic form

$$u = a(x)y + b(x)y^2 + c(x)y^3 + d(x)y^4 + e(x)y^5 \quad \cdots \quad (3.31)$$

is assumed for u.

This value of u is made to satisfy the boundary conditions (3.14), a being treated as arbitrary. Thus b, c, d and e are determined in terms of a, U and λ by (3.14).

A form

$$a = a_0 + a_1\lambda, \quad \cdots \quad (3.32)$$

where a_0 and a_1 are numerical constants, is assumed for a. Separation is found to occur when $\lambda = -\dfrac{a_0}{a_1} = \lambda_s$ say.

The solution for $\lambda = 0$, i.e., U = constant, is known and a_0 is chosen to make the solution of the present method in good agreement with the known solution in this case.

Leaving the value of a_1 undetermined we find, proceeding exactly as in the ordinary method,

$$\frac{dz}{dx} = \frac{P(x, z)}{UQ(\lambda)} \qquad \ldots \quad (3.33)$$

where P and Q involve a_1. To prevent $\frac{dz}{dx}$ becoming infinite $Q(\lambda)$ must not vanish. As in the original method the equation

$$Q(\lambda) = 0 \qquad \ldots \quad (3.34)$$

is quadratic in λ. It is found impossible to make the roots complex by suitable choice of a_1. Denoting by λ_1 and λ_2 the roots of (3.34) where $|\lambda_1| > |\lambda_2|$, a_1 is chosen to make the ratio $\left|\frac{\lambda_2}{\lambda_s}\right|$ as large as possible.

When this condition is applied we find $\lambda_1 = 48\cdot52$, $\lambda_2 = -30\cdot89$, $\lambda_s = -17\cdot18$ and $\lambda = 4\cdot365$ when $x = 0$.

Thus, the modified method gives

$$\frac{u}{U} = (1\cdot89 + 0\cdot11\lambda)\frac{y}{\delta} - \frac{\lambda}{2}\left(\frac{y}{\delta}\right)^2 + (-1\cdot34 + 0\cdot84\lambda)\left(\frac{y}{\delta}\right)^3$$

$$+ (0\cdot12 - 0\cdot62\lambda)\left(\frac{y}{\delta}\right)^4 + (0\cdot33 + 0\cdot17\lambda)\left(\frac{y}{\delta}\right)^5 \quad \ldots \quad (3.35)$$

and

$$\frac{dz}{dx} = \frac{p(\lambda)}{U} + z^2 q(\lambda) U'' \qquad \ldots \quad (3.36)$$

where

$$p(\lambda) = \frac{0\cdot8[-59051\cdot9 + 13783\cdot3\lambda - 53\cdot93\lambda^2 - \lambda^3]}{(-1500\cdot63 - 17\cdot6337\lambda + \lambda^2)}$$

and

$$q(\lambda) = \frac{0\cdot8(14\cdot6947 - \lambda)}{(-1500\cdot63 - 17\cdot6337\lambda + \lambda^2)}$$

$p(\lambda)$ and $q(\lambda)$ are tabulated in Table 4.

This modification gives a method which is applicable to a wider range of problems than the original. The modified method is not quite so accurate as the original, when the latter does not break down, but it seems to give a valuable approximation for many cases in which the original collapses.

Leaving the value of a_1 undetermined we find, proceeding exactly as in the ordinary method,

$$\frac{dz}{dx} = \frac{P(x, z)}{UQ(\lambda)} \quad \ldots \quad (3.33)$$

where P and Q involve a_1. To prevent $\frac{dz}{dx}$ becoming infinite $Q(\lambda)$ must not vanish. As in the original method the equation

$$Q(\lambda) = 0 \quad \ldots \quad (3.34)$$

is quadratic in λ. It is found impossible to make the roots complex by suitable choice of a_1. Denoting by λ_1 and λ_2 the roots of (3.34) where $|\lambda_1| > |\lambda_2|$, a_1 is chosen to make the ratio $\left|\frac{\lambda_2}{\lambda_s}\right|$ as large as possible.

When this condition is applied we find $\lambda_1 = 48 \cdot 52$, $\lambda_2 = -30 \cdot 89$, $\lambda_s = -17 \cdot 18$ and $\lambda = 4 \cdot 365$ when $x = 0$.

Thus, the modified method gives

$$\frac{u}{U} = (1 \cdot 89 + 0 \cdot 11\lambda)\frac{y}{\delta} - \frac{\lambda}{2}\left(\frac{y}{\delta}\right)^2 + (-1 \cdot 34 + 0 \cdot 84\lambda)\left(\frac{y}{\delta}\right)^3$$

$$+ (0 \cdot 12 - 0 \cdot 62\lambda)\left(\frac{y}{\delta}\right)^4 + (0 \cdot 33 + 0 \cdot 17\lambda)\left(\frac{y}{\delta}\right)^5 \quad \ldots \quad (3.35)$$

and

$$\frac{dz}{dx} = \frac{p(\lambda)}{U} + z^2 q(\lambda) U'' \quad \ldots \quad (3.36)$$

where

$$p(\lambda) = \frac{0 \cdot 8\,[-59051 \cdot 9 + 13783 \cdot 3\lambda - 53 \cdot 93\lambda^2 - \lambda^3]}{(-1500 \cdot 63 - 17 \cdot 6337\lambda + \lambda^2)}$$

and

$$q(\lambda) = \frac{0 \cdot 8\,(14 \cdot 6947 - \lambda)}{(-1500 \cdot 63 - 17 \cdot 6337\lambda + \lambda^2)}$$

$p(\lambda)$ and $q(\lambda)$ are tabulated in Table 4.

This modification gives a method which is applicable to a wider range of problems than the original. The modified method is not quite so accurate as the original, when the latter does not break down, but it seems to give a valuable approximation for many cases in which the original collapses.

An alternative method, which can be compared with this modification when the ordinary method breaks down, is indicated in paragraph 8. This comparison has been made by the present writer in a particular problem and the agreement found to be good. It is hoped to publish the results of this comparison in a later paper.

4. *Bairstow and Green's Method.*—The principle of the method was suggested by Bairstow and elaborated by Green (*loc. cit.*).

Summary.—The method is to expand ψ, the stream function, as a power series in y whose coefficients are functions f_r of x. Successive values of f_r are found by recurrence formulae, involving differentiation of the preceding values, except in the cases of f_0 and f_1; f_1 is determined by the pressure gradient. The difficulty in this method lies in finding f_0, which is related to the skin-friction, and its evaluation seems to involve much hard labour. In any case, the determination of the coefficients after f_1 involves numerical differentiation, a process which at best is not very accurate.

The method is closely related to that of the preceding paragraph. Moreover, having determined the skin-friction by the method of the last paragraph, the method of the present paragraph could probably be applied to give a closer approximation to the velocity in the middle of the boundary layer.

Again, this method gives a possible means of continuing the solution of paragraph 2; the skin-friction which is required for starting the solution of the present paragraph being immediately obtainable from the solution of paragraph 2.

As presented by Green, the drawback to the method is the trial and error step-by-step method of finding f_0 and the special difficulties which occur in the neighbourhood of the point of separation. It is anticipated that in a case when the method of paragraph 2 ceases to be of value owing to the magnitude of the derivatives of U then this step-by-step method will become too cumbersome to be of value.

The pressure distribution used by Green is nearly the same as that given by Hiemenz for about 50° from the forward stagnation point. This latter distribution was used by the present writer (*see* paragraph 10) for the calculations of the velocity distributions for various other methods. Hence, as far as 50° from the forward stagnation point, Green's results have been included with the others in Figs. 2, 3 and 4. Unfortunately, the two pressure distributions are not in good agreement in the neighbourhood of the point of separation and hence Green's solution cannot be compared, immediately, with the others in this important region. Owing to the labour involved, it was not deemed necessary to carry out Green's calculations using Hiemenz's experimental results.

Details.—Put $x = dx'$, $y = dy'$, $u = Vu'$, $v = Vv'$, $U = VU'$ and $p = \rho V^2 p'$, where d is a representative length and V a representative velocity of the system considered.

Equations (1.1) and (1.3) then become

$$u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=-\frac{dp'}{dx'}+\frac{1}{R}\frac{\partial^2 u'}{\partial y'^2}, \quad \ldots \quad \ldots \quad \ldots \qquad (4.1)$$

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.2)$$

where $R = Vd/\nu$.

The method could be applied without reducing the equations to these non-dimensional forms, but the subsequent work could not be expressed so conveniently.

The dashes in equations (4.1) and (4.2) can now be left off without confusion.

We try to obtain a solution of (4.1) in the form

$$R\psi = f_0\frac{y^2}{2!}+f_1\frac{y^3}{3!}+\ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.3)$$

where the f's are functions of x only.

Then

$$Ru = f_0 y + f_1\frac{y^2}{2!}+\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.4)$$

$$-Rv = f_0'\frac{y^2}{2!}+f_1'\frac{y^3}{3!}+\ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.5)$$

where dashes denote differentiations with respect to x.

Substituting from (4.4) and (4.5) in (4.1)

$$\sum_{\beta=1}^{\infty}\frac{y^\beta}{\beta!}f_{\beta-1}\sum_{\gamma=1}^{\infty}\frac{y^\gamma}{\gamma!}f'_{\gamma-1}-\sum_{\beta=1}^{\infty}\frac{y^{\beta+1}}{(\beta+1)!}f'_{\beta-1}\sum_{\gamma=1}^{\infty}\frac{y^{\gamma-1}}{(\gamma-1)!}f_{\gamma-1}$$

$$=-R^2\frac{dp}{dx}+\sum_{\alpha=2}^{\infty}\frac{y^{\alpha-2}}{(\alpha-2)!}f_{\alpha-1} \quad \ldots \quad \ldots \quad \ldots \qquad (4.6)$$

$$\text{i.e. } \sum_{\alpha=2}^{\infty}y^\alpha\sum_{\beta=1}^{\alpha-1}f_{\beta-1}f'_{\alpha-\beta-1}\frac{\alpha-2\beta+1}{\beta!(\alpha-\beta+1)!}=-R^2\frac{dp}{dx}$$

$$+\sum_{\alpha=0}^{\infty}y^\alpha\frac{f_{\alpha-1}}{\alpha!} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.7)$$

Therefore, equating coefficients of powers of y we find when

$$\alpha=0 \qquad f_1 = R^2\,dp/dx \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.8)$$

$$\alpha=1 \qquad f_2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.9)$$

$$\alpha>1 \qquad \sum_{\beta=1}^{\alpha-1}f_{\beta-1}f'_{\alpha-\beta-1}\frac{\alpha-2\beta+1}{\beta!(\alpha-\beta+1)!}=\frac{1}{\alpha!}f_{\alpha+1}. \qquad (4.10)$$

In particular

$$\left.\begin{aligned}\frac{1}{2!}f_3 &= \frac{1}{1!\,2!}f_0 f_0' \\ \frac{1}{3!}f_4 &= \frac{2}{1!\,3!}f_0 f_1'\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4.11)$$

Theoretically f_0 has to be found by applying the boundary conditions at the edge of the boundary layer,

i.e.

$$u = \mathrm{U}, \quad \frac{\partial u}{\partial y} = 0\,.$$

Assuming, as in paragraph 3, a finite thickness for the boundary layer, i.e. that the two last named conditions are satisfied for a finite value $\bar{y}$ of y, we have

$$\left.\begin{aligned}\mathrm{RU} &= f_0\bar{y} + f_1\frac{\bar{y}^2}{2!} + f_2\frac{\bar{y}^3}{3!} + \ldots \\ 0 &= f_0 + f_1\bar{y} + f_2\frac{\bar{y}^2}{2!} + \ldots\end{aligned}\right\} \quad \ldots \qquad (4.12)$$

Elimination of $\bar{y}$ between these equations gives a condition for finding f_0.

For purposes of calculation a finite number of terms only, of these series, can be used. Consider the case when n terms of the series for u are used. This corresponds to satisfying the boundary conditions for u, $\frac{\partial^2 u}{\partial y^2}$, ... $\frac{\partial^n u}{\partial y^n}$ at $y = 0$ in virtue of equations (4.8), (4.9) and (4.10) and equations (4.12) correspond to satisfying the conditions $u = \mathrm{U}$ and $\frac{\partial u}{\partial y} = 0$ at $y = \delta$.

To determine f_0 *at the origin for flow past a circle*, Green uses the series with three terms only, viz.:

$$\left.\begin{aligned}\mathrm{RU} &= f_0\bar{y} + f_1\bar{y}^2/2! \\ 0 &= f_0 + f_1\bar{y}\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \qquad (4.13)$$

Thus, since f_2 is zero, three terms of each of the series are used. This corresponds to satisfying three conditions at $y = 0$ and two conditions at $y = \bar{y}$ as compared with Pohlhausen's two conditions at $y = 0$ and three at $y = \bar{y}$.

Eliminating $\bar{y}$ between equations (4.13) gives

$$f_0^2 = -2f_1\,\mathrm{RU}\,. \quad \ldots \quad \ldots \quad \ldots \quad . \qquad (4.14)$$

The skin-friction in the neighbourhood of the origin given by the methods of paragraphs 2, 3 and 4 may now be compared by considering

$$\lim_{x \to 0} \frac{1}{U} \sqrt{\nu \Big/ \frac{dU}{dx}} \left(\frac{\partial u}{\partial y}\right)_{y=0}.$$

The three values are 1·233, 1·20 and 1·414 respectively (the value 1·233 given by the method of paragraph 2 is, of course, the correct one.)

Differentiating (4.14) gives the derivatives of f_0, apart from the difficulty of numerical differentiation, if $f_1 = R^2 dp/dx$ is a tabulated function. Thus the values of the f's at the origin can be found subject only to the limitations of numerical differentiation and the approximation in (4.13). Of course, this latter approximation could be improved by using more terms in (4.13) and solving the subsequent differential equation for f_0, or alternatively using the correct value of f_0 as given by the method of paragraph 2. The latter method is recommended.

Green then developed a step-by-step method of calculating f_0, starting from its value near the forward stagnation point, and correcting an extrapolated value at each section, so that calculation of U at the section based on f_0 should agree with the correct value of U. Some details are explained below.

Green's Step-by-Step Method for Determining f_0.—For the case of the circle the method consists in proceeding by steps of 5° around the circumference. We may suppose for example that f_0, f_0' and f_0'' are known at a point 40° from the forward stagnation point but that owing to the difficulties of numerical differentiation the value of f_0''' is only known approximately. It is required to correct this approximate value. Using this approximate value in the series

$$\left.\begin{aligned} (f_0)_{45°} &= (f_0)_{40°} + s\,(f_0')_{40°} + \frac{s^2}{2!}(f_0'')_{40°} + \frac{s^3}{3!}(f_0''')_{40°} \\ (f_0')_{45°} &= (f_0')_{40°} + s\,(f_0'')_{40°} + \frac{s^2}{2!}(f_0''')_{40°} \\ (f_0'')_{45°} &= (f_0'')_{40°} + s\,(f_0''')_{40°} \end{aligned}\right\} \quad .. \quad (4.15)$$

the corresponding values of f_0, f_0' and f_0'' at 45° can be obtained. From these values $f_3, f_4 \ldots\ldots f_8$ at 45° are determined from the recurrence relations. Also $f_1 = R^2 dp/dx$ and $f_2 = 0$.

