

### Summary

Response of a metallic capsule to a transient internal pressure is analyzed adopting two models. The first is a two-dimensional (2-D) model with finite extent in the plane of the capsule's cross-section and infinite length normal to that plane. The second is a three-dimensional (3-D) axisymmetric model consisting of a short thin cylinder capped by thin disks. In both models, transfer matrices relate state vectors of tractions and displacements at two segment boundaries. The time interval from pressure initiation to failure of its material measures the capsule's effectiveness as a confining boundary.

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Transient response of a metallic capsule  
to an internal pressure pulse

Michael El-Raheb

## 1. Introduction

Ceramic materials are used to harden vehicles and structures against ballistic events. These materials benefit from a higher compressive strength compared to metals yet their tensile strength is reduced by brittle failure. Following impact, a ceramic material shocks the projectile initiating damage. As time progresses and tensile release waves form, micro-cracks propagate and multiply in the material culminating in total failure. Strain energy in the virgin material then converts into kinetic energy ejecting the debris away from the projectile's path. Prolonging the time interval when comminuted material remains in this path after failure is one way to extend the material's utility in eroding the projectile.

In what follows a quantitative account is given of an experimental project duplicated in two laboratories and aimed at evaluating ballistic performance of stacked configurations. This account is interesting as unplanned differences in experimental setups revealed the strong effect of confinement on penetration.

Controlled ballistic experiments on periodic stacks of ceramic layers bonded by thin polymer films were performed at two laboratories named "A" and "B" for shortness (Weber and Hohler (1997), Weber, et. al. (1999)). Ballistic performance was measured by the residual penetration  $p_r$  of the projectile into a thick Aluminum backing block. In this way  $p_r$  is a measure of the projectile's residual kinetic energy. All experiments consisted of stacks made of 100mm x 100mm square Aluminum Nitride (AlN) tiles and 38mm total thickness of either: 1 x 38mm monolith, 2 x 19mm layers, 3 x 12.7mm layers, or 6 x 6.35mm layers. All layers were bonded by 0.254mm uniform polyurethane film, adhering to the tiles by heating.

In **Series 3c**, the stacks in the "B" setup were confined laterally by 14cm x 14cm square steel tubing 12.5mm thick, and axially by a 5mm thick steel cover plate perforated centrally with a 30mm diameter hole allowing the projectile to strike the stack (Fig. 1(a)). The stack rested on a 60mm Aluminum backing block to serve as witness plate for  $p_r$  measurements. The "A" setup was laterally confined was achieved by welding four 6.35mm plates along the edges to form a box keeping a 5mm clearance around the stack, and covered by a 6.35mm plate. The cylindrical projectile was made of Tungsten alloy with 8.33mm diameter and length to diameter ratio of 6, weighing 50g, with velocity near 1170 m/s. Fig. 1(b) plots  $p_r$  versus number of layers in

the stack for all "A" and "B" stacks tested. In this Figure, data from each series of experiments is labeled by a different symbol like  $\bullet$ ,  $\Delta$ ,  $\square$ ,  $\diamond$  and  $\times$ . At first, results from "A" and "B" did not match. "A" measured  $p_r=30\text{mm}$  for the 1.5" monolith, and  $p_r=0\text{mm}$  for the 1/2" layer stack ( $\bullet$  Fig. 1(b)). "B" measured  $p_r=0\text{mm}$  for the 38mm monolith and 6mm for the 12.7mm layer stack (not shown in Fig. 1(b)). A review of the "A" experimental setup revealed that lateral confinement failed at the corner welds, ejecting the side plates laterally immediately following impact thus negating the effect of confinement. This confirmed that the principal cause for the substantially lower  $p_r$  in the "B" setup was the strong confinement that delayed removal of comminuted material from the projectile's path, increasing its erosion. This effect masked the real ballistic performance of ceramic.

The "A" and "B" setups repeated **Series 3c** experiments with the following changes. The size of the confining tube in "B" was increased allowing a 5 mm clearance around the stack for lateral expansion of the damaged ceramic without interference from the confining tube. Also, a 3mm gap was left between cover plate and top face of the stack. In the "A" setup, the gap between stack, sidewalls and cover plate was increased to 20mm practically eliminating the effect of confinement. Indeed, results for the 12.7mm layer stack agreed: "B" measured  $p_r=20\text{mm}$  and  $23\text{mm}$  ( $\Delta$  Fig. 1(b)), and "A" measured  $p_r=20\text{mm}$  and  $26\text{mm}$  ( $\bullet$  Fig. 1(b)). For the 38mm monolith the mismatch in  $p_r$  persisted: "B" measured  $p_r=6.8\text{mm}$  and  $0\text{mm}$  ( $\Delta$ ) pointing to a possible problem in the "B" setup. "B" was repeated in 3 more tests on the 38mm monolith yielding  $p_r=11\text{mm}$ ,  $33\text{mm}$  and  $46\text{mm}$  ( $\diamond$  Fig. 1(b)). This unusual scatter among "B" results for the 38mm monolith confirmed that a fundamental difference existed between the two setups meriting a closer look at the "B" setup. Examining the "B" cover plates of the last 3 tests revealed that the cover plate corresponding to  $p_r=11\text{mm}$  suffered substantial plastic deformation, while that corresponding to  $p_r=46\text{mm}$  had very little plastic deformation. This suggested that the large scatter in  $p_r$  was caused by how the cover plate confines ejecta in the direction of impact. The 3mm gap

between cover plate and stack in "B" was small enough to confine ejecta at times when they did not escape through the central hole in the cover plate remaining in the path of the projectile. This increased resistance, reducing  $p_r$  and creating sufficient pressure to cause the cover plate to yield. When ejecta escaped through the hole, the projectile encountered less resistance increasing  $p_r$  and keeping the cover plate intact. This demonstrated that even the weak confinement of the cover plate randomly affected ballistic performance by retaining or allowing ejecta to pass through the hole.

