RADIATION RESISTANCE AND EFFICIENCY OF MULTITURN LOOP ANTENNAS

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Investigation of High Frequency Aircraft Antennas

Subject of Report Radiation Resistance and Efficiency of Multiturn Loop Antennas

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ABSTRACT

Details are presented of an approximate analysis which yields expressions for the components of the distant electric field, radiation resistance, loss resistance and efficiency of a small, circular loop antenna. The equations for radiation resistance and loss resistance are then generalized to the case of a small multiturn loop antenna of arbitrary shape. The basic assumption used in the analysis is that the current is sinusoidally distributed on the antenna.

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RADIATION RESISTANCE AND EFFICIENCY OF MULTITURN LOOP ANTENNAS

I. INTRODUCTION

Because of space limitations, an aircraft antenna at HF must be relatively small electrically. This generally implies an undesirable impedance characteristic and low efficiency. A new HF antenna system has been proposed[1] in which an attempt is made to overcome these problems by purposely exciting currents on the aircraft structure in order to make use of a larger radiating surface. Currents on a conducting surface are associated with an external tangential magnetic field that is maximum at the conducting surface. Such a magnetic field can generally be established more effectively by a loop element than by a stub or dipole element. Hence a loop antenna has been chosen as the basic element of the new HF antenna system.

The small single-turn loop antenna is too inefficient to be used as the basic radiating element in a feasible HF antenna system. However, the efficiency of the basic loop element can be greatly enhanced by increasing the number of turns.

The small multiturn loop antenna has turns which are electrically small in area but sufficient in number such that the total wire length may be a significant portion of a wavelength. The geometry of the multiturn loop antenna is illustrated in Fig. 1.

The basic assumption used by most authors[2,3] in the analysis of the small single-turn loop antenna is that the current on the loop is uniform and in-phase at all points on the loop. In the case of the multiturn loop, the assumption of uniform in-phase current is not valid if the number of turns is such that the total length of wire in the loop is a significant portion of a wavelength. For this reason, the results of the usual analyses of small loop antennas are useless for predicting the behavior of the type of small multiturn loop antenna discussed here.

A more realistic description of the multiturn loop antenna is obtained if the current on the loop is assumed to be a superposition of two oppositely directed uniform traveling-wave currents of equal amplitude (i.e., sinusoidal current distribution). An analysis based on this assumed form of the current distribution has been performed and the far-zone electric field, radiation resistance, loss resistance, and efficiency have been calculated. The details of this analysis are the subject of this report.

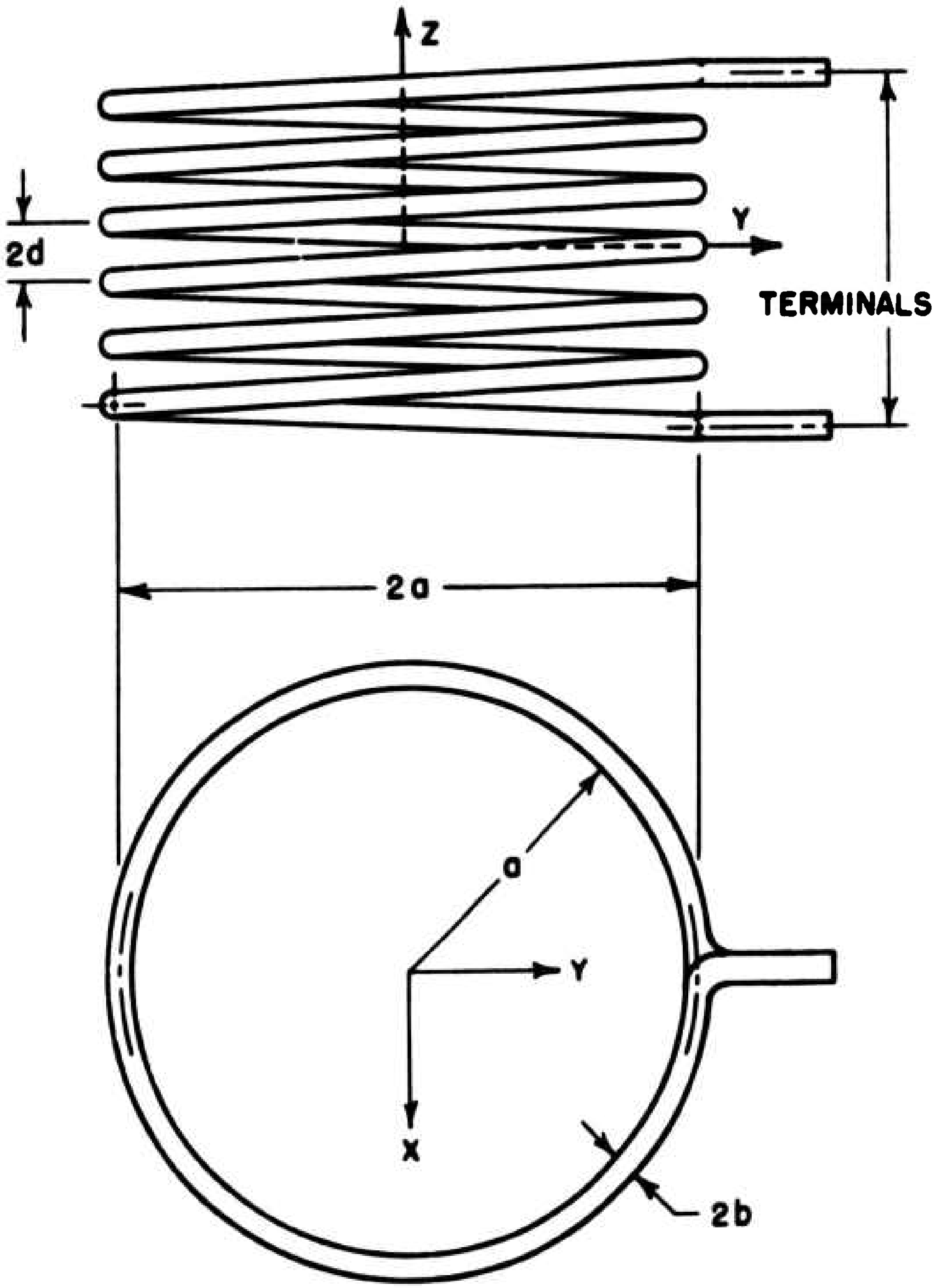

Fig. 1. Geometry of a multiturn loop antenna.

II. ANALYSIS

A. Distant Electric Fields

The far-zone magnetic field from a ring source is given by[4]

$$\overline{H} = \frac{jka}{4\pi r} e^{-jkr} \int \overline{J}(\phi') \times \hat{r}' \, e^{jka \sin\theta \cos(\phi-\phi')} d\phi' \quad . \tag{1}$$

For a current distribution of the form $\overline{J}(\phi') = J(\phi)\hat{\phi}'$, it is found, with the usual far-field approximations, that

$$\overline{J}(\phi') \times \hat{r}' \simeq \overline{J}(\phi') \times \hat{r} = J(\phi')\hat{\phi}' \times \hat{r}$$