Now

$$\frac{R}{f_0}\frac{\partial u}{\partial y} = 1 + \frac{f_1}{f_0}y + \frac{f_2}{f_0}\frac{y^2}{2!} + \ldots \quad .. \quad .. \quad .. \quad .. \quad (4.16)$$

and hence the value of $\left[\frac{R}{f_0}\frac{\partial u}{\partial y}\right]$ at 45° is determined.

The corresponding value of $\left[\frac{\mathrm{RU}}{f_0}\right]_{45°}$ can then be determined by integration. Since $[\mathrm{U}_{45°}]$ is known, the corresponding value of $(f_0)_{45°}$ can be determined. The assumed value of $(f_0'')_{40°}$ is correct when the value of $(f_0)_{45°}$ thus found and the one originally found from (4.15) using this value of $(f_0'')_{40°}$ are the same.

The reader is referred to the original paper for further details of this step-by-step method and for the special method required in the neighbourhood of the point of separation.

It is interesting to note that the condition $\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=\bar{y}} = 0$ is not satisfied by using a finite number of terms of (4.12). This is probably connected with the difficulty experienced by Green in making the velocity pass smoothly over into the correct value as given by the velocity in the mainstream. Green himself smoothed off the curves by eye, and this is probably sufficiently accurate for practical purposes. More accurately, a solution of the equations could probably be found on the assumptions valid near the edge of the boundary layer, that u and U are nearly equal, and v is small; and Green's solution joined up with this one.

5. *The First Step-by-Step Method.—Summary.*—The method makes use of the expression

$$\frac{\partial \psi'}{\partial x} = \frac{\nu\,[\psi'' - \psi''(0)] - \frac{1}{\varrho}\frac{dp}{dx}y}{\psi'}$$

$$+ 2\psi' \int_0^y \frac{\psi''\left[\{\psi'' - \psi''(0)\}\,\nu - \frac{1}{\varrho}\frac{dp}{dx}y\right]dy}{\psi'^3} \quad \ldots \qquad (5.1)$$

(when the argument of any function is not specified it is to be taken as y, and dashes denote differentiations with respect to y) to provide a basis for step-by-step calculations.

Equation (5.1) theoretically solves the problem, but in practice it cannot be used straightforwardly owing to the inaccuracies introduced by numerically differentiating a tabulated function twice—a difficulty which is inherent in all step-by-step processes.

For purposes of this method we can suppose ψ'', ψ' and ψ are given as tabulated functions of y up to and including the point $x = a$. An approximate value for ψ'' at $x = a + \Delta a$ can be found by extrapolation and the corresponding values of ψ' and ψ found by integration. Briefly, the method is to correct the extrapolated values of ψ'' by means of a trial and error method, using (5.1) and the relation

$$\psi(a+\triangle a) - \left(\frac{\partial \psi}{\partial x}\right)_{a+\triangle a} \triangle a = \psi(a) \quad \ldots \quad \ldots \quad (5.2)$$

Details.—To obtain equation (5.1) above, we notice that equation (1.1), on making use of the stream function, becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = -\frac{1}{\varrho}\frac{dp}{dx} + \nu\frac{\partial^3 \psi}{\partial y^3} \quad \ldots \quad \ldots \quad \ldots \quad (5.3)$$

Therefore

$$\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\Big/\frac{\partial \psi}{\partial y}\right) = \left(\nu\frac{\partial^3 u}{\partial y^3} - \frac{1}{\varrho}\frac{dp}{dx}\right)\Big/\left(\frac{\partial \psi}{\partial y}\right)^2 \quad \ldots \quad (5.4)$$

Therefore

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y}\int_0^y \nu \frac{\left(\frac{\partial^3 \psi}{\partial y^3} - \frac{1}{\varrho}\frac{dp}{dx}\right) dy}{\left(\frac{\partial \psi}{\partial y}\right)^2}, \quad \ldots \quad (5.5)$$

since $\frac{\partial \psi}{\partial x} = 0$ when $y = 0$.

Equation (5.5) can be written

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y}\int_0^y \frac{\nu \frac{\partial}{\partial y}\left\{\frac{\partial^2 \psi}{\partial y^2} - \left(\frac{\partial^2 \psi}{\partial y^2}\right)_{y=0} - \frac{1}{\varrho}\frac{dp}{dx}y\right\} dy}{\left(\frac{\partial \psi}{\partial y}\right)^2}.$$

Integrating by parts, in order to reduce the third order differential coefficient which occurs under the integral sign to a second order one, we obtain equation (5.1) above, since

$$\lim_{y \to 0} \frac{\nu[\psi''(y) - \psi''(0)] - \frac{1}{\varrho}\frac{dp}{dx}y}{\psi'(y)} = 0. \quad \ldots \quad \ldots \quad (5.6)$$

[Both the numerator and the denominator in (5.6) vanish, and the limit is given by differentiating them both, *i.e.*, the limit is

$$\left[\left(\nu\frac{\partial^2 u}{\partial y^2} - \frac{1}{\varrho}\frac{dp}{dx}\right)\Big/\frac{\partial u}{\partial y}\right]_{y=0} = \left[\frac{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial y}}\right]_{y=0}$$

from equation (1.1) and is zero since u and v vanish, and $\partial u/\partial y$ does not, at the wall (except at the point of separation).]

If the constant term $\psi''(0)$ had not been introduced the result of integrating by parts would have been an integral which diverged at $y = 0$ and a function which was infinite there. Introducing this constant overcomes this difficulty.

ψ'', ψ' and ψ being given up to and including a point $x = a$, extrapolation will give an approximate value for $\psi''_{a+\Delta a}$ and then by integration the corresponding values of $\psi'_{a+\Delta a}$, and $\psi_{a+\Delta a}$ can be found. The method consists in correcting the approximate values of $\psi''_{a+\Delta a}$ first when $y = 0$, then when $y = \Delta\beta, 2\Delta\beta, \ldots.$ where $\Delta\beta$ denotes a small step.

The value of $\psi''_{a+\Delta a}(0)$ has to be corrected first of all. Difficulty arises at this point, and the simple method which can be applied to the remaining points is not applicable here. If we differentiate (5.1) twice with respect to y and put $y = 0$, we find

$$\frac{\partial}{\partial x}[\psi''(0)] = \nu\, \psi^{(5)}(0)/\psi''(0) \quad \ldots \quad (5.7)$$

Thus

$$\psi''_{a+\Delta a}(0) = \psi''_a(0) + \nu . \Delta a .\ \psi_a^{(5)}(0)/\psi''_a(0) \quad \ldots \quad (5.8)$$

Hence we require to determine $\psi_a^{(5)}(0)$ from the data given at $x = a$. For the initial step $\psi_a^{(5)}(0)$ could be found, of course, from the solution of paragraph 2 by repeated differentiation of the differential equations for the f's, g's, etc. (this process involves no loss of accuracy since the determination of the third and higher derivatives is a matter of algebra) but this method is not applicable to any step beyond the first.

A method which is applicable to any section can be obtained in the following manner. Goldstein (*loc. cit.* p. 2) gives a method of continuing any given velocity distribution

$$u = a_1 y + a_2 y^2 + \ldots . \text{ at } x = 0.$$

He states that the conditions for the absence of singularities in the continued solutions are

$$\begin{aligned} &p_0/\varrho + 2\nu a_2 = 0, \quad a_3 = 0, \\ &(5!)\nu^2 a_5 + p_1 a_1/\varrho = 0 \\ &(6!)\nu^3 a_6 = 2\, p_0 p_1/\varrho^2, \text{ etc.} \end{aligned} \quad \ldots \quad (5.9)$$

where

$$-\frac{dp}{dx} = p_0 + p_1 x + p_2 x^2 + \ldots .$$

For any value of x the method of paragraph 2 gives the velocity u as a power series in x whose coefficients are functions of y. These coefficients can be expressed as power series in y, for sufficiently small values of y, and by derangement of the resulting series for u, u can be obtained as a power series in y whose coefficients are functions of x. As might be expected this series satisfies the conditions for the absence of singularities. This has been algebraically verified in detail by the present writer.

Hence, using the conditions for the absence of singularities we can write for all values of x

$$\psi'' = \psi''(0) - \frac{p_0}{\nu\varrho} y + \psi^{(5)}(0)\frac{y^3}{3!} - \frac{p_1}{\nu^2\varrho}\psi''(0)\,y^4 + \frac{2p_0p_1}{\nu^3\varrho^2}y^5 + \ldots \qquad (5.10)$$

where

$$p_0 = -\,dp/dx \text{ and } p_1 = -\,d^2p/dx^2$$

i.e.,

$$\psi^{(5)}(0) = \frac{6}{y^3}\left[\psi''(y) - \psi''(0) + \frac{p_0}{\nu\varrho}y + \frac{p_1}{\nu^2\varrho}\psi''(0)\,y^4 - \frac{2p_0p_1}{\nu^3\varrho^2}y^5 + \ldots\right]. \qquad (5.11)$$

Given p_0, p_1 and a table of values of ψ'' this series determines the value of $\psi^{(5)}(0)$ to one place of decimals less than $\psi''(0)$.*

Thus (5.8) and (5.11) determine the correct value of $\psi''_{a+\triangle a}(0)$.

Next, we correct the value of $\psi''_{a+\triangle a}(\triangle\beta)$. From (5.1) we notice that

$$\left(\frac{\partial\psi}{\partial x}\right)_{a+\triangle a,\,\triangle\beta} = \frac{\nu\left[\psi''_{a+\triangle a}(\triangle\beta) - \psi''_{a+\triangle a}(0)\right] - \frac{1}{\varrho}\left(\frac{dp}{dx}\right)_{a+\triangle a}\cdot\triangle\beta}{\psi'_{a+\triangle a}(\triangle\beta)}$$

$$+\,2\psi'_{a+\triangle a}(\triangle\beta)\int_0^{\triangle\beta}\frac{\psi''_{a+\triangle a}(y)\left[\nu\left\{\psi''_{a+\triangle a}(y) - \psi''_{a+\triangle a}(0)\right\} - \frac{1}{\varrho}\left(\frac{dp}{dx}\right)_{a+\triangle a}y\right]dy}{\left[\psi'_{a+\triangle a}(y)\right]^3} \qquad (5.12)$$

---

* Theoretically, the result used is

$$\psi^{(5)}(0) = \lim_{y\to 0}\frac{6}{y^3}\left[\psi''(y) - \psi''(0) + \frac{p_0 y}{\nu\varrho} + \frac{p_1}{\nu^2\varrho}\psi''(0)\,y^4 - \frac{2p_0p_1}{\nu^3\varrho^2}y^5\right]$$

In practice, the limit is found by substituting fairly small values of y in the expression on the right-hand side of equation (5·11) until a constant result is obtained.

The integral can be evaluated by the Trapezium Rule since $\triangle\beta$ is supposed small. Symbolically this rule can be written

$$\int_0^{\triangle\beta} \phi(y)\, dy = \left[\frac{\phi(\triangle\beta) + \phi(0)}{2}\right] \triangle\beta \qquad \ldots \quad \ldots \quad \ldots \qquad (5.13)$$

$\phi(\triangle\beta)$ is immediately determined from the extrapolated value of $\psi''_{a+\triangle a}(\triangle\beta)$

$$\phi(0) = \lim_{y \to 0} \frac{\psi''_{a+\triangle a}\left[\nu\{\psi''_{a+\triangle a} - \psi''_{a+\triangle a}(0)\} - \frac{1}{\varrho}\left(\frac{dp}{dx}\right)_{a+\triangle a} y\right]}{\psi'^3_{a+\triangle a}}$$

$$= \left[\frac{\nu\,\psi^{(5)}(0)}{6\,[\psi''(0)]^3}\right]_{a+\triangle a} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (5.14)$$

The term in the integral arising from $\phi(0)$ is, in general, very small and it is sufficient to use an extrapolated value of $\psi^{(5)}_{a+\triangle a}(0)$ in determining it.

In use, for the initial step $\psi^{(5)}_{a-3\triangle a}(0)$, $\psi^{(5)}_{a-2\triangle a}(0)$, $\psi^{(5)}_{a-\triangle a}(0)$, $\psi^{(5)}_{a}(0)$ were found from the solution of paragraph 2 and $\psi^{(5)}_{a+\triangle a}(0)$ found by extrapolation from these values.

Thus $\int_0^{\triangle\beta} \phi(y) dy$ is determined, and with it the value of $\left(\frac{\partial\psi}{\partial x}\right)_{a+\triangle a,\, \triangle\beta}$. If the extrapolated value of ψ'' at $a + \triangle a$ were correct (and hence the corresponding value of ψ found by integration) the value of $\left(\frac{\partial\psi}{\partial x}\right)_{a+\triangle a}$ found by the above method would satisfy

$$\psi(a + \triangle a) - \left(\frac{\partial\psi}{\partial x}\right)_{a+\triangle a} \triangle a = \psi(a) \qquad \ldots \quad \ldots \qquad (5.15)$$

The value of $\psi''_{a+\triangle a}(\triangle\beta)$ is varied until this equality is satisfied. Both terms in the right-hand side of (5.12) are very sensitive to slight variations in $\psi''_{a+\triangle a}(\triangle\beta)$ and the required correction can, in general, be found at the first or second trial.

Thus the values of $\psi''_{a+\triangle a}$, $\psi'_{a+\triangle a}$, $\psi_{a+\triangle a}$ are corrected at $\triangle\beta$.

The process is repeated to obtain the corrected values at $2\triangle\beta$, $3\triangle\beta$, ..., the only difference arising from the evaluation of the integral.

We consider the integral $I = \int_0^{n\triangle\beta} \phi(y)\, dy$. We can suppose that the values of $\psi''(r\triangle\beta)$ and $\psi(r\triangle\beta)$ for $r = 1, 2, \ldots (n-1)$ have already been corrected.