An explanation was still needed to account for the match in results and reduced scatter in the "B" data of **Series 3c** for the multiple layer configurations in spite of the cover plate's closeness. That explanation lies in the concept of phase velocity  $c_p$ . For the monolith, the rate of formation of comminuted material in ejecta is approximately the compression speed of sound, which for AlN is 10km/s. For a stack made of 12.7mm ceramic layers bonded by 0.254mm polyurethane film,  $c_p \sim \sqrt{(E_b h_c) / (\rho_c h_b)} = 1.7\text{km/s}$  where  $\rho_c, h_c$  are ceramic mass density and layer thickness, and  $E_b, h_b$  are film modulus and thickness (El-Raheb (1997)). The reduction in  $c_p$  caused by dispersion from weak coupling reduces the rate of ejecta formation by a factor of 6. This raises the probability of ejecta escaping from the hole and in turn diminishes scatter. In fact, the reduction in  $p_r$  scatter for stacks with thinner ceramic layers is consistent with the corresponding reduction in  $c_p$ .

To bring the two setups into equivalence required adjusting the "B" setup in **Series 3d** by increasing the distance between cover plate and stack from 3mm to 50mm. This allowed sufficient space for the ejecta to expand without restriction from confinement. With this final setup, experiments measured a higher  $p_r$  in two tests with the 38mm monolith: 29mm and 41mm (□ Fig. 1(b)). Faced with this unexpectedly strong effect of confinement, the confining tubing in the "B" setup was increased from 14cm to 19.5cm and 2 more tests were performed for each stack reducing the scatter to less than 15% in **Series 3e**. For the 38mm monolith,  $p_r$  = 25mm and 31mm (X Fig. 1(b)).

Note the large scatter in  $p_r$  for the monolith when gap between cover plate and stack was 3mm (  $\Delta$  and  $\Diamond$  Fig. 1(b)), and the reduced scatter of all data when that gap was increased to 50mm (  $\square$  and  $\times$  Fig. 1(b)). The scatter diminishes as layer thickness in the stack is reduced.

Examining the experimental results presented above, it is evident that metallic encapsulation of ceramic tiles may extend the time interval of active resistance against the projectile also known as dwell time. One drawback is added weight implying that an optimization process is needed to determine the appropriate material properties and thickness of the capsule wall for maximum specific performance (performance/weight ratio). One measure of increased performance is the time interval it takes from total comminuting of ceramic to failure of the capsule allowing confined comminuted material to extend projectile erosion.

This work focuses on the transient response of a hollow capsule excited internally by a time dependent pressure pulse. Two models of the capsule are developed

- (a) A 2-D model of 4 strips connected at the edges forming a hollow rectangle, with finite extent in the plane of the cross-section and infinite in the direction normal to it.
- (b) A 3-D axisymmetric model of a short thin cylinder capped by thin disks at the cylinder's ends.

In the 2-D model, the transfer matrix approach is utilized to relate state vectors at the two ends of a 2-D strip. In the 3-D axisymmetric model, a transfer matrix is also used for the cylinder, while each disk is modeled as dynamic impedances relating force to displacement vectors along the boundary.

The internal pressure generated by the expanding comminuted material is approximated by a hyperbolic cosine bell shaped distribution with origin at the center of the faceplates simulating peak pressure at impact and pressure decay remote from impact. This approximation is consistent with both a hydro-acoustic model of the comminuted material based on the small amplitude acoustic equation and a numerical distribution determined by "EPIC" a general purpose "hydrocode".

Presented in a forthcoming paper, the acoustic model for internal pressure requires the compliance of the capsule walls. An appropriate measure for compliance is dynamic mobility of the capsule that relates

internal pressure to normal acceleration at the boundary. Dynamic mobility is derived from frequency response adopting modal method.

Finally, a parametric analysis compares histories from the 2-D and 3-D models with emphasis on peak stress and its location. Also, frequency response of driving point mobility yields its frequency-averaged radial distribution serving as an input to the acoustic model for internal pressure.

Since the 3-D axisymmetric model combines disk and cylinder, a literature survey focuses on each component separately as analysis on structures combining both is rare as the preferred approach relies on general-purpose computer programs.

A large body of references exists treating static and dynamic problems of isotropic, orthotropic, stiffened and rotating disks. A major portion concerns frequency response while a smaller portion treats transient response. Soamidas and Ganesan (1991) analyze variable thickness polar orthotropic disks. Xiang et al. (1996) treat linear axisymmetric frequency response with concentric stiffeners. Karunasena et al. (1997) treat the static axisymmetric response of a disk with annular supports adopting Mindlin's plate equations. Liew and Yang (2000) solve the three-dimensional free vibration problem of solid and annular plates for symmetric and asymmetric modes by an approximate polynomial-Ritz method. El-Raheb and Wagner (1987), (1994) study the axisymmetric wave propagation in a disk from impulse of short duration adopting a plate flexural model and a 3-D elasticity model. El-Raheb and Wagner (2001) then extend the axisymmetric flexure model to asymmetric response from off-center impact.

For thin cylinders, Wang et al. (1997) utilize the Ritz method in the modal response of ring-stiffened cylinders. Um and Kitipornchai (1998) derive 3-D elasticity solutions to frequency response of open cylinders. Lin and Jen (2003) adopt the Chebyshev collocation method in the analysis of laminated anisotropic cylinders. El-Raheb and Wagner (1989) treat wave propagation in a thin cylinder with concentrated masses attached to its surface adopting a modal approach. El-Raheb and Wagner (1985), (1986) and (1989) adopt transfer matrices to treat frequency response of cylinders connected to toroidal shells stiffened by discrete rings and annular disks and damped by constrained visco-elastic layers applied to the surface.