or

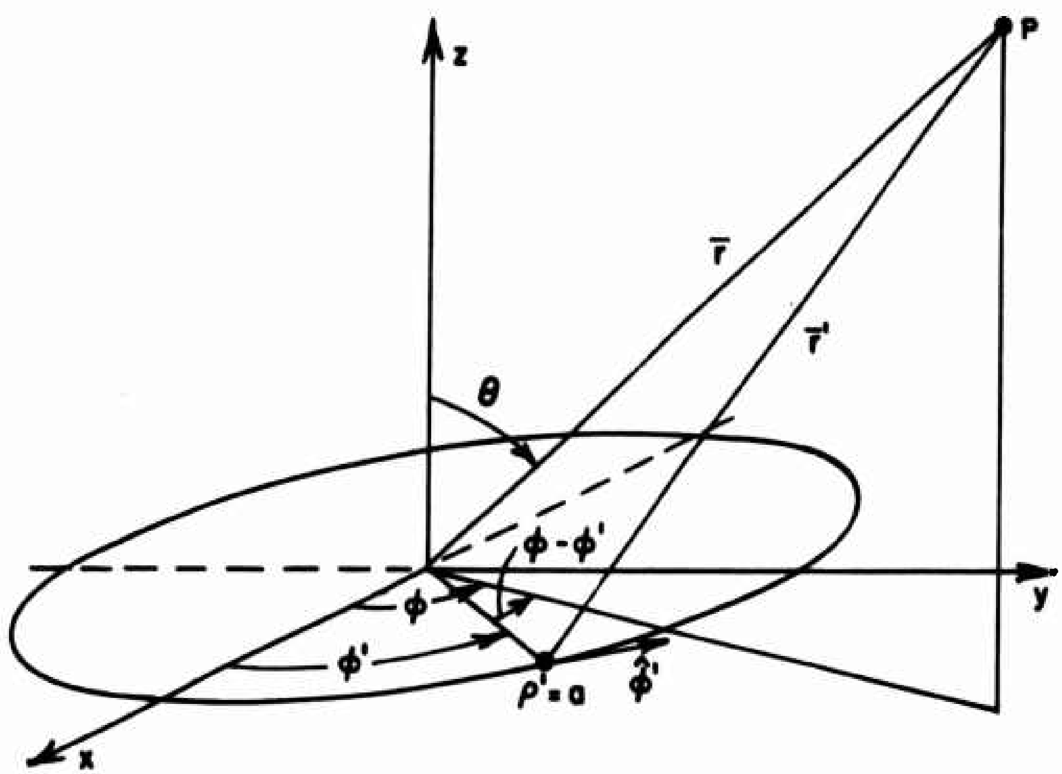

Fig. 2. Geometry of a ring source.

(2) $$\overline{J}(\phi') \times \hat{r}' \simeq J(\phi')\,[\hat{\theta}\cos(\phi-\phi') - \hat{\phi}\cos\theta\sin(\phi-\phi')] \quad .$$

Substituting Eq. (2) into Eq. (1) and introducing the notation $z = ka\sin\theta$ gives

(3) $$\overline{H} = \frac{jka}{4\pi r}\,e^{-jkr}\int J(\phi')\,[\hat{\theta}\cos(\phi-\phi') - \hat{\phi}\cos\theta\sin(\phi-\phi')]\,e^{jz\cos(\phi-\phi')}d\phi' \; .$$

The far-zone electric field is given by

$$E_\theta = \frac{k}{\omega\mu} H_\phi \; , \quad E_\phi = -\frac{k}{\omega\mu} H_\theta$$

or

(4) $$E_\theta = -\frac{j\omega\mu a}{4\pi r}\,e^{-jkr}\cos\theta \int J(\phi')\sin(\phi-\phi')e^{jz\cos(\phi-\phi')}d\phi' \; ,$$

and similarly

(5) $$E_\phi = -\frac{j\omega\mu a}{4\pi r}\,e^{-jkr}\int J(\phi')\cos(\phi-\phi')e^{jz\cos(\phi-\phi')}\,d\phi' \quad .$$

It is assumed that the multiturn loop antenna is fed from a balanced source. Thus, the current along the wire will be symmetrical about the midpoint of the loop. With the midpoint of the wire in the loop located at $\phi' = 0$, (see Fig. 3), the current distribution is assumed to be

$$I(\phi') = I_o \cos ka\phi'$$

where

(6) $$-(2N+1)\pi \leq \phi' \leq (2N+1)\pi$$

The limits for ϕ' in Eq. (6) correspond to a multiturn loop antenna consisting of (2N+1) turns.

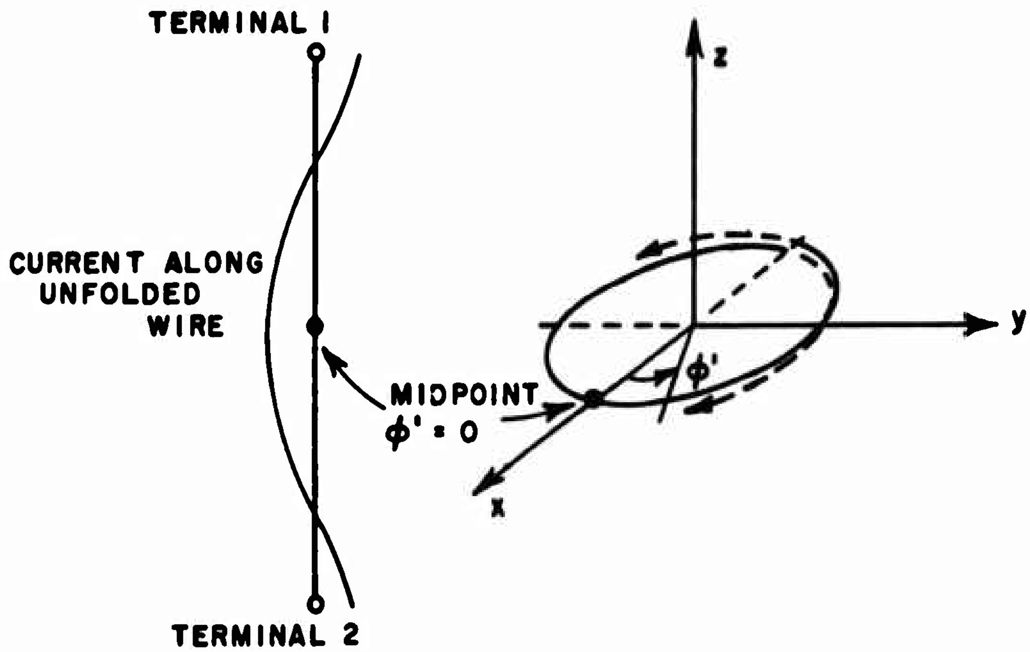

Fig. 3. Current distribution for a multiturn loop antenna.

Alternatively Eq. (6) can be written as

$$I(\phi') = \frac{I_o}{2}\left[e^{jka\phi'} + e^{-jka\phi'}\right], \quad -(2N+1)\pi \leq \phi' \leq (2N+1)\pi \tag{7}$$

i.e., consisting of a left and a right traveling wave. In the following, the left and right traveling waves will be kept separated throughout the calculations. Thus, substituting the left traveling wave from Eq. (7) into Eq. (4) gives

$$E_\theta^L = \frac{j\omega\mu a I_o}{8\pi r}\, e^{-jkr} \cos\theta \int e^{jka\phi'} \sin(\phi'-\phi)\, e^{jz\cos(\phi'-\phi)}\, d\phi' \ .$$

Substituting $\phi' - \phi = \phi'' + \frac{\pi}{2}$ yields

$$E_\theta^L = \frac{j\omega\mu a I_o}{8\pi r}\, e^{-jkr} e^{jka\left(\phi + \frac{\pi}{2}\right)} \cos\theta \int e^{jka\phi''} \cos\phi''\, e^{-jz\sin\phi''}\, d\phi'' \ . \tag{8}$$

The limits for the integral in Eq. (8) are such that the total coil is "scanned". More specificly, the integral in Eq. (8) may be written as the following sum (where the integrand is $e^{jk_o\phi''} \cos\phi'' e^{-jz\sin\phi''} d\phi''$):

$$(9) \qquad \int e^{jka\phi''} \cos\phi'' e^{-jz\sin\phi''} d\phi'' =$$

$$\left[\int_{L_1(\phi)}^{-(2N+1)\pi} + \int_{-(2N+1)\pi}^{-(2N-1)\pi} + \cdots\right.$$

$$\left. \cdots + \int_{-3\pi}^{-\pi} + \int_{-\pi}^{\pi} + \int_{\pi}^{3\pi} + \cdots \int_{(2N+1)\pi}^{(2N+1)\pi} - \int_{(2N+1)\pi}^{L_2(\phi)}\right] .$$

The first and last integral in Eq. (9) will have limits depending on the position of the field point P, i.e., will depend on ϕ. However, since their sum corresponds to the field from a portion of a wire smaller than 1/2 loop and since the present problem deals with the multiturn loop, the contribution is assumed to be negligible and will be ignored in the following.