Using the assumed value of $\psi''(n\triangle\beta)$ the value of I is required. The easiest method of evaluation is to write

$$I = \int_0^{(n-1)\triangle\beta} \phi(y)\,dy + \int_{(n-1)\triangle\beta}^{n\triangle\beta} \phi(y)\,dy \quad .. \quad .. \quad (5.16)$$

The first of these integrals is supposed known and the second can be obtained from the Trapezium Rule. Alternatively, greater accuracy is obtained by writing

$$I = \int_0^{(n-2)\triangle\beta} \phi(y)\,dy + \int_{(n-2)\triangle\beta}^{n\triangle\beta} \phi(y)\,dy \quad .. \quad .. \quad (5.17)$$

and using Simpson's Rule to evaluate the second integral, *i.e.*,

$$\int_{(n-2)\triangle\beta}^{n\triangle\beta} \phi(y)\,dy = \frac{\triangle\beta}{3}\left[\phi(n\triangle\beta) + \phi(\overline{n-2}\triangle\beta) + 4\phi(\overline{n-1}\triangle\beta)\right]$$
$$.. \quad .. \quad .. \quad .. \quad .. \quad (5.18)$$

This latter method can be used conveniently to determine ψ' from ψ''. The easiest and most accurate method of obtaining ψ from ψ'' and ψ' seems to be by means of the Euler-Maclaurin Formula; this can be written

$$\Big[\psi\Big]_{\overline{n-1}\triangle\beta}^{n\triangle\beta} = \int_{(n-1)\triangle\beta}^{n\triangle\beta} \psi'\,dy = \frac{\triangle\beta}{2}\left[\psi'(n\triangle\beta) + \psi'(\overline{n-1}\triangle\beta)\right]$$
$$- \frac{(\triangle\beta)^2}{12}\left[\psi''(n\triangle\beta) - \psi''(\overline{n-1}\triangle\beta)\right] + 0(\{\triangle\beta\}^4) \quad (5.19)$$

The Length of the Step.—The magnitude of the step $\triangle\alpha$, at any section, is determined by the accuracy required and the maximum value of $\partial^2\psi/\partial x^2$ at the section considered.

From equation (5.2) above the error in the value of ψ at $x = \alpha + \triangle\alpha$ is

$$0\left[(\triangle\alpha)^2\left(\frac{\partial^2\psi}{\partial x^2}\right)_{x=\alpha}\right]$$

and the corresponding error in u is

$$0\left[(\triangle\alpha)^2\left(\frac{\partial^2 u}{\partial x^2}\right)_{x=\alpha}\right].$$

Thus, if $u\varepsilon$ is the maximum error permitted in u in the step $\triangle\alpha$, $\triangle\alpha$ is determined from the relation

$$\frac{1}{u}\cdot(\triangle\alpha)^2\left(\frac{\partial^2 u}{\partial x^2}\right)_{x=\alpha} \leq \varepsilon \quad .. \quad .. \quad (5.20)$$

The value of $\partial^2 u/\partial x^2$ at the initial step is, of course, determined from the solution in paragraph 2. At any other section, for a particular value of y, say, β, $\partial^2 u/\partial x^2$ can be obtained numerically (roughly) from a table of values of u for $y = \beta$, and thus a rough idea of the maximum value of $\partial^2 u/\partial x^2$ at any section can be found.

It will be seen from (5.20) above that the length of the step, at any section, varies as the square root of the error permitted. Unfortunately, the errors committed at each step may be cumulative —a drawback inherent in all step-by-step methods. The principal drawback to the method is that the steps $\Delta\alpha$ which can be taken are rather small.

The case of flow past a circle does not provide an adequate example of the use of this method since the method of paragraph 2 gives the velocity distribution to within $0\cdot2$ per cent. as far as the point of separation. This accuracy is as great as would be required for practical purposes. Should, however, greater accuracy be required the solution of paragraph 2 could be stopped, say, at 60° from the forward stagnation point, where the velocity is given to within $0\cdot05$ per cent., and the method of continuation used. To be of value the error permitted per step would have to be less than or equal to $0\cdot05$ per cent. and this would involve proceeding by steps of 1°. This, however, is not a case when the method of the present paragraph is recommended. Generally an accuracy of 1 per cent. will be sufficient; in the example just mentioned this would allow steps of $4\cdot5°$ to be taken, starting from 60°.

In the case of flow past a thin elliptic cylinder of eccentricity $0\cdot9860$ placed at 3° incidence to a steady stream, it was found that the method of paragraph 2 was of no value at all, since the step it gave was less than 1/20th of the chord. In this region $\partial^2 u/\partial x^2$ is $0(10^3)$, and hence the step given by the method of continuation is also negligible.

It seems probable that, between the thin elliptic cylinder and the circular cylinder, an elliptic cylinder will exist such that the method of paragraph 2 is insufficient to carry the solution as far as the point of separation and such that the steps given by the method of the present paragraph make the point of separation attainable.

For the case of an aerofoil, also, the method of paragraph 2 will not be of value far from the forward stagnation point owing to the rapidity with which $\partial u/\partial x$, $\partial^2 u/\partial x^2$, ... change in that neighbourhood.

6. *Von Mises's Method.—Summary.*—This method gives a means of continuing a solution of the boundary layer equations. It is an alternative to the method given in the last paragraph. Since, however, the step which can be taken by this method is no longer than that of the preceding method, and the work involved per step is greater than that required in the preceding method, it is not recommended.

The method involves solving the differential equation

$$\frac{d^2 z_{\alpha+\Delta\alpha}}{d\psi^2} = \frac{z_{\alpha+\Delta\alpha} - z_\alpha}{\nu u_\alpha \Delta\alpha}$$

at each step, where $z = U^2 - u^2$ and ψ is the stream function. $\psi = 0$ is a singular point of each integral.

Details.—The method of paragraph 2 gives ψ and u as tabulated functions of y up to a point $x = a$. We may, therefore, suppose u given as a function of ψ as far as this point. To obtain u tabulated at equal intervals of ψ, however, requires interpolation. We find, if we denote by the suffix 1 differential coefficients when x and y are independent variables, and by the suffix 2 differential coefficients when x and ψ are independent variables

$$\left(\frac{\partial u}{\partial x}\right)_1 = \left(\frac{\partial u}{\partial x}\right)_2 + \left(\frac{\partial u}{\partial \psi}\right)_2 \left(\frac{\partial \psi}{\partial x}\right)_1 = \left(\frac{\partial u}{\partial x}\right)_2 - \left(\frac{\partial u}{\partial \psi}\right)_2 v \qquad (6.1)$$

$$\left(\frac{\partial u}{\partial y}\right)_1 = \left(\frac{\partial u}{\partial \psi}\right)_2 \left(\frac{\partial \psi}{\partial y}\right)_1 = \left(\frac{\partial u}{\partial \psi}\right)_2 u \quad \ldots \quad \ldots \quad \ldots \qquad (6.2)$$

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_1 = \left[u \frac{\partial}{\partial \psi}\left(u \frac{\partial u}{\partial \psi}\right)\right]_2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (6.3)$$

By (6.1) and (6.2)

$$u\left(\frac{\partial u}{\partial x}\right)_1 + v\left(\frac{\partial u}{\partial y}\right)_1 = u\left(\frac{\partial u}{\partial x}\right)_2.$$

Equation (1.1) then takes the form

$$u \frac{\partial u}{\partial x} = U \frac{dU}{dx} + \nu \frac{\partial}{\partial \psi}\left(u \frac{\partial u}{\partial \psi}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (6.4)$$

when x and ψ are independent variables.

Now, if we write

$$z = U^2 - u^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (6.5)$$

we find

$$\frac{\partial z}{\partial x} = \nu u \frac{\partial^2 z}{\partial \psi^2} = \nu \sqrt{U^2 - z}\, \frac{\partial^2 z}{\partial \psi^2}. \quad \ldots \quad \ldots \quad \ldots \qquad (6.6)$$

Further we notice that

$$\frac{1}{2}\frac{\partial z}{\partial \psi} = - u \frac{\partial u}{\partial \psi} = - \frac{\partial u}{\partial y} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (6.7)$$

The boundary conditions are

(i) $z = 0$ when $\psi = \infty$.
(ii) $z = U^2$ when $\psi = 0$.
(iii) $z = z_0(\psi)$ when $x = 0$.

By use of (6.6), the approximate equation

$$z_{a+\Delta a}(\psi) - z_a(\psi) = \Delta a \left(\frac{\partial z}{\partial x}\right)_{x=a} \quad \ldots \quad \ldots \quad \ldots \qquad (6.8)$$

may be replaced by

$$z_{a+\Delta a}(\psi) - z_a(\psi) = \nu\, \Delta a \sqrt{U^2_a - z_a}\, \frac{\partial^2 z_a}{\partial \psi^2} \quad \ldots \qquad (6.9)$$

where $z_{a+\Delta a}$ and z_a denote the values of z at $x = a + \Delta a$ and $x = a$ respectively.

As it stands (6.9) involves double numerical differentiation of a tabulated function to obtain any step from the previous one. Hence it cannot be used repeatedly for numerical calculation. Alternatively, (6.8) could be replaced by

$$z_{a+\Delta a}(\psi) - z_a(\psi) = \Delta a \left(\frac{\partial z}{\partial x}\right)_{x=a+\Delta a} \quad \cdots \quad (6.10)$$

and then as in (6.9) we could write

$$z_{a+\Delta a}(\psi) - z_a(\psi) = \nu\, \Delta a \sqrt{U^2_{a+\Delta a} - z_{a+\Delta a}}\, \frac{\partial^2 z_{a+\Delta a}}{\partial \psi^2} \quad \cdots \quad (6.11)$$

(6.11) is a nonlinear second order differential equation for $z_{a+\Delta a}$. Luckert (*loc. cit.* page 253) takes an easier form by writing

$$z_{a+\Delta a}(\psi) - z_a(\psi) = \nu\, \Delta a \sqrt{U^2_a - z_a}\, \frac{\partial^2 z_{a+\Delta a}}{\partial \psi^2} \quad \cdots \quad (6.12)$$

(6.12) is a linear second order differential equation for $z_{a+\Delta a}$ since z_a is a known function of ψ. The error in (6.12) would not be expected to be any greater than the errors in (6.10) and (6.11).

The boundary conditions are

(i) $z_{a+\Delta a} = 0$ when $\psi = \infty$.

(ii) $z_{a+\Delta a} = U^2_{a+\Delta a}$ when $\psi = 0$.

Thus, when $\psi = 0$ the equation has a singular point. This difficulty has to be overcome by using the series mentioned in paragraph 5, viz. :—

$$\psi_{a+\Delta a} = \psi''_{a+\Delta a}(0)\frac{y^2}{2!} - \frac{p_0(a+\Delta a)}{\nu \varrho\, 3!} y^3 + 0(y^5) \quad \cdots \quad (6.13)$$

$$u_{a+\Delta a} = \psi''_{a+\Delta a}(0)\, y - \frac{p_0(a+\Delta a)}{\nu \varrho\, 2!} y^2 + 0(y^4) \quad \cdots \quad (6.14)$$

$$\frac{\partial u_{a+\Delta a}}{\partial y} = \psi''_{a+\Delta a}(0) - \frac{p_0(a+\Delta a)}{\nu \varrho} y + 0(y^3) \quad \cdots \quad (6.15)$$

where dashes denote differentiations with respect to y, and $-\dfrac{p_0(a+\Delta a)}{\varrho} = \left[U \dfrac{dU}{dx}\right]_{a+\Delta a}$ (the terms in y^4, y^3 and y^2 in (6.13), (6.14) and (6.15) respectively, vanish). Then $\psi^{(5)}_a(0)$ can be found from the table of values of ψ''_a, as in the last paragraph using (5.11) and then (5.8), *i.e.*, $\psi''_{a+\Delta a}(0) = \psi''_a(0) + \Delta a \cdot \nu \cdot \psi^{(6)}_a(0)/\psi''_a(0)$, gives $\psi''_{a+\Delta a}(0)$. Thus, (6.13), (6.14) and (6.15) give the initial values of ψ, z $(= U^2 - u^2)$ and $\partial z/\partial \psi \left(= -\dfrac{1}{2}\dfrac{\partial u}{\partial y}\right)$ for starting the integration of the equation. Choosing the initial values in this way, from series whose asymptotic expressions satisfy the boundary conditions at infinity, causes the conditions at infinity to be satisfied automatically.

This will give z, and therefore u, as functions of ψ at $x = a + \triangle a$. In order to express the results in the usual way, u should be expressed as a function of y. Since $u = \partial\psi/\partial y$, and u and y vanish together,

$$y = \int_0^u \frac{d\psi}{u},$$

and so this step involves an ordinary integration to find y.

Hence, at each step a numerical integration of a second order differential equation and a numerical evaluation of an integral are required. Moreover, the steps which can be taken by means of this method are no longer than those in the method of the preceding paragraph, and the work involved here at each step is greater. Hence the method of the preceding paragraph seems definitely preferable.

7. *Thom's First Method.*—Thom has given a method of solving the boundary layer equation to any accuracy required, but from a practical point of view the method is extremely laborious—too laborious for repeated application.

He considers a rectangle ABCD within the boundary layer, having its sides AD, BC of length $2x$ parallel to the wall, whilst AB and CD are of length $2y$.

If 0 is the centre of this rectangle and u, v and dp/dx are the velocity components and pressure gradient there respectively, he finds on using the boundary layer equations that, approximately

$$\begin{aligned} u = u_m &- k_1 u\,(u_A + u_B - u_C - u_D) \\ &- k_2 v\,(u_A + u_D - u_B - u_C) + k_3 \qquad (7.1) \end{aligned}$$

where $k_1 = y^2/8x$, $k_2 = y/8$, $k_3 = -(y^2 dp/dx)/2$ and

$$u_m = \frac{u_A + u_B + u_C + u_D}{4}, \text{ using an obvious notation.}$$

The method then is to divide the boundary layer into a rectangular net and *assume* values of the velocity u at the corners of the rectangles. The velocity at the centre of each rectangle can then be calculated by means of (7.1) above. The values of the velocities at the centres of the rectangles being taken as new " corner velocities " the values of the velocities at the centres of these rectangles can be found from (7.1). But the centres of these rectangles coincide with the vertices of the original rectangles, and the velocities as found should agree with those originally assumed.

The process has to be repeated until agreement is obtained to the required degree of accuracy.

Thom (*loc. cit.* page 6) uses the method on the flow past a circle for an angular distance of 20° from the forward stagnation point. He makes the remark that many cycles had to be calculated to obtain the result.

8. *Falkner and Skan's Method.*—As an introduction, the authors (*loc. cit.* page 4) give a complete mathematical solution of the boundary layer equations when the velocity at the edge of the boundary layer is expressible in the form $U = kx^m$. This solution contains, as special cases, three solutions which had been given previously.

(i) When $m = 1$, the flow in the neighbourhood of a stagnation point is given. The solution of this problem has been given in paragraph 2 of the present paper.

(ii) When $m = 0$, the flow past a straight wall is given.

(iii) When $m = -1$, the flow between two converging walls is given.

The first of these is the one of interest in the problem under consideration, but the range in which a relation $U = kx$ holds is generally very small. The solution when $U = kx^m$ is used as a basis for two approximations for larger values of x.

If we write $\zeta = \psi x^{-\frac{1}{2}} U^{-\frac{1}{2}} \nu^{-\frac{1}{2}}$ and $\xi = y x^{-\frac{1}{2}} U^{+\frac{1}{2}} \nu^{-\frac{1}{2}}$ the boundary layer equation (1.1) can be expressed as a partial differential equation with ζ as dependent variable and ξ and x as independent variables. In the case when $U = kx^m$, where m is constant, this partial differential equation simplifies and ζ is given as the dependent variable of an *ordinary* differential equation whose independent variable is ξ. The coefficients in this equation are functions of m.