## 2. 2-D model



Fig. 2(a) depicts the capsule geometry in the 2-D model. In this model, a rectangular plate with finite extent along the local  $x$  axis and infinite length along the local  $z$  axis is termed a 2-D strip. A schematic of strips joining at corners through mobilities is shown in Fig. 2(b) where reference coordinate system  $(X,Y)$  is shown as dashed lines. Corner " $i$ " (enclosed in a circle) joins three branches  $j, j+1$  and  $j+2$ , through linkage mobilities  $\mathbf{y}_{i,j}$ ,  $\mathbf{y}_{i,j+1}$  and  $\mathbf{y}_{i,j+2}$  respectively where  $(i,j)$  stands for the corner-branch dyad. In the frequency domain where time dependence is sinusoidal with frequency  $\omega$

$$\mathbf{y}_{i,j} = [\mathbf{z}_{i,j}]^{-1} \quad (1)$$

$$\mathbf{z}_{i,j}^{(\alpha)} = (-\hat{m}\omega^2 + \hat{k} + i\omega\hat{\zeta})_{i,j}^{(\alpha)}, \quad \alpha=1,2, \quad i=\sqrt{-1}$$

and  $\mathbf{z}_{i,j}$  is a  $2 \times 2$  diagonal impedance matrix with components the one-degree of freedom oscillators with mass  $\hat{m}$ , spring  $\hat{k}$ , viscous damper  $\hat{\zeta}$ , along the local coordinates  $(x,y)$  of the  $j^{\text{th}}$  branch. Let  $\mathbf{A}_{i,j}$  be the  $4 \times 4$  transformation matrix relating local to reference state vectors  $\mathbf{S}_{loc} \rightarrow \mathbf{S}_{ref}$

$$\mathbf{S}_{i,j}^{ref} = \mathbf{A}_{i,j} \mathbf{S}_{i,j}^{loc}, \quad \mathbf{S}_{i,j} = \{\mathbf{f}, \mathbf{g}\}_{i,j}^T, \quad \mathbf{A}_{i,j} = \begin{bmatrix} \mathbf{a}_{i,j} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_{i,j} \end{bmatrix} \quad (2)$$

where  $\{\mathbf{f}, \mathbf{g}\}_{i,j}^T$  are force and displacement vectors at the end of branch  $(i,j)$

and  $\mathbf{a}_{i,j}$  is the  $2 \times 2$  transformation sub-matrix. Define  $\mathbf{f}$  and  $\mathbf{g}$  as

$$\mathbf{f} = \{f_x, q_y, m_z\}^T, \quad \mathbf{g} = \{u_x, u_y, \psi_z\}^T \quad (3)$$

$f_x, q_y, m_z$  are extensional force, shear force and moment, and  $u_x, u_y, \psi_z$  are corresponding displacements and rotation along local axes depicted in Fig. 2(c). Let  $l_{i,j}$  be branch length then the state vector at  $x=l_{i,j}$  is related to that at  $x=0$  by the transfer matrix

$$\mathbf{S}_{i,j}(l) \equiv \begin{Bmatrix} \mathbf{f}(l) \\ \mathbf{g}(l) \end{Bmatrix}_{i,j} = \mathbf{T}_{i,j}(0 \rightarrow l) \mathbf{S}_{i,j}(0) \equiv \begin{bmatrix} \mathbf{t}_{11} & \mathbf{t}_{12} \\ \mathbf{t}_{21} & \mathbf{t}_{22} \end{bmatrix}_{i,j} \begin{Bmatrix} \mathbf{f}(0) \\ \mathbf{g}(0) \end{Bmatrix}_{i,j} \quad (4)$$

For a 2-D strip the transfer matrix is derived in Appendix I. Since linkage mobilities  $\mathbf{y}$  and corner impedances  $\mathbf{z}$  connect branch ends to rigid corners,

$\tilde{\mathbf{S}}$  at a rigid corner is related to  $\mathbf{S}$  at a branch end by

$$\begin{Bmatrix} \tilde{\mathbf{f}} \\ \tilde{\mathbf{g}} \end{Bmatrix}_{i,j} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \varepsilon_e \mathbf{y} & \mathbf{I} \end{bmatrix}_{i,j} \begin{bmatrix} \mathbf{I} & \varepsilon_e \mathbf{z} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}_{i,j} \begin{Bmatrix} \mathbf{f} \\ \mathbf{g} \end{Bmatrix}_{i,j}, \quad \varepsilon_e = \begin{cases} 1 & \text{at } x=0 \\ -1 & \text{at } x=l_{i,j} \end{cases} \quad (5)$$

At each corner, equilibrium of  $\tilde{\mathbf{f}}$  and compatibility of  $\tilde{\mathbf{g}}$  take the form

$$\begin{aligned} \sum_{j=1}^{n_{bi}} \mathbf{a}_{i,j} \tilde{\mathbf{f}}_{i,j} \varepsilon_e &= \mathbf{0}, \quad i=1, n_c \\ \mathbf{a}_{i,j} \tilde{\mathbf{g}}_{i,j} &= \mathbf{a}_{i,1} \tilde{\mathbf{g}}_{i,1}, \quad j=2, n_{bi}, \quad i=1, n_c \end{aligned} \quad (6)$$

where  $n_{bi}$  is number of branches joined at corner  $i$  and  $n_c$  is total number of corners. Substituting (5) in (6) and expressing  $\mathbf{S}_{i,j}(l_{i,j})$  in terms of  $\mathbf{S}_{i,j}(0)$  using (4) produces  $6n_c$  simultaneous equations in the global  $\mathbf{S}_G(0)$ , the ensemble of all  $\mathbf{S}_{i,j}(0)$  in the configuration

$$\mathbf{M}_G \mathbf{S}_G(0) = \mathbf{0}, \quad \mathbf{S}_G(0) = \{\mathbf{S}_{i,j}(0)\}^T, \quad (j=1, n_{bi}, \quad i=1, n_c) \quad (7)$$