The "center" integral in Eq. (9) may be written as

$$(10) \qquad \int_{-\pi}^{\pi} e^{jka\phi''} \cos\phi'' e^{-jz\sin\phi''} d\phi''$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \left[e^{j(ka+1)\phi''} + e^{j(ka-1)\phi''}\right] e^{-jz\sin\phi''} d\phi'' .$$

From the definition of Anger functions (see Appendix for details) it is seen that

$$(11) \qquad \int_{-\pi}^{\pi} e^{jka\phi''} \cos\phi'' e^{-jz\sin\phi''} d\phi'' = \pi \left[\mathbf{J}_{ka+1}(z) + \mathbf{J}_{ka-1}(z)\right] .$$

As shown in the Appendix, making the simple transformation $\phi'' = \phi''' + \pi$ gives

$$(12) \qquad \int_{\pi}^{3\pi} e^{jka\phi''} \cos\phi'' e^{-jz\sin\phi''} d\phi'' = \pi\, e^{j2\pi ka}\left[\mathbb{J}_{ka+1}(z) + \mathbb{J}_{ka-1}(z) \right],$$

and in general

$$(13) \qquad \int_{(2n-1)\pi}^{(2n+1)\pi} e^{jka\phi''} \cos\phi'' e^{-jz\sin\phi'} = \pi\, e^{jn2\pi ka}\left[\mathbb{J}_{ka+1}(z) + \mathbb{J}_{ka-1}(z) \right].$$

By application of Eq. (13) it is found for Eq. (9) that

$$(14) \qquad \int e^{jka\phi''} \cos\phi'' e^{-jz\sin\phi''} d\phi'' = \pi \left[\mathbb{J}_{ka+1}(z) + \mathbb{J}_{ka-1}(z) \right]$$

$$\cdot \left[e^{-jN2\pi ka} + \text{---} + e^{-j2\pi ka} + 1 + e^{j2\pi ka} + \text{----} + e^{N2\pi ka} \right]$$

$$= \pi \left[\mathbb{J}_{ka+1}(z) + \mathbb{J}_{ka-1}(z) \right] \frac{\sin\left[(2N+1)\, ka\pi\right]}{\sin ka\pi} \quad .$$

Substituting Eq. (14) into Eq. (8) yields

$$(15) \qquad E_{\theta}^{L} = \frac{j\omega\mu a I_o}{8r} e^{-jkr} e^{jka\left(\phi+\frac{\pi}{2}\right)} \cos\theta \quad \frac{\sin\left[(2N+1)ka\pi\right]}{\sin ka\pi}$$

$$\left[\mathbb{J}_{ka+1}(z) + \mathbb{J}_{ka-1}(z) \right] \quad .$$

Similarly, for the right traveling wave

$$(16) \qquad E_{\theta}^{R} = \frac{j\omega\mu a I_o}{8\pi r} e^{-jkr} \cos\theta \int e^{-jka\phi'} \sin(\phi'-\phi)\, e^{jz\cos(\phi'-\phi)} d\phi' .$$

By substituting $\phi' - \phi = -\phi'' + \frac{\pi}{2}$ in the integral of Eq. (16) there results

$$(17)\qquad \int e^{-jka\phi'} \sin(\phi'-\phi)\, e^{jz\cos(\phi'-\phi)}\, d\phi'$$

$$= e^{-jka\left(\phi+\frac{\pi}{2}\right)} \int e^{jka\phi'} \cos\phi'\, e^{jz\sin\phi'}\, d\phi' ,$$

where the direction of integration is from negative to positive values of ϕ'. Thus, since the only difference between the integral in Eqs. (17) and (14) is the sign of z, it is readily seen that for Eq. (17)

$$(18)\qquad \int e^{-jka\phi'} \sin(\phi'-\phi) e^{jz\cos(\phi'-\phi)}\, d\phi'$$

$$= e^{-jka\left(\phi+\frac{\pi}{2}\right)} \pi \left[\mathbb{J}_{ka+1}(-z) + \mathbb{J}_{ka-1}(-z) \right] \cdot \frac{\sin[(2N+1)ka\pi]}{\sin ka\pi} .$$

Substituting Eq. (18) into Eq. (16) gives

$$(19)\qquad E_\theta^R = \frac{j\omega\mu a I_o}{8r}\, e^{-jkr}\, e^{-jka\left(\phi+\frac{\pi}{2}\right)} \cos\theta\, \frac{\sin[(2N+1)ka\pi]}{\sin ka\pi}$$

$$\cdot \left[\mathbb{J}_{ka+1}(-z) + \mathbb{J}_{ka-1}(-z) \right] .$$

Equations (15) and (19) give , for the cosinusoidal (standing) wave:

$$(20)\qquad E_\theta^S = \frac{j\omega\mu a I_o}{8r}\, e^{-jkr} \cos\theta\, \frac{\sin[(2N+1)ka\pi]}{\sin ka\pi}$$

$$\cdot \left[e^{jka\left(\phi+\frac{\pi}{2}\right)} \left(\mathbb{J}_{ka+1}(z) + \mathbb{J}_{ka-1}(z) \right) \right.$$

$$\left. + e^{-jka\left(\phi+\frac{\pi}{2}\right)} \left(\mathbb{J}_{ka+1}(-z) + \mathbb{J}_{ka-1}(-z) \right) \right] . \qquad -\pi < \phi < \pi$$

For small loops i.e., $a << \lambda$ the phase term can be ignored and by formulas in the Appendix, Eq. (20) eventually reduces to:

$$E_{\theta}^{S} \simeq \frac{j\omega\mu a I_o}{4r} e^{-jkr} \frac{\sin[(2N+1)ka\pi]}{\sin ka\pi} \cot g\theta \left[\mathbf{J}_{ka}(z) - \mathbf{J}_{-ka}(z)\right] \quad a << \lambda . \tag{21}$$

The expansion of $\left[\mathbf{J}_{ka}(z) - \mathbf{J}_{-ka}(z)\right]$ is given in the Appendix.

Substituting the left traveling wave $\frac{I_o}{2} e^{jka\phi'}$ into Eq. (5) yields

$$E_{\phi}^{L} = -\frac{j\omega\mu a I_o}{8\pi r} e^{-jkr} \int e^{jka\phi'} \cos(\phi'-\phi) e^{jz\cos(\phi'-\phi)} d\phi' , \tag{22}$$

where as before the integration is extended over all turns of the loop. Substituting

$$\phi' - \phi = \phi'' + \frac{\pi}{2}$$

into the integral of Eq. (22) gives

$$\int e^{jka\phi'} \cos(\phi'-\phi) e^{jz\cos(\phi'-\phi)} d\phi' \tag{23}$$

$$= - e^{jka\left(\phi + \frac{\pi}{2}\right)} \int e^{jka\phi''} \sin\phi'' e^{-jz\sin\phi''} d\phi''$$

$$= - e^{+jka\left(\phi + \frac{\pi}{2}\right)} \left[\int_{L_1(\phi)}^{-(2N+1)\pi} + \int_{-(2N+1)\pi}^{-(2N-1)\pi} + \cdots \int_{-3\pi}^{-\pi} \right.$$

$$\left. + \int_{-\pi}^{\pi} + \int_{\pi}^{3\pi} + \cdots + \int_{(2N-1)\pi}^{(2N+1)\pi} - \int_{(2N+1)\pi}^{L_2(\phi)} \right] .$$

As shown in the Appendix

$$(24)\qquad \int_{(2n-1)\pi}^{(2n+1)\pi} e^{jka\phi''} \sin\phi''\, e^{-jz\sin\phi''}\, d\phi''$$

$$= -j\, e^{jn2\pi ka}\, \pi \left[\mathbb{J}_{ka+1}(z) - \mathbb{J}_{ka-1}(z) \right] .$$

Application of Eq. (24) in Eq. (23) yields, by neglecting the terminal contribution as previously,

$$(25)\qquad \int e^{jka\phi'} \cos(\phi'-\phi)\, e^{jz\cos(\phi'-\phi)}\, d\phi'$$

$$= j\, e^{jka\left(\phi + \frac{\pi}{2}\right)}\, \pi\, \frac{\sin[(2N+1)ka\pi]}{\sin ka\pi} \left[\mathbb{J}_{ka+1}(z) - \mathbb{J}_{ka-1}(z) \right] .$$