The first approximation is given by assuming that the solution for any particular value of x can be obtained by putting m equal to the value of $(U'x/U)$, corresponding to this value of x, in the ordinary differential equation just mentioned. This corresponds to the omission of some terms from the original partial differential equation. In effect, this approximation is equivalent to replacing the original partial differential equation by an ordinary differential equation in ζ and ξ whose coefficients are functions of $(U'x/U)$, i.e. of x.

An attempt is made to obtain a better approximation—the second—by altering the coefficients in the ordinary differential equation of the first approximation. The coefficients are assumed to be functions of x and are chosen to make the neglected terms in the original partial differential equation smaller than the neglected terms in the first approximation.

From the conditions imposed on these functions two methods of attack may be developed. The first of these is the one given by Falkner and Skan and the second is due to the present writer. It will be shown, however, that in general one of the conditions imposed ceases to be valid in the neighbourhood of the point of separation and therefore the methods should not be used in that neighbourhood. This involves a serious limitation to the value of the methods.

Details.—Equation (1.3) implies the existence of a stream function defined by $u = \partial \psi/\partial y$, $v = -\partial \psi/\partial x$.

Transforming equation (1.1) by using variables x, ζ and ξ instead of x, ψ and y, where $\zeta = \psi x^{-\frac{1}{2}} U^{-\frac{1}{2}} \nu^{-\frac{1}{2}}$, $\xi = y x^{-\frac{1}{2}} U^{\frac{1}{2}} \nu^{-\frac{1}{2}}$ we find

$$x\left[\left(\frac{\partial \zeta}{\partial \xi}\right)\left(\frac{\partial^2\zeta}{\partial \xi\, \partial x}\right) - \left(\frac{\partial^2\zeta}{\partial \xi^2}\right)\left(\frac{\partial \zeta}{\partial x}\right)\right]$$

$$+\left[l\left(\frac{\partial \zeta}{\partial \xi}\right)^2 - \frac{l+1}{2}\,\zeta\,\frac{\partial^2\zeta}{\partial \xi^2} - \frac{\partial^3\zeta}{\partial \xi^3} - l\right] = 0 \quad .. \quad (8.1)$$

where $l = U'x/U$.

In terms of ζ and ξ we find

$$u = J\partial \zeta/\partial \xi \quad .. \quad .. \quad .. \quad .. \quad (8.2)$$

$$-v = x^{\frac{1}{2}} U^{\frac{1}{2}} \nu^{\frac{1}{2}} \left[\frac{\partial \zeta}{\partial x} + \tfrac{1}{2}\zeta\left\{\frac{1+l}{x}\right\} + \tfrac{1}{2}\frac{\partial \zeta}{\partial \xi}\,\xi\left\{\frac{l-1}{x}\right\}\right]. \quad .. \quad (8.3)$$

The boundary conditions are

$$\left.\begin{array}{ll} \text{(i)} & u = v = 0 \text{ at } y = 0 \\ \text{(ii)} & u = U \text{ at } y = \infty \\ \text{(iii)} & u = U = 0 \text{ at } x = 0 \text{ when } m > 0 \end{array}\right\} \quad .. \quad .. \quad (8.4)$$

In terms of the new variables these become

$$\left.\begin{array}{ll} \text{(i)} & \zeta = \partial\zeta/\partial\xi = 0 \text{ at } \xi = 0 \\ \text{(ii)} & \partial\zeta/\partial\xi = 1 \text{ at } \xi = \infty \\ \text{(iii)} & \partial\zeta/\partial\xi = 1 \text{ at } \xi = \infty \end{array}\right\} \quad .. \quad .. \quad .. \quad (8.5)$$

Owing to the transformation used (ii) and (iii) are evidently the same; hence the reason for using this transformation.

Special Case.—In particular when $U = kx^m$, $l = m$ and is constant.

In this case a solution in which ζ is a function of ξ only, can be obtained. (Indeed a dimensional argument, similar to one given by Blasius (*loc. cit.* page 1), can be used to prove that ζ is a function of ξ only.)

If ζ is a function of ξ only, equation (8.1) becomes

$$m\left(\frac{d\zeta}{d\xi}\right)^2 - \frac{m+1}{2}\,\zeta\,\frac{d^2\zeta}{d\xi^2} - \frac{d^3\zeta}{d\xi^3} - m = 0 \quad .. \quad .. \quad (8.6)$$

with the boundary conditions

$$\zeta = \frac{d\zeta}{d\xi} = 0 \text{ at } \xi = 0.$$

$$\frac{d\zeta}{d\xi} = 1 \text{ at } \xi = \infty.$$

The authors show that, for a certain range of values of m, this equation has a solution satisfying the boundary conditions. Hence, within this range of values of m this solution is the solution of equation (8.1).

General Case.—For the special case $U = kx^m$, discussed in the last paragraph, we found that ζ was a function of ξ only. Moreover, for small values of x we can suppose the velocity distribution at the edge of the boundary layer to be of the form $U = kx$. Hence for small values of x in any problem, ζ is a function of ξ only. Thus, the first part of equation (8.1) vanishes for small values of x.

The first approximation is given by assuming that the first part of equation (8.1) may be neglected, even when l is no longer constant.

Equation (8.1) then reduces to

$$l\left(\frac{\partial \zeta}{\partial \xi}\right)^2 - \frac{l+1}{2}\,\zeta\,\frac{\partial^2 \zeta}{\partial \xi^2} - \frac{\partial^3 \zeta}{\partial \xi^3} - l = 0 \qquad \text{(8.7)}$$

where l is a known function of x.

For any particular value of x, (8.7) reduces to the form (8.6) and can be solved numerically, thus giving the velocity distribution at the point x considered.

By putting $\zeta = a'_2\,\xi^2 + a'_3\,\xi^3 + \ldots.$ .. in (8.1)

and $\zeta = a''_2\,\xi^2 + a''_3\,\xi^3 + \ldots.$ .. in (8.7)

(since $\zeta = \dfrac{\partial \zeta}{\partial \xi} = 0$ at $\xi = 0$) it can be seen that

$$6a'_3 = -\,l$$
$$6a''_3 = -\,l$$

Thus, this approximation is equivalent to making the value of $\dfrac{\partial^3 \zeta}{\partial \xi^3}$ correct at $\xi = 0$ for all values of x, i.e., to making the curvature of the velocity profile correct at the solid boundary and to making $\dfrac{\partial u}{\partial y}, \dfrac{\partial^2 u}{\partial y^2} \ldots.$ vanish at the edge of the boundary layer.

This approximation may be compared with the method of paragraph 3. In effect, the method of the present paragraph gives an ordinary differential equation whose independent variable contains y as compared with the ordinary differential equation in x obtained by Pohlhausen. The boundary conditions satisfied by u in the two methods are the same at the solid boundary and roughly the same at infinity. The reason why the method of paragraph 3 gives better results than this first approximation is due to the neglected terms in the approximate forms of the boundary layer equations used by Pohlhausen being less than in the present method.

The second approximation is given by assuming that equation (8.1) may be written in the form :—

$$F_1(x)\left(\frac{\partial \zeta}{\partial \xi}\right)^2 - F_2(x)\,\zeta\,\frac{\partial^2 \zeta}{\partial \xi^2} - \frac{\partial^3 \zeta}{\partial \xi^3} - F_1(x) = 0 \quad \ldots \quad (8.8)$$

where F_1 and F_2 are functions of x only. The form of equation (8.8) is compatible with the boundary condition $\frac{\partial \zeta}{\partial \xi} = 1$ when $\frac{\partial^2 \zeta}{\partial \xi^2} = \ldots = 0$. Once F_1 and F_2 have been determined (8.8) reduces to an ordinary differential equation in ζ and ξ for any particular value of x. For any pair of numerical values for F_1 and F_2, the particular solution of (8.8) satisfying the boundary conditions can be obtained. This has been done by Falkner and Skan* and the solutions exhibited graphically.

To determine F_1 we make, as in the first approximation, the curvature of the velocity profile correct at the solid boundary, i.e., the values of $\left(\frac{\partial^3 \zeta}{\partial \xi^3}\right)_{\xi=0}$ given by (8.1) and (8.8) are made the same.

If we put

$$\zeta = a'_2\xi^2 + a'_3\xi^3 + \ldots . \quad \ldots \quad (8.9)$$

in (8.8)

and

$$\zeta = a''_2\xi^2 + a''_3\xi^3, \quad \ldots \quad (8.10)$$

in (8.1), we find

$$6a'_3 = -F_1(x)$$

$$6a''_3 = -l.$$

Thus, the condition $a'_3 = a''_3$ yields $F_1(x) = l$ immediately.

The criterion to determine $F_2(x)$ is more difficult to obtain. Ideally it would be chosen to make $a'_2 = a''_2$, i.e., to make the skin-friction correct in the approximate solution. This condition is very difficult to handle analytically since a'_2 and a''_2 are defined by the condition $\frac{\partial \zeta}{\partial \xi} = 1$ when ξ is infinite. The series representations (8.9) and (8.10) are not adequate for this purpose.

If, however, (8.9) is the solution of (8.8) the terms neglected in the original partial differential equation (8.1) are

$$x\left[\frac{\partial \zeta}{\partial \xi}\frac{\partial^2 \zeta}{\partial x\,\partial \xi} - \frac{\partial \zeta}{\partial x}\frac{\partial^2 \zeta}{\partial \xi^2}\right] + \left[F_2(x) - \frac{1+l}{2}\right]\zeta\frac{\partial^2 \zeta}{\partial \xi^2} \qquad (8.11)$$

* Falkner and Skan indicate a region where they say imaginary solutions may occur. The present writer has found real solutions occurring in certain parts of this region.

For the condition to determine $F_2(x)$, Falkner and Skan choose the condition that the coefficient of the lowest power of ξ, i.e. ξ^2, in this error should vanish.

This gives

$$F_2(x) - \frac{1+l}{2} = \frac{x}{a'_2}\frac{da'_2}{dx} \quad \ldots \quad (8.12)$$

If we consider the neighbourhood of the point of separation we see, from graphs given by Falkner and Skan relating F_1, F_2 and a'_2 in the solution of (8.8), that the value of F_2 corresponding to a value of $F_1 = l$ usually associated with separation and to small values of a'_2 is finite and roughly about 5. Equation (8.12) therefore implies that da'_2/dx and a'_2 vanish together. It can be seen from the exact solution of paragraph 2 for the flow past a circle that this condition is not even approximately satisfied. Therefore, in the neighbourhood of the point of separation the condition (8.12) should not be used.

Further insight to the meaning of (8.12) can be obtained by substituting from (8.9) in (8.8) and from (8.10) in (8.11). We find that the condition that

$$a'_5/a'^2_2 = a''_5/a''^2_2 \quad \ldots \quad (8.13)$$

is

$$F_2(x) = \tfrac{1}{2}\left(1 + l - \frac{2x}{a''_2}\frac{da''_2}{dx}\right) \quad \ldots \quad (8.14)$$

where a''_2 is the correct value of $\left(\dfrac{\partial^2 \zeta}{\partial \xi^2}\right)_{\xi=0}$

Thus, if $a'_2 \doteqdot a''_2$ then (8.12) implies that $a'_5 \doteqdot a''_5$.

This remark is true whichever two solutions of (8.1) and (8.8) are chosen. Let us suppose, for a moment, that the correct value of a''_2 is known and that the corresponding value of $F_2(x)$ is obtained from (8.14). The derivation of (8.14) gives us no reason for supposing that when this value of $F_2(x)$ is used in (8.8) the value of a'_2 corresponding to the particular solution satisfying the condition $\dfrac{\partial \zeta}{\partial \xi} = 1$ at infinity is even approximately correct.

Thus, logically, the application of the condition (8.12) seems to be without foundation. It appears to be a matter for trial whether the application of the condition (8.12) gives a better approximation than the first.

Theoretically, this difficulty could be overcome by using the condition that the mean square error should vanish. For, in the solution of (8.8) ζ can be expressed as a function of F_2, the corresponding value of a'_2 and ξ. The integral equation

$$\int_0^\infty \left\{ x\left[\frac{\partial \zeta}{\partial \xi}\frac{\partial^2 \zeta}{\partial x\,\partial \xi} - \frac{\partial^2 \zeta}{\partial \xi^2}\frac{\partial \zeta}{\partial x}\right] - \left[F_2(x) - \frac{1+l}{2}\right]\zeta\frac{\partial^2 \zeta}{\partial \xi^2}\right\}^2 d\xi = 0 \quad \ldots \quad (8.15)$$

gives a relation between F_2 and a'_2 for making the error a minimum. Unfortunately, this expression seems to be too complicated to be of use.

We proceed now to the discussion of the method when the condition (8.12) is used. The application of this condition suggested by the present writer is immediate. The solutions of (8.8) satisfying the condition at infinity express F_2 as a function of F_1 and a'_2, i.e., as a function of x and a'_2. The graphs given by Falkner and Skan show the exact nature of this relationship. Thus, (8.12) is an ordinary differential equation in x for a'_2 and can be solved by the usual methods. This gives the skin-friction immediately and velocity profiles can be obtained from the corresponding values of F_1 and F_2 and graphs given by Falkner and Skan.

The method can be used, either to continue another solution, or to give a complete solution. In the latter case, the initial value for starting the integration is given by the accurate solution when $l = 1$ as $a'_2 = 0{\cdot}616$. In the former case, the initial value for starting the integration may be supposed known.

Summary of Procedure.—(i) Tabulate $l = U'x/U$ for a series of values of x, say, $x_1, x_2, \ldots$.

(ii) From the value of $F_1 = l$ at $x = x_r$ and any value of a'_2 obtain F_2 from graphs published by Falkner and Skan.

(iii) From this value of F_2 and equation (8.12) obtain the corresponding value of da'_2/dx. Thus, da'_2/dx can be obtained for any pair of values of a'_2 and x, and the solution completed in the customary graphical way for a first order linear differential equation.

The method has been used by the present writer as an alternative method for comparison with Dryden's modification of Pohlhausen's method, when the ordinary method due to Pohlhausen breaks down (*see* paragraph 3). The method of the present paragraph was started from a value of a'_2, given by the ordinary Pohlhausen method, some distance before the breakdown occurred. It was used to continue the solution through the region where the difficulty occurred. Since the method of the present paragraph should not be used in the neighbourhood of the point of separation, i.e., for negative values of U′, the ordinary Pohlhausen solution was joined on as soon as the region of difficulty was passed. (This region of difficulty is essentially confined to positive values of U′.)