Transient response is determined by modal analysis. The eigenset of (7) derives from

$$\det[\mathbf{M}_G] = 0 \quad \Rightarrow \quad \{\Phi; \omega\}_k \quad (8)$$

$\{\Phi; \omega\}_k$  is the  $k^{th}$  eigenset formed of the eigenvector

$$\Phi_k = \{\varphi, \eta, \psi\}_k^T \quad (8a)$$

and corresponding eigenvalue  $\omega_k$ . The modal analysis proceeds by expanding the global displacement vector  $\mathbf{g}_G(x, t)$  in terms of the orthogonal set (8a)

$$\mathbf{g}_G(x, t) = \sum_k b_k(t) \Phi_k(x) \quad (9)$$

Substituting (9) in the forced elasto-dynamic equations of motion and enforcing orthogonality of the  $\{\Phi_k(x)\}$  set yields uncoupled ordinary differential equations in  $b_k(t)$

$$\begin{aligned} \ddot{b}_k(t) + \omega_k^2 b_k(t) &= N_{fk} f(t) / N_{kk} \\ N_{fk} &= \langle \mathbf{p}(x) | \rho \Phi_k(x) \rangle, \quad N_{kk} = \langle \Phi_k(x) | \rho \Phi_k(x) \rangle \end{aligned} \quad (10)$$

( $\dot{\phantom{x}}$ ) is time derivative,  $\mathbf{p}(x)$  is spatial vector of external excitation,  $f(t)$  is its time dependence, and  $\rho$  is density. Equation (10) admits the solution

$$b_k(t) = -\frac{N_{fk}}{\omega_k N_{kk}} \int_0^t f(\tau) \sin(\omega_k(t-\tau)) d\tau \quad (11)$$

### 3. 3-D axisymmetric model

The 3-D axisymmetric model consists of two thin disks capping the two ends of a short cylinder (see Fig. 3(a)). The cylinder transfer matrix is derived in Section 3a. Dynamic impedance of the disk at its boundary is derived in Section 3(b). Finally, the procedure of coupling motions of cylinder and disks is derived in Section 3c.

#### 3a. Cylinder transfer matrix

The cylinder has radius  $a_c$ , length  $l_c$ , plate thickness  $h_c$ , modulus  $E_c$ , Poisson ratio  $\nu_c$ , and mass density  $\rho_c$ . Forces and displacements of a cylinder element are shown in Fig. 3(b). For periodic motions in time with frequency  $\omega$ , the coupled axisymmetric elasto-dynamic equations of the cylinder are

$$\begin{aligned}\partial_x N_{xx} &= -\rho_c h_c \omega^2 u_c \\ \partial_x Q_x + \frac{N_{\theta\theta}}{a_c} &= -\rho_c h_c \omega^2 w_c\end{aligned}\tag{12}$$

$$Q_x = \partial_x M_{xx} - \frac{\rho_c h_c^3}{12} \chi \omega^2 \partial_x w_c, \quad \chi = 1 + \frac{2\kappa}{1-\nu_c}$$

$(x, \theta)$  are axial and circumferential coordinates,  $(u_c, w_c)$  are axial and radial displacement, the factor  $\chi$  multiplying rotatory inertia accounts for shear deformation and  $\kappa$  is shear constant (El-Raheb and Wagner (1998)), and  $(N_{xx}, Q_x, M_{xx})$  are axial, shear and moment resultants relating to displacements by the constitutive law

$$\begin{aligned}N_{xx} &= \frac{E_c h_c}{1-\nu_c^2} \left[ \partial_x u_c - \nu_c \frac{w_c}{a_c} \right] \\ N_{\theta\theta} &= \frac{E_c h_c}{1-\nu_c^2} \left[ \nu_c \partial_x u_c - \frac{w_c}{a_c} \right] \\ M_{xx} &= -\frac{E_c h_c^3}{12(1-\nu_c^2)} \partial_{xx} w_c\end{aligned}\tag{13}$$

Substituting (13) in (12) leads to a sixth order system of ordinary differential equations in  $(u_c, w_c)$

$$\mathbf{D}_c \begin{Bmatrix} u_c \\ w_c \end{Bmatrix} = \mathbf{0}\tag{14}$$

Since the operator  $\mathbf{D}_c$  has constant coefficients, the solution is in terms of exponentials

$$\begin{Bmatrix} u_c(x) \\ w_c(x) \end{Bmatrix} = \begin{Bmatrix} u_{cj} \\ w_{cj} \end{Bmatrix} e^{\lambda_{cj}x} \quad (15)$$

Substituting (15) in (14) yields a homogeneous set of simultaneous equations

$$\mathbf{L}_c \begin{Bmatrix} u_c \\ w_c \end{Bmatrix} = \mathbf{0} \Rightarrow \det[\mathbf{L}_c] = 0 \quad (16a)$$

The implicit eigenvalue problem (16a) yields the eigenvalues  $\lambda_{cj}$ ,

eigenvectors  $\{C_{cij}, C_{cwj}\}^T$ , and solutions for  $u_c, w_c$

$$u_c(x) = \sum_{j=1}^6 C_{cij} e^{\lambda_{cj}x}, \quad w_c(x) = \sum_{j=1}^6 C_{cwj} e^{\lambda_{cj}x} \quad (16b)$$

Define the state vector  $\mathbf{S}_c(x)$  as

$$\mathbf{S}_c(x) = \begin{Bmatrix} \mathbf{f}_c(x) \\ \mathbf{g}_c(x) \end{Bmatrix} \quad (17)$$

$$\mathbf{f}_c = \{N_{xx}, Q_{xx}, M_{xx}\}^T, \quad \mathbf{g}_c = \{u_c, w_c, \partial_x w_c\}^T$$