Finally, substituting Eq. (25) into Eq. (22) gives

$$(26)\qquad E_\phi^L = \frac{\omega\mu a I_o}{8r}\, e^{-jkr}\, e^{jka\left(\phi + \frac{\pi}{2}\right)}\, \frac{\sin[(2N+1)ka\pi]}{\sin ka\pi} \left[\mathbb{J}_{ka+1}(z) - \mathbb{J}_{ka-1}(z) \right]$$

or by application of the recursive relationship given in the Appendix,

$$(26)\qquad E_\phi^L = -\frac{\omega\mu a I_o}{4r}\, e^{-jkr}\, e^{jka\left(\phi + \frac{\pi}{2}\right)}\, \frac{\sin[(2N+1)ka\pi]}{\sin ka\pi}\, \mathbb{J}'_{ka}(z) .$$

For the right traveling wave $\frac{I_o}{2}\, e^{-jka\phi'}$, Eq. (5) gives

$$(27)\qquad E_\phi^R = -\frac{j\omega\mu a I_o}{8\pi r}\, e^{-jkr} \int e^{-jka\phi'} \cos(\phi'-\phi)\, e^{jz\cos(\phi'-\phi)}\, d\phi'$$

where the integration is extended over the whole multiturn loop. Substituting

$$\phi' - \phi = -\phi'' + \frac{\pi}{2}$$

into Eq. (27) yields for the integral alone

$$(28) \qquad \int e^{-jka\phi'} \cos(\phi'-\phi)\, e^{jz\cos(\phi'-\phi)} d\phi'$$

$$= e^{-jka\left(\phi + \frac{\pi}{2}\right)} \int e^{jka\phi''} \sin\phi''\, e^{jz\sin\phi''} d\phi'' ,$$

where the integration goes from negative to positive values of ϕ''. Thus by application of Eqs. (23) and (24) in Eq. (28),

$$(29) \qquad \int e^{-jka\phi'} \cos(\phi'-\phi)\, e^{jz\cos(\phi'-\phi)}\, d\phi'$$

$$= -j\, e^{-jka\left(\phi + \frac{\pi}{2}\right)} \frac{\sin[(2N+1)ka\pi]}{\sin ka\pi} \left[\mathbb{J}_{ka+1}(-z) - \mathbb{J}_{ka-1}(-z) \right] .$$

Substituting Eq. (29) into Eq. (27) yields:

$$E_\phi^R = -\frac{\omega\mu a I_o}{8r} e^{-jkr} e^{-jka\left(\phi + \frac{\pi}{2}\right)} \frac{\sin[(2N+1)ka\pi]}{\sin ka\pi} \left[\mathbb{J}_{ka+1}(-z) - \mathbb{J}_{ka-1}(-z) \right]$$

and finally by application of the recursive relationship given in the Appendix,

$$(30) \qquad E_\phi^R = \frac{\omega\mu a I_o}{4r} e^{-jkr} e^{-jka\left(\phi + \frac{\pi}{2}\right)} \frac{\sin[(2N+1)ka\pi]}{\sin ka\pi} \mathbb{J}'_{ka}(-z) .$$

From Eqs. (26) and (30), the total field for the cosinosoidal (standing wave) distribution is

$$(31) \quad E_\phi^S = -\frac{\omega\mu a I_o}{4r} e^{-jkr} \frac{\sin[(2N+1)ka\pi]}{\sin ka\pi} \left[e^{jka\left(\phi + \frac{\pi}{2}\right)} \mathbb{J}'_{ka}(z) + e^{-jka\left(\phi + \frac{\pi}{2}\right)} \mathbb{J}'_{-ka}(z) \right] . \qquad -\pi < \phi < \pi$$

For $a << \lambda$ Eq. (31) reduces to

$$E_{\phi}^{S} \simeq -\frac{\omega\mu a I_{o}}{4r} e^{-jkr} \frac{\sin(2N+1)ka\pi}{\sin ka\pi}\left[\mathbf{J}'_{ka}(z) + \mathbf{J}'_{-ka}(z)\right] . \tag{32}$$

The expansion of $\left[\mathbf{J}'_{ka}(z) + \mathbf{J}'_{-ka}(z)\right]$ is given in the Appendix.

Equations (20) and (31) give the total θ-polarized and ϕ-polarized components of the far-zone electric field of a multiturn loop antenna of arbitrary size. These equations reduce to Eqs. (21) and (32) for small multiturn loop antennas, i.e., loops for which $a << \lambda$.

B. Radiation Resistance R_R

In regard to calculating the radiated power it is interesting to note that E_{θ}^{S} and E_{ϕ}^{S} are in phase quadrature. Thus, no cross-products between the two fields occur, and calculations can be performed very simply. In the present work, special attention will be paid to the case where $a << \lambda$. From the expressions for E_{θ}^{S} and E_{ϕ}^{S} in Eqs. (21) and (32), respectively, and utilizing the expansions Eqs. (82) and (84), it is clear that in this case

$$E_{\theta}^{S} << E_{\phi}^{S} ,$$

due to the sin $ka\pi/2$ and cos $ka\pi/2$ factors. Thus, in the following, only the radiated power due to E_{ϕ}^{S} will be calculated.

Equations (32) and (84) yield

$$E_{\phi}^{S} = -\frac{\omega\mu a I_{o}}{2r} e^{-jkr} \frac{\sin(2N+1)ka\pi}{\sin ka\pi} \cos\frac{ka\pi}{2} \sum_{m=1}^{\infty} \frac{(-1)^{m} m\left(\frac{z}{2}\right)^{2m-1}}{\Gamma\left(m+1-\frac{ka}{2}\right)\Gamma\left(m+1+\frac{ka}{2}\right)} . \tag{33}$$

Taking only the first term in Eq. (33) and using the approximations $\cos ka\pi/2 \simeq 1$ and $\Gamma(2 - ka/2)\,\Gamma(2 + ka/2) \simeq 1$ gives

$$(34) \qquad \frac{|E_\phi^S|^2}{2Z_o} \simeq \frac{1}{2}\,\frac{\omega^2\mu^2a^2\,I_o^2}{4r^2\,Z_o}\,\frac{\sin^2(2N+1)ka\pi}{\sin^2 ka\pi}\,\frac{z^2}{4} \quad ,$$

and by substituting $z = ka\sin\theta$, Eq. (34) becomes

$$(35) \qquad \frac{|E_\phi^S|^2}{Z_o} = \frac{1}{2}\,\frac{(ak)^4 I_o^2}{16r^2}\,Z_o\,\frac{\sin^2(2N+1)ka\pi}{\sin^2 ka\pi}\,\sin^2\theta \ .$$

The total radiated power is

$$(36) \qquad P_R = \int_0^{2\pi} d\phi \int_0^{\pi} \frac{|E_\phi^S|^2}{2Z_o}\, r^2 \sin\theta\, d\theta$$

$$= \frac{1}{2}\,\frac{\pi(ak)^4 I_o^2}{8}\,Z_o\,\frac{\sin^2(2N+1)ka\pi}{\sin^2 ka\pi} \int_0^{\pi} \sin^3\theta\, d\theta$$

$$P_R = \frac{\pi}{12}\,(ak)^4 I_o^2\, Z_o\,\frac{\sin^2(2N+1)ka\pi}{\sin^2 ka\pi} \quad .$$