Falkner and Skan apply the condition (8.12) differently. They assume an expansion

$$F_2(x) = \tfrac{1}{2}\left(1 + l + c_1 \frac{dl}{dx} x + c_2 \frac{d^2 l}{dx^2} x^2 + \ldots\right) \quad \ldots \quad (8.16)$$

and the condition (8.12) gives

$$-\frac{2x}{a'_2}\frac{da'_2}{dx} = \left(c_1 \frac{dl}{dx} + c_2 x \frac{d^2 l}{dx^2} + \ldots\right) \quad \ldots \quad (8.17)$$

An approximation, not mentioned by the authors in their paper, is involved in the method of determining the c's from (8.17). If, following Falkner and Skan, we assume that each term in the expansion in (8.17) is small compared with the previous one we find, omitting the terms in x^3 and higher powers of x, that

$$\frac{-2}{a'_2}\frac{da'_2}{dx} = c_1 \frac{dl}{dx}$$

$$\text{i.e.,} \quad c_1 = \left(\frac{-2}{a'_2}\frac{da'_2}{ldx}\right)_{x=0} = \left(\frac{-2}{a'_2}\frac{da'_2}{dl}\right)_{l=1} \quad \ldots \quad (8.18)$$

Falkner and Skan state that the coefficients $c_1 = \left(\frac{-2}{a'_2}\frac{da'_2}{dl}\right)_{l=1}$, $\left(\frac{dc_1}{dl}\right)_{l=1}$ can be evaluated from the known solution of (8.1) as $x \to 0$. However, for small values of x, $U = u_1 x$, $l = 1$ and this gives no information about $\left(\frac{da'_2}{dl}\right)_{l=1}$, $\left(\frac{dc_1}{dl}\right)_{l=1}$, ...

In stating that $\left(\frac{da'_2}{dl}\right)_{l=1} = \left(\frac{da'_2}{dm}\right)_{m=1}$, where a'_2 is regarded as a function of m in the solutions of (8.1) with $U = kx^m$ for constant values of m, Falkner and Skan are, in effect, determining the constants c from the first approximation.

Thus, the value of $F_2(x)$ determined in this way satisfies the condition (8.12) approximately and not accurately.

It is interesting to compare the value $c_1 = -0{\cdot}925$ given by this approximate method with those obtainable from the accurate solution of paragraph 2.

If $U = u_1 x + u_2 x^2 + \ldots$ where u_1 and u_2 are $\neq 0$.
$c_1 = -0{\cdot}8356$.

If $U = u_1 x + u_3 x^3 + \ldots$ where u_1 and u_3 are $\neq 0$.
$c_1 = -0{\cdot}8515$.

In the case of c_2 the approximate method gives

$$c_2 = K_2 \left[\left(\frac{dl}{dx}\right)^2 \Big/ \frac{d^2l}{dx^2}\right]_{x=0} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8.19)$$

where K_2 is a numerical constant.

The exact solution gives

$$c_2 = K'_2 + K''_2\left[\left(\frac{dl}{dx}\right)^2 \Big/ \frac{d^2l}{dx^2}\right]_{x=0} \quad \ldots \quad \ldots \quad \ldots \quad (8.20)$$

where K'_2 and K''_2 are numerical constants.

The way to use the original method is clear. The quantities $l, \frac{dl}{dx} \ldots$ are given in any problem as functions of x; c_1, K_2 etc. are numerical constants given by Falkner and Skan and thus the functions F_1 and F_2 can be tabulated as functions of x. The graphs published by Falkner and Skan then determine a'_2 and the ratio u/U for $\xi = 0{\cdot}25, 0{\cdot}50, 0{\cdot}75$ and $1{\cdot}00$, thus giving the skin-friction and points on the velocity profile.

Using the approximate values of the c's it is evident, in the case of symmetrical flow, that $c_2 = c_3 = \ldots = 0$ and therefore

$$F_2(x) = \tfrac{1}{2}\left[1 + l + c_1 \frac{dl}{dx} x\right] \quad \ldots \quad (8.21)$$

This value of $F_2(x)$ determined from the condition (8.12) should become infinite at the point of separation. The fact that it does not become infinite for any value of x means that the approximate values for the c's are not sufficiently accurate to satisfy the condition (8.12) as far as the point of separation. Indeed it will be seen that the coefficient c_2 does not vanish, that similarly c_3, c_4, . . . also do not vanish, and, therefore, that the value of $F_2(x)$ given by (8.21) is quite wrong for the larger value of x owing to the neglect of these terms.

In the case of the circle, for which diagrams (*see* figs. 2–6) have been given, using the original method and the expression (8.21) for $F_2(x)$, it will be seen that the skin-friction is given in good agreement with the exact solution of paragraph 2 as far as the point of separation. This must be taken as coincidence since (8.21) is not an adequate representation of the condition from which it is derived, as far as the point of separation.

The quantity a'_2 has been calculated using the method suggested by the present writer. It is shown in fig. 7 compared with the exact solution of paragraph 2, the solution of paragraph 3, and Falkner and Skan's first approximation. It will be seen that the second approximation, calculated in this way, is in good agreement with the exact solution as far as $x = 6$, i.e., for positive values of U'. For negative values of U' and the correspondingly small values of a'_2 the agreement is poor. The breakdown of the approximate method of the present paragraph for small values of a'_2 was anticipated earlier from a consideration of the condition (8.12).

More evidence, in the shape of comparisons with other methods for various velocity distributions, is desirable, but it seems probable that the approximation considered will give reliable results so long as U' remains positive. The method should not be used for negative values of U'.

The present writer can find no theoretical grounds for using (8.21) as distinct from (8.12). The good approximation it gives for the case of the circle seems to be the sole justification for using (8.21).

9. *Thom's Second Method.—Summary.*—The method is an approximate one, the first approximation being obtained by assuming that u/U is a function of y only. Now in paragraph 2 we obtained the velocity distribution through the boundary layer in the form

$$u = u_1 f'_1 x + 3u_2 f'_2 x^2 + \ldots .$$

where the velocity U at the edge of the boundary layer is given by

$$U = u_1 x + u_2 x^3 + \ldots .$$

The u's are constants and the f's are functions of y only. Therefore u/U is a function of y only provided that

$$u_2 = u_3 = \ldots . = 0$$

since the f's are all unequal.

Hence, the first approximation holds only when the velocity is linearly connected with the distance from the forward stagnation point. In general, such a relation will only hold for a small range of values of x.

The second approximation consists in estimating the neglected terms by means of the first approximation, and would be expected to be true only when the velocity has not deviated far from the form $u_1 x$.

Thom applied this method to the case of the circle. In this case it will be seen later that u_3/u_1 and u_5/u_1 are very small, and hence the example is favourable to the use of this method. The solution given by this method is compared (*see* paragraph 10) with the more accurate one given by the method of paragraph 2 in figs. 2, 3 and 4. The method will be seen to give a good approximation as far as 50° from the forward stagnation point. Near 70° from the forward stagnation point, where U′ is very small and the velocity profile is considerably different from a linear function of the distance from the forward stagnation point, the method is valueless.

Details.—First Approximation.—Write $u = Uf$ .. (9.1)
where f is a function of y only.

Equation (1.1) on neglecting the term $v\,\dfrac{\partial u}{\partial y}$, gives

$$U\left(\frac{dU}{dx}\right)f^2 = U\left(\frac{dU}{dx}\right) + \nu U f'' \quad .. \quad .. \quad .. \quad .. \qquad (9.2)$$

On integration this gives

$$y = [2\nu/(dU/dx)]^{\frac{1}{2}}\, F(f) \quad .. \quad .. \quad .. \qquad (9.3)$$

where

$$F(f) = \int_0^f \frac{\sqrt{3}\, df}{\sqrt{2}\,(f^3 - 3f + 2)} = \log \frac{(\sqrt{3} - \sqrt{2})\sqrt{1-f}}{\sqrt{3} - \sqrt{f+2}} \quad .. \qquad (9.4)$$

Equation (9.3) expresses y as a function of f; by giving a series of values to f, lying between 0 and 1, we can find the corresponding value of y, and hence the velocity distribution at any given section.

Second Approximation.—With f no longer restricted to be a function of y only, but permitted to involve x also, a closer approximation can be obtained by using (9.3) to estimate the value of the neglected term $v\frac{\partial u}{\partial y}$ in the equation of motion, and the term $U\frac{\partial f}{\partial x}$ in $\frac{\partial u}{\partial x}$, whe u is given by (9.1).

Using (9.3) we find

$$\text{(i)} \;..\; \left(\frac{\partial f}{\partial x}\right)_y = -\left(\frac{\partial y}{\partial x}\right)_f \Big/ \left(\frac{\partial y}{\partial f}\right)_x = \tfrac{1}{2}\frac{F(f)}{F'(f)}\frac{\dfrac{d^2U}{dx^2}}{\dfrac{dU}{dx}} \quad .. \quad .. \qquad (9.5)$$

where $F(f)$ is given by (9.4) and thus

$$\frac{\partial u}{\partial x} = \frac{dU}{dx}f + \tfrac{1}{2}\frac{F(f)}{F'(f)}\frac{\dfrac{d^2U}{dx^2}}{\dfrac{dU}{dx}}U \quad .. \quad .. \quad .. \quad .. \qquad (9.6)$$

$$..\;\text{(ii)}\; v\frac{\partial u}{\partial y} = -U\frac{dU}{dx}\phi(f) \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (9.7)$$

where

$$\phi(f) = \frac{1}{F'(f)}\int_0^f fF'(f)\,df.$$

Then equation (1.1) becomes

$$Uf^2\frac{dU}{dx} + \tfrac{1}{2}U^2 f\frac{F(f)}{F'(f)}\frac{\dfrac{d^2U}{dx^2}}{\dfrac{dU}{dx}} - U\frac{dU}{dx}\phi(f) = U\frac{dU}{dx} + \nu Uf'' \qquad (9.8)$$

Integrating gives

$$y = \left(\frac{3\nu}{2}\Big/\frac{dU}{dx}\right)^{\frac{1}{2}}\int_0^f \frac{df}{\sqrt{f^3 - 3f + 2 - \frac{3}{2}\left\{\left(U\frac{d^2U}{dx^2}\right)\Big/\left(\frac{dU}{dx}\right)^2\right\}\int_f^1\frac{fF(f)}{F'(f)}df + 3\int_f^1\phi(f)\,df}}$$

$$.. \quad .. \quad .. \quad .. \quad .. \qquad (9.9)$$

thus giving the particular value of y to associate with a given value of u, at any section.

Owing to the special nature of the first approximation the second approximation cannot be expected to give good results unless the assumption originally made is roughly true.

10. *Conclusion*.—The methods of paragraphs 2, 3, 4, 8 and 9 have been graphically compared for the flow past a circle using the experimental values of Hiemenz. The results are given in Figs. 2, 3, 4, 5 and 6. For a circle of radius 4·87 cms. in a fluid of kinematic viscosity $\nu = 0{\cdot}01$ and the velocity at infinity of 19·2 cms./sec. (i.e., $R = Ud/\nu = 18544{\cdot}5$) Hiemenz (*loc. cit.*) gives

$$U = u_1 x + u_3 x^3 + u_5 x^5 \ldots \qquad (10.1)$$

where $u_1 \equiv 7{\cdot}151$, $u_3 = -0{\cdot}04497$ and $u_5 = -0{\cdot}0003300$ for $0 \leq x \leq 8$. The corresponding pressure distribution is shown in Fig. 1.

We consider, first, the accuracy with which the method of paragraph 2, together with Table 1, gives the velocity distribution and the point of separation by estimating errors as in (*c*) of paragraph 2. The velocity distribution is given by equation (2.55).

The term in x^7 is $8\left(\frac{u_3 u_5}{u_1} h'_7 + \frac{u_3^3}{u_1^2} k'_7\right) x^7 \leq 0{\cdot}06$ for $0 \leq x \leq 7$.

Thus, for a range of values of x, $0 \leq x \leq 7$ (corresponding to an angular displacement of 82° from the forward stagnation point), the possible error in the velocity is 0·2 per cent.

The point of separation is given by equation (2.56). Denote by $\phi(x)$ the right-hand side of this equation, by $\theta(x)$ the sum of the terms in x, x^3 and x^5 and by $R(x)$ the remaining terms. Arguing as in (*c*) of paragraph 2 we see that the term in x^7 remains less than 0·15 for x in the range $0 \leq x \leq 7{\cdot}1$.

Therefore $|R| < 0{\cdot}15$ + terms small in comparison with 0·15. Hence we expect $|R|$ certainly to be less than 0·2. Now

$$\theta(6{\cdot}85) = 0{\cdot}3, \quad \theta(7{\cdot}05) \doteq -0{\cdot}2.$$

Therefore $\phi(6{\cdot}85) = 0{\cdot}3 + R(6{\cdot}85)$.
Thus $\phi(6{\cdot}85)$ lies between 0·1 and 0·5 since $|R| < 0{\cdot}2$.
Also $\phi(7{\cdot}05) = -0{\cdot}2 + R(7{\cdot}05)$.
Thus $\phi(7{\cdot}05)$ lies between 0·0 and $-0{\cdot}4$. Hence $\phi(x)$ changes sign in the interval 6·85 and 7·05. Therefore, $\phi(x) = 0$ has a root between 6·85 and 7·05, i.e., the point of separation lies between $x = 6{\cdot}85$ and $x = 7{\cdot}05$, i.e., between 81° and 83° from the forward stagnation point.

Pohlhausen (*loc. cit.*) attacks Hiemenz's result for the point of separation as being obtained from an insufficient number of terms of a slowly converging series, as all terms of higher power than the fifth in x were neglected. It will be seen that this attack was, to some extent, unjustified, since we have shown that Hiemenz's result for the position of the point of separation holds good to within two degrees, whereas Pohlhausen suggests that it was more by good fortune than mathematics that Hiemenz was anywhere near the correct answer.

It should be pointed out again that this example of flow past a circle is rather a special case inasmuch as, when the velocity at the edge of the boundary layer is expressed in the form $(u_1x + u_3x^3 + u_5x^5)$ for a representation holding right from the forward stagnation point to the point of separation, the ratios u_3/u_1 and u_5/u_1 are very small.

The application of the remaining methods to this problem needs no further remark, except it should be pointed out again, that owing to the amount of work involved the method of Bairstow and Green was not applied to the pressure distribution given by Hiemenz. The velocity distributions for this method were found by cross-plotting from the velocity distributions given by Green (*loc. cit.*). The pressure distribution used by Green for the calculation of his results and that corresponding to equation (10.1) above are compared in Fig. 1, and it will be seen that the two pressure distributions are considerably different in the neighbourhood of the point of separation. Green's results are not, therefore, included beyond 50° from the forward stagnation point.

It will be seen from the figures and the foregoing discussion that if the method of paragraph 2 (together with Table 1 or 2) is sufficient to give the solution as far as is required, it is certainly the best method to use. If the solution cannot be carried sufficiently far by this means the method of paragraph 3 is recommended, though many more comparisons between this method and that of paragraph 2 are required before the former can be used with great confidence owing to the difficulty in estimating the error.