Substituting (16b) in (13) then in (17)

$$\mathbf{S}_c(x) = \mathbf{B}_c(x) \mathbf{C}_c, \quad \mathbf{C}_c = \{C_{cu1}, C_{cu2}, \dots, C_{cw1}, C_{cw2}, \dots\}^T \quad (18)$$

$\mathbf{B}_c(x)$  is a matrix of exponentials. Evaluating (18) at the two boundaries of the cylinder  $x=0, l_c$  then eliminating  $\mathbf{C}_c$

$$\mathbf{S}_c(l_c) = \mathbf{B}_c(l_c) \cdot \mathbf{B}_c^{-1}(0) \cdot \mathbf{S}_c(0) \equiv \mathbf{T}_c(0 \rightarrow l_c) \mathbf{S}_c(0) \quad (19a)$$

$\mathbf{T}_c(0 \rightarrow l_c)$  is the cylinder transfer matrix relating  $\mathbf{S}_c$  at its two boundaries. Eq. (19a) can be expressed in terms of four sub-matrices

$$\begin{Bmatrix} \mathbf{f}_c(l_c) \\ \mathbf{g}_c(l_c) \end{Bmatrix} = \begin{bmatrix} \mathbf{t}_{c11} & \mathbf{t}_{c12} \\ \mathbf{t}_{c21} & \mathbf{t}_{c22} \end{bmatrix} \begin{Bmatrix} \mathbf{f}_c(0) \\ \mathbf{g}_c(0) \end{Bmatrix} \quad (19b)$$

### 3b. Disk impedance matrix

Each disk has radius  $a_c$  which is the same as the cylinder's, plate thickness  $h_d$ , and modulus  $E_d$ , Poisson ratio  $\nu_c$  and mass density  $\rho_d$ . Forces and displacements of a disk element are shown in Fig. 3(c). For periodic

motions in time with frequency  $\omega$ , the in-plane axisymmetric equation of the disk is

$$\partial_r N_{rr} + \frac{(N_{rr} - N_{\theta\theta})}{r} = \rho_d h_d \omega^2 u_d \quad (20)$$

$(r, \theta)$  are radial and circumferential coordinates,  $u_d$  is radial displacement, and  $(N_{rr}, N_{\theta\theta})$  are radial and circumferential stress resultants relating to displacements by the constitutive law

$$\begin{aligned} N_{rr} &= \frac{E_d h_d}{1 - \nu_d^2} \left[ \partial_r u_d + \nu_d \frac{u_d}{r} \right] \\ N_{\theta\theta} &= \frac{E_d h_d}{1 - \nu_d^2} \left[ \nu_d \partial_r u_d + \frac{u_d}{r} \right] \end{aligned} \quad (21)$$

Substituting (21) in (20) yields

$$\left[ \nabla_1^2 + \rho_d h_d \omega^2 \right] u_d = 0, \quad \nabla_n^2 \equiv \partial_{rr} + \frac{1}{r} \partial_r - \frac{n^2}{r^2} \quad (22)$$

with solution

$$\begin{aligned} u_d(r) &= C_{du} J_1(k_{d\epsilon} r) \\ k_{d\epsilon} &= \frac{\omega}{c_{d\epsilon}}, \quad c_{d\epsilon} = \sqrt{\frac{E_d}{(1 - \nu_d^2) \rho_d}} \end{aligned} \quad (23)$$

Mindlin's axisymmetric flexural plate equations of the disk are (El-Raheb and Wagner (2001))

$$\begin{aligned} \frac{\tilde{D}}{2} \left[ (1 - \nu_d) \nabla_1^2 \psi_d + (1 + \nu_d) \partial_{rr} \psi_d \right] - \kappa G_d h_d (\psi_d + \partial_r w_d) &= -\frac{\rho_d h_d^3}{12} \omega^2 \psi_d \\ \kappa G_d h_d (\nabla^2 w_d + \partial_r \psi_d) &= -\rho_d h_d \omega^2 w_d, \quad \tilde{D} = \frac{E_d h_d^3}{12(1 - \nu_d^2)} \end{aligned} \quad (24)$$

with constitutive relations

$$M_{rr} = \tilde{D} \left( \partial_r \psi_d + \nu_d \frac{\psi_d}{r} \right), \quad Q_r = \kappa G_d h_d (\partial_r w_d + \psi_d) \quad (25)$$

$w_d, \psi_d$  are axial displacement and rotation,  $Q_r, M_{rr}$  are shear and moment resultants, and  $\kappa$  is shear constant. Eliminating  $\psi_r$  in (24) yields the 4th order equation in  $w_d$

$$\left[ \left( \nabla_0^2 + \frac{\omega^2}{c_{ds}^2} \right) \left( \nabla_0^2 + \frac{\omega^2}{c_{ds}^2} \right) - \frac{12\omega^2}{c_{ds}^2 h_d^2} \right] w_d = 0, \quad c_{ds} = \sqrt{\frac{\kappa G_d}{\rho_d}} \quad (26)$$

with general solution in terms of Bessel functions

$$w_d(r) = C_{dw} J_0(\lambda_r r) \quad (27)$$

Substituting (27) in (26) yields the dispersion relation

$$(-\lambda_r^2 + k_{d\varepsilon}^2)(-\lambda_r^2 + k_{ds}^2) - \frac{12k_{d\varepsilon}^2}{h_d^2} = 0 \quad (28)$$

$$k_{d\varepsilon} = \omega / c_{d\varepsilon}, \quad k_{ds} = \omega / c_{ds}$$

The four roots  $\lambda_{rj}$  of (28) correspond to four primitives where only the two regular at  $r=0$  are retained

$$w_d(r) = \sum_{j=1}^2 C_{dwj} J_0(\lambda_{rj} r) \quad (29)$$

$$\psi_d(r) = \sum_{j=1}^2 C_{d\psi j} J'_0(\lambda_{rj} r), \quad C_{d\psi j} = -\frac{1}{\lambda_{rj}} (\lambda_{rj}^2 + k_{rs}^2) C_{dwj}$$

( $'$ ) is derivative with respect to the argument.