From the definition of the radiation resistance, R_R°, with respect to a current maximum, i.e.,

$$(37) \qquad P_R = \frac{1}{2}\, R_R^\circ\, I_o^2 \ ,$$

and using Eq. (36) in Eq. (37), we obtain

$$R_R^\circ = \frac{\pi}{6}\,(ak)^4\, Z_o\,\frac{\sin^2(2N+1)ka\pi}{\sin^2 ka\pi}$$

or alternatively

$$(38)\qquad R_R^{*} = \frac{1}{2}\, 640\, \pi^6 \left(\frac{a}{\lambda}\right)^4 \frac{\sin^2(2N+1)ka\pi}{\sin^2 ka\pi}$$

$$(39)\qquad = \frac{64\pi^6}{2.81}\, 10^{-7}\, f_{MHz}^4\, a^4\, \frac{\sin^2(2N+1)ka\pi}{\sin^2 ka\pi}$$

$$= \frac{32\pi^6 \cdot 10^{-7}}{81}\, f_{MHz}^4\, a^4\, \frac{\sin^2(2N+1)ka\pi}{\sin^2 ka\pi} .$$

with respect to current maximum I_o

Since $a << \lambda$, $\sin ka\pi \simeq ka\pi$. Thus, from Eq. (39)

$$(40)\qquad R_R^{*} = \frac{8}{9}\,\pi^2 \cdot 10^{-3}\, f_{MHz}^2\, a^2 \sin^2(2N+1)ka\pi$$ (with respect to current maximum I_o)

Since the input current according to Eq. (6) is

$$(41)\qquad I_{Terminals} = I_o \cos(2N+1)ka\pi ,$$

it is found from Eqs. (40) and (41) that the input resistance is

$$(42)\qquad R_R^{inp} = \frac{R_R^{*}}{\cos^2(2N+1)ka\pi} = \frac{8}{9}\,\pi^2 \,.10^{-3} f_{MHz}^2\, a^2 \tan^2(2N+1)ka\pi$$

(with respect to input current)

C. Loss Resistance R_L and Efficiency Eff.

Denote the conductor resistance per unit length by R_o. Then the total power lost in the multiturn loop antenna is

$$P_L = \frac{1}{2}\int_{-(2N+1)\pi}^{(2N+1)\pi} R_o\, I^2(\phi')\, a\, d\phi'$$

$$= \frac{1}{2}\, R_o\, I_o^2 \int_{\phi' = -(2N+1)\pi}^{(2N+1)\pi} \cos^2 ka\phi'\, d\, a\phi'$$

$$P_L = \frac{R_o I_o^2}{4k} \left[ak\phi' + \frac{1}{2} \sin 2ak\phi' \right]_{\phi' = -(2N+1)\pi}^{(2N+1)\pi}$$

$$= \frac{1}{4} R_o I_o^2 \left[2(2N+1)\pi a + \frac{1}{k} \sin\left[(2N+1)2\pi ka\right] \right]$$

Thus

$$P_L = \frac{1}{4} R_o I_o^2 (2N+1)2\pi a \left[1 + \frac{\sin\left[(2N+1)2\pi ka\right]}{(2N+1)\, 2\pi ka} \right] \quad (43)$$

From the definition of loss resistance:

$$P_L = \frac{1}{2} R_L^{*} I_o^2$$

and from Eq. (43),

$$R_L^{*} = \frac{1}{2} R_o (2N+1)2\pi a \left[1 + \frac{\sin\left[(2N+1)2\pi ka\right]}{(2N+1)2\pi ka} \right] \quad (44)$$

with respect to current maximum .

For a copper wire, according to Schelkunoff,[5]

$$R_o = \frac{R_s}{\pi d_m} = \frac{2.61 \cdot 10^{-7} \sqrt{f}}{\pi d_m} \quad \text{ohm/m}$$

or

$$R_o = \frac{8.32 \cdot 10^{-8}}{d_m} \sqrt{f_{MHz}} \quad (d_m = \text{wire diameter in meter}). \quad (45)$$

Substituting Eq. (45) into Eq. (44):

$$(46) \qquad R_L^{\bullet} = \frac{4.16 \cdot 10^{-5}}{d_m} (2N+1)2\pi a \sqrt{f_{MHz}} \left[1 + \frac{\sin[(2N+1)2\pi ka]}{(2N+1)2\pi ka}\right]$$

(with respect to current maximum)

For the input current, as mentioned previously, (see Eq. (41))

$$I_{Terminals} = I_o \cos(2N+1) ka\pi \; .$$

Thus, for the loss resistance as seen at the terminals

$$(47) \qquad R_L = \frac{R_L^{\bullet}}{\cos^2(2N+1)ka\pi}$$

$$= \frac{4.16 \cdot 10^{-5}}{d_m} \; \frac{(2N+1)2\pi a \sqrt{f_{MHz}}}{\cos^2(2N+1)ka\pi} \left[1 + \frac{\sin[(2N+1)2\pi ka]}{(2N+1)2\pi ka}\right] \; .$$

(For a copper wire antenna)

The radiation efficiency of an antenna is customarily defined as[6]

$$(48) \qquad \text{Eff.} = \frac{P_R}{P_R + P_L} \; .$$

The radiated power and loss power are related to the radiation resistance, loss resistance and terminal current as follows:

$$(49) \qquad P_R = \frac{1}{2} R_R I^2$$

$$(50) \qquad P_L = \frac{1}{2} R_L I^2 \; .$$

Using Eqs. (49) and (50) in Eq. (48) yields

$$\text{Eff.} = \frac{\frac{1}{2} R_R I^2}{\frac{1}{2} R_R I^2 + \frac{1}{2} R_L I^2}$$

or

$$\text{Eff.} = \frac{R_R}{R_R + R_L} \quad . \tag{51}$$

When the values of R_R and R_L from Eqs. (42) and (47) are used in Eq. (51), the efficiency is obtained.

D. An Example Calculation

To illustrate the type of behavior predicted by Eqs. (42), (47), and (51) for the small multiturn loop antenna, a set of calculations has been made for a particular model. The parameters of the model are as follows.

$N = 2$ (5-turn loop)

$a = 0.2$ meters

$d_m = 0.00159$ meters (1/16 inch) .

The calculations were made for a model with the given parameters at frequencies between 0 and 150 MHz, and are presented in Figs. 4, 5, and 6. Figure 4 shows the radiation resistance and loss resistance separately, while Fig. 5 depicts the total input resistance. Measured input resistance for this antenna is shown in Fig. 6. Measured and calculated efficiency for the 5-turn loop is shown in Fig. 7. For a description of the impedance and efficiency measurements, see Reference 7.

E. Loops of Noncircular Shape

Equations (42) and (47) can be generalized to the case of noncircular loops by recognizing that

$$2\pi a = \text{perimeter of one turn of the loop in meters} = \rho_T$$

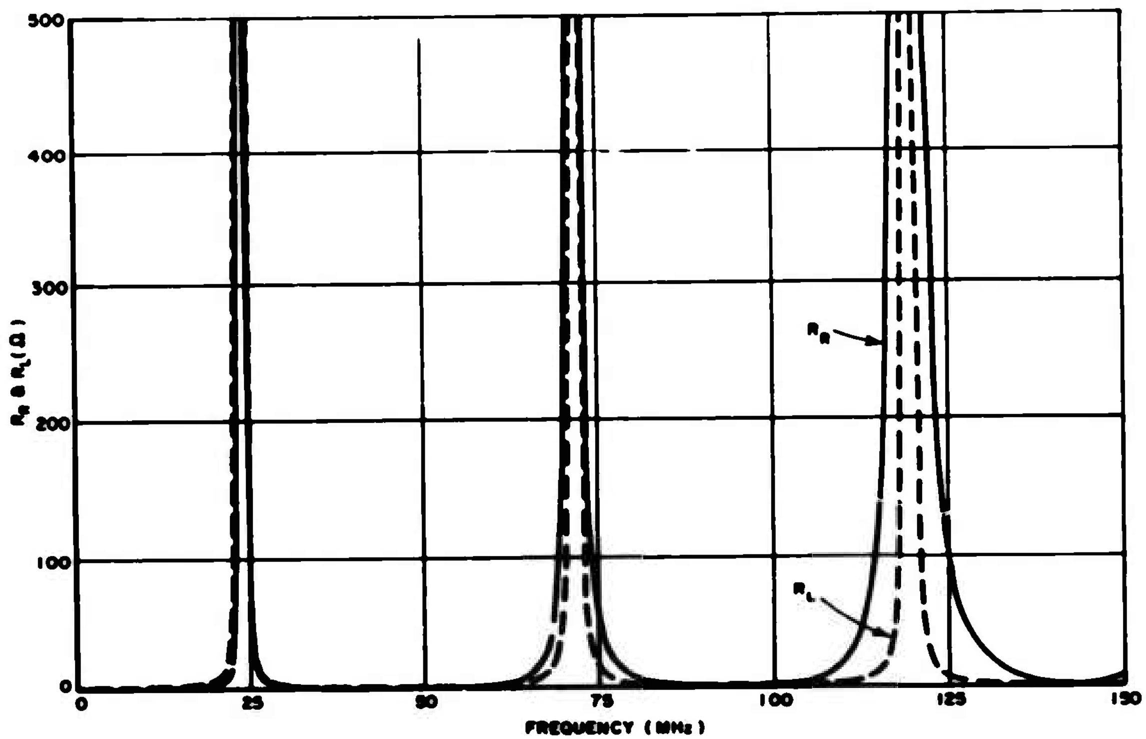

Fig. 4. Radiation resistance and loss resistance of a 5-turn loop antenna.

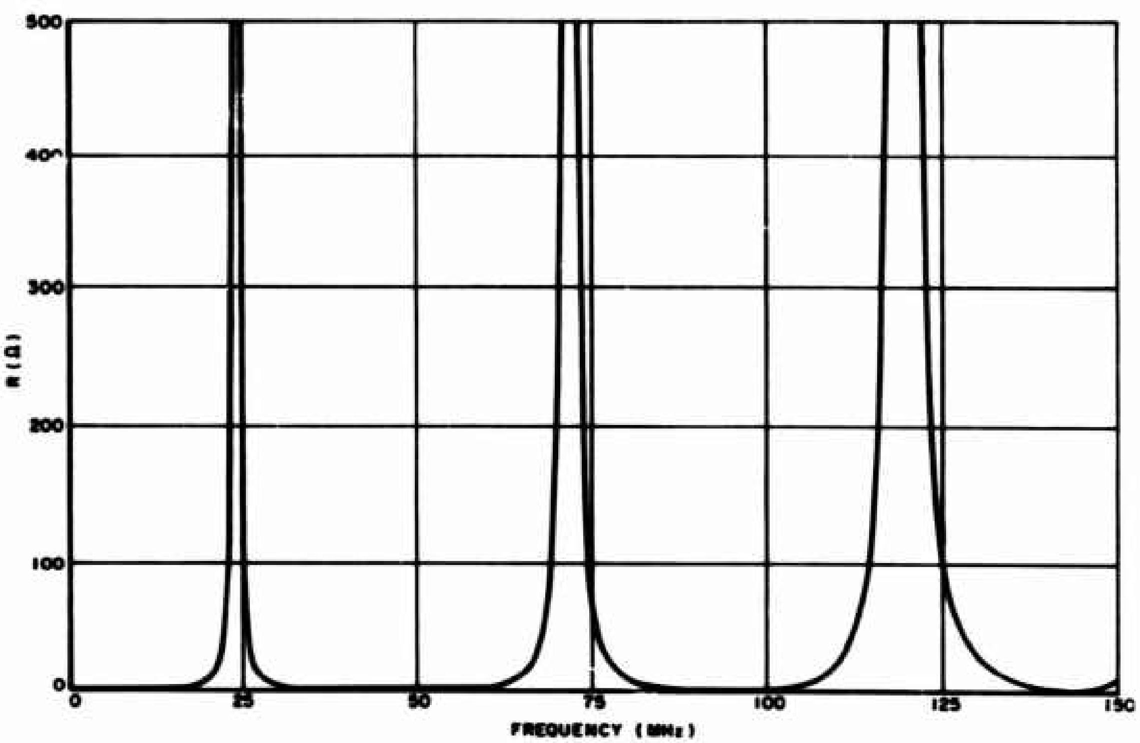

Fig. 5. Input resistance of a 5-turn loop antenna.

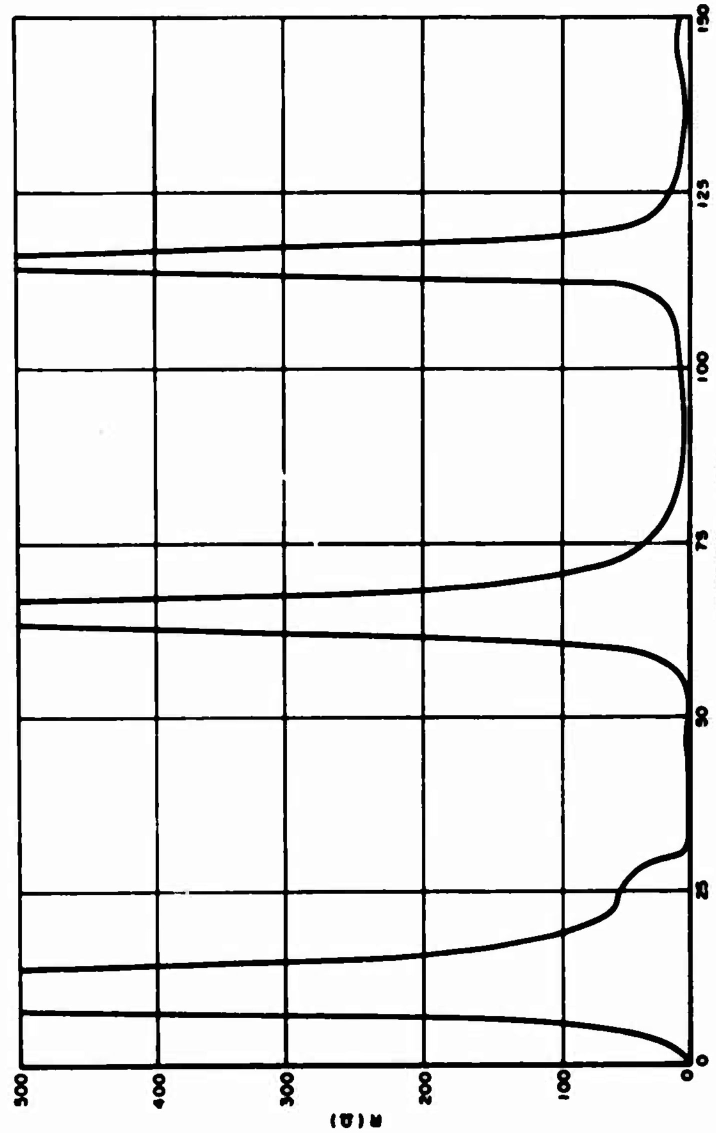

Fig. 6. Measured input resistance of a 5-turn loop antenna.

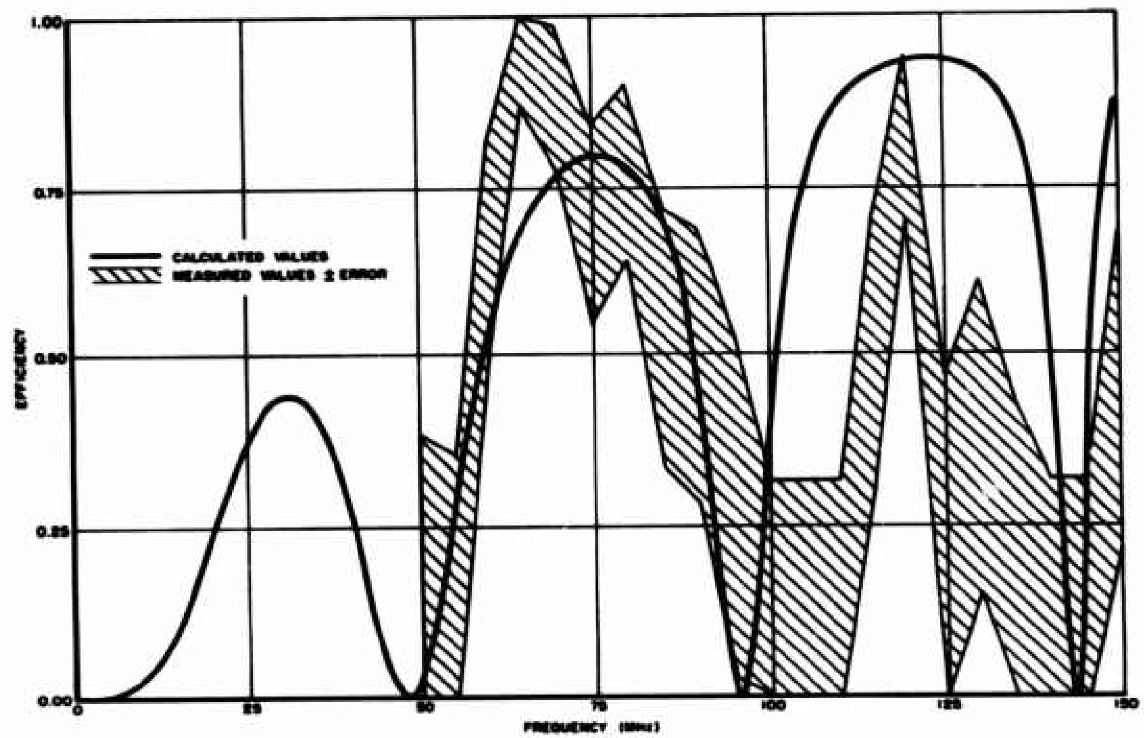

Fig. 7. Calculated and measured efficiency of a 5-turn loop antenna.

πd_m = perimeter of the conductor in meters = ρ_C

πa^2 = area of one turn of the loop in $(\text{meters})^2$ = A_T

$2N+1$ = number of turns = n .

If the antenna is constructed of some metal other than copper, the loss resistance is modified by the factor[8]

$$\sqrt{\frac{\mu_r}{\sigma_r}}$$

where σ_r is the relative conductivity of the metal and μ_r is the relative permeability of the metal, i.e.,

$$\sigma_r = \frac{\text{conductivity of metal}}{\text{conductivity of copper}}$$

$$\mu_r = \frac{\text{permeability of metal}}{\text{permeability of copper}} \quad .$$

When these generalizations are made in Eqs. (42) and (47), the results are

$$R_R = 2.79 \cdot 10^{-3}\, f^2_{MHz}\, A_T \tan\left(\frac{kn\rho_T}{2}\right)\ \Omega \tag{52}$$

$$R_L = 1.31.10^{-4}\, \frac{n\rho_T}{\rho_C}\, \frac{\sqrt{\dfrac{\mu_r f_{MHz}}{\sigma_r}}}{\cos^2\left(\dfrac{kn\rho_T}{2}\right)} \left[1 + \frac{\sin(kn\rho_T)}{kn\rho_T}\right]\ \Omega \tag{53}$$

where

ρ_T = perimeter of one turn in meters

ρ_C = perimeter of the conductor in meters

A_T = area of one turn in $(\text{meters})^2$

n = number of turns

k = free space propagation constant in radians/meter

f_{MHz} = frequency in MHz

σ_r = conductivity of loop material relative to that of copper

μ_r = permeability of loop material relative to that of copper.

III. SUMMARY AND CONCLUSIONS

The present work has dealt with a theoretical investigation of the multiturn loop antenna. Proceeding on the assumption that the current distribution on the loop antenna is sinusoidal, expressions were developed for the components of the far-zone electric field of a circular multiturn loop antenna of arbitrary size. These expressions were then specialized to the case of a multiturn loop antenna of electrically small radius and an approximate expression for antenna radiation resistance at the antenna terminals was obtained. The assumption of sinusoidal current distribution was also used in calculating the loss resistance at the antenna terminals, for an antenna constructed of copper wire. The radiation resistance and loss resistance were then used to calculate the antenna radiation efficiency.

A comparison of theoretical and experimental results for a circular 5-turn loop antenna demonstrated the validity of the analysis.

Finally, the expressions for radiation resistance and loss resistance were generalized to apply to small multiturn loop antennas of arbitrary shape and constructed of wire of arbitrary cross section and arbitrary material.

APPENDIX
FORMULAS FOR ANGER AND WEBER FUNCTIONS

Since the Anger and Weber functions are used in the previous calculations, it is pertinent to state here a few facts about those functions. The principal reference sources are: Watson,[9] Bateman,[10] Bernard and Ishimaru,[11] and Nielsen.[12]

Anger and Weber functions, respectively, are defined by

$$(54) \qquad \mathbf{J}_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\phi - z\sin\phi)d\phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\nu\phi - z\sin\phi)}d\phi$$

and

$$(55) \qquad \mathbf{E}_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(\nu\phi - z\sin\phi)d\phi \quad .$$

From Eqs. (54) and (55):

$$(56) \qquad \mathbf{J}_\nu(z) \pm i\,\mathbf{E}_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{\pm j(\nu\phi - z\sin\phi)}d\phi \quad .$$

For ν equal to an integer n, it is seen from Eq. (54) and the well known formula for Bessel functions[13]

$$J_n(z) = \frac{j^{-n}}{2\pi}\int e^{j(n\phi + z\cos\phi)}d\phi = \frac{1}{\pi}\int_0^\pi \cos(n\phi - z\sin\phi)d\phi$$

that

$$\mathbf{J}_n(z) \equiv J_n(z)$$

i.e., for integer order, Anger and Bessel functions are identical.

Some functional relationships for the Anger and Weber functions are

$$(57) \qquad \mathbf{J}_\nu(-z) = \mathbf{J}_{-\nu}(z)$$

(58) $$\mathbf{J}_{-\nu}(-z) = \mathbf{J}_{\nu}(z)$$

(59) $$\mathbf{E}_{\nu}(-z) = - \mathbf{E}_{-\nu}(z)$$

(60) $$\mathbf{E}_{-\nu}(-z) = - E_{\nu}(z)$$

(61) $$\sin \nu\pi \; \mathbf{J}_{\nu}(z) = \cos \nu\pi \, \mathbf{E}_{\nu}(z) - \mathbf{E}_{-\nu}(z)$$

(62) $$\sin \nu\pi \, \mathbf{E}_{\nu}(z) = \mathbf{J}_{-\nu}(z) - \cos \nu\pi \, \mathbf{J}_{\nu}(z)$$

From Eqs. (61) and (62), the following relationships may be derived:

(63) $$\mathbf{J}_{\nu}(z) - j \, \mathbf{E}_{\nu}(z) = e^{-j\nu\pi}[\mathbf{J}_{-\nu}(z) - j \, \mathbf{E}_{-\nu}(z)]$$

(64) $$\mathbf{J}_{\nu}(z) + j \, \mathbf{E}_{\nu}(z) = e^{j\nu\pi}[\mathbf{J}_{-\nu}(z) + j \, \mathbf{E}_{-\nu}(z)] \quad .$$

Recursive relationship for the Anger and Weber function are given by

(65) $$\mathbf{J}_{\nu-1}(z) - \mathbf{J}_{\nu+1}(z) = 2 \, \mathbf{J}'_{\nu}(z)$$

(66) $$z\mathbf{J}'_{\nu}(z) + \nu \, \mathbf{J}_{\nu}(z) = z \, \mathbf{J}_{\nu-1}(z) + \frac{\sin \nu\pi}{\pi}$$

(67) $$z \, \mathbf{J}'_{\nu}(z) - \nu \, \mathbf{J}_{\nu}(z) = - z \, \mathbf{J}_{\nu+1}(z) - \frac{\sin \nu\pi}{\pi}$$

(68) $$\mathbf{J}_{\nu-1}(z) + \mathbf{J}_{\nu+1}(z) = \frac{2\nu}{z} \, \mathbf{J}_{\nu}(z) - \frac{2\sin \nu\pi}{\pi z}$$

(69) $$\mathbf{E}_{\nu-1}(z) - \mathbf{E}_{\nu+1}(z) = 2 \, \mathbf{E}'_{\nu}(z)$$

(70) $$z \, \mathbf{E}'_{\nu}(z) + \nu \, \mathbf{E}_{\nu}(z) = z \, \mathbf{E}_{\nu-1}(z) + \frac{1-\cos \nu\pi}{\pi}$$

(71) $$z\,\mathbf{E}'_{\nu}(z) - \nu\,\mathbf{E}_{\nu}(z) = -z\,\mathbf{E}_{\nu+1}(z) - \frac{1-\cos\nu\pi}{\pi}$$