I am indebted to Mr. Falkner for pointing out to me that the analysis given in the preceding pages covers the case of a round-nosed obstacle only, and that the case of a cylinder, whose section has a sharp point in the neighbourhood of the nose, has not been mentioned.

The theoretical solution for this case seems to be possible only when the forward stagnation point coincides with the sharp point. In this case the velocity distribution in the main stream in the neighbourhood of the forward stagnation point can be written in the form $U = kx^m$ where $m \neq 1$. Mr. Falkner remarks, further, that the only method, so far published, which specifically includes this case is due to Miss Skan and himself. It is, indeed, evident from the description of this method given in paragraph 8 that its application is quite independent of the choice of the value 1 for m in the neighbourhood of the origin.

The method of paragraph 3 can, however, immediately be extended to the case considered. The only difference lies in the choice of the initial value for starting the integration of equation (3.1). For small values of x, U has the form kx^m and equation (3.5) can then be written

$$d\lambda/dx = \frac{mf(\lambda) + (m-1)\,h(\lambda)}{x} \quad \text{.. .. .. (10.2)}$$

As in paragraph 3 we use the criterion that the thickness of the boundary layer should be finite at $x = 0$. This condition gives

$$mf(\lambda) + (m-1)\,h(\lambda) = 0, \text{ at } x = 0 \ldots \quad .. \quad .. \qquad (10.3)$$

This equation seems to have been noticed in the first place by Dryden (*loc. cit.*) when he carried out a series of comparisons between Pohlhausen's method, his own modification of it, and the accurate solution for the case when $U = kx^m$ holds for all values of x. In this simple case the equation (10.2) can be integrated immediately and the solution obtained from the initial value given by (10.3).

In the more general case considered the initial values for λ and $\frac{d\lambda}{dx}$ have to be determined from (10.3) and (10.2) respectively and the general form (3.1) used for the integration.

Equation (10.3) is cubic in λ for any particular value of m, and therefore there are three possible values for the initial value of λ. It seems very probable that, as in the case $m = 1$ discussed in paragraph 3, it should be possible to eliminate two of these values as giving either imaginary solutions or solutions which necessarily break down before the point of separation is reached and that the remaining value gives a solution which does not necessarily break down before the point of separation is reached. This has been verified by the present writer for the cases $m = 0$ and 2. The value of this method, or Dryden's modification of it, lies in the solution extending to the point of separation.

I wish to acknowledge my indebtedness to Dr. Goldstein for suggesting this paper and also for the many helpful suggestions he has made.

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8. R. & M. 1176.

TABLE 1

η	f_1	f'_1	f''_1	f_3	f'_3	f''_3
0·0	0·0000	0·0000	1·23264	0·000	0·000	0·7246
0·1	0·0060	0·1183	1·1328	0·004	0·068	0·625
0·2	0·0233	0·2266	1·0345	0·013	0·125	0·529
0·3	0·0510	0·3252	0·9386	0·028	0·174	0·438
0·4	0·0881	0·4144	0·8463	0·048	0·213	0·354
0·5	0·1336	0·4946	0·7583	0·071	0·245	0·278
0·6	0·1867	0·5662	0·6751	0·096	0·269	0·211
0·7	0·2466	0·6298	0·5973	0·124	0·287	0·153
0·8	0·3124	0·6859	0·5251	0·154	0·300	0·104
0·9	0·3835	0·7350	0·4586	0·184	0·308	0·063
1·0	0·4592	0·7778	0·3980	0·215	0·313	0·029
1·1	0·5389	0·8149	0·3431	0·246	0·314	+0·008
1·2	0·6220	0·8467	0·2937	0·277	0·313	−0·017
1·3	0·7081	0·8739	0·2498	0·309	0·311	−0·031
1·4	0·7966	0·8968	0·2109	0·340	0·307	−0·041
1·5	0·8873	0·9161	0·1769	0·370	0·302	−0·048
1·6	0·9798	0·9324	0·1473	0·400	0·298	−0·051
1·7	1·0738	0·9457	0·1218	0·429	0·293	−0·052
1·8	1·1688	0·9569	0·0999	0·458	0·288	−0·051
1·9	1·2650	0·9659	0·0814	0·487	0·283	−0·048
2·0	1·3619	0·9732	0·0658	0·515	0·278	−0·045
2·1	1·4596	0·9792	0·0528	0·543	0·273	−0·040
2·2	1·5577	0·9841	0·0420	0·570	0·269	−0·036
2·3	1·6563	0·9876	0·0332	0·597	0·266	−0·032
2·4	1·7552	0·9905	0·0260	0·623	0·263	−0·028
2·5	1·8543	0·9928	0·0202	0·649	0·259	−0·023
2·6	1·9537	0·9946	0·0156	0·676	0·258	−0·020
2·7	2·0533	0·9960	0·0119	0·701	0·257	−0·017
2·8	2·1529	0·9971	0·0091	0·726	0·256	−0·013
2·9	2·2528	0·9979	0·0068	0·751	0·254	−0·011
3·0	2·3525	0·9985	0·0051	0·777	0·253	−0·010
3·1	2·4523	0·9988	0·0036	0·802	0·252	−0·008
3·2	2·5522	0·9992	0·0027	0·828	0·251	−0·007
3·3	2·6521	0·9994	0·0023	0·853	0·251	−0·005
3·4	2·7521	0·9996	0·0019	0·879	0·251	−0·004
3·5	2·8520	0·9997	0·0014	0·904	0·250	−0·003
3·6	2·9520	0·9998	0·0010	0·929	0·250	−0·002
3·7	3·0519	0·9999	0·0008	0·954	0·250	−0·002
3·8	3·1518	0·9999	0·0004	0·979	0·250	−0·001
3·9	3·2518	0·9999	0·0003	1·004	0·250	−0·001
4·0	3·3518	1·0000	0·0002	1·029	0·250	−0·001
4·1	3·4518	1·0000	0·0001	1·054	0·250	0·000
4·2	3·5518	1·0000	0·0001	1·079	0·250	0·000
4·3	3·6518	1·0000	0·0000	1·104	0·250	0·000

TABLE 1—*continued*

η	g_6	g'_6	g''_6	h_6	h'_6	h''_6	k_7	k'_7	k''_7
0·0	0·00	0·00	0·637	0·00	0·00	0·12	0·00	0·00	0·012
0·1	0·00	0·06	0·54	0·00	0·01	0·07	0·00	0·00	0·01
0·2	0·01	0·11	0·44	0·00	0·01	+0·03	0·00	0·00	0·02
0·3	0·03	0·15	0·35	0·00	0·02	−0·01	0·00	0·00	0·02
0·4	0·04	0·18	0·27	0·01	0·01	−0·04	0·00	0·01	0·02
0·5	0·06	0·20	0·20	0·01	0·01	−0·07	0·00	0·01	0·03
0·6	0·08	0·22	0·14	0·01	+0·00	−0·08	0·00	0·01	0·04
0·7	0·10	0·23	0·09	0·01	−0·01	−0·08	0·00	0·01	0·05
0·8	0·13	0·24	0·05	0·00	−0·02	−0·08	0·00	0·01	0·05
0·9	0·15	0·24	+0·02	0·00	−0·03	−0·08	0·00	0·02	0·05
1·0	0·17	0·24	−0·01	+0·00	−0·03	−0·07	0·01	0·02	0·05
1·1	0·20	0·24	−0·03	−0·01	−0·04	−0·06	0·01	0·03	0·05
1·2	0·22	0·24	−0·04	−0·01	−0·05	−0·04	0·01	0·03	0·04
1·3	0·25	0·23	−0·05	−0·02	−0·05	−0·03	0·01	0·04	0·03
1·4	0·27	0·23	−0·06	−0·02	−0·05	−0·02	0·02	0·04	0·02
1·5	0·29	0·22	−0·06	−0·03	−0·05	−0·01	0·02	0·05	0·02
1·6	0·31	0·21	−0·07	−0·03	−0·05	0·00	0·02	0·05	0·01
1·7	0·34	0·21	−0·06	−0·04	−0·05	+0·01	0·03	0·05	0·01
1·8	0·36	0·20	−0·06	−0·04	−0·05	0·02	0·03	0·05	+0·01
1·9	0·38	0·19	−0·06	−0·05	−0·04	0·02	0·04	0·05	0·00
2·0	0·39	0·19	−0·06	−0·05	−0·04	0·03	0·04	0·05	0·00
2·1	0·41	0·19	−0·05	−0·05	−0·04	0·03	0·05	0·05	−0·01
2·2	0·43	0·18	−0·04	−0·05	−0·03	0·03	0·05	0·04	−0·02
2·3	0·45	0·18	−0·04	−0·05	−0·03	0·02	0·06	0·04	−0·03
2·4	0·46	0·18	−0·03	−0·05	−0·03	0·02	0·07	0·04	−0·04
2·5	0·48	0·18	−0·03	−0·05	−0·02	0·02	0·08	0·03	−0·04
2·6	0·49	0·18	−0·02	−0·05	−0·02	0·02	0·09	0·02	−0·03
2·7	0·51	0·18	−0·01	−0·05	−0·01	0·01	0·09	0·01	−0·02
2·8	0·53	0·17	−0·01	−0·05	0·00	0·01	0·09	0·00	−0·01
2·9	0·54	0·17	0·00	−0·05	0·00	0·00	0·09	0·00	−0·01
3·0	0·56	0·17	0·00	−0·05	0·00	0·00	0·09	0·00	0·00
3·1	0·58	0·17	0·00	−0·05	0·00	0·00	0·09	0·00	0·00

TABLE 2

η	f_1	f'_1	f''_1	f_2	f'_2	f''_2
0·0	0·0000	0·0000	1·23264	0·000	0·000	0·7982
0·1	0·0060	0·1183	1·1328	0·004	0·075	0·699
0·2	0·0233	0·2266	1·0345	0·015	0·140	0·602
0·3	0·0510	0·3252	0·9386	0·032	0·195	0·509
0·4	0·0881	0·4144	0·8463	0·054	0·242	0·423
0·5	0·1336	0·4946	0·7583	0·080	0·280	0·344
0·6	0·1867	0·5662	0·6751	0·109	0·311	0·273
0·7	0·2466	0·6298	0·5973	0·142	0·336	0·210
0·8	0·3124	0·6859	0·5251	0·175	0·353	0·156
0·9	0·3835	0·7350	0·4586	0·212	0·367	0·109
1·0	0·4592	0·7778	0·3980	0·249	0·375	0·070
1·1	0·5389	0·8149	0·3431	0·287	0·381	0·037
1·2	0·6220	0·8467	0·2937	0·325	0·384	+0·012
1·3	0·7081	0·8739	0·2498	0·363	0·384	—0·007
1·4	0·7966	0·8968	0·2109	0·402	0·382	—0·021
1·5	0·8873	0·9161	0·1769	0·440	0·380	—0·032
1·6	0·9798	0·9324	0·1473	0·477	0·377	—0·039
1·7	1·0738	0·9457	0·1218	0·515	0·373	—0·042
1·8	1·1688	0·9569	0·0999	0·552	0·368	—0·043
1·9	1·2650	0·9659	0·0814	0·588	0·364	—0·043
2·0	1·3619	0·9732	0·0658	0·625	0·361	—0·041
2·1	1·4596	0·9792	0·0528	0·661	0·357	—0·038
2·2	1·5577	0·9841	0·0420	0·696	0·353	—0·034
2·3	1·6563	0·9876	0·0332	0·731	0·350	—0·030
2·4	1·7552	9·9905	0·0260	0·766	0·346	—0·026
2·5	1·8543	0·9928	0·0202	0·801	0·344	—0·022
2·6	1·9537	0·9946	0·0156	0·835	0·342	—0·019
2·7	2·0533	0·9960	0·0119	0·870	0·340	—0·016
2·8	2·1529	0·9971	0·0091	0·904	0·338	—0·014
2·9	2·2528	0·9979	0·0068	0·938	0·337	—0·011
3·0	2·3525	0·9985	0·0051	0·972	0·336	—0·009
3·1	2·4523	0·9988	0·0036	1·006	0·335	—0·007
3·2	2·5522	0·9992	0·0027	1·040	0·335	—0·006
3·3	2·6521	0·9994	0·0023	1·073	0·335	—0·005
3·4	2·7521	0·9996	0·0019	1·106	0·334	—0·003
3·5	2·8520	0·9997	0·0014	1·139	0·334	—0·003
3·6	2·9520	0·9998	0·0010	1·172	0·334	—0·002
3·7	3·0519	0·9999	0·0008	1·205	0·334	—0·001
3·8	3·1518	0·9999	0·0004	1·238	0·333	—0·001
3·9	3·2518	0·9999	0·0003	1·271	0·333	—0·001
4·0	3·3518	1·0000	0·0002	1·304	0·333	—0·001
4·1	3·4518	1·0000	0·0001	1·337	0·333	0·000
4·2	3·5518	1·0000	0·0001	1·370	0·333	0·000
4·3	3·6518	1·0000	0·0000	1·403	0·333	0·000

TABLE 2—*continued*

η	g_3	g'_3	g''_3	h_3	h'_3	h''_3	h_4	h'_4	h''_4
0·0	0·000	0·000	0·725	0·00	0·00	0·166	0·00	0·00	—0·019
0·1	0·004	0·068	0·625	0·00	0·01	0·12	0·00	0·00	—0·02
0·2	0·013	0·125	0·529	0·00	0·02	0·07	0·00	0·00	—0·02
0·3	0·028	0·174	0·438	0·01	0·03	+0·03	0·00	—0·01	—0·01
0·4	0·048	0·213	0·354	0·01	0·03	—0·01	0·00	—0·01	—0·01
0·5	0·071	0·245	0·278	0·01	0·03	—0·04	0·00	—0·01	0·00
0·6	0·096	0·269	0·211	0·01	0·02	—0·05	0·00	—0·01	0·00
0·7	0·124	0·287	0·153	0·02	0·02	—0·07	—0·01	0·00	+0·01
0·8	0·154	0·300	0·104	0·02	+0·01	—0·07	—0·01	0·00	0·02
0·9	0·184	0·308	0·063	0·02	0·00	—0·07	—0·01	0·00	0·03
1·0	0·215	0·313	0·029	0·02	—0·01	—0·07	—0·01	0·00	0·03
1·1	0·246	0·314	+0·003	0·02	—0·01	—0·06	—0·01	0·00	0·04
1·2	0·277	0·313	—0·017	0·01	—0·02	—0·05	—0·01	+0·01	0·04
1·3	0·309	0·311	—0·031	0·01	—0·02	—0·04	0·00	0·01	0·04
1·4	0·340	0·307	—0·041	0·01	—0·03	—0·03	0·00	0·01	0·04
1·5	0·370	0·302	—0·048	+0·01	—0·03	—0·02	0·00	0·02	0·03
1·6	0·400	0·298	—0·051	0·00	—0·03	0·00	0·00	0·02	0·03
1·7	0·429	0·293	—0·052	0·00	—0·03	0·00	0·00	0·02	0·02
1·8	0·458	0·288	—0·051	0·00	—0·03	+0·01	+0·01	0·03	0·01
1·9	0·487	0·283	—0·048	—0·01	—0·03	0·02	0·01	0·03	+0·01
2·0	0·515	0·278	—0·045	—0·01	—0·03	0·03	0·01	0·03	0·00
2·1	0·543	0·273	—0·040	—0·01	—0·02	0·03	0·02	0·03	—0·01
2·2	0·570	0·269	—0·036	—0·01	—0·02	0·03	0·02	0·03	—0·01
2·3	0·597	0·266	—0·032	—0·01	—0·02	0·03	0·02	0·02	—0·02
2·4	0·623	0·263	—0·028	—0·01	—0·02	0·04	0·02	0·02	—0·02
2·5	0·649	0·259	—0·023	—0·02	—0·01	0·03	0·03	0·02	—0·03
2·6	0·676	0·258	—0·020	—0·02	—0·01	0·03	0·03	0·02	—0·03
2·7	0·701	0·257	—0·017	—0·02	—0·01	0·02	0·03	0·02	—0·03
2·8	0·726	0·256	—0·013	—0·02	—0·01	0·02	0·03	0·01	—0·02
2·9	0·751	0·254	—0·011	—0·02	0·00	0·02	0·03	0·01	—0·02
3·0	0·777	0·253	—0·010	—0·02	0·00	0·02	0·03	0·00	—0·01
3·1	0·802	0·252	—0·008	—0·02	0·00	0·02	0·03	0·00	0·00
3·2	0·828	0·251	—0·007	—0·02	0·00	0·01	0·03	0·00	0·00
3·3	0·853	0·251	—0·005	—0·02	0·00	0·01	0·03	0·00	0·00
3·4	0·879	0·251	—0·004	—0·02	0·00	0·00	0·03	0·00	0·00
3·5	0·904	0·250	—0·003	—0·02	0·00	0·00	0·03	0·00	0·00
3·6	0·929	0·250	—0·002	—0·02	0·00	0·00	0·03	0·00	0·00
3·7	0·954	0·250	—0·002	—0·02	0·00	0·00	0·03	0·00	0·00
3·8	0·979	0·250	—0·001	—0·02	0·00	0·00	0·03	0·00	0·00
3·9	1·004	0·250	—0·001	—0·02	0·00	0·00	0·03	0·00	0·00
4·0	1·029	0·250	—0·001	—0·02	0·00	0·00	0·03	0·00	0·00
4·1	1·054	0·250	0·000	—0·02	0·00	0·00	0·03	0·00	0·00

k''_4
−0·019
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TABLE 3

λ	$f(\lambda)$	$g(\lambda)$	$h(\lambda)$
12·0	∞	∞	∞
11·0	−62·20	0·4395	64·18
10·0	−27·25	0·2133	31·33
9·0	−13·88	0·1375	20·14
8·0	−5·80	0·0994	14·36
7·8	−4·46	0·0939	13·51
7·6	−3·20	0·0889	12·73
7·4	−1·99	0·0843	12·02
7·2	−0·831	0·0801	11·35
7·052	0·00	0·0772	10·89
6·8	+1·37	0·0727	10·16
6·6	2·43	0·0693	9·62
6·4	3·46	0·0662	9·11
6·2	4·47	0·0633	8·63
6·0	5·45	0·0606	8·18
5·0	10·19	0·0492	6·23
4·0	14·76	0·0404	4·65
3·0	19·40	0·0334	3·30
2·0	24·00	0·0275	2·11
+1·0	28·88	0·0225	+1·023
0·0	34·05	0·0180	0·000
−1·0	39·61	0·0140	−0·986
−2·0	45·56	0·0102	−1·96
−3·0	52·33	0·0065	−2·94
−4·0	59·77	+0·0029	−3·95
−5·0	68·17	−0·00074	−5·02
−6·0	77·81	−0·0045	−6·16
−7·0	88·95	−0·0085	−7·42
−8·0	102·28	−0·0131	−8·84
−9·0	118·36	−0·0183	−10·48
−10·0	138·31	−0·0244	−12·44
−11·0	163·88	−0·0319	−14·51
−12·0	198·00	−0·0417	−18·00

TABLE 4

λ	$p(\lambda)$	$q(\lambda)$
48·52	∞	∞
45·00	—1072·615	0·090078
40·00	— 451·491	0·033407
35·00	— 281·739	0·018195
30·00	— 197·521	0·010839
25·00	— 143·535	0·006262
20·00	— 102·961	0·002920
15·00	— 68·463	+0·000158
10·00	— 36·723	—0·002382
5·00	— 4·293	—0·004960
4·365	0·000	—0·005302
4·00	+ 2·4926	—0·005501
3·00	9·4343	—0·006057
2·00	16·326	—0·006630
0·00	31·481	—0·007834
— 5·00	74·490	—0·011356
—10·00	131·532	—0·016136
—15·00	216·269	—0·023390
—17·18	271·854	—0·028253

$$\left(\frac{dz}{dx}\right)_v = -\left(\frac{U''}{U'^2}\right)_v 3\cdot793$$

(23340) Wt. 184/505 500 7/35 Hw. G.377/1

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FIG. A.

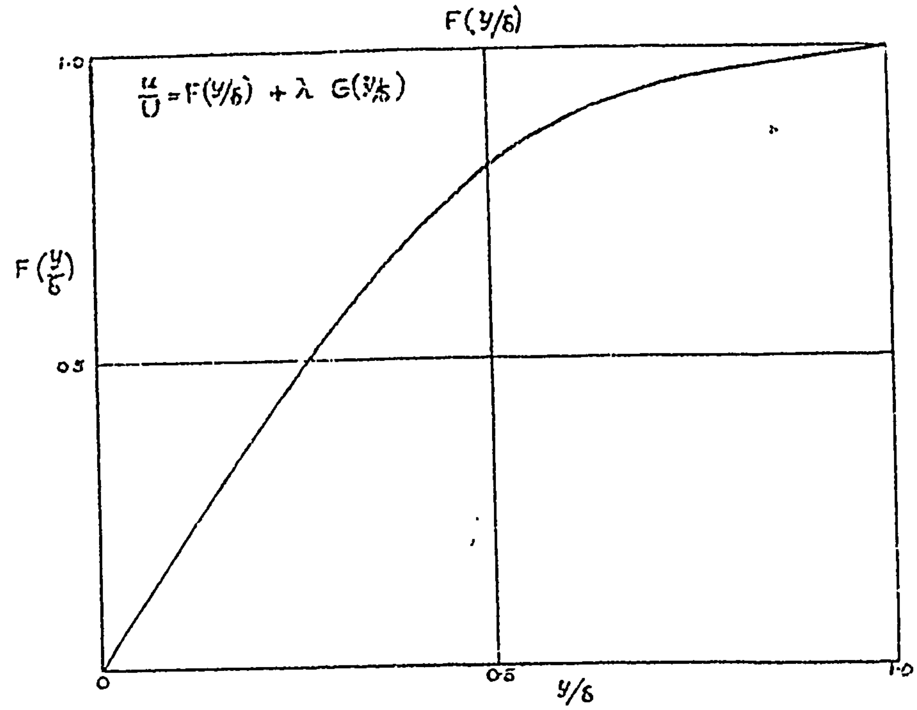

FIG. B.

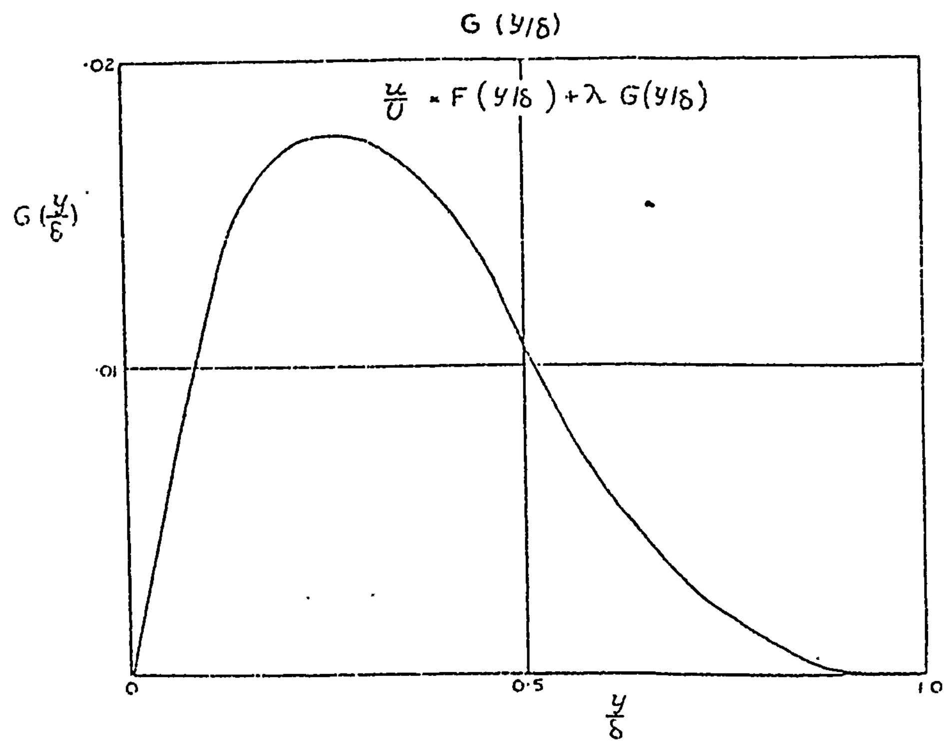

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Fig. 1.

Pressure Distributions.

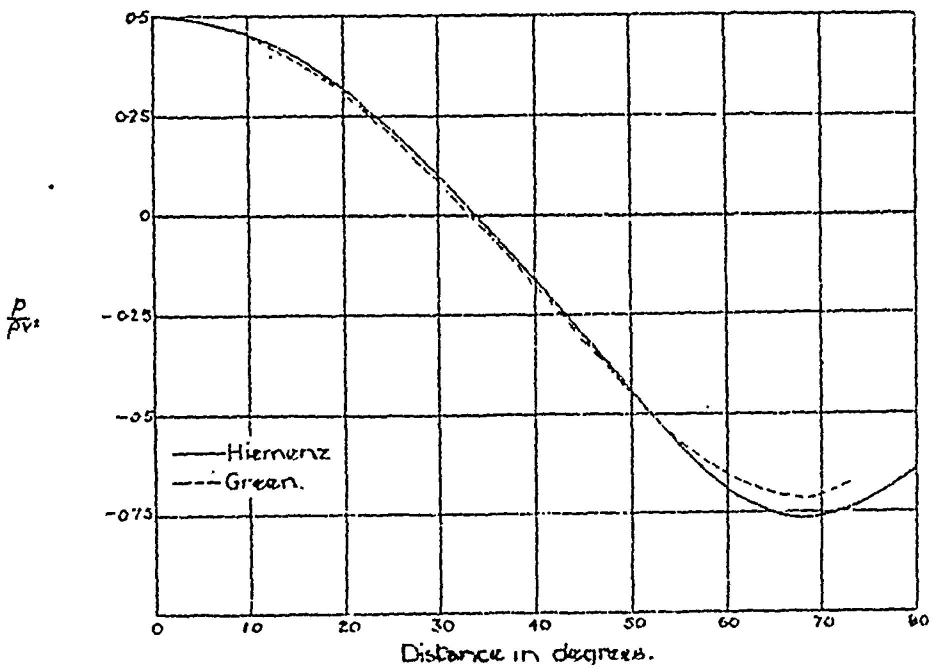

Fig. 2.

Section at x=1 (11·76° from forward stagnation point).

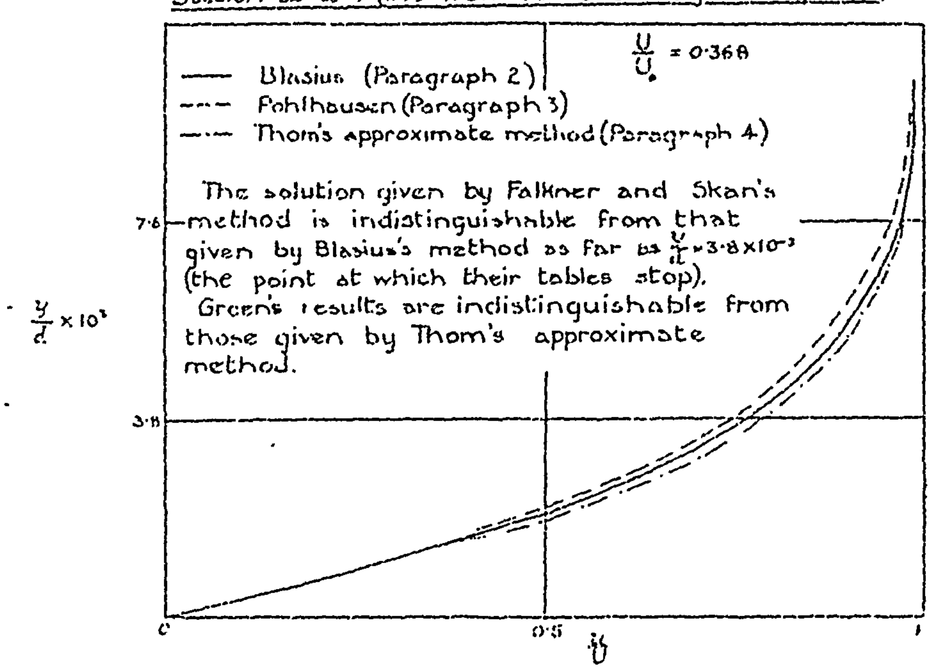

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FIG. 3.