Define the state vector  $\mathbf{S}_d$  combining in and out of the plane variables

$$\mathbf{S}_d = \{\mathbf{f}_d, \mathbf{g}_d\}^T \quad (30)$$

$$\mathbf{f}_d = \{N_r, Q_r, M_r\}^T, \quad \mathbf{g}_d = \{u_d, w_d, \psi_d\}^T$$

Substituting (23) and (29) in (21) and (25) yields

$$\mathbf{f}_d(r) = \mathbf{B}_{fd}(r) \mathbf{C}_d, \quad \mathbf{g}_d(r) = \mathbf{B}_{gd}(r) \mathbf{C}_d \quad (31)$$

$$\mathbf{C}_d = \mathbf{B}_{gd}^{-1}(a_c) \mathbf{g}_d(a_c), \quad \mathbf{C}_d = \{C_{du}, C_{dw1}, C_{dw2}\}^T$$

$\mathbf{B}_{fd}$  and  $\mathbf{B}_{gd}$  are  $3 \times 3$  matrices. Eliminating  $\mathbf{C}_d$  from (31) determines the disk impedance matrix

$$\mathbf{f}_d(r) = \mathbf{B}_{fd}(r) \mathbf{B}_{gd}^{-1}(r) \mathbf{g}_d(r) \equiv \mathbf{Z}_d(r) \mathbf{g}_d(r) \quad (32)$$

$\mathbf{Z}_d(a_c) = \mathbf{B}_{fd}(a_c) \cdot \mathbf{B}_{gd}^{-1}(a_c)$  is the impedance matrix of the disk at the boundary.

### 3c.. Coupling cylinder and disks

Continuity of  $\mathbf{S}$  at the cylinder-disk interface  $x = x_e \equiv 0, l_c$  and  $r = a_c$  requires expressing  $\{\mathbf{f}_d, \mathbf{g}_d\}$  in the coordinate system of  $\{\mathbf{f}_c, \mathbf{g}_c\}$  by means of a re-ordering matrix

$$\mathbf{f}_c(x_e) = \mathbf{R}_f \mathbf{f}_d(a_c), \quad \mathbf{g}_c(x_e) = -\mathbf{R}_f \mathbf{g}_d(a_c) \Rightarrow \mathbf{g}_d(a_c) = -\mathbf{R}_f^{-1} \mathbf{g}_c(x_e)$$

$$\mathbf{R}_f = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (33)$$

Substituting (32) in (33) then eliminating  $\mathbf{g}_d$

$$\mathbf{f}_c(0) = \mathbf{Z}_c \mathbf{g}_c(0), \quad \mathbf{f}_c(l_c) = -\mathbf{Z}_c \mathbf{g}_c(l_c)$$

$$\mathbf{Z}_c = -\mathbf{R}_f \mathbf{Z}_d(a_c) \mathbf{R}_f^{-1} \quad (34)$$

$\mathbf{Z}_c$  is disk impedance in the cylinder coordinate system. Applying (34) at the two boundaries of the cylinder and relating  $\mathbf{f}_c(l_c), \mathbf{g}_c(l_c)$  to  $\mathbf{f}_c(0), \mathbf{g}_c(0)$  using (19b) produces the homogeneous equations

$$\mathbf{M}_c \begin{Bmatrix} \mathbf{S}_c(0) \\ \mathbf{S}_c(l_c) \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}, \quad \mathbf{M}_c = \begin{bmatrix} \mathbf{I} & -\mathbf{Z}_c & \mathbf{O} & \mathbf{O} \\ \mathbf{t}_{c11} & \mathbf{t}_{c12} & -\mathbf{I} & \mathbf{O} \\ \mathbf{t}_{c21} & \mathbf{t}_{c22} & \mathbf{O} & -\mathbf{I} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} & \mathbf{Z}_c \end{bmatrix} \quad (35)$$

$\mathbf{M}_c$  is a  $12 \times 12$  matrix,  $\mathbf{I}$  and  $\mathbf{O}$  are  $3 \times 3$  unit and null matrices, and  $\mathbf{0}$  is the null vector of order 6. A non-trivial solution of (35) yields the implicit eigenproblem

$$\det[\mathbf{M}_c] = 0 \Rightarrow \{\Phi_c, \omega\}_k \quad (36a)$$

Eq. (31) and (33) determine the disk eigenfunction  $\Phi_d(r)$ . The global eigenfunction of cylinder and conjoined disks then follows

$$\Phi_{Gk}(s) = \{\Phi_c(x), \Phi_d(r)\}_k^T \quad (36b)$$

$s$  is an intrinsic coordinate along the boundary;  $s \equiv x$  on the cylinder and  $s \equiv r$  on the disks. Modal analysis proceeds by expanding the global

displacement vector  $\mathbf{g}_G(s, t) = \{\mathbf{g}_c(x), \mathbf{g}_d(r)\}^T$  in terms of the orthogonal set

(36b)

$$\mathbf{g}_G(s, t) = \sum_k b_k(t) \Phi_{Gk}(s) \quad (37)$$

Substituting (37) in the forced elasto-dynamic equations of motion of cylinder and disks and enforcing orthogonality of the  $\{\Phi_{Gk}(s)\}$  set yields uncoupled ordinary differential equations in  $b_k(t)$

$$\begin{aligned} \ddot{b}_k(t) + \omega_k^2 b_k(t) &= N_{fk} f(t) / N_{kk} \\ N_{fk} &= \langle \mathbf{p}(s) | \rho \Phi_{Gk}(s) \rangle, \quad N_{kk} = \langle \Phi_{Gk}(s) | \rho \Phi_{Gk}(s) \rangle \end{aligned} \quad (38)$$

( $\dot{\phantom{x}}$ ) is time derivative,  $\mathbf{p}(s)$  is spatial vector of external excitation,  $f(t)$  is its time dependence, and  $\rho$  is density. Equation (38) admits the solution

$$b_k(t) = -\frac{N_{fk}}{\omega_k N_{kk}} \int_0^t f(\tau) \sin(\omega_k(t-\tau)) d\tau \quad (39)$$

For frequency response,  $\ddot{b}_k(t) = -\omega^2 b_{k0} e^{i\omega t}$  reducing (38) to

$$b_k(t) = N_{fk} e^{i\omega t} / (N_{kk} (-\omega^2 + \omega_k^2)) \quad (40)$$

$\omega$  is frequency of excitation.