(72) $$\mathbf{E}_{\nu-1}(z) + \mathbf{E}_{\nu+1}(z) = \frac{2\nu}{z}\,\mathbf{E}_{\nu}(z) - \frac{2(1-\cos\nu\pi)}{\pi z}$$

The integral

$$\frac{1}{2\pi}\int_{(2n-1)\pi}^{(2n+1)\pi} e^{j(\nu\phi - z\sin\phi)}d\phi$$

may be evaluated by substituting

$$\phi = \phi' + 2\pi n$$

yielding

$$\frac{1}{2\pi}\int_{(2n-1)\pi}^{(2n+1)\pi} e^{j(\nu\phi - z\sin\phi)}d\phi = \frac{1}{2\pi}\,e^{jn2\pi\nu}\int_{-\pi}^{\pi} e^{j(\nu\phi' - z\sin\phi')}d\phi'$$

and by application of Eq.(54),

(73) $$\frac{1}{2\pi}\int_{(2n-1)\pi}^{(2n+1)\pi} e^{j(\nu\phi - z\sin\phi)}d\phi = e^{jn2\pi\nu}\,\mathbf{J}_{\nu}(z) \quad .$$

Similarly, from Eq. (56),

(74) $$\frac{1}{\pi}\int_{2N\pi}^{(2N+1)\pi} e^{\pm j(\nu\phi - z\sin\phi)}d\phi = e^{\pm jN2\pi\nu}[\mathbf{J}_{\nu}(z) \pm j\,\mathbf{E}_{\nu}(z)] \quad .$$

Furthermore, by application of Eq. (54),

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos\phi \; e^{j(\nu\phi - z\sin\phi)} d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[e^{j(\nu+1)\phi} + e^{j(\nu-1)\phi} \right] e^{-jz\sin\phi} \, d\phi \quad .$$

$$= \mathbb{J}_{\nu+1}(z) + \mathbb{J}_{\nu-1}(z) \quad .$$

Thus

$$\text{(75)} \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\phi \; e^{j(\nu\phi - z\sin\phi)} d\phi = \mathbb{J}_{\nu+1}(z) + \mathbb{J}_{\nu-1}(z) \; ,$$

which may be generalized to

$$\text{(76)} \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} \cos p\phi \; e^{j(\nu\phi - z\sin\phi)} d\phi = \mathbb{J}_{\nu+p}(z) + \mathbb{J}_{\nu-p}(z) \quad .$$

Also by Eq. (54)

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin p\phi \; e^{j(\nu\phi - z\sin\phi)} d\phi = \frac{1}{2i\pi}\int_{-\pi}^{\pi} \left[e^{j(\nu+p)} - e^{j(\nu-p)} \right] e^{-jz\sin\phi} d\phi$$

or

$$\text{(77)} \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} \sin p\phi \; e^{j(\nu\phi - z\sin\phi)} = -j \left[\mathbb{J}_{\nu+p}(z) - \mathbb{J}_{\nu-p}(z) \right] \; ,$$

and by the transformation

$$\phi = \phi' + 2\pi n$$

$$\frac{1}{\pi}\int_{(2n-1)\pi}^{(2n+1)\pi} \cos p\phi \; e^{j(\nu\phi - z\sin\phi)} d\phi$$

$$= \frac{1}{2\pi}\int_{(2n-1)\pi}^{(2n+1)\pi} \left[e^{j(\nu+p)\phi} + e^{j(\nu-p)\phi} \right] e^{-jz\sin\phi} d\phi$$

$$= \frac{1}{2\pi} e^{jn2\pi\nu} \int_{-\pi}^{\pi} \left[e^{j(\nu+p)\phi'} + e^{j(\nu-p)\phi'} \right] e^{-jz\sin\phi'} d\phi'$$

or

$$\text{(78)} \qquad \frac{1}{\pi}\int_{(2n-1)\pi}^{(2n+1)\pi} \cos p\phi \; e^{j(\nu\phi - z\sin\phi)} d\phi = e^{jn2\pi\nu} \left[\mathbf{J}_{\nu+p}(z) + \mathbf{J}_{\nu-p}(z) \right] ,$$

and similarly

$$\text{(79)} \qquad \frac{1}{\pi}\int_{(2n-1)\pi}^{(2n+1)\pi} \sin p\phi \; e^{j(\nu\phi - z\sin\phi)} = -j \; e^{jn2\pi\nu} \left[\mathbf{J}_{\nu+p}(z) - \mathbf{J}_{\nu-p}(z) \right] .$$

Series expansions of Anger and Weber functions are given by

$$\text{(80)} \qquad \mathbf{J}_{\nu}(z) = \cos\frac{\nu\pi}{2} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m}}{\Gamma\left(m+1-\frac{\nu}{2}\right)\Gamma\left(m+1+\frac{\nu}{2}\right)}$$

$$+ \sin\frac{\nu\pi}{2} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{2m+1}}{\Gamma\left(m+\frac{3-\nu}{2}\right)\Gamma\left(m+\frac{3+\nu}{2}\right)}$$

or alternatively

$$= \frac{\sin \nu\pi}{\nu\pi}\left[1 - \frac{z^2}{2^2-\nu^2} + \frac{z^4}{(2^2-\nu^2)(4^2-\nu^2)} - \frac{z^6}{(2^2-\nu^2)(4^2-\nu^2)(6^2-\nu^2)} + \cdots\right]$$

$$+ \frac{\sin \nu\pi}{\pi}\left[\frac{z}{1^2-\nu^2} - \frac{z^3}{(1^2-\nu^2)(3^2-\nu^2)} + \frac{z^5}{(1^2-\nu^2)(3^2-\nu^2)(5^2-\nu^2)} - \cdots\right].$$

(81) $$\mathbf{E}_\nu(z) = \sin\frac{\nu\pi}{2}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m}}{\Gamma\left(m+1-\frac{\nu}{2}\right)\Gamma\left(m+1+\frac{\nu}{2}\right)}$$

$$-\cos\frac{\nu\pi}{2}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+1}}{\Gamma\left(m+\frac{3-\nu}{2}\right)\Gamma\left(m+\frac{3+\nu}{2}\right)}$$

or alternatively

$$= \frac{1-\cos \nu\pi}{\nu\pi}\left[1 - \frac{z^2}{2^2-\nu^2} + \frac{z^4}{(2^2-\nu^2)(4^2-\nu^2)} - \cdots\right]$$

$$- \frac{1+\cos \nu\pi}{\pi}\left[\frac{z}{1^2-\nu^2} - \frac{z^3}{(1^2-\nu^2)(3^2-\nu^2)} + \frac{z^5}{(1-\nu^2)(3^2-\nu^2)(5^2-\nu^2)} - \cdots\right].$$

From Eq (80),

(82) $$\mathbf{J}_\nu(z) - \mathbf{J}_{-\nu}(z) = 2\sin\frac{\nu\pi}{2}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+1}}{\Gamma\left(m+\frac{3-\nu}{2}\right)\Gamma\left(m+\frac{3+\nu}{2}\right)}.$$

Nielsen[12] denotes

$$\mathbf{J}_\nu(z) - \mathbf{J}_{-\nu}(z) = 2X^\nu(z)$$

and has also derived Eq. (82).[12] Similarly

$$J_\nu(z) + J_{-\nu}(z) = 2\cos\frac{\nu\pi}{2}\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m}}{\Gamma\left(m+1-\frac{\nu}{2}\right)\Gamma\left(m+1+\frac{\nu}{2}\right)} \tag{83}$$

where Nielsen[12] denotes

$$J_\nu(z) + J_{-\nu}(z) = 2\,\overset{\nu}{\pi}(z).$$

By differentiation of Eq. (83):

$$J'_\nu(z) + J'_\nu(z) = 2\cos\frac{\nu\pi}{2}\sum_{m=0}^{\infty}\frac{(-1)^m\, m\left(\frac{z}{2}\right)^{2m-1}}{\Gamma\left(m+1-\frac{\nu}{2}\right)\Gamma\left(m+1+\frac{\nu}{2}\right)} \tag{84}$$

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