Section at $x = 2{\cdot}5$ (35·28° from forward stagnation point)

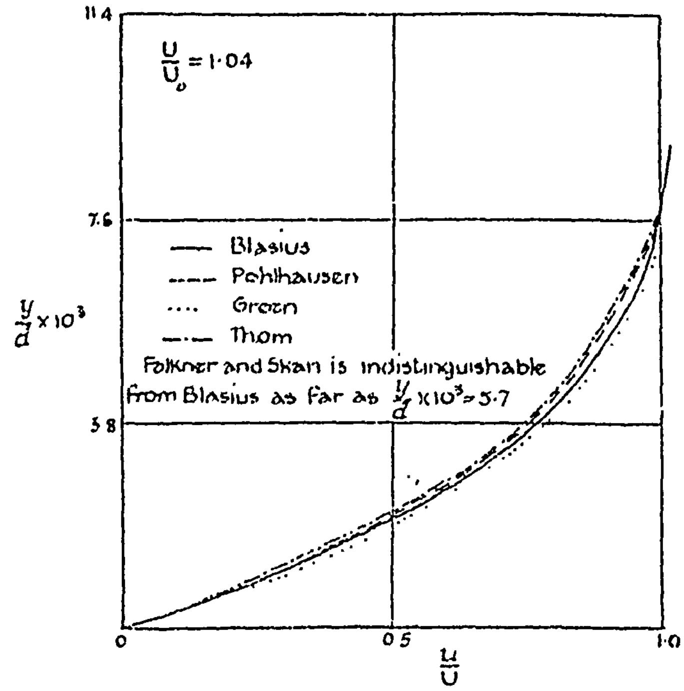
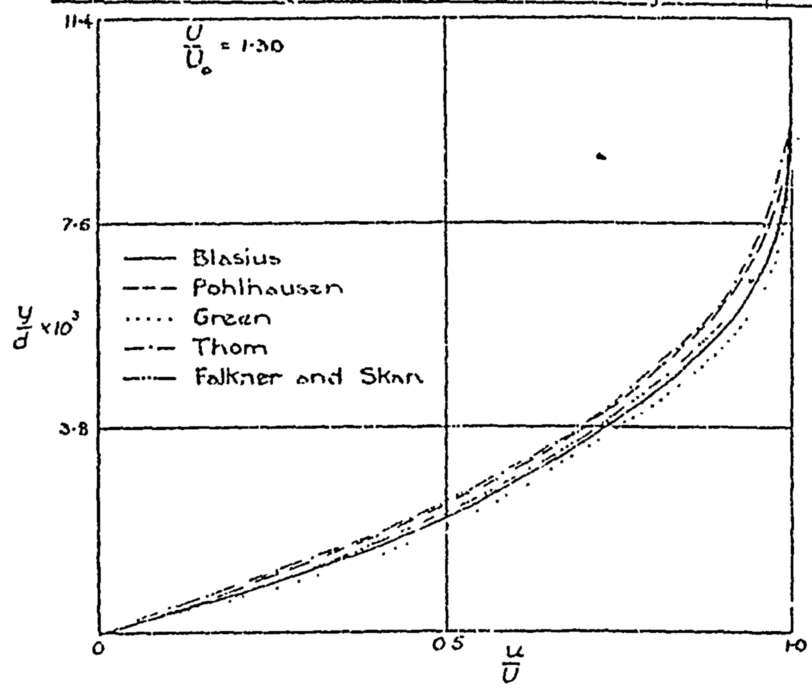


FIG. 4.

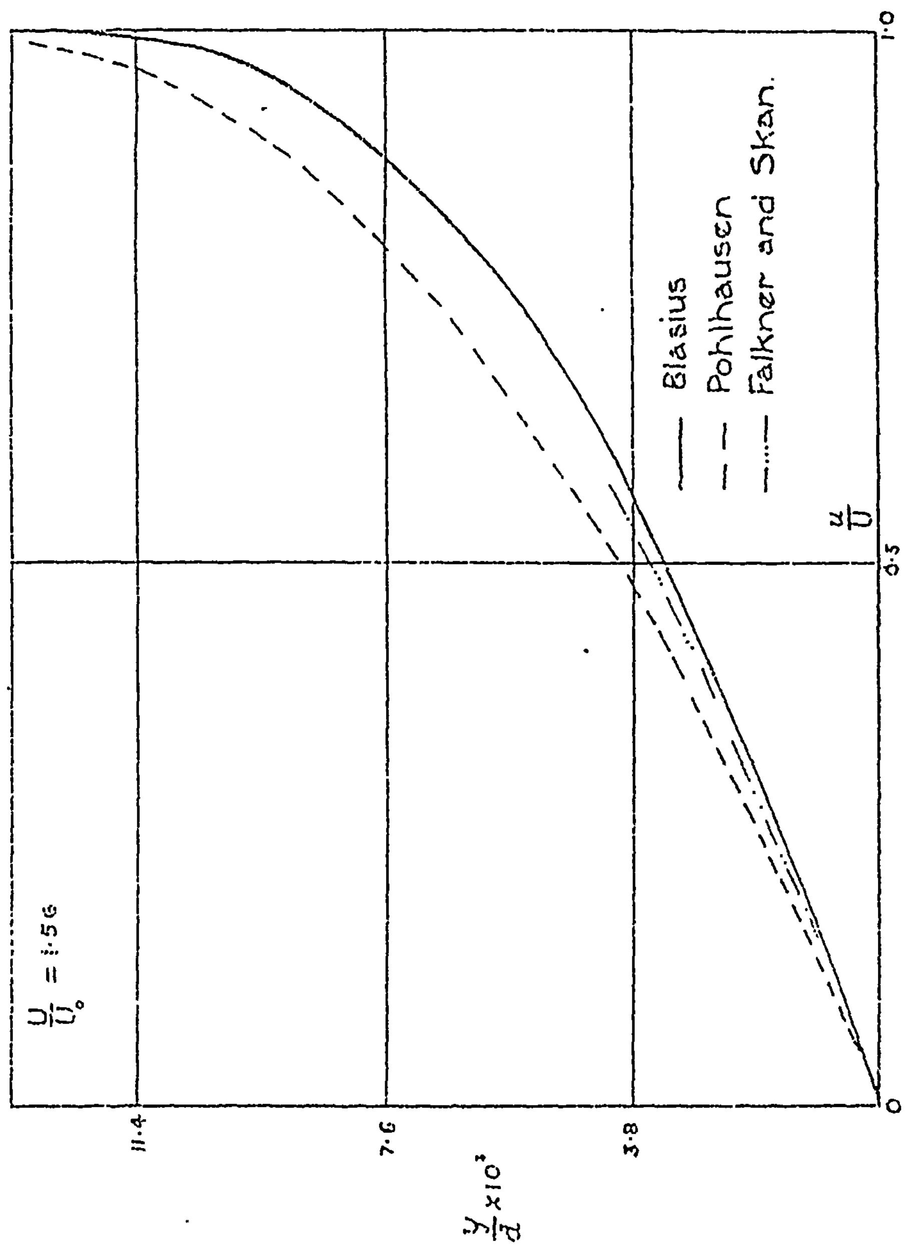
Section at $x = 4$ (47·04° from forward stagnation point)



R.& M. 1632.

Fig 5.

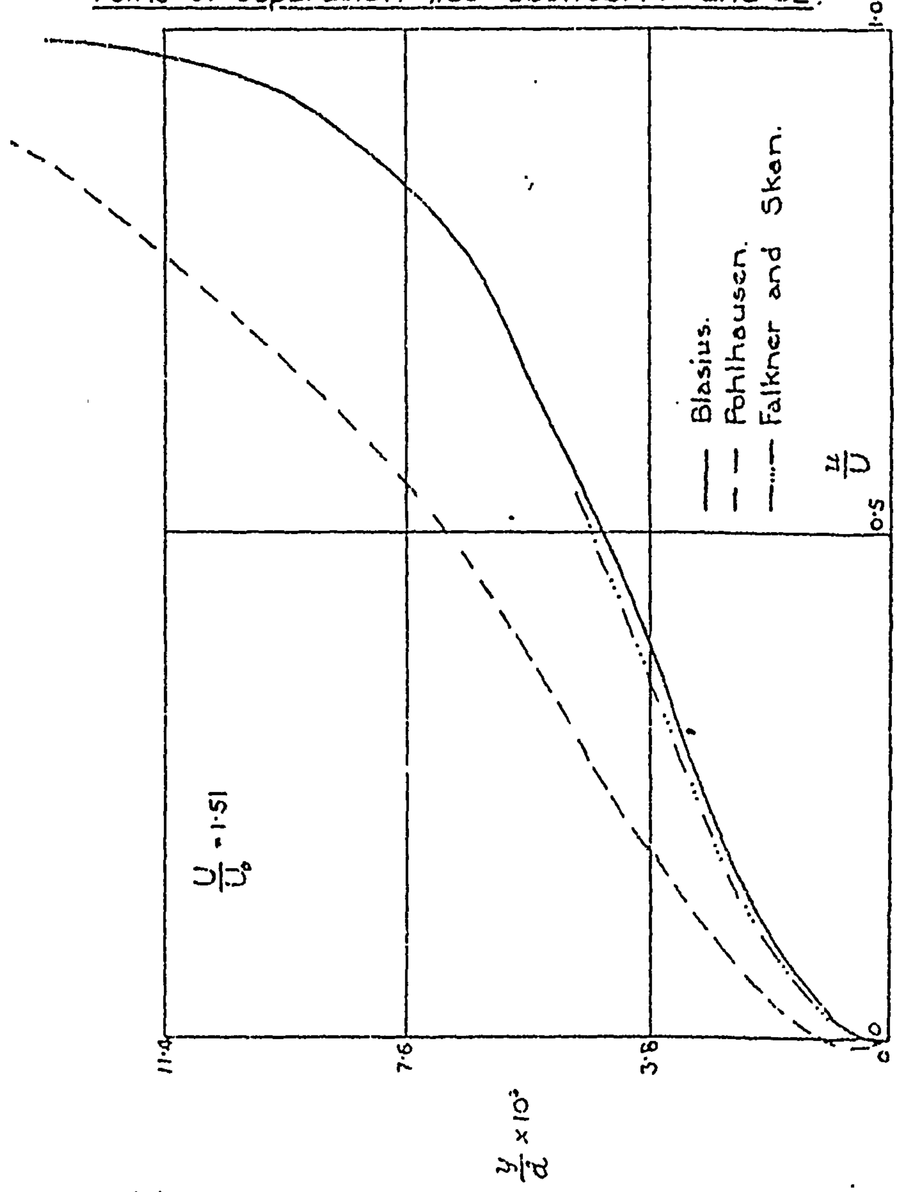
Section at x = 6 (70·56° from the forward stagnation point)



R.&M. 1632.

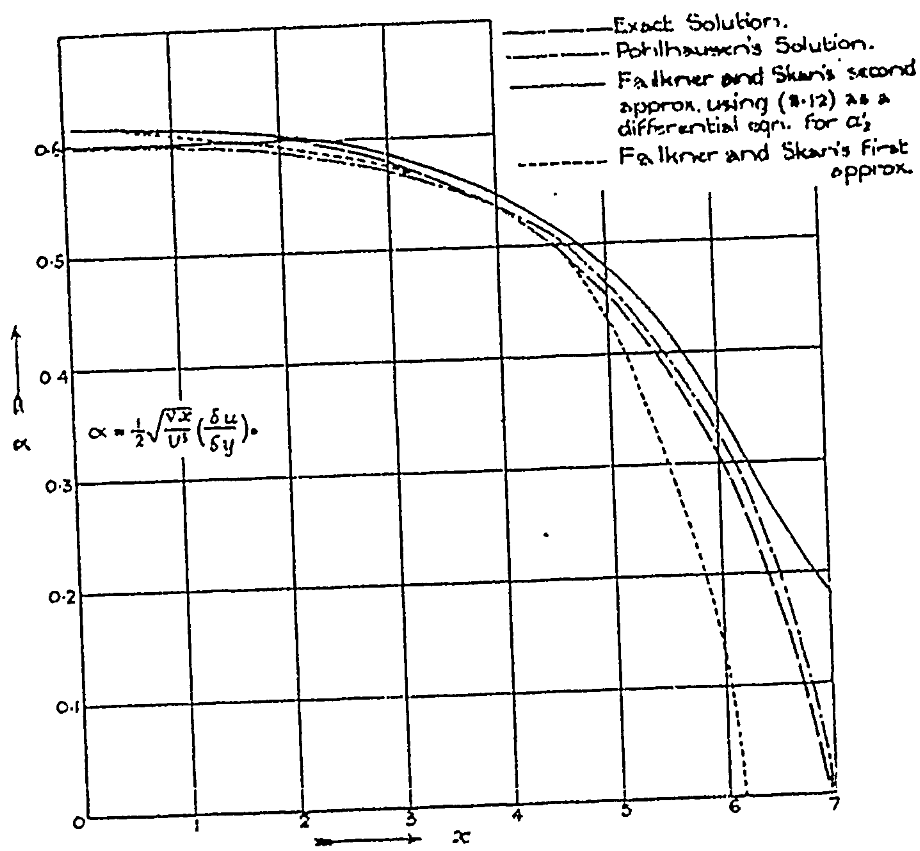
Fig. 6.

Section at x-7 (82·32° from forward stagnation point)
Point of separation lies between 80 and 82°.



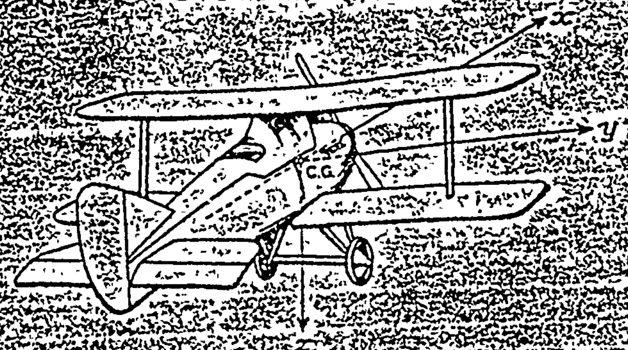
R. & M. 1632.

FIG. 7.



Exact Solution.
Pohlhausen's Solution.
Falkner and Skan's second approx. using (8·12) as a differential eqn. for α_2
Falkner and Skan's First approx.
0·6
0·5
0·4
0·3
0·2
0·1
0
1
2
3
4
5
6
7
α
x
$\alpha = \frac{1}{2}\sqrt{\frac{\nu x}{U^3}}\left(\frac{\delta u}{\delta y}\right)_0$

SYSTEM OF AXES.



Axes	Symbol Designation	x longitudinal	y lateral	z normal
	Positive direction	forward	starboard	downward
Force	Symbol	X	Y	Z
Moment	Symbol Designation	L rolling	M pitching	N yawing
Angle of Rotation	Symbol	ϕ	θ	ψ
Velocity	Linear	u	v	w
	Angular	p	q	r
Moment of Inertia		A	B	C

Components of linear velocity and force are positive in the positive direction of the corresponding axes. Components of angular velocity and moment are positive in the cyclic order y to z about the axis of x, z to x about the axis of y, and x to y about the axis of z.

The angular movement of a control surface (elevator or rudder) is governed by the same convention, the elevator angle being positive downwards and the rudder angle positive to port. The aileron angle is positive when the starboard aileron is down and the port aileron is up. A positive control angle normally gives rise to a negative moment about the corresponding axis. The symbols for the control angles are :—

ξ aileron angle
η elevator angle
η_T tail setting angle
ζ rudder angle

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Vol. II. Stability and Control, Spinning, Materials, Engines, etc.	37	6
TECHNICAL REPORT of the Aeronautical Research Committee for the year 1931–32, with Appendices—		
Vol. I. Aerodynamics, Airships and Meteorology	35	0
Vol. II. Stability and Control, Spinning, Strength of Construction, Materials, etc.	27	6
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List B.—Revised to March 31, 1933.
B.1. Air Ministry Publications.
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