#### 4. Forcing pressure

In the 3-D axisymmetric model, a spatial distribution of the applied pressure is approximated by the bell-shaped function

$$p(r) = p_0 + p_1 \operatorname{sech}(p_2 r), \quad 0 \leq r \leq a_c \quad (41)$$

where the triad  $(p_0, p_1, p_2)$  of constants characterizes translation, magnification and maximum slope of the profile. Different triads are assigned to top and bottom disks (see Fig. 3(a)). A typical normalized distribution is shown in Fig. 4.

In the 2-D model, assuming that pressure acts over a distance  $l_x$  along  $z$  where  $l_x$  is length of capsule along top or bottom faces (see Fig. 2(a)), a profile similar to (41) is used but with  $p_0, p_1$  scaled to conserve total force

$$\tilde{p}(x) = \tilde{p}_0 + \tilde{p}_1 \operatorname{sech}(p_2(x - 0.5l_x)), \quad 0 \leq x \leq l_x \quad (42)$$

The condition of force conservation yields

$$l_x \int_0^{l_x} \tilde{p}(x) dx = 2\pi \int_0^{a_c} p(r) r dr, \quad l_x = 2a_c \quad (43)$$

Assuming that  $p_0/p_1 = \tilde{p}_0/\tilde{p}_1$ , then Eq. (43) suffices to determine the dyad  $(\tilde{p}_0, \tilde{p}_1)$ .



## 6. Results

The 2-D capsule has a rectangular cross-section made of 4 plates as shown in Fig. 5:

- (1) top plate joining corners 1 through 5
- (2) bottom plate joining corners 6 through 10
- (3) left plate joining corners 1 and 10
- (4) right plate joining corners 5 and 6

A global coordinate system is defined for both the 2-D and 3-D models with origin at corner 1 such that  $\{X: 0 \leq X \leq l_x\} \otimes \{Y: 0 \leq Y \leq w_y\}$  where  $(l_x, w_y)$  are length and width of the capsule. The following geometry and material properties are considered

$$l_x = 100 \text{ mm}, w_y = 12.5 \text{ mm}, h = 2.54 \text{ mm}$$

$$E = 2 \times 10^{12} \text{ dyn/cm}^2, \rho = 8 \text{ g/cm}^3, \nu = 0.3$$

where  $h$  is uniform plate thickness and  $(E, \rho, \nu)$  are Young's modulus, density and Poisson ratio. In the 2-D model, 3 stresses are monitored along each segment's local  $x$ -axis: flexural stress  $\sigma_{xxf}$ , shear stress  $\tau_{xz}$  and extensional axial stress  $\sigma_{xxe}$ . In the 3-D model 4 stresses are monitored; either  $\sigma_{xxf}, \tau_{xz}, \sigma_{xxe}, \sigma_{\theta\theta e}$  along the cylinders  $x$ -axis which is the global  $Y$ -axis, where  $\sigma_{\theta\theta e}$  is the extensional circumferential stress, or else  $\sigma_{rrf}, \tau_{rz}, \sigma_{rre}, \sigma_{\theta\theta e}$  along the disk's radius which is the global  $X$ -axis. Stress at a point with global coordinates  $(X, Y)$  will be referred to as  $\sigma(X, Y)$ .

The first 10 mode shapes and corresponding resonant frequencies are shown in Figures 6(a-j). The fundamental at 1.063 KHz (Fig. 6(a)) is symmetric about the axis of bilateral symmetry. This corresponds to the first symmetric flexural mode of the two faces. The second mode (Fig. 6(b)) is the first anti-symmetric flexural mode of the faces. Higher flexural modes with larger wave numbers are shown in Fig. 6(c-f) and then the fundamental symmetric extensional mode at 21.72 KHz (Fig. 6(g)), followed by the first anti-symmetric extensional mode (Fig. 6(h)) at 24.54 KHz. At higher frequencies, (Fig. 6(i-j)) the flexural modes exhibit noticeable motion of the side plates, which were evanescent for frequencies below the fundamental symmetric extensional mode in Fig. 6(g). For each plate, modes belong to one of 4 groups depending on symmetry of the mode (symmetric or anti-symmetric) and type of mode (flexural or extensional).

In the 3-D axisymmetric model, a spatial distribution of the applied pressure on top and bottom plates is approximated by the bell-shaped function in Eq. (38) where the triad of constants  $(p_0, p_1, p_2)$  characterizes translation, magnification and maximum normalized slope of the profile. Different triads are assigned to top and bottom plates (see Fig. 4). The normalized peak pressure for the axisymmetric excitation is unity. In the 2-D model, the equivalent bell shaped distribution with  $(\tilde{p}_0, \tilde{p}_1)$  that conserves total force is determined applying Eq. (39). The resulting  $(\tilde{p}_0, \tilde{p}_1)$  is listed in Table I showing that in the 2-D model, peak applied pressure on the top plate is  $p_{mx} \equiv \tilde{p}_0 + \tilde{p}_1 = 0.4$ . The top plate is divided into 4 equidistant intervals of 12.5mm along  $X$  while the side plate is divided into 2 equidistant intervals of 6.25mm along  $Y$ . The time dependence of the pressure pulse is trapezoidal lasting 25  $\mu s$  with 5  $\mu s$  rise and fall times and a 15  $\mu s$  plateau.

Fig. 7 plots stress histories on the left plate for  $0 \leq Y \leq 12.5mm$ . The peak of  $\sigma_{xyf}(0,0)$  reaches  $38 p_{mx}$  (Fig. 7(a)) pointing to a location where material may yield and ultimately fail.  $\sigma_{xze}$  (Fig. 7(c)) is relatively small since it reacts to the difference between integrated pressure over the top plate ( $\approx p_{mx} l_x / 4 \equiv O(1)$ ) and its inertia. Fig. 8 plots stress histories on the top plate for  $12.5 \leq X \leq 50mm$  at 12.5mm intervals. The peak of  $\sigma_{xyf}(50,0)$  (Fig. 8(a)) reaches  $23 p_{mx}$  pointing to another location of considerable flexural stress. Fig. 8(b) plots histories of shear stress  $\tau_{xz}$  with peaks not exceeding  $p_{mx}$ . Fig. 8(c) plots histories of extensional stress  $\sigma_{xze}$  in the plane of the plate with a peak of  $\sigma_{xze}(50,0)$  reaching  $5 p_{mx}$ . It is evident that the two most vulnerable locations with high  $\sigma_{xyf}$  are at the interface of top and side plates and at the top plate's center. Extensional and shear stresses do not contribute to yielding and failure of the material unless deformation is sufficient to promote additional stress from nonlinear stretching.

Geometry and material properties in the 3-D model replicate those in the 2-D model. The first 6 mode shapes are plotted in the left column of Fig. 9 while the corresponding modes from the 2-D model are plotted in the

right column for comparison. The fundamental mode 1 at 2.075 KHz is the first symmetric flexural mode of the disks while mode 2 at 4.714 KHz is the first anti-symmetric. Flexural modes 3,4 and 6 then occur before the fundamental symmetric extensional mode 5 occurs at 29.74 KHz. The frequency ratio  $\omega_{3-D}/\omega_{2-D}$  for modes 1,2 and 4 is approximately 2 (Fig. 9). This ratio drops to less than 1.3 for all following modes suggesting that the difference in response between 3-D and 2-D models is largest at low frequencies. It also implies that for a pulse of short duration compared to the fundamental period, transient response from the two models may be comparable since then response would be controlled by modes with short wave length.

Fig. 10 plots stress histories on the cylinder for  $0 \leq Y \leq 12.5\text{mm}$ . The peak of  $\sigma_{xyf}(0,0)$  (Fig. 10(a)) is half that in 2-D while the peak of  $\sigma_{xxe}$  (Fig. 10(c)) is almost the same as that in 2-D.  $\sigma_{\theta\theta e}$  (fig. 10(d)) is approximately  $1/4\sigma_{xyf}$  and has no counterpart in 2-D. Fig. 11 plots stress histories on the top disk for  $0 \leq X \leq 50\text{mm}$  at 12.5mm intervals. The peak of  $\sigma_{rf}(50,0)$  reaches the same magnitude as that of  $\sigma_{xyf}(0,0)$  in 2-D, and the peak of  $\sigma_{rf}(0,0)$  reaches the same magnitude as that of  $\sigma_{xyf}(50,0)$  in 2-D. Fig. 11(b) plots histories of shear stress  $\tau_{xz}$  with a peak  $1/3$  that in 2-D.  $\sigma_{rre}$  and  $\sigma_{\theta\theta e}$  (Fig. 11(c),(d)) have approximately the same magnitude reaching  $1/5$  that of  $\sigma_{xxe}$  in 2-D. Consequently in 3-D, magnitude of peak flexural stress is approximately the same as in 2-D but locations are reversed; highest stress is at the disk's center followed by that at the edge. It is surprising that in spite of the different dimensionality, histories of flexural stress from the two models agree qualitatively and quantitatively. The discrepancy between the two models is in the extensional stress that is substantially lower than flexural stress.

Fig. 12 follows the evolution of deformation of the 3-D capsule for  $10 \leq t \leq 80\mu\text{s}$ . The dominant mode of deformation is a symmetric bulging that is larger on the top plate than on the bottom due to the difference in magnitude of the applied pressure.

## 7. Conclusion

Two models are presented for transient response of a thin walled capsule forced by a time dependent internal pressure. The 2-D model has finite dimensions in the plane of the cross-section and extends to infinity normal to this plane. The 3-D axisymmetric model includes a short thin cylinder capped by two thin disks. Both models adopt transfer matrices relating state variables at two ends of a segment and transient response based on modal analysis. Noteworthy results are

- 1) For low wave numbers, eigen-frequencies of the 3-D model are twice those of the 2-D model. The difference diminishes as wave number rises.
- 2) The assumed bell-shaped internal pressure distribution agrees with that determined from an acoustic model and from a numerical general purpose program.
- 3) In both the 2-D and 3-D models, peak flexural stress occurs at the junction of cylinder and disks as well as center of disks.
- 4) Peak extensional and shear stresses are one order of magnitude smaller than peak flexural stress except for circumferential stress in the cylinder with peak 1/4 that of flexural stress.
- 5) In spite of the different dimensionality, histories from the two models are in good agreement.

## Appendix A. Transfer matrix of 2-D strip

For sinusoidal time dependence with frequency  $\omega$ , the linear elastodynamic equations of a 2-D strip are

$$\begin{aligned} \frac{\partial^2 u_s}{\partial x^2} &= -k_{s\epsilon}^2 u_s, \quad k_{s\epsilon} = \omega / c_{s\epsilon}, \quad c_{s\epsilon} = \sