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Friction Laws and Energy Transfer in Circular Flow, Part I - The Law of Shear
Stresses in Circular Flow Part II - Energy Transfer in Circular Flow and Possible   (None)

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While the laws of Newton give a well established base upon which to determine the shear stresses in a
straight flow, there is, at the present time, an almost complete lack of agreement as to the treatment of
circular flow. Two controversial theorems exist. The discrepancy between the results of the two theories,
namely the general theory of laminar friction and the momentum-transfer theory, is eliminated as explained
in the text. The second part deals with the theory of energy transfer, which is compared with test results
carried out by Hilsch, and hints on applications of the Hilsch device are given. The Hilsch effect gives an
explanation of a phenomenon which often occurs when a gas expands in a valve or nozzle. Usually, immedi
ately behind the nozzle, the pipe becomes very cold, so that formation of ice can be observed.

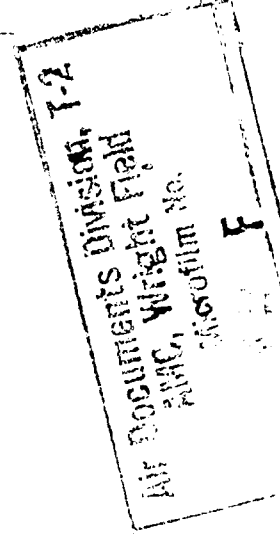Possible Applications (Explanation of the Hilsch or Ranque Effect)

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FRICTION LAWS AND ENERGY TRANSFER IN CIRCULAR FLOW

(Project No. LP-259)

Rudolph Kassner
and
Eugen Knoernschild

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FRICTION LAWS AND ENERGY TRANSFER IN CIRCULAR FLOW

Part I - The Laws of Shear Stresses in Circular Flow

by

Rudolf Kassner

Part II - Energy Transfer in Circular Flow and Possible Applications
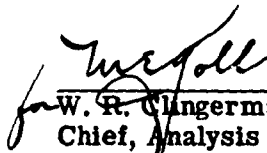(Explanation of the Hilsch or Ranque Effect)

by

Eugen Knoernschild

Propulsion Section, Analysis Division,
      Intelligence Department
Project No. LP-259
Project Engineer: Allen H. McCormick

Approved by:

W. R. Clingerman, Colonel, USAF
Chief, Analysis Division
Intelligence Department

For the Commanding General:

H. M. McCoy, Colonel, USAF
Chief of Intelligence

<u>Friction Laws and Energy Transfer in Circular Flow</u>
(Project No. LP-259)

Part I - The Laws of Shear Stresses in Circular Flow

Rudolf Kassner

ABSTRACT

While the laws of Newton and Prandtl give a well-established base upon which to determine the shear stresses in a straight flow, there is, at the present time, an almost complete lack of agreement as to the treatment of circular flow. There exist two controversial theorems: In accordance with the general theory of laminar friction of fluid, shear stresses are proportional to the shear velocity; on the other hand, there have been made derivations based on the momentum-transfer theory and leading to the conclusion that shear stresses in laminar as well as in turbulent motion are proportional to the rotation in the fluid.

In this report, the discrepancy between the results of the general theory of laminar friction and the momentum-transfer theory is eliminated. It is shown that these discrepancies are only due to the fact that the method of applying the momentum-transport theory to circular motion was incomplete. If it is applied in the accurate way used in the kinetic theory of gases, taking into account particles (molecules) with all velocity directions, the result is in accordance with the general theory of laminar friction. A corresponding calculation carried out for turbulent motion leads to a similar result. Furthermore, a comparison with the vorticity transport theory by G. F. Taylor is given.

Part II - Energy Transfer in Circular Flow and Possible Applications
(Explanation of the Hilsch or Ranque Effect)

Eugen Knoernschild

ABSTRACT

When a gas streams tangentially into a tube, a flow which complies with the law of constant angular momentum (irrotational flow) will be built up. This type of flow will be changed under the influence of viscosity. While in the midst of the flow the viscous forces are in equilibrium; this equilibrium cannot be maintained at the boundaries of the flow, which means in the center of the tube and at the walls. From there, a conversion of the flow starts, progressing from the boundaries to the inner parts of the flow and leading to a type of flow which has no shear stress, i.e., the rotational flow. Obviously, by this conversion the central parts of the flow decrease their tangential velocity, while the velocity of the outer parts is increased.

The velocity conversion influences the temperature distribution insofar as the

F-TR-2198-ND

adiabatic temperature distribution of the irrotational flow changes (by the effect of turbulent heat exchange) over to the adiabatic temperature distribution of rotational flow, while the mean temperature of both distributions remains the same. As the temperature distribution of the rotational flow means lower temperatures in the center than at the outer parts of the flow, we find that, by the conversion from irrotational to rotational flow, not only the velocity but also the temperature at the inner parts of the flow is lower than at the outer parts. By separating those two parts of the flow, a cold- and a hot-air stream can be produced. The theory of energy transfer is compared with test results of Hilsch, and hints on applications of the Hilsch device are given.

BIOGRAPHICAL NOTES

Rudolf Kassner, 1/ born 1911 in Berlin, Germany, studied mechanical engineering at the Technical University of Berlin. From 1934 to 1936, as assistant to Professor Herbert Wagner, at the Institute of Aircraft Research, Technical University, Berlin, he engaged in extensive work on wing flutter.

He was associated with the Junkers Werke, Dessau, from 1937 to 1938, working on aerodynamic and supersonic problems. From 1938 he devoted himself to the field of jet propulsion at Junkers; later on at the Heinkel Works, Rostock and Stuttgart, Zuffenhausen (Hirth - Plant), he was in charge of basic research on all phases of thermodynamics and fluid dynamics, and also on problems of controlling turbojets.

He is the author of four publications on wing flutter and fundamental principles of controlling turbojets.

Eugen Knoernschild, 1/ born 1906 in Germany, studied mechanical engineering at the Technical University of Munich. Since 1934, he has been associated with the DVL, Berlin. Until 1936 he worked in the field of high-altitude aero engines. From 1936 to 1945 he worked on all sorts of problems pertaining to gas turbines and jet propulsion. Since 1939 he had been vice-director of the Institute of Applied Thermo, Hydro, and Aerodynamics, Compressors and Turbines of the DVL. He especially devoted himself to research in the field of high-temperature gas- and exhaust gas-turbine performance, high-speed midget turbo-compressors, turbine-blade cooling, and to an extensive experimental study of series of turbine blades. He is the author of about twenty publications on these subjects.

1/ Mr. R. Kassner acknowledges some valuable suggestions from Dr. H. v. Ohain for Part I of this report.

Dr. Knoernschild acknowledges valuable suggestions from Mr. R. Kassner concerning Part II of this report.

TABLE OF CONTENTS

PART I - THE LAWS OF SHEAR STRESSES IN CIRCULAR FLOW

PART II - ENERGY TRANSFER IN A CIRCULAR FLOW AND POSSIBLE APPLICATIONS (THE EXPLANATION OF THE HILSCH OR RANQUE EFFECT)

LIST OF ILLUSTRATIONS

Page

LIST OF ILLUSTRATIONS (CONT'D)

PART I - THE LAWS OF SHEAR STRESSES IN CIRCULAR FLOW

LIST OF SYMBOLS

l	Free path length of molecule or Prandtl's mixture length of fluid portion
m	Mass of molecule or fluid portion
M	Shear moment
n, n^*	Number of molecules
p	Pressure
r	Radius
s	Path travelled by a particle
t	Time of travel
U	Velocity of mean flow
v	Velocity of fluid particle
x	Coordinate, generally in direction of mean flow
y	Coordinate normal to x
ε	Turbulent exchange rate
μ	Viscosity of fluid
τ	Shear stress
ρ	Density of fluid
Ω	Angular velocity
ξ, η	Velocity components of a particle relative to mean flow

FRICTION LAWS AND ENERGY TRANSFER IN CIRCULAR FLOW

Part I - The Laws of Shear Stresses in Circular Flow

1. Introduction

Friction stresses in a straight flow, e.g., in the boundary layer along a straight wall, can be evaluated by means of the laws of Newton for laminar friction and of the formula of Prandtl for turbulent friction. According to these formulas, friction stresses depend on the velocity gradient normal to the flow. However, there is almost complete lack of agreement as to the treatment of circular flow, with which mostly we have to deal in flow machines.

According to the general theory of laminar friction of fluid flow, which is applicable to any kind of laminar flow, and therefore including circular flow, the stresses caused in the fluid by friction correspond to the stresses caused by elastic deformations in a solid body, differing only in that, instead of the deformations themselves, the rate of change of the deformations has to be considered. Accordingly, the shear stress, which with solid bodies is proportional to the shear angle, in viscous flow is proportional to what we shall call shear velocity, i.e., to the rate with which the angle between two surfaces of an originally rectangular element of fluid changes.

Contrary to this, there may be found, even in very renowned works, the opinion that the shear stress in circular motion is proportional to the rotation (i.e., to the mean angular velocity of the rotating elements of fluid) if laminar as well as turbulent friction is regarded.

Neither theorem contradicts the other or the Newtonian law, so long as they are applied to the straight motion of fluid. If applied to a circular motion, however, the first relation means there would be no shear stresses in a fluid which is rotating according to the law $U = \Omega r$ (i.e., like a rigid body), because in this case there is no relative gliding or shearing; according to the second theorem, however, in this case shear stresses generally would occur, whereas no shear stresses would exist in the case of irrotational flow.

Thus, e.g., the following can be found in [1] Traupel's book, page 38, "Neue allgemeine Theorie der Mehrstufigen Axialen Turbomaschine": "If a flow is irrotational, no shear stresses occur in it even if the fluid is viscous. Also a superposed turbulent motion in this case does not produce an exchange of momentum, i.e., no shear stresses, so long as the disturbances are not so predominant as to change completely the character of the flow."

The assumption that shear stress is proportional to the rotation is generally derived from the conception that friction is caused by molecules, or fluid portions (in the turbulent case) exchanged normally to the mean flow. Accordingly, it is conceived that friction in circular flow is caused by the moment of momentum exchanged per unit of time between two circular layers. As the mean moment of momentum or the circulation respectively in irrotational flow is constant along the radius, it has been concluded that in this case no exchange of moment of momentum, and consequently no friction stresses, can result.

The evaluation of turbulent friction through the momentum transport has been established by Prandtl, and thus the theorem that friction stress in turbulent circular motion must be proportional to the rotation is ascribed to Prandtl. 2/ G. I. Taylor has repeatedly pointed out that the result of this theorem is contrary to experience, and he had (even in advance of Prandtl) derived a formula based on a "vorticity transport theory," according to which

2/ For example, see Wattendorf [2] who refers to Prandtl 3 .

turbulent friction is determined by the shear velocity. It is Taylor's opinion that the momentum transport would not yield a correct result.

In this report it will be shown that the momentum transport theory, if correctly applied, yields a result which is similar for laminar and for turbulent friction and which, with laminar friction, is in absolute accordance with the general theory of laminar fluid friction.

In the case of turbulent friction it is similar to Taylor's formula. It will be shown, however, that in this case both - momentum transport and vorticity transport - have to be considered, in order to get correct results.

2. The Basic Problem

A flow with circular motion is considered, the velocity of fluid being given as a function of the radius r (Fig. I-1). The relations which govern the shear stress in such types of flow are being sought for laminar as well as for turbulent motion.

The laminar as well as the turbulent friction in a straight boundary layer can be explained by a momentum transport normal to the streamlines. With laminar flow of a gas, it is the irregular molecular motion which causes the molecules of the slow-flowing layers near the wall to penetrate into the faster layers and to exert a retarding effect. Every particle of the mass m maintains its velocity v until it collides within the adjacent layer and transfers its momentum $m \cdot v$. With turbulent motion, this momentum transport is done by portions of fluid traveling from one layer to the other.

Applying the same considerations to the flow with circular motion, by regarding particles (molecules or entire fluid portions) moving normal to the path of the circular flow, it is possible to calculate the moment exerted from one circular layer to the adjacent one by adding the moments of momentum of the particles transferred per unit of time.

Difficulties arise from the following assumptions: According to the above, two particles, m_1 and m_2, which belong to the two different flow layers marked by r_1 and r_2 are considered. Their circumferential velocity component may be equal to the velocity U_1 and U_2 of their respective layers. They may differ from the mean flow by radial velocities which enable particle 1 to penetrate the path of particle 2 and to transfer its moment of momentum, while particle 2 acts reversely. The transferred moment of momentum equals

$$m_1 \ U_1 \ r_1 \ - \ m_2 \ U_2 \ r_2$$

or

$$m \ (U_1 \ r_1 \ - \ U_2 \ r_2)$$

if the particles have the same mass $m_1 = m_2 = m$.

We shall extend this calculation to the total number n of the particles transferred in radial direction per unit of area and time. Thus we obtain an equation of the moment of the shear force acting per unit of area.

$$\mathcal{T} \ r = n \ m \ (U_1 \ r_1 - U_2 \ r_2)$$

$$= n \ m \ \triangle r \cdot \frac{d}{dr} \ (U \cdot r)$$

and hence the shear stress itself:

$$\tau = n \, m \, \Delta r \cdot \left(\frac{dU}{dr} + \frac{U}{r} \right) \tag{1}$$

with Δr as the mean path travelled in radial direction by the particles.

In the special case of irrotational flow yielding the equation $U \cdot r = $ const, this would lead to the conclusion that there is no shear stress in the flow. On the other hand, the formula would yield to a considerable shear stress in the case or rotational flow, i.e., a flow with constant angular velocity which rotates like a rigid body.

Both results are incompatible with experience, which shows that an irrotational flow is unstable, and that under the influence of viscosity a change of its velocity profile is taking place such that it is converted - beginning from the inside layers - into a rotational flow. 3/ On the other hand, if a rotational flow once exists, the inner friction stresses do not show a tendency to change its velocity profile.

However, the preceding conclusions are not only incompatible with experience, but also, so far as laminar motion is concerned, opposite to the result of the general theory of laminar fluid friction. The implications of this theory, which has long been known and is contained in most works on fluid mechanics, can be explained by considering a straight flow.

In a straight flow, Newton's law of friction yields the simple equation $\tau = \mu \frac{\partial U}{\partial y}$ only if the coordinates x and y are chosen in directions parallel and normal to the flow. (See Fig. 2.)

In changing over to a coordinate system the x-axis of which forms an angle with the direction of the flow, the system of coordinates is turned so that there exist not only values of U_x and $\frac{\partial U_x}{\partial y}$ but also of U_y and $\frac{\partial U_y}{\partial x}$, and three components of stress in two-dimensional flow (nine if the consideration is extended to a three-dimensional flow) will be obtained. The stresses in the fluid can then be compared with the stresses in solid bodies, but with the difference that the stress is proportional to the rate of change of deformation and not proportional to the deformation itself.

In this general case of a straight flow forming an angle with the x-axis, the shear stress is

$$\tau_{xy} = \mu \left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \right) \tag{2}$$

Besides that, there exists the tensile and the compressive stresses

$$\tau_x = 2\mu \frac{\partial U_x}{\partial x} \qquad \text{and} \qquad \tau_y = 2\mu \frac{\partial U_y}{\partial y} .)$$

We comprehend the meaning of $\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \right)$ if we consider that $\frac{\partial U_x}{\partial y}$ and $\left(-\frac{\partial U_y}{\partial x} \right)$ are the angular velocities, in the clockwise direction, of two surfaces of a fluid particle originally parallel to the x- and y-axis (Fig. I-3). So the expression $\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}$ designates the difference of the angular velocities, i.e., it determines the velocity with which the

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3/ This must not be confused with the fact (proved by tests) that an irrotational flow of a viscous fluid can be maintained if the friction losses in the fluid are compensated by providing energy from outside. In other words, this is the case with the flow between two concentric cylindrical walls, one of which is rotated by means of an external force. For this, see par. 6 of this report, Steady Circular Flow of Viscous Fluids.

angle between the two surfaces changes. We shall call it the "shear velocity."

Contrary to this, the <u>rotation</u> of the particles is defined as the <u>sum</u> of the angular velocities of the two surfaces:

$$\frac{\partial U_x}{\partial y} + \left(-\frac{\partial U_y}{\partial x}\right)$$

So there is a discrepancy between the two equations. The first equation was derived by applying the Prandtl theory of a momentum transfer to circular motion. According to it, friction is proportional to the rotation

$$\frac{\partial U_x}{\partial y} - \frac{\partial U_y}{\partial x} = \frac{dU}{dr} + \frac{U}{r} \; .$$

On the other hand, there exists the well-established formula of the general theory of laminar friction which also can be derived from momentum transfer theories. According to these formulas, equation (2), friction is proportional to the "shear velocity" $\dfrac{\partial U_x}{\partial y} + \dfrac{\partial U_y}{\partial x}$,

which in circular motion would be $\dfrac{dU}{dr} - \dfrac{U}{r}$.

The most important objections to the theory that shear stress is proportional to the rotation may be obtained from a consideration of the energy of the fluid. The dissipation of energy in a unit of volume and per unit of time can be expressed by the work done by the shear stresses at the surfaces of this element

$$A = \oint \tau \, ds = \tau_{xy}\left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}\right) dx \, dy \, dz. \tag{3}$$

The energy dissipation is proportional to the product, shear stress times "shear velocity." So, if the shear stress were proportional to the rotation, there could not be an energy dissipation either with the irrotational flow (because there would be no shear stress) nor in the rotational flow (because there is no shear velocity). We can express the last statement referring to rotational flow in another manner. In the rotational flow, no energy dissipation is possible because the work of the moments of shear forces, being in equilibrium at an element of fluid, vanishes as the element rotates like a rigid body. This is in accordance with the following consideration:

We shall consider a flow with circular motion as being only under the influence of internal friction forces, while external friction forces acting from the walls will be eliminated by assuming that the walls move with the fluid. Because of lack of an external moment, the velocity profile can be converted by friction only into profiles which have the same amount of moment of momentum. It may easily be shown that, out of all the possible velocity profiles, rotational flow has the minimum kinetic energy. Consequently, the friction force will always tend to build a rotational flow. Therefore, friction forces must be active in any flow which does not comply with the rotational flow.

If rotational flow could be changed by friction, this would mean an increase of kinetic energy at the expense of disordered energy. Such a process would be contrary to the natural laws. A solution of the problem of eliminating the existing discrepancies will be shown in the following chapters.

3. <u>Derivation of the Exact Equation for Laminar Circular Motion From the Kinetic Theory of Gases</u>

We shall derive first the equation for laminar circular motion by means of the method of momentum transport.

Contrary to the preliminary considerations (par. 1), we shall use a more accurate procedure, corresponding to the methods used for the derivations of the Newtonian law of inner friction, in the works on kinetic theory of gases. 4/

Again we shall calculate the tangential shear stress from the tangential momentum, transferred per unit of time through the unit area. However, we shall not assume that the exchanged particles differ from the mean motion by small radial velocities, but rather we shall take as a basis the irregular movement of molecules according to the kinetic theory of gases, where particles move in all directions and with different velocities.

Imagine a cube of unit volume erected on the area through which the exchange of momentum takes place. From all the molecules contained in the cube we take only those having velocity components v_x and v_y which lie in the range between v_x and $v_x + \Delta v_x$, respectively v_y and $v_y + \Delta v_y$ (Fig. I-4). Let us designate their number by n^*. n^* is a function of v_x, v_y, Δv_x, Δv_y, being proportional to the assumed Δv_x and Δv_y.

Out of this select number there will $n^* v_y$ particles pass the control area per unit time and unit area. Consequently the tangential impulse transported by them through the unit area is $n^* m v_x v_y$

$$\text{with } m = \text{mass of one molecule.} \tag{4}$$

We extend this process to all particles contained in the volume, by integrating over the whole range of (v_x, v_y), and obtain:

$$\tau = -\Sigma \quad n^* m \, v_x \, v_y \quad \underline{5}/ \tag{5}$$

for the whole momentum transport, which is equivalent to a shear stress, as pointed out before.

Since the control area has been assumed tangential to the mean motion of flow, the same number of particles with positive and negative v_y will always pass through the area per unit time.

This yields $\quad \Sigma \, n^* m \, v_y = 0.$ $\tag{6}$

A shear stress, according to equation (5), will therefore result only if the molecules passing the area in one direction have on the average, for instance, higher v_x - component than those which pass in the other direction. There can be two causes for such an effect:

(1) There is a velocity gradient like the one in the boundary layer of a straight flow

(2) The direction of the velocity of the mean flow is locally different, as in the case with circular motion. A clear conception of this effect will be obtained in the next chapter.

First we shall give a mathematical derivation of the law of shear stress. Let us trace a molecule, having passed the control area in point P (Fig. I-5), back to the point P_1, where it underwent the last change of its direction by the collision with another molecule.

4/ See, e.g., HAAS, 4

5/ In a more accurate manner of writing: $-\int n^* m v_x \, v_y \, dv_x \, dv_y$

Relative to the mean flow, the molecule has traveled a distance $1 = v_m t$, which is of the order of its free-path length, v_m denoting the velocity of the molecule and t the time since it last collided. In fact, however, the molecule has traveled during this time from P_1 to P, traveling from the radius r_1 to the radius r on a straight path with a constant velocity.

Let the components of the mean velocity of flow be

$$U_{x_1}, \quad U_{y_1} \quad \text{in point } P_1$$

$$U_x, \quad U_y \quad \text{in point } P$$

then the velocity components of the molecule in these points can be written

$$v_x = v_{x_1} = U_{x_1} + \xi 1 \tag{7}$$

$$v_y = v_{y_1} = U_{y_1} + \eta \tag{8}$$

where $\xi,$, $\eta,$ denote the differences between the components of velocity of the molecule and the mean flow at point P_1. Now the velocity components of the mean flow in P are $U_x = U, U_y = 0$; while those in P_1 can be expressed by

$$U_{x,} = U + (y_1 - y) \frac{\partial U}{\partial y} \tag{9}$$

$$U_{y,} = U \frac{x - x_1}{r} \tag{10}$$

if small terms of higher order are neglected.

In equation (9) we see the influence of the velocity gradient $\frac{\partial U}{\partial y} = \frac{dU}{dr}$, whereas the y - component U_y in equation (10) is due to the change in velocity direction.

Substituting $$x - x_1 = u_x \, t \tag{11}$$

$$y - y_1 = u_y \, t \tag{12}$$

with t = time which the molecule needs to travel from

$$P_1 \quad \text{to } P$$

we finally obtain from equations (7) to (12)

$$V_x = U + \xi, - \frac{dU}{dr} \eta, t \tag{13}$$

$$V_y = \eta, + \frac{U^2 + U\xi,}{r} t \tag{14}$$

if again small terms of second order are neglected.

Equation (13) contains the influence of the velocity gradient in exactly the same way as when dealing with the problem of straight flow; equation (14), however, contains the influence of the curvature in circular motion.

With equations (13) and (14) we get from equation (5) the expression for the shear stress

$$\tau_{xy} = -m \sum n^* \left(u + \xi_1 - \frac{du}{dr}\eta_1 t\right)\left(\eta_1 + \frac{u^2 + u\xi_1}{r}t\right) \tag{15}$$

where n* means the number of molecules per unit volume having velocity components between

$$\xi_1, \quad \eta_1 \quad \text{and} \quad \xi_1 + \Delta\xi_1, \quad \eta_1 + \Delta\eta_1 .$$

The calculation of $n^* = n^*(\xi_1, \eta_1)$ in the circular flow involves an additional effect - the density change in radial direction, which, in the usual macroscopic way of treating the subject, is derived from the radial pressure gradient.

Particles with positive η_1, which pass the control area in the direction from inside to outside, come from regions with less density than those moving in the opposite direction. The number n^* (ξ_1, η_1) is proportional to the density of the regions from which the molecules come.

Hence, if without a radial density change the velocity distribution of the molecules in the flow were given by $n_o^*(\xi, \eta)$, we have to take into account the radial density

increase by writing

$$n^*(\xi_1, \eta_1) = n_o^*(\xi, \eta)\cdot(1 - b\eta t) \tag{16}$$

where

$$b = \frac{1}{n}\frac{dn}{dr} = \frac{1}{\rho}\frac{d\rho}{dr} = \frac{1}{p}\frac{dp}{dr} \tag{17}$$

is the relative radial density or pressure change 6/ and ηt the radial path traveled by the molecule.

Having introduced relation (16) into equation (15), we may dispense with the index 1 which marks the particles with respect to their original location:

$$\tag{18}$$

The radial density change now can be found as we use the methods of molecular physics, simply, from equation (6), stating that there is no resulting mass flow through the control area:

$$\sum n^*(\xi_1, \eta_1)v_y = 0$$

which with (14) and (16) yields

$$\sum n_o^*(1 - b\eta t)(\eta + \frac{u^2 + u\xi}{r}t) = 0 \tag{19}$$

6/ Pressure and density can be assumed proportional because of constant temperature.

Moreover, from the fact that the resulting velocity of all molecules must be equal to the mean velocity of flow, we can derive the equation:

$$\Sigma \, n_o{}^* \eta = 0 \tag{20}$$

$$\Sigma \, n_o{}^* \xi = 0 \tag{21}$$

$$\Sigma \, n_o{}^* \xi \, \eta = 0 \tag{22}$$

Substituting equations (20) - (22) in (19), we obtain

$$t_m \left[-b \, \Sigma \, n_o{}^* \eta^2 + \frac{U^2}{r} \, \Sigma \, n_o{}^* \right] = 0 \tag{23}$$

with t_m = mean value of travel time t of the molecules.

With $\quad b = \frac{1}{\rho} \frac{d\rho}{dr} \quad$ (17), this equation is in fact identical with the well-known equation

$$-\frac{d\rho}{dr} + \frac{U^2}{r} \rho = 0 \tag{24}$$

which determines the radial pressure change in a circular motion. Since $m \, \Sigma \, n_o{}^* \eta^2$

equals the normal component of momentum which hits the unit of surface per unit of time, that means it is equal to the pressure p in the fluid. $m \, \Sigma \, n^*$ equals the mass of molecules per unit volume and therefore is equal to the density.

Substituting equation (23) in equation (18), we obtain the equation for the shear stress

$$\tau = -\Sigma \, n_o{}^* m \, V_x \, V_y = m \left[\frac{dU}{dr} \Sigma \, n_o{}^* \eta^2 t - \frac{U}{r} \Sigma \, n_o{}^* \xi^2 t \right] \tag{25}$$

Now, considering that in the irregular motion of molecules no velocity direction (ξ , η) is preferred, we apply the well-known relation
$$\Sigma \, n_o{}^* \xi^2 t = \Sigma \, n_o{}^* \eta^2 t$$

Furthermore, as in equation (1), we introduce the viscosity as the sum of all mass particles exchanged per unit time and unit area, multiplied by their distance of travel:

$$\mu = \Sigma \, m \cdot \Delta r = m \, \Sigma \, n_o{}^* \eta \cdot \eta \, t \tag{26}$$

Thus, we finally obtain from equation (25)

$$\tau = \mu \left(\frac{dU}{dr} - \frac{U}{r} \right) \tag{27}$$

Applying this result to irrotational flow, where $\quad U \cdot r$ = const or

$$\frac{dU}{dr} + \frac{U}{r} = 0, \tag{28}$$

we get the friction stress $\quad \tau = -\mu \cdot 2 \frac{U}{r}$

$$\tag{29}$$

This result is indeed opposite to the result of the usual consideration given in par. 2, The Basic

Problem, (compare equation (27) with equation (1)). Consequently, opposite to this consideration a transfer of moment of momentum must be accomplished from one circular flow filament to the other, in spite of the fact that the mean angular moment of momentum might be the same in all layers. The effect can be explained, however, as derived in the above equations, by considering that the individual molecules have a different angular momentum.

If we take any of the concentric cylinders separating two circular layers as a control area, it means that the particles penetrating this cylinder from the inside to the outside have in the average a higher amount of angular momentum than those traveling in the opposite direction.

·4.   Shear Stress in Circular Flow With Turbulent Motion

In the case of turbulent friction, instead of molecules we have to deal with portions of fluid which are in irregular motion relative to the mean flow, and are continuously changing their shape until they cannot be retraced or, in other words, lose their identity. Instead of considering this continuous change of shape and velocity, we simplify the problem and make the usual assumption that the fluid portions travel a certain distance, the "mixing length," without affecting each other, and then lose their velocity relative to the mean flow.

Unlike the molecules, however, which are not acted upon by any forces while moving between two collisions, the particles of the turbulent flow, as parts of the fluid, are under the influence of the radial pressure gradient of the mean flow.

This time we shall use a system of coordinates, moving with the mean flow, the abscissa of which is circumferential and the ordinate normal to it, while the origin moves with a point P of the mean flow.

In a circular flow the system of coordinates will rotate with the angular velocity

$$\Omega = \frac{U}{r} .$$

If $\frac{dU}{dr}$ denotes the radial velocity gradient of the mean flow, then the velocity distribution near point P relative to the rotating system can be described by

$$U_x = \left(\frac{dU}{dr} - \frac{U}{r}\right) y$$

(30)

$$U_y = 0$$

(31)

In order to calculate the motion of particles relative to the rotating system, we have to assume that the particles are under the influence of d'Alembert's forces, which are

(1) the force of Coriolis
(2) the centrifugal force.

The forces resulting from the radial pressure gradient being in equilibrium with the centrifugal force, we shall consider only the force of Coriolis. This force results in an acceleration $a_c = 2 \Omega v$, which is normal to the velocity v of the particle relative to the moving system. Thus a particle, after traveling a small distance P, $P = s = vt$, has an additional velocity $a_c t = a_c \frac{s}{v} = 2 \Omega v \frac{s}{v} = 2 \frac{U}{r} s$ normal to the original direction of v.

Figure I-6 shows the path of a particle with a slight curvature due to the acceleration of Coriolis: The velocity v in P differs from the original velocity v_i in P_i by $\frac{u}{r}s$, which for small distances s can be assumed normal to v_i .

This means:

$$v_x = v_{x_i} + 2\frac{u}{r}y_i \tag{32}$$

$$v_y = v_{y_i} - 2\frac{u}{r}x_i \tag{33}$$

Now we may express again, as in par. 3, the velocity of a particle at any point by the velocity of the mean flow, and the velocity difference of the particle relative to the mean flow, denoted by the components ξ , η in x and y directions. A particle starting its path at a point with the velocity differences ξ_i , η_i , according to equations (30) and (31) has the resulting velocity relative to the rotating system of coordinates

$$v_{x_i} = u_{x_i} + \xi_i = \left(\frac{du}{dr} - \frac{u}{r}\right)y_i + \xi_i \tag{34}$$

$$v_{y_i} = u_{y_i} + \eta_i = \eta_i \tag{35}$$

if small terms of higher order are neglected.

Under the influence of Coriolis' force, the velocity of a particle changes during its travel, so that a particle arriving in P at this point will have, according to equations (32), (33), the velocity components

$$v_x = v_{x_i} + 2\frac{u}{r}y_i = \xi_i + \left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)y_i \tag{36}$$

$$v_y = v_{y_i} - 2\frac{u}{r}x_i = \eta_i - 2\frac{u}{r}x_i \tag{37}$$

Neglecting again small terms of higher order, and substituting

$$x_i = -v_x \tau \tag{38}$$

$$y_i = -v_y \tau \tag{39}$$

we obtain

$$v_x = \xi_i - \eta_i \tau \left(\frac{du}{dr} + \frac{u}{r}\right) \tag{40}$$

$$v_y = \eta_i + \xi_i \tau \cdot 2\frac{u}{r} \tag{41}$$

From these equations the following essential facts may be seen (Fig. I-7):

(1) A particle starting its way with the relative velocity component ξ_i = 0, i.e., a particle with only a radial component η_i relative to the mean flow, as it has been considered in par. 1, will arrive in P with a tangential component $v_x = -\eta_i \tau \left(\frac{du}{dr} + \frac{u}{r}\right)$ Indeed,

in the case of irrotational flow U · r = const, such particles arrive at P with V_x = 0, i.e., they do not carry a tangential momentum.

(2) Particles differing from the velocity of mean flow by a component ξ_i only (i.e., particles with η_i = 0) arrive at point P with an additional radial velocity $v_y = 2\frac{u}{r}\xi_i \tau$. This means that in the case of irrotational flow a tangential momentum is carried from one layer to the other by all particles except those considered under (1).

(3) The tangential momentum carried per second through the unit of area yields

$$\Sigma n^* m\, V_x\, V_y = \Sigma n^* m\, [\xi_i \eta_i + \xi_i^2\, t\, \frac{2U}{r} - \eta_i^2 t\, (\frac{dU}{dr} + \frac{U}{r})] \qquad (42)$$

Now, in turbulent mixed flow there is no preference given to any direction, and, in consequence of the continuity equation, any movement of fluid portions in one direction will produce movements normal to it of the same order of magnitude. Therefore, if the integration (equation (42) is carried out, the first term $\Sigma n^* m\, \xi_i \eta_i$ vanishes, while in the second and

third terms we can substitute $\Sigma n^* m\, \xi_i^2\, t = \Sigma n^* m\, \eta_i^2\, t = \rho\, \frac{\widetilde{v}^2}{3}\, \frac{l}{\widetilde{v}} = \rho\, \frac{\widetilde{v} l}{3}$ with
l = mixture length; t = time needed by the particle for traveling the mixture length; and

$\widetilde{v} = \frac{l}{t}$ = average velocity of the fluid portions relative to the mean flow. With these substitutions we get from equation (42) the friction stress

$$\mathcal{T} = -\Sigma n^* m\, V_x\, V_y = -\Sigma n^* m\, \frac{\widetilde{v} l}{3}\, (\frac{dU}{dr} - \frac{U}{r}) \qquad (43)$$

This result is similar to the result derived in the previous paragraphs for laminar friction, insofar as the friction stress is determined by $(\frac{dU}{dr} - \frac{U}{r})$ It has been shown

that this term, the "shear velocity," is identical with the velocity gradient of the circular flow relative to the system of coordinates rotating with the angular velocity $\frac{U}{r}$ of the radius vector. In the rotational flow where, relative to the rotating system, no point of the mean flow has a velocity, no shear stresses exist. Evidently the influence of the Coriolis's acceleration has vanished by the above integration, for the result would have been the same without considering Coriolis's acceleration.

If we write $\mathcal{T} = \mathcal{E}\, (\frac{dU}{dr} - \frac{U}{r})$

$$\qquad (44)$$

with $\mathcal{E}$ as "turbulent exchange rate" or "virtual viscosity," we have to keep in mind that $\mathcal{E}$ will not be a constant along the flow, as l will change in the fluid and $\widetilde{v}$ will depend on the velocity gradient itself. 7/ This consideration might lead to the assumption

that shear stress is proportional to $\rho l^2 (\frac{dU}{dr} - \frac{U}{r})^2$ as it is proportional to $\rho l^2 (\frac{\partial U}{\partial y})^2$ within straight flow.

It would lead too far to investigate these relations, which even in the straight-flow case treated by Prandtl are only considered as approximate, and we confine ourselves to the statement

that $\mathcal{T}$ is a function of $(\frac{dU}{dr} - \frac{U}{r})$

We now finally have the answer to the question raised in par. 2.

Equation (42) shows that the equation (1), which usually is ascribed to Prandtl, is based on an error, insofar as only particles starting with $\xi_i = 0$ have been considered. Actually these particles having an angular momentum equal to the average of their layer are the only ones which, in case of irrotational flow, do not cause a momentum transport. All other particles transport a momentum. All

---

7/ $\mathcal{E}$ may be assumed as constant in the special case of irrotational circular flow as long as l is proportional to the radius r while $\widetilde{v}$ is proportional to $l\, (\frac{dU}{dr} - \frac{U}{r}) \sim \frac{r}{r^2} \sim \frac{1}{r}$.

particles with positive ζ', (all particles moving faster than the mean flow) are, on the average, deviated due to Coriolis's forces into outer layers, and transfer to these layers their excess of momentum. So they exert an accelerating force on these layers. On the other hand, particles moving slower are in the average deviated into inner layers and there exert a decelerating force.

5. Comparison with the Vorticity Theory by G. I. Taylor

G. I. Taylor [5] [6] early developed a theory for the mechanism of turbulent friction, according to which friction is not the result of a transfer of momentum but of vorticity. Taylor derives from this theory a formula which, so far as the influence of velocity distribution on friction is concerned, is in accordance with the theory herein described. In the case of rotational flow, where the vorticity is constant over the whole mean flow, there can not be a vorticity transport, and consequently no shear stresses.

By this assumption, Taylor succeeded in making his calculations independent of the local pressure differences, which in the eddying motion are caused by the irregular movement of the fluid portions towards each other. Those pressure differences continuously change the momentum of the portions but, according to Kelvin's circulation theorem, they cannot affect their vorticity.

Taylor concludes that the reason why Prandtl's theory does not apply to circular flow is that he neglects the effect of the local pressure changes on the momentum. Contrary to Taylor's conclusion, the author wants to express another opinion. It has been proved by the result of the last paragraph, that the momentum transfer theory, if applied to all particles and to all velocity directions instead of a selected group with a certain velocity direction, yields a formula which is in accordance with the vorticity transport theory. This means that, at least in this consideration dealing with all particles, there is no error due to neglect of local pressure differences.

The influence of these pressure differences caused by the irregular turbulent motion is the same as the influence of the irregular collisions in the molecular motion. It tends to produce a motion where no direction is preferred. Even if a motion with turbulence only in a radial direction could be imagined, a turbulent transfer in all directions would originate after a short time.

So the discrepancy of the usual Prandtl formula for circular motion is only in an indirect way due to neglect of local pressure differences, for the turbulent transfer in radial direction only, which in deriving this so-called "Prandtl formula" obviously had been considered, is incompatible with the effect of these pressure differences.

It can be shown further that turbulent friction cannot be explained exclusively by either the vorticity transport or by the momentum transport, but that both are contributing to it.

The moment exerted from one circular layer to the adjacent layer corresponds to the moment of momentum, $\int dm \, v_u \cdot r$ [8], transferred between these layers per unit of time. In case of turbulent motion, where we consider portions of fluid instead of molecules (as in the laminar case), the moment of momentum of every portion consists of two parts:

    (1) the part resulting from the motion of the center of gravity of this portion

    (2) the part resulting from the rotation of this portion around its center of gravity.

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[8] v_u = circumferential velocity component of particles.

As in turbulent motion, the diameter of the portion and the mixing-path length are of the same order of magnitude, the above two parts are also of the same order of magnitude. (The velocity differences of the particles and their circumferential velocities due to the rotation are of the same order.)

The momentum transport theory makes allowance only with regard to the first part, while in the vorticity transport theory only the second part is considered. Actually the results of the two theories have to be combined to determine the friction. As both are proportional to

$\frac{dU}{dr} - \frac{U}{r}$, the sum as well is proportional to this term. Only the factor of

proportionality of $\left(\frac{dU}{dr} - \frac{U}{r}\right)$ and the friction will be influenced by changing over from either theory to a combination of both. This may be of importance, if the mixing length is evaluated from friction measurements.

6. **Steady Circular Flow of Viscous Fluids**

The unsteady change of a velocity profile in circular motion under the influence of friction forces will be the subject of another report. [9]

Here we shall confine ourselves to the possible types of steady circular flow. The flow is limited by cylindrical walls of radii r_i and r_e, and the velocities U_i and U_e of the flow at the walls outside of the boundary layers may be known. We are looking for the velocity distribution within the flow.

The condition for steady flow equilibrium for each annular element is that the moments of the shear stresses on its inner and its outer circumference are alike, so this moment must be the same in the entire flow:

$$\tau \cdot 2\pi r \cdot r = const.$$ (45)

or, substituting τ according to equation (44) and considering the virtual viscosity ε as a constant:

$$\left(\frac{dU}{dr} - \frac{U}{r}\right) r^2 = const$$ (46)

or

$$\frac{d}{dr}\left(\frac{U}{r}\right) = \frac{const}{r^3}$$ (47)

The general solution of this differential equation can be written

$$U = \frac{c_1}{r} + c_2 r$$ (48)

Thus the velocity profile of steady flow may be determined by superposing an irrotational and a rotational circular flow.

The importance of the irrotational type of viscous flow lies in the fact that so far as ε can be considered as constant, it represents the flow with constant moment of friction stresses, which is the condition for steady motion. The rotational flow is also a solution, since in it

[9] A remarkable application of the above derived friction laws is given in Part II of this report.

F-TR-2198-ND

there occur no friction stresses at all. The superposition of the two types of flow can be imagined by putting an irrotational flow on a rotating turntable, which does not influence the stresses, as already pointed out by G. I. Taylor.

The friction moment M, which acts in the irrotational flow, must be induced from the boundaries (the walls) rotating with Ω_i and Ω_e, respectively. Thus the Power $M(\Omega_i - \Omega_e)$ has to be furnished, in order to compensate the friction losses in the flow. If the driving moment exerted by the rotating walls ceases, the velocity profile changes until at last there remains only the velocity profile of an irrotational flow.

REFERENCES FOR PART I

[1] W. Traupel, "Neue allgemeine Theorie der mehrstufigen axialen Turbo-maschine," Verlag Leemann und Co., Zuerich 1942.

[2] Frank L. Wattendorf, "A Study of the Effect of Curvature on Fully Developed Flow," (Proc. Roy. Soc. London A Vol. 148 p. 565).

[3] L. Prandtl, "Vortraege aus dem Gebiet des Aerodynamik Kr. Aachen," (1929), p. 1.

[4] Haas, "Einfuehrung in die theoretische Physik," Berlin u Leipzig 1930 (de Gruyter) Vol. II, pp. 268 - 270.

[5] G. I. Taylor, "Fluids in Turbulent Motion," (Proc. Roy. Soc. London A Vol. 135 p. 697).

[6] G. I. Taylor, "Distribution of Velocity and Temperature Between Concentric Rotating Cylinders," (Proc. Roy. Soc. London A, Vol. 151, p. 494).
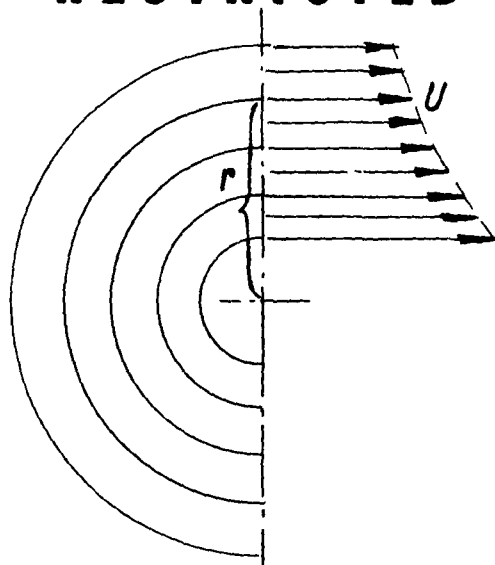
Fig. I - 1 Velocity Pattern U(r) of Circular Flow
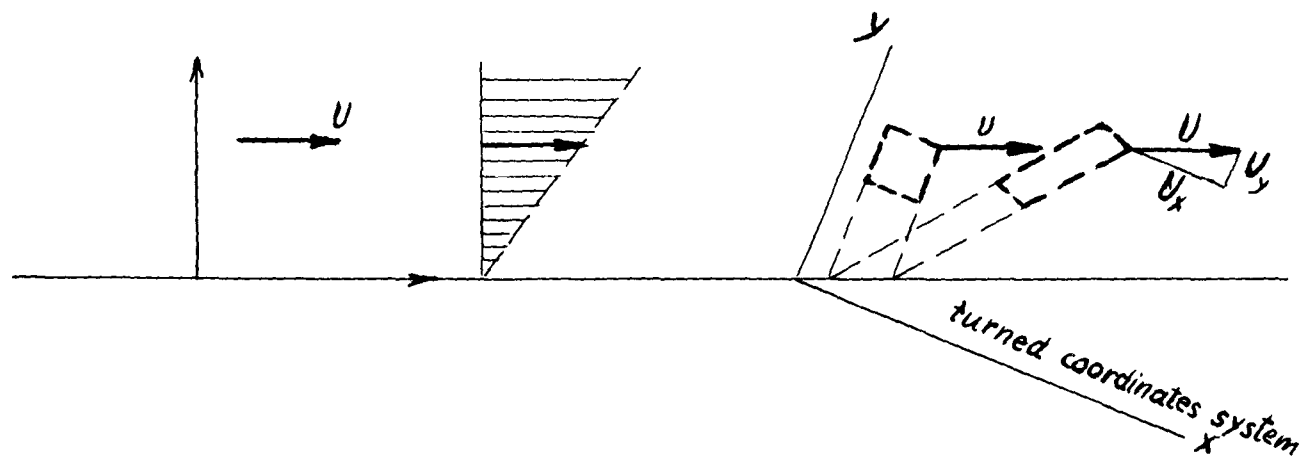
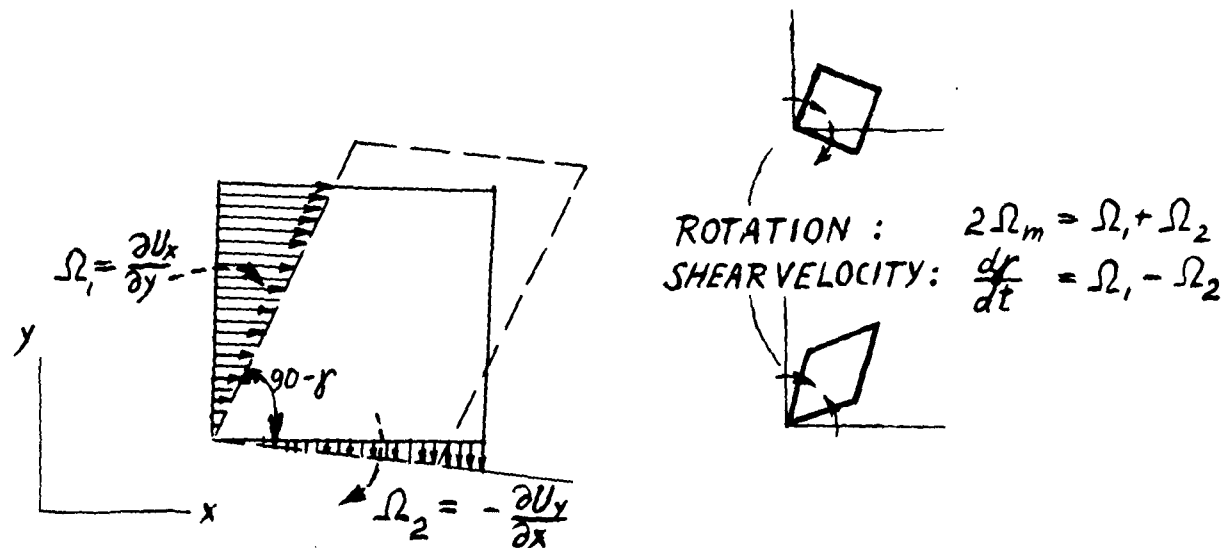Fig. I - 2 Deformation in Straight Laminar Boundary Layer

ROTATION : $2\Omega_m = \Omega_1 + \Omega_2$

SHEAR VELOCITY: $\dfrac{d\gamma}{dt} = \Omega_1 - \Omega_2$

$\Omega_1 = \dfrac{\partial U_x}{\partial y}$

$\Omega_2 = -\dfrac{\partial U_y}{\partial x}$

Fig. I - 3 Rotation and Shear Velocity
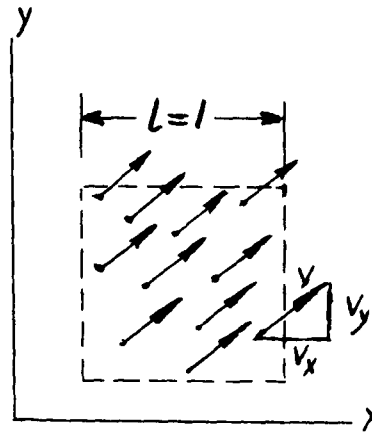
Fig. I - 4 Particles with Equal Velocity v_x, v_y,
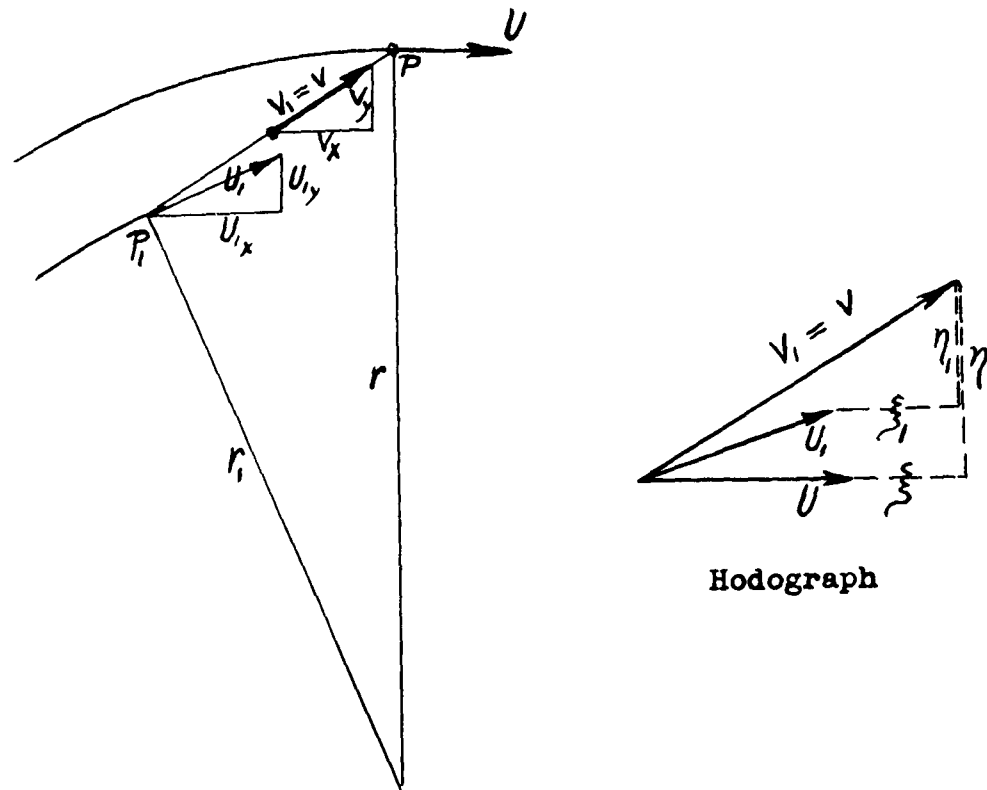Differing at Most by $\Delta v_x, \Delta v_y$

Hodograph
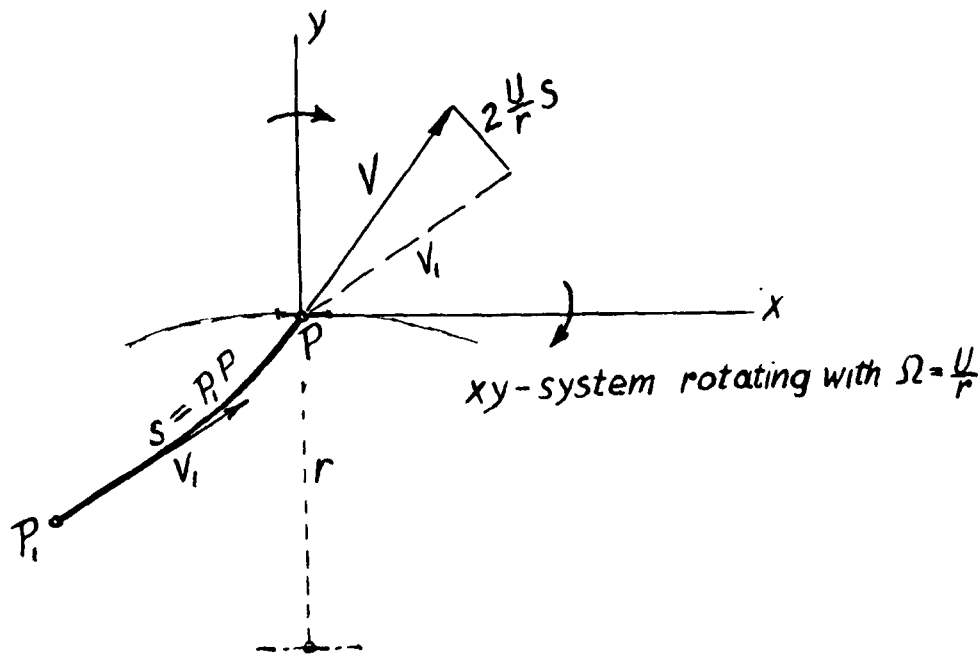
Fig. I - 5 Path of a Molecule From P_1 to P

Fig. I - 6 Path and Velocity Change of a Fluid
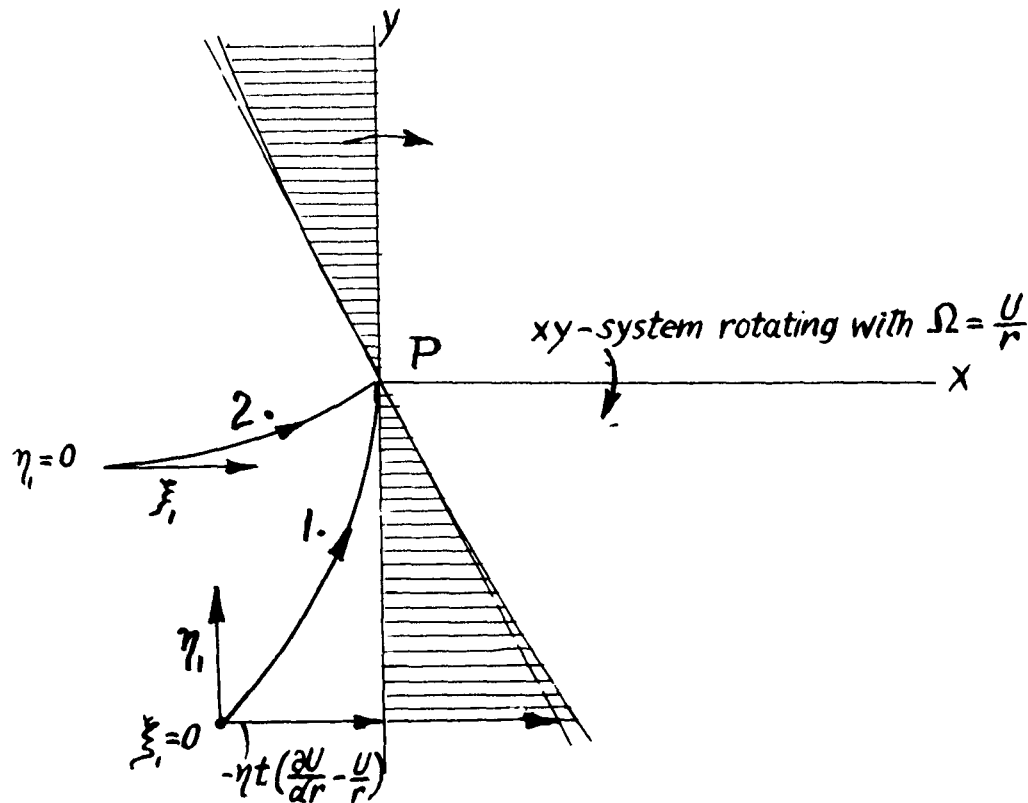Portion Under the Effect of Coriolis Forces

Fig. I - 7 Path of Fluid Portions Starting
With $\xi_1 = 0$ or $\eta_1 = 0$ Respectively

PART II - ENERGY TRANSFER IN CIRCULAR FLOW AND POSSIBLE APPLICATIONS (EXPLANATION OF THE HILSCH OR RANQUE EFFECT)

SYMBOLS

B = moment of momentum

c_p = specific heat

e = coefficient of performance

E = energy of flow

F = shear force

H = heat removed

I = inertia

l = Prandtl's mixture length

L = energy loss of flow

m = mass of molecule or fluid portion

M = shear moment

p = pressure

r = radius

t = time

T = temperature

U = velocity of mean flow

W = work required

y = coordinate normal to the flow

γ = ratio of mass of cold air to total mass of air

ε = exchange rate

ϑ = ratio of temperature drop to maximum adiabatic temperature drop

μ = viscosity of fluid

τ = shear stress

ρ = density of fluid

Ω = angular velocity

Part II - Energy Transfer in a Circular Flow and Possible Applications (The
Explanation of the Hilsch or Ranque Effect).

1. Experiments of Hilsch

In 1946 a treatise on "Expansion of a Gas in the Centrifugal Field and Application for
Refrigeration" was given by Rudolf Hilsch, (ref. [1] , [2]) Germany. Hilsch referred to an
experiment which Ranque [3] conducted in France in 1933. The device which Hilsch used
with his experiments is shown in Fig. II-1. It consists of a tube "a" about 2 in. in diameter
and about 12 in. in length, a nozzle "b" leading tangentially into the tube, an orifice "c" and a
valve "d."

Air of about 60 psi gage and 68°F is expanded through the nozzle and rotates inside the
tube, while a small axial velocity which enables the flow to proceed to the valve is maintained.
As long as the valve is fully open, the whole mass of air which enters through the nozzle passes
out of the valve, and external air is sucked into the right-hand part of the tube. By partially
closing the valve, part of the air of the central parts of the tube takes a direction opposite to
the flow near the tube walls, and leaves through the right-hand exit. Now, while this part of
the air is of noticeably lower temperature than that of the input, the air which flows near the
walls of the tube and leaves through the valve becomes hotter. The surface of the tube on the
left side is very hot, while that of the right side forms ice. By changing the area of the valve
and of the orifice, optimum values of the process can be obtained. Figure II-2 shows test results
of Hilsch which indicate that if the ratio of cold effluent gas to total mass of gas is about 30%,
lowest temperatures of about -100°F are attained. Naturally, when half of the air leaves
through the cold exit and the other half through the hot exit, and no heat exchange with surround-
ing parts takes place, the temperature of the cold air is as much below initial temperature as
the hot part is above.

Many attempts to explain the phenomenon have been made (Refs. [3] , [4] , [5] , [6] , [7] , [8])
but, so far as the author is aware, there was, until now, no exact explanation. In the following
pages, an explanation will be presented which accounts for at least a substantial part of the
phenomenon.

2. Posing of the Problem

The preceding experiments indicate that there must be some sort of energy transfer from
one part of the flow to the other, thus decreasing the total temperature of the one part and
increasing the temperature of the other one.

Now it is known that any viscous flow has the tendency to equalize its velocity distribution.
For instance, this is the case with the mixture of two streams of different velocities, or if
there is a velocity gradient normal to the flow. The reason is that the viscous friction in the
flow tends to cause a velocity distribution with the least shear stress. So, layers of high velocity
will be retarded and will transfer their energy to the layers with low velocity.

On the other hand, it is known that this transfer of kinetic energy is combined with a transfer
of heat energy. This means that temperature differences as well, will be equalized, and even
more quickly than the velocity differences, as can be shown in Ref. [9] .

The problem with which we must deal is: How can an energy transfer, which no doubt takes
place in the Hilsch device, be possible without an equalization of the temperature? We shall
see in the following paragraphs that this actually happens in the Hilsch device.

3. Velocity Distribution and Shear Stress Within a Fluid with Rotary Motion

When the compressed air passes the nozzle and enters the circular tube, a velocity profile is built up, inside the tube, which complies with the law of constant angular momentum (c.a.m.) or of irrotational flow, and is shown in Fig. II-3. From the law of constancy of energy in the fluid, the temperature distribution can be calculated, and is shown in Fig. II-4. The inner elements of the flow, which have greater velocity than the outer ones, show a correspondingly greater temperature drop. We shall now consider whether or not this velocity profile can be maintained. The shear stress in a viscous fluid may, according to Newton, be expressed by:

$$\tau = \mu \frac{dU}{dy} \tag{1}$$

where τ = shear stress, μ = viscosity of the fluid, U = velocity of flow, y = length normal to the flow.

As we have to deal with rotary motion, the velocity gradient $\frac{dU}{dy}$ of the straight motion has to be replaced by the term $\frac{dU}{dr} - \frac{U}{r}$

$$\underline{1}/ \tag{2}$$

The term $\frac{dU}{dr} - \frac{U}{r}$ indicates the velocity gradient with respect to a flow which underlies

no shear stress. In such a flow, no relative motion of the particles occurs; it rotates like a rigid body, its angular velocity being constant; i.e.,

$$\Omega = \frac{U}{r} = \text{constant} \tag{3}$$

The shear force F, which works on the annular element and tends to reduce the velocity difference between two elements, is:

$$F = \mu \left(\frac{dU}{dr} - \frac{U}{r} \right) 2 \pi r \tag{4}$$

for the unity of axial extension.

The friction moment around the center of the motion therefore will be:

$$M = \mu \left(\frac{dU}{dr} - \frac{U}{r} \right) 2 \pi r^2 \tag{5}$$

Equations (4) and (5) refer to the purely laminar flow. A similar relation exists for turbulent flow (Refs. [9] and [10]); the turbulent shear stress can, in case of circular motion, be expressed in the form:

$$\tau_{turb} = \varepsilon \left(\frac{dU}{dr} - \frac{U}{r} \right) \tag{6}$$

with

$$\varepsilon = \rho l^2 \left(\frac{dU}{dr} - \frac{U}{r} \right) \tag{7}$$

(ε = exchange rate; 1 = Prandtl's mixture length; ρ = density of fluid)

Now we have to calculate the velocity gradient of the flow with c.a.m. Its velocity distribution is

$$U \times r = \text{constant} = k \tag{8}$$

$\underline{1}/$ As to the derivation of shear stress in circular flow see Kassner [10] .

hence:

$$\frac{du}{dr} = -\frac{K}{r^2}, \text{ and } \quad \frac{u}{r} = \frac{K}{r^2} \tag{8a}$$

and the velocity gradient yields

$$\frac{du}{dr} - \frac{u}{r} = -\frac{2K}{r^2} \qquad \text{(See Fig. 5.)} \tag{8b}$$

As it is very likely that the mixture length, 1, is almost proportional to the radius, 2/ this would lead to the assumption that ε = constant or almost constant along the radius. So, comparing equations (3) and (6), it appears that, when considering the effect of turbulence on this problem, the only difference between laminar and turbulent motion is that ε is some order of magnitude higher than μ .

We now can assume that if shear stresses are to influence the velocity distribution of the irrotational flow, the results will be basically the same with the assumption of either laminar or turbulent friction.

The moment of shear stress may therefore be expressed by:

$$M = 2\pi r^2 \varepsilon \frac{2K}{r^2} = 4\pi \varepsilon K = \text{Constant} \tag{9}$$

As equation (9) shows, this moment is constant along the radius. In other words: Between two annular elements of the flow with c.a.m., there is no difference in moment, and therefore, no acceleration force, which might act in changing the velocity distribution. In order to maintain this distribution, it is only necessary to supply the energy loss due to the friction.

Contrary to this, in a flow with Ω = constant, i.e., in a rotational flow, there is, because of $\frac{du}{dr} = \frac{u}{r}$ and $\tau = 0$, no friction moment. It may be of interest that, as can be shown mathematically, there exists a flow with a constant moment between that of the irrotational and that of the rotational flow. Such a flow can be imagined by superposing the irrotational and the rotational flow.

4. The Change of Velocity Distribution of a Rotating Flow

While the above considerations make the assumption of infinite extension in radial direction, the real flow is passing between boundaries. Let us consider, for example, the boundary which is given by the fact that the flow does not fill up the whole area of the tube, but keeps a central part almost undisturbed. We regard the innermost annular element of the flow, which has, according to c.a.m., the highest velocity. This velocity may be calculated from the pressure drop between the nozzle and the external pressure, which is likely to be maintained in the central part of the tube. While this element, a (Fig. II-6), exerts a dragging effect equal to the constant-friction moment of the flow on the adjacent outer element, b, there is no such dragging effect (given by the dotted arrow) exerted on the inner boundary of the element, a. Thus, the equilibrium of this element is disturbed, and the change of moment per unit time causes a deceleration of this element. This will stop only when the frictional moment between a and b has vanished, that is, when a and b have the same rotation ($\Omega_a = \Omega_b$). Now the equilibrium of the annular

2/ V. Karman made a similar assumption (See Ref. [12]).

element, b, is disturbed and the above process repeats. Obviously this process will continue until the flow with the frictional moment, $M_F = 0$, that is, a rotational velocity distribution, is reached.

On the outer boundary of the flow, a similar process takes place. On the surface of the annular element, c, there is a constant friction moment working towards the adjacent element, d, while at the outside surface the equilibrium is disturbed, as the force given by the dotted arrow does not exist. Therefore the element, c, yields to the dragging force of element, d, and accelerates, as long as the frictional moment between c and d is working. Then the process changes over to element d and so on.

Another explanation of the disturbance of equilibrium of the flow can be given by the following consideration: any viscous flow with a velocity gradient, e.g., an irrotational flow, underlies a shear stress. On account of the equilibrium of the flow, the shear forces in tangential direction must be equal to those in radial direction.

Consider now a particle in the boundary of the flow (Fig. II-7). There is still a moment of the radial shear-components, while tangentially there acts only a stress on the inside surface of the particle; an equivalent stress outside is missing. So, the momentum equilibrium of the particle is disturbed, and a conversion to another type of flow, i.e., the rotational flow, where no shear force is working, ensues.

On the whole it is apparent that, by the process just explained, the flow distribution of c.a.m. is converted into that of the rotational flow, in the way shown in Fig. II-8. Here four different velocity distributions, which occur one after the other, are to be seen.

While on the inside boundary of the flow the preceding process is likely to correspond with what actually happens in the tube; on the outside boundary, there is the wall of the tube. Here, near the wall, the well-known boundary-layer effect acts, so that the velocity distribution will be similar to curve A (Fig. II-9). This means that, instead of no force acting on the outside element, there is a force by wall friction which, to a certain amount, tends to decelerate this element. If the force exerted by wall friction is assumed smaller than that of turbulent friction in the flow (and this is presumably the case), the velocity distribution will be similar to Fig. II-10. The wall friction will convert the profile 4 to the forms 5 and 6. It should be noted, that the velocity distribution on the outer part of the flow is only of small importance to our problem.

In order to investigate the velocity and energy distribution analytically, we shall first consider the flow conversion, without regarding the effect of wall friction.

In the absence of wall friction there is no external moment acting on the flow, so that we can compute the rotational velocity distribution, which is formed out of the irrotational distribution, from the fact that the moment of momentum of irrotational flow must be equal to that of the rotational flow.

The moment of momentum can be expressed by $B = \int dI \cdot \Omega$ where dI = inertia of an annular element and Ω = angular velocity of the element.

In the case of irrotational flow, there is

$$dI = r^2 \, dm \quad \text{when} \quad dm = 2\pi r \rho \, dr = \text{mass of annular element}$$

and $\Omega = \dfrac{U}{r}$, while $U = \dfrac{K}{r}$

By integration we obtain:

$$B_I = \pi \rho K (r_e^2 - r_i^2) \qquad \underline{3/} \qquad (10)$$

where r_e and r_i are outer and inner radius of the flow.

For the rotational flow, the moment of momentum yields

$$B_{II} = \frac{\pi}{2} \rho \, \Omega (r_e^4 - r_i^4) \qquad (11)$$

Equalizing the two expressions, equations (10) and (11), for rotational and irrotational flow, we obtain an equation for the velocities on the outer boundaries U_{eII} and U_{eI}.

$$U_{eII} = 2 U_{eI} \frac{r_e^2}{r_e^2 - r_i^2} \qquad (12)$$

Equation (12) shows that $U_{eII} > U_{eI}$

in case $r_i \longrightarrow 0$, $U_{eII} \longrightarrow 2 U_{eI}$, or the velocity on the outer boundary increases, while the conversion from irrotational to rotational flow takes place to double the original value. Fig. II-11 shows, for example, the velocity distribution before and after the conversion.

With the knowledge of the velocity distribution, we can proceed to the comparison of the energy of the two kinds of flow.

5. The Energy of a Flow with Rotary Motion

The energy E of the flow with c.a.m. must be found in the energy E_2 of the rotational flow, if allowance is made for the friction losses L, which occur, while the flow conversion proceeds. Thus we have:

$$E_2 = E_1 - L \qquad (13)$$

The energy is

$$E = 1/2 \cdot \int dm \, U^2 \qquad (14)$$

by substituting $U^2 = \frac{K^2}{r^2}$ for irrotational flow, we obtain:

$$E_I = \pi \rho K^2 \ln r_e/r_i \qquad (15)$$

where ρ = density of fluid

In the same way we compute the rotational flow:

$$E_{II} = 1/2 \int dm \, \Omega^2 r^2 \qquad (16)$$

$\underline{3/}$ For simplifying this and the following equations, a constant density ρ in the flow was assumed.

hence

$$E_{II} = \pi \rho \, K^2 \, \frac{r_e^2 - r_i^2}{r_e^2 + r_i^2} \tag{17}$$

As E_{II} is less than E_I, a certain amount of energy must have been lost, and an over-all efficiency can be computed in the form

$$\eta = \frac{E_{II}}{E_I} = \frac{(r_e/r_i)^2 - 1}{(r_e/r_i)^2 + 1} \cdot \frac{1}{\ln r_e/r_i} \tag{18}$$

Figure II-12 shows this efficiency as a function of the ratio $r_i : r_e$. The less the velocity differences between the two forms of flow, the less the energy loss, which appears as friction heat; but also, the less are the possibilities for cooling effects.

6. Computation of Energy Loss Along the Radius

For an exact description of the energy transfer, the distribution of the friction losses along the radius must be found. For this purpose we use again the impulse-momentum equation in the form

$$M = \frac{dB}{dt} \tag{19}$$

where M = moment around the center of motion, and $\frac{dB}{dt}$ = change of angular momentum per unit of time.

Equation (19) yields to

$$\int M \, dt = \int dB = B_2 - B_1 \tag{20}$$

Let us consider first the change of angular momentum on the inner side of the flow between r_i and r_m. As described in par. 4, "The Change of Velocity Distribution of a Rotating Flow," the velocity profile changes over from a to b (Fig. II-13). There is a constant friction moment acting in those parts of the flow where the distribution a is maintained. When the distribution changes to b, starting from the radius r_i and progressing to r_m, the moment in those parts of the flow where the conversion has taken place vanishes. So the fluid elements in the middle parts of the flow, on which the friction moment acts during a longer time, have higher friction losses. If we assume that the velocity profile is converted, as shown in Fig. II-13, and designate the conversion point by r_i^*, the angular momentum after the conversion is a function of r_i^*.

The angular momentum before the conversion was

$$B_1 = \pi \rho \, K \left(r_m^2 - r_i^2 \right) \tag{21}$$

After the conversion:

$$B_2 = \frac{\pi}{2} \rho K \, \frac{r_i^{*4} - r_i^4}{r_i^{*2}} + \pi \rho K \left(r_m^2 - r_i^{*2} \right) \tag{22}$$

The acting moment, which gave rise to the change $B_1 - B_2$ is:

From (9), $\qquad\qquad M = 4 \pi \varepsilon K = $ Constant

$$\int M \; dt = .4\pi\varepsilon\kappa t = B_1 - B_2$$

hence:

$$\varepsilon t = \frac{\rho}{4}\left(r_i^{*2} - r_i^2\right) - \frac{\rho}{8}\frac{r_i^{*4} - r_i^4}{r_i^{*2}}$$

$$\varepsilon t = \frac{\rho}{8} r_i^2 \frac{\left[(r_i^*/r_i)^2 - 1\right]^2}{(r_i^*/r_i)^2} \tag{23}$$

A similar equation can be found for the outer part of the flow. When the exchange rate ε is known, the time while an element is undergoing a friction can be determined from equation (23). Figure II-14 shows εt as a function of r as an example.

The energy loss along the radius can now be calculated. The energy per second, transferred from the inner to the adjoining outer element by action of the friction moment, is

$$\frac{\partial E}{\partial t} = M\Omega \tag{24}$$

The energy change per element $2\pi r \; dr$ is: $\frac{\partial E}{\partial r}$

$$\frac{\partial E}{2\pi r \partial r} = \frac{1}{2\pi r}\frac{\partial}{\partial r}\int (M\Omega) \, dt \tag{25}$$

as M = constant, $\int \frac{d}{dr}(M\Omega)$ simplifies to

$$\int M \frac{d\Omega}{dr} \, dt$$

Hence

$$\frac{\partial\Omega}{\partial r} = \frac{\partial}{\partial r}\left(\frac{U}{r}\right) = \frac{1}{r}\left(\frac{dU}{dr} - \frac{U}{r}\right) \tag{26}$$

Equation (26) yields, along the curve U x r = K

$$\frac{\partial\Omega}{\partial r} = -\frac{1}{r^*}\frac{2K}{r^{*2}} \tag{27}$$

Thus we have:

$$\frac{\partial E}{2\pi r^* \partial r^*} = \int \frac{4\varepsilon K^2}{r^{*4}} \, dt = 4\varepsilon K^2 t \frac{1}{r^{*4}} \tag{28}$$

Substituting equation (23) we obtain

$$\frac{\partial E}{2\pi r^* \partial r^* \rho} = \frac{K^2}{2 r_i^2}\left(\frac{r_i}{r_i^*}\right)^2\left[1 - \left(\frac{r_i}{r_i^*}\right)^2\right]^2 \tag{29}$$

A similar calculation can be made with respect to the outer parts of the flow, yielding

$$\frac{\partial E}{2\pi r^* \partial r^* \rho} = \frac{K^2}{2 r_e^2}\left(\frac{r_e}{r_e^*}\right)^2\left[1 - \left(\frac{r_e}{r_e^*}\right)^2\right]^2 \tag{30}$$

To check equations (29) and (30), we integrate this equation from r_i to r_m, and from r_e to r_m, and add the integrals. We obtain

$$E = E \bigg/_{r_i}^{r_m} + E \bigg/_{r_m}^{r_e} = \pi \rho K^2 \left[\ln \frac{r_e}{r_i} - \frac{1 - \left(\frac{r_i}{r_e}\right)^2}{1 + \left(\frac{r_i}{r_e}\right)^2} \right] \qquad (31)$$

which is equivalent to the energy loss, computed in par. 5, "The Energy of a Flow with Rotary Motion." So we conclude that the assumptions made for the distribution of the energy along the radius have been chosen correctly.

It must be mentioned that, because of the exact differential equation which controls the preceding described mechanism, the conversion from the velocity distribution of irrotational flow to that of rotational flow is not progressing like a wave from the outer parts while the inner parts are at rest (curve b, Fig. II-15), until the conversion takes place. Actually, there will be a velocity distribution, as curve c shows, but the difference between b and c certainly will only slightly affect our problem.

7. Influence of Heat Transfer

Experiments and calculations show that the momentum transfer and the heat transfer are of the same order of magnitude [13] and [14] . We shall proceed to investigate how this heat transfer affects the temperature distribution.

For this, let us first consider the possibilities of heat transfer in the irrotational flow. A particle of fluid which moves from an annular element A to B (Fig. II-16) undergoes a pressure and temperature change along its path which in case no heat is added or subtracted, yields to the adiabatic law. As in the irrotational flcw, the pressures and temperatures along the radius are distributed according to the adiabatic law; a heat equilibrium between the particles is maintained, which means that a particle moving from element A to B does not differ from those to be found there. On the contrary, if there is a temperature distribution which differs from the adiabatic law, the presence of a strong turbulent motion in the fluid is apt to re-establish the adiabatic distribution.

As is known, there is besides the turbulent motion, a molecular heat transport to be considered, which tends to equalize the temperature differences, 4/. This transport is steadily going on, but as the turbulent heat transport is by far more efficient than the molecular heat transport, the adiabatic temperature distribution will virtually be maintained.

During the conversion of irrotational to rotational flow, the temperature is still fairly like that of the irrotational flow, while the velocity and therefore the pressure differs very much from that flow. Again, particles moving from element A to B will be subject to the pressure changes along the radius, and will subsequently undergo an adiabatic temperature change. As the particles in A will be different from those in B, to such an extent as is the difference between the adiabatic temperature before and after the conversion from irrotational to rotational flow,

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4/ A similar process takes place in our atmosphere. In the absence of heat transfer to the universe or to the earth, the temperature of the atmosphere would be constant along the altitude. But in the troposphere, the rather strong turbulent motion (currents) tends towards an adiabatic distribution. So finally we have a temperature distribution which is between the adiabatic and the isothermic (Ref. [15]).

a heat transfer will ensue. For instance, assuming A as being hotter than B, particles from A entering the element B will add to the temperature of that element, while equivalent particles of B entering element A will reduce the temperature of this element. This will stop only when a new adiabatic temperature distribution, corresponding to the pressure distribution of the rotational flow, is attained.

So we have, after the conversion from irrotational to rotational flow, a velocity distribution according to $\sqrt{2}$ = constant, and a temperature distribution which, assuming complete heat exchange by turbulent motion, is equivalent or almost equivalent to the adiabatic temperature distribution in a flow with constant angular velocity.

Figure II-17 shows the temperature distribution of the irrotational flow (curve a), the energy loss during the conversion from irrotational to rotational flow (vertical distance between curves a and b), calculated from equations 29 and 30. The losses are zero at the boundaries where the conversion begins, and have a maximum at the point where the conversions of the velocity profile, proceeding from both boundaries meet.

The mean temperature of the irrotational flow which - again neglecting density changes - is equal to the integral $\frac{1}{\pi r^2} \int_a 2\pi r \, T \, dr$, is given by line T_{m1} , the mean temperature with regard to the temperature rise by friction during the conversion is given by line T_{m2} , equivalent to $\frac{1}{\pi r^2} \int_b 2\pi r \, T \, dr$. The temperature distribution of the rotational flow, the mean temperature of which must be equal to T_{m2} , is given by curve c. Adding the temperature rise by impact of the rotational velocity, we obtain the total temperature represented by curve d. Necessarily, the integral $\frac{1}{\pi r^2} \int_d 2\pi r \, T \, dr$ of curve d must be equal to the initial temperature T_o . This fact may be used as a check.

In Fig. 18, the total energy of the flow after the conversion from irrotational to rotational flow for the ratio $r_i : r_e$ = 0.2, 0.4, 0.6 and 0.8 is to be seen. The lowest temperatures are connected by a dashed line. Apparently a minimum temperature, which means the highest temperature drop, can be attained with the ratio $r_i : r_e \sim 0.6$. This means the dimensions of the orifice of the tube should be so chosen that the irrotational flow fills an annular cross section which extends from the walls to about 60% of the tube radius.

While Fig. II-18 showed the temperature distribution after the conversion of the velocity profile, the question arises concerning the temperature distribution during the process of conversion. An evaluation shows that with an increasing amount of conversion, the total temperature, consisting the static temperature and the temperature rise due to the impact of the innermost converted layer, rapidly approaches the lowest attainable value. This means that, almost immediately after the conversion has started, this layer has nearly the maximum attainable temperature drop. A rough sketch of the temperature distribution is to be seen in Fig. II-19, the progress with the time marked by Nos. 1 - 4. It shows that the temperature gradient along the radius is very steep at the beginning of the conversion. We shall see in par. 8, "Pressure Equilibrium," that a considerable part of the Hilsch effect can be explained by this phenomenon, shown in Fig. II-19.

8. Pressure Equilibrium

Let us now consider how the distribution of pressure affects the process. Immediately after the expansion of the air through the nozzle, there is a pressure distribution as shown in

Fig. II-20, right part, Section I. The pressure in the central parts, where a rotational flow builds up, is near to that of the surrounding atmosphere. The conversion from irrotational to rotational flow implies a change in the pressure distribution, since in the latter the pressure rise along the radius is proportional to the square of the radius, while in the former a hyperbolic distribution prevails.

While the flow is progressing helically in the tube from the orifice to the valve, the pressure difference between the central parts and the wall becomes smaller. Consequently, there must be a pressure rise in the central parts of the flow; this means that there exists a pressure gradient in the central parts toward the orifice, where the pressure of the surrounding atmosphere prevails. Thus, while the conversion from irrotational to rotational flow goes on, the innermost layers of the flow, which have the lowest temperature, separate from the flow, and under the influence of the pressure gradient, accelerate in axial direction towards the orifice and flow out through this orifice. Now the mass of fluid flowing in the direction of the valve having decreased, parts of the flow which were at the outer layers progress to the center, to fill the space left by the out-flowing cold fluid. Again they come to regions of lower pressure and, their angular momentum being unchanged, their velocity increases and their temperature drops. Again, by the friction of the flow with constant angular momentum, a conversion to rotational flow takes place, the kinetic energy being transferred to the outer parts of the flow. Having reached the region where the pressure drop in the direction of the orifice prevails, the inner layers of this flow start to flow out through the nozzle. This process goes on, as long as the conversion from irrotational to rotational lasts. So we see that by this progress not only an infinitely small, but a rather large amount of the out-flowing fluid attains a very low temperature. This temperature is, even if only a very small layer of the irrotational flow has been converted, almost equal to the lowest temperature of the process, as pointed out in par. 7. It does not matter that the temperature gradient along the radius is very steep, as only the innermost layers separate from the flow and leave through the orifice.

Let us summarize now the whole process of producing cold air: By the expansion of a gas in a tube, we first have a temperature distribution which complies with the law of constant energy of every particle. This means a strong temperature drop in the central parts of the flow. By the shear stress working in the irrotational flow, the conversion into rotational flow takes place, and the kinetic energy is transferred to the outer parts of the flow. With the heat transfer of turbulent motion, a new adiabatic temperature distribution which complies with, or almost with, the new pressure distribution of the rotational flow ensues. As this distribution still involves lower temperature in the central parts than at the walls of the tube, we find that, by the conversion from irrotational to rotational flow, even considering full effect of turbulent heat exchange in the flow, not only the kinetic energy but also the static temperature is lower in the center than at the walls. In addition, the pressure distribution in the tube affects the outflow of the gas in such a way that the fluid, before leaving the tube through the orifice, undergoes a further pressure and temperature drop, so that a great part of the cold flow leaves at the minimum temperature of the rotational flow.

The preceding is the explanation of the cooling effect of the Hilsch device but we may formulate it also in the following way:

In any viscous flow with a velocity gradient, there is a kinetic energy transfer and a heat transfer which work against each other. If there is an additional pressure gradient normal to the flow, a resulting transfer of total energy is possible. Without such a pressure gradient, kinetic and heat-energy transfer are of equal magnitude, and no resulting transfer of total energy ensues.

9. Comparison With Test Results

With the knowledge of the temperature distribution, we may proceed to calculate the mean

temperature of cold air as a function of the ratio μ of the mass of cold fluid to total mass of fluid.

As Hilsch adjusted his device to get best results, we too shall take the ratio $r_i : r_e = 0.6$, which gave maximum temperature drop. The process of the successive progressing of air towards the center of the tube makes it difficult to compute exactly the mean temperature of the cold air. We shall take account of this effect by making the simplifying assumption that the inner parts of the flow add more, namely inverse to the radius, to the temperature of the cold air. That means, we do not take the mean temperature, which in a circular flow is

$$\frac{1}{\pi r^2} \int_{r_i}^{r_e} 2\pi r \, T \, dr \; , \text{ but the value } \; \frac{1}{r} \int T \, dr$$

The evaluation of this integral is shown in Fig. II-21 as a function of the ratio μ for a value $r_i/r_e = 0.6$ and a pressure in front of the nozzle of 142 lb/sq in. Compared with this curve is the temperature distribution according to Hilsch's test (dotted curve of Fig. II-2).

It is apparent that the theoretical calculation agrees rather well with the test results. So we can say that the above-made assumptions are chosen rather correctly. Only at very low values of μ , is there a discrepancy, which is due to the fact that, if the discharge of cold air is very small, the effect of being heated up by surrounding hotter parts influences the temperature very much. These temperature losses will become relatively small, if greater amounts of cold air ($\mu > 0.2$) are discharged.

Presumably, the compressed air of Hilsch's device had about 100% humidity. So the heat transferred from the vapor to the air, while the vapor changes over to a liquid and finally to snow or hail, may amount to about 15 BTU/lb. This would be equal to a temperature increase, of the air, of about 30°F, and leads to the assumption that a further temperature drop may be gained by using dry air. We may suppose, therefore, that the effect of pressure distribution, explained in par. 7, would be still greater.

Hilsch noticed a bubbling noise when the tube was working best, i.e., when the temperature drop was highest. This noise is probably caused by water drops or grains of ice, which formed in the central parts of the flow and are centrifuged towards the tube walls. It is obvious that by this process the cold-air mass is very dry, while the hot-air mass, which evaporates the ice grains, becomes humid.

10. Applications

Let us first consider the application of the Hilsch tube as a refrigeration device. The coefficient of performance of a refrigerator is the ratio e of heat H removed to work W required. We shall consider here the case of a continuous cold air stream.

For the Carnot process this ratio yields:

$$e = \frac{H}{W} = \frac{T_0}{T_0 - T_1} \tag{34}$$

where T_0 = temperature of surrounding air, T_1 = cooling temperature. It is ∞ at $T_0 - T_1 = 0$. The same formula is valid for the expansion turbine with adiabatic compression.

With the Hilsch tube, the removal of heat equals the amount μ of cold air times the temperature drop, which can be expressed as percentage of the maximum adiabatic temperature

F-TR-2198-ND

drop of the process η $(T_0 - T_1)$. So we have:

$$H = \varphi \cdot \eta \cdot (T_0 - T_1) \cdot c_p \tag{35}$$

The work which is required is given by the adiabatic compression of the air, and amounts to

$$W = (T_0 - T_1) \frac{T_0}{T_1} \cdot c_p \tag{36}$$

In case the energy of the hot air can be regained, only the compression of cold air has to be taken into consideration and the work will be

$$W' = \varphi (T_0 - T_1) \frac{T_0}{T_1} \cdot c_p \tag{37}$$

For those two cases, the coefficient of performance yields:

$$e = \varphi \cdot \eta \frac{T_1}{T_0} \tag{38}$$

$$e = \eta \frac{T_1}{T_0} \tag{39}$$

An ideal performance of the device can be assumed if the temperature drop equals the adiabatic, i.e., $\eta = 1$.

Hence:

$$e_0 = \frac{T_1}{T_0} \tag{40}$$

Cooling of the hot part $(1 - \varphi)$ of the gas until the temperature T_0 is reached, and mixing it with the cold part φ, yields again equation (35). In this case, no regain of energy being possible, e is to be expressed by equation (38).

In Fig. II-22 the coefficient of performance is plotted against the temperature drop $T_0 - T_1$. The coefficient is zero if the temperature drops to absolute zero. With the temperature drop becoming smaller, the coefficient of performance for Carnot and the expansion turbine is increasing rapidly, becoming infinite at $T_0 - T_1 = 0$, while that of the Hilsch tube is increasing linearly with decreasing $T_0 - T_1$, and reaches the value 1.0 at $(T_0 - T_1) = 0$. The dotted line is for $\varphi \eta = 0.4$, a value which may be reached with further improvements. It is clearly understood, from Fig. II-22, that the Hilsch tube can compete with a normal refrigeration device only when very great temperature drops are to be considered.

Another application of the Hilsch device may be found in gas liquefaction, as here the use of a reciprocating expansion has been retarded by mechanical difficulties of lubrication at the low temperatures. P. Kapitza overcame these difficulties by the use of a small radial expansion turbine. This turbine may be replaced by the Hilsch device. It may even be possible to regain part of the energy which is transferred from the central parts to the outer parts of the fluid by providing a diffusor-like exit for the outer part, instead of the valve now used.

In fast flying aircraft, the device may be used to cool the air which is taken into the cabin, or the control device of the guided missile, the necessary pressure in front of the nozzle being supplied by the impact of the flight velocity. Though the efficiency of this device is less than that of an expansion turbine, it has the advantage that no movable parts are required.

The Hilsch effect gives an explanation of a phenomenon which often occurs when a gas expands in a valve or nozzle. Usually, immediately behind the nozzle, the pipe becomes very cold, so that formation of ice can be observed. Figure II-23 shows what probably happens in the flow. When the fluid flows through the contraction caused by the nozzle, the flow filaments, with the exception of the central one, assume a circular motion, converging in front of the nozzle and diverging behind the nozzle. In this circular motion the velocity distribution of the irrotational flow builds up near the edge of the nozzle (Fig. II-23), which changes over, at least in the layers near the edge, by the effect of shear stress to a rotational flow. Consequently, these layers are subject to a drop of total temperature. As the vortex, which usually builds downstream of the nozzle, consists of these layers, it is evident that it will have the temperature of these layers. By way of heat exchange through the wall of the pipe, heat is extracted from the surface of the pipe, forming ice when the surrounding air is humid.

Similar effects may result when the flow in turbines or compressors is to be considered. There will, no doubt, be an energy transfer in the curved flow through a series of compressor or turbine blades, in such a way that the energy from the inner parts of the curved path will be transferred to the outer parts. Hence, the total temperature on the concave side of the blade will be increased, while that of the convex side will be decreased. The energy transfer may to some degree influence the heat transfer from the fluid to the walls, and the separating of the boundary layer on curved walls. As the conversion from irrotational to rotational flow depends on the velocity gradient and the time, in most cases of a flow around a curved body only the innermost layers of the flow will be converted during the short time the flow passes the body. Only in case the radius of the curved path is very small may a more considerable part of the flow be converted.

REFERENCES FOR PART II

[1] Rudolf Hilsch "Die Expansion von Gasen im Zentrifugalfeld als Kaelteprozess," Zeitschr. Naturf. I (1946) p. 208.

[2] Rudolf Hilsch "The Use of the Expansion of a Gas in a Centrifugal Field as a Cooling Process" Review of Scientific Instruments, Vol. 18/No. 2 1947 p. 108 (unabridged translation of No. 1)

[3] G. Ranque "Experiences sur la detente giratoire avec productions simultanees d'un echappement d'air chaud et d'air froid." Le journal de physique et le Radium 1933, suppl. p. 112.

[4] U. S. Patent 1,952,281 by G. Ranque "Method and Apparatus for Obtaining From a Fluid Under Pressure Two Currents of Fluids at Different Temperature."

[5] "The Maxwellian Demon," Industrial and Engineering Chemical Reports on the Chemical World Today. Technology p. 5, May 1946.

[6] "Low Temperature Research," Industrial and Engineering Chemistry. Dec. 1946, pp. 5 - 14.

[7] "The Maxwellian Demon," Popular Science Monthly, May 1947.

[8] Miscellaneous Developments in German Science and Industry p. 29, Report No. 68. Joint Int. Objectives Agency, Washington, D. C.

[9] L. Prandtl "Stroemungslehre" III, par. 4.

[10] R. Kassner "The Laws of Shear Stress in Circular Flow."(Part I of this report.)

[11] L. F. Wattendorf "A Study of the Effect of Curvature on a Fully Developed Flow," Proc. Roy. Soc. London A Vol. 148, p. 565.

[12] V. Karman "Aerodyn Theory," Vol. III, p. 132

[13] L. Prandtl "Stroemungslehre" V par. 15.

[14] G. I. Taylor "The Application of O. Reynolds Theory of Heat Transfer to Flow Through a Pipe." Proc. Roy. Soc. London A Vol. 129 (1930) p. 25.

[15] Bjerkness "Physikalische Hydrodynamik," Berlin, Springer/1933 p. 685.
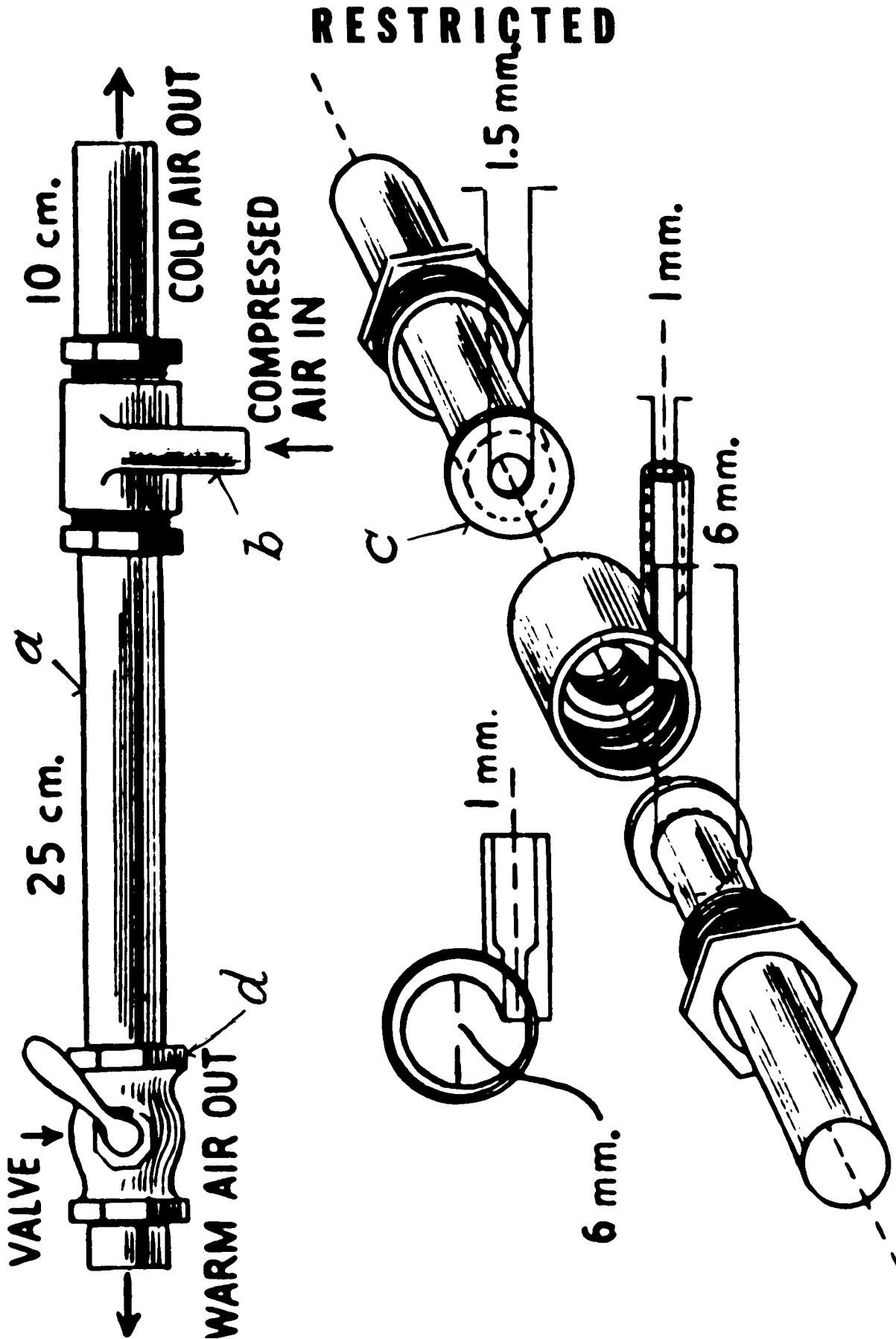
Fig. II – 1   Drawing of Hilsch's Tube

COLD AIR OUT

10 cm.

COMPRESSED
AIR IN

25 cm.

VALVE

WARM AIR OUT

1.5 mm.

1 mm.

6 mm.

1 mm.

6 mm.
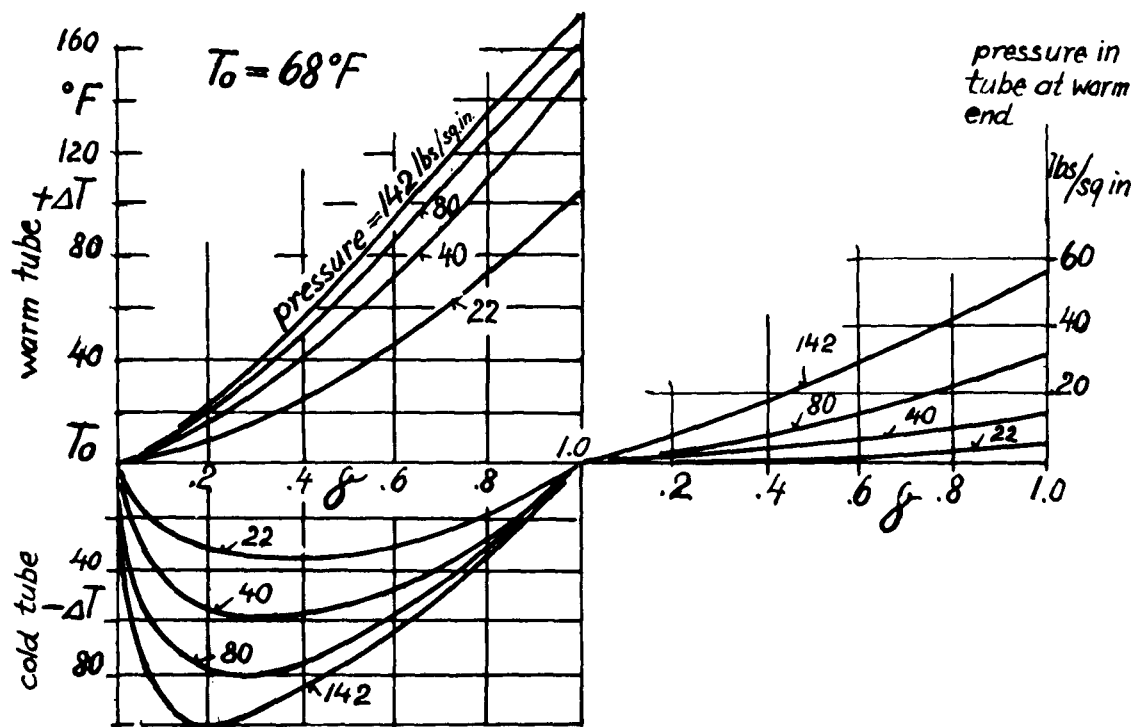
Fig. II - 2 Test Results of Hilsch (tube diam. = 0.18 in., nozzle diam. = 0.023 in. orifice diam. = 0.07 in., γ = ratio of mass of cold air to total mass of air)
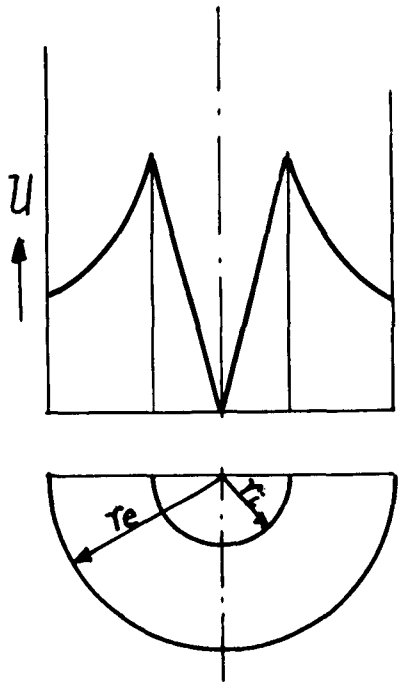
Fig. II - 3 Distribution of the circum-
ferential Velocity in a Tube
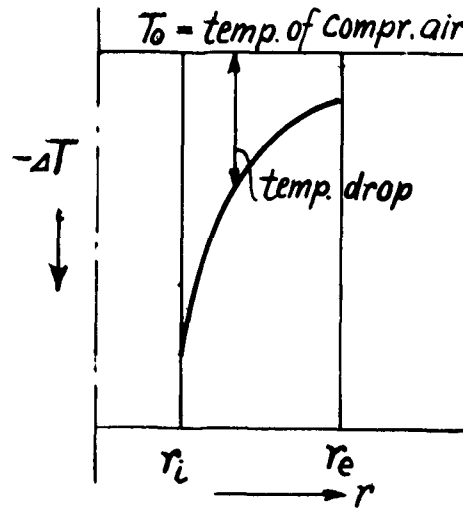
Fig. II - 4 Temperature **T** as a
Function of Radius
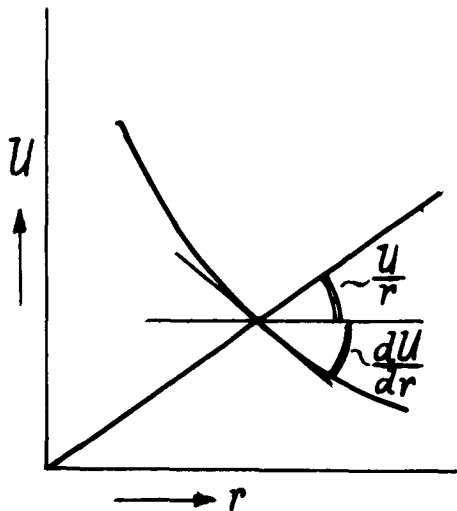
Fig. II - 5 Computation
of $\frac{dU}{dr}$ and $\frac{U}{r}$
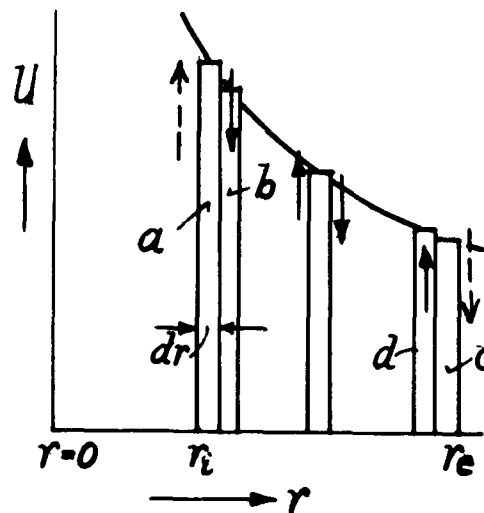
Fig. II - 6 Equilibrium of
Irrotational Flow
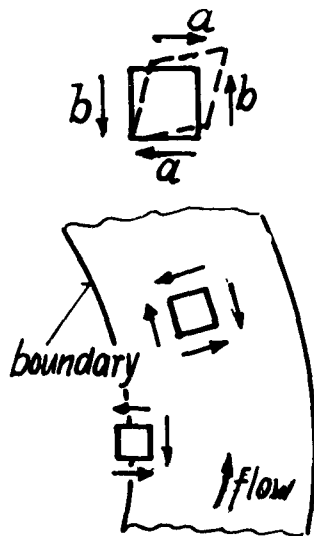
Fig. II - 7 Shear Stress in Irrotational Flow
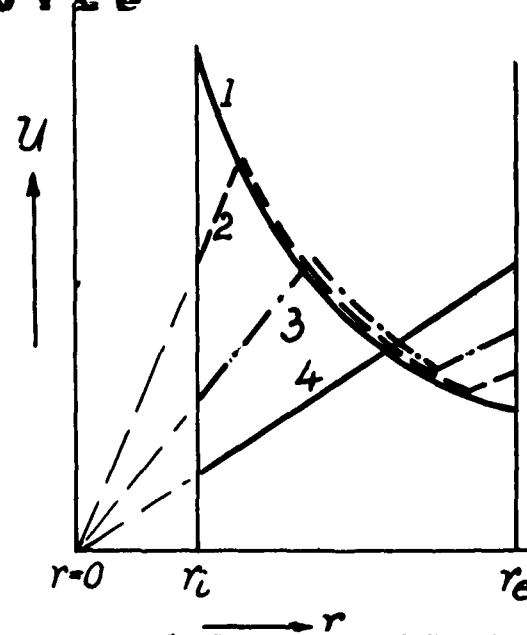
Fig. II - 8 Conversion of Irrotational to Rotational Flow
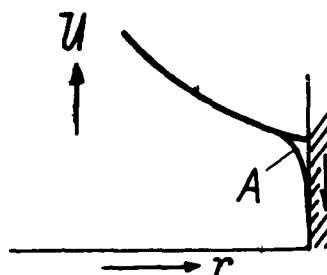
Fig. II - 9 Wall Friction
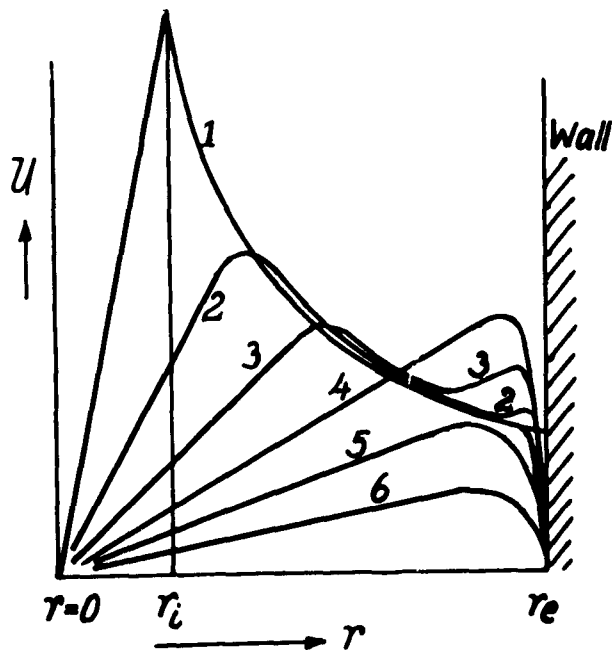
Fig II - 10 Conversion of Flow Shown for 6 Intervals
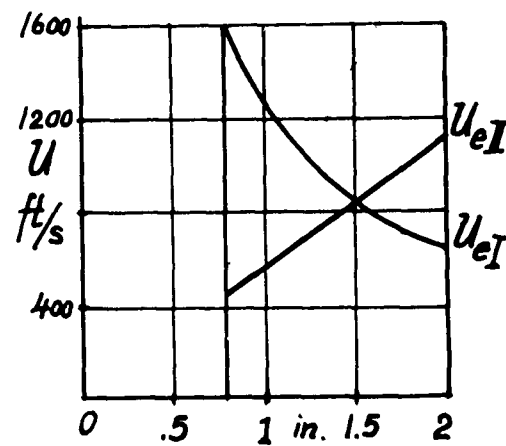
Fig. II - 11 Conversion of Flow With r_i/r_e = .4. Tube diam. = 2 in.
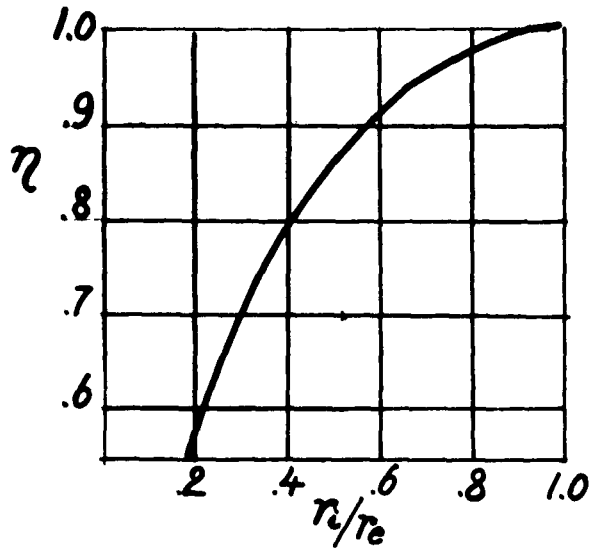
Fig. II - 12 Energy Loss
of Conversion
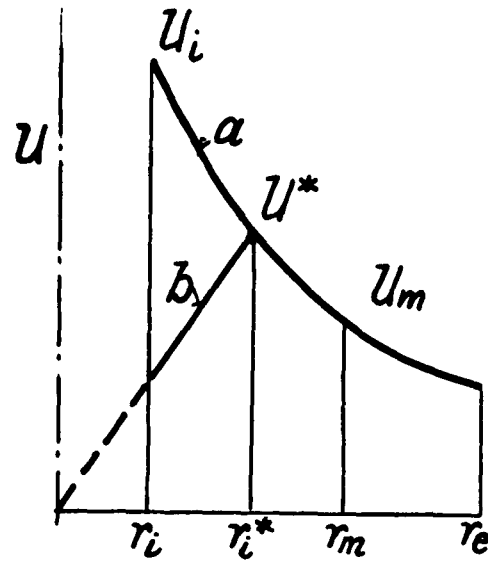
Fig. II - 13 Process of
Conversion
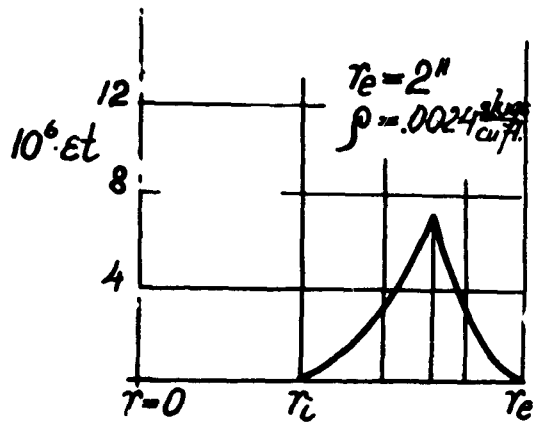
Fig. II - 14 Time of Conversion
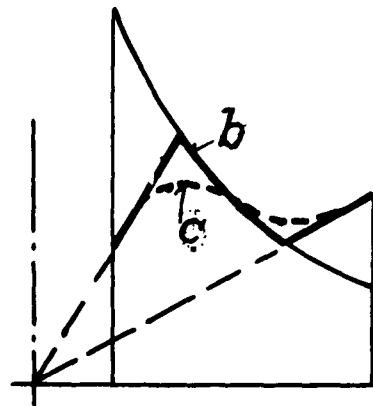
Fig. II - 15 Actual Velocity Change
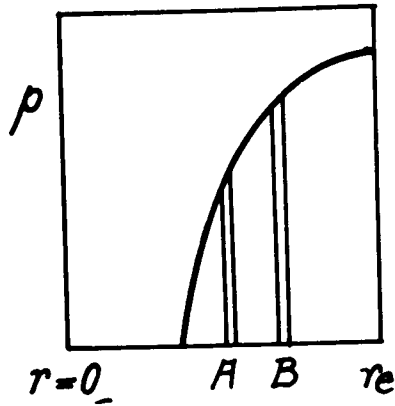
Fig. II - 16 Equilibrium of
Irrotational Flow

$P_o = 145 \, psi$

Fig. II - 18 Computation of
Optimum Ratio r_i/r_e

Fig. II - 17 Temperature Conversion
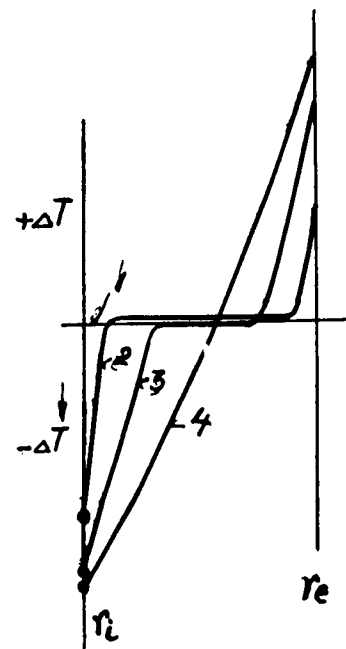
Fig. II - 19 Distribution of Total
Temperature During Conversion
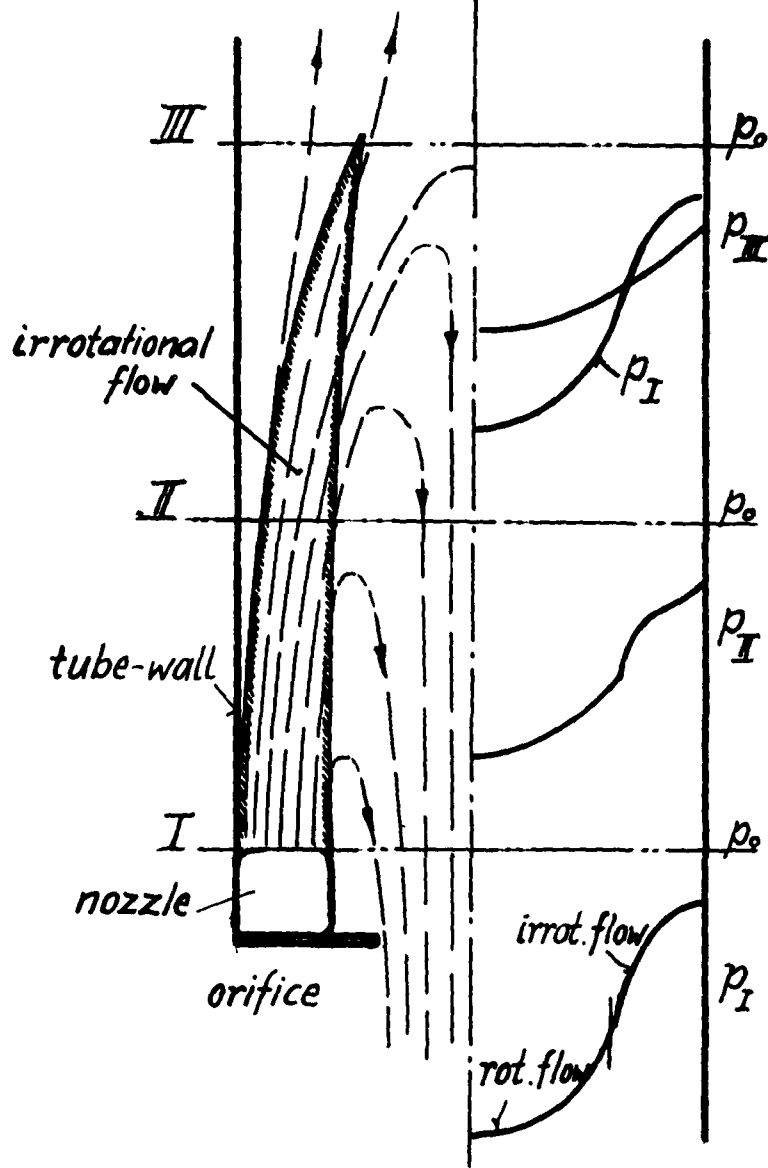
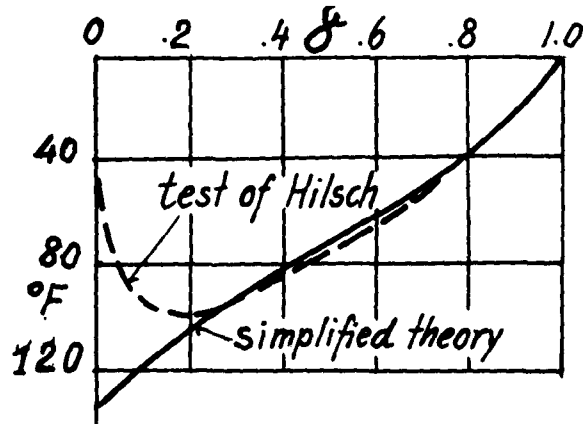Fig. II - 20 Pressure Distribution and Flow Pattern in the Tube

Fig. II - 21 Comparison of Hilsch's Measurements With Theory

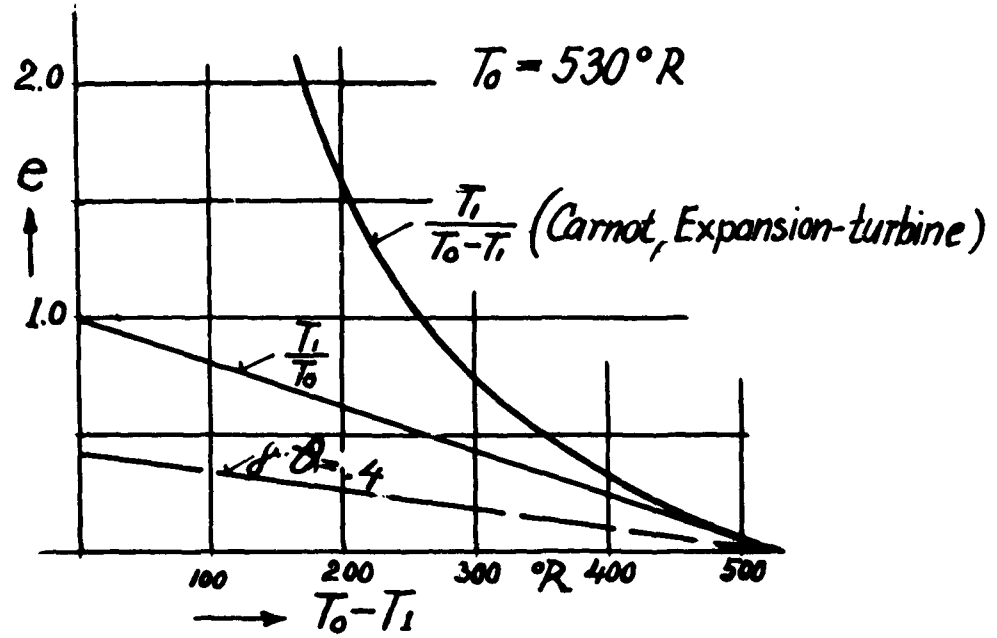Fig. II - 22 Coefficient of Performance

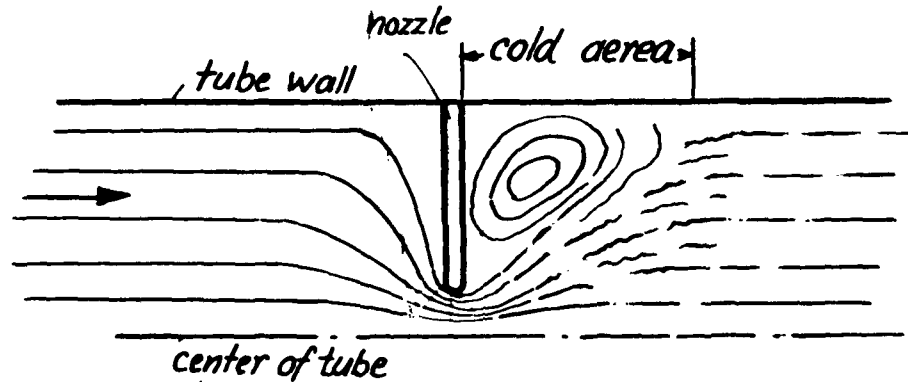Fig. II - 23 Cooling Effect by Throttling

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  * Possible Applications (Explanation of the Hilsch or Ranque Effect)

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AD-A800 229

TECHNICAL INDEX   Wright-Patterson Air Force Base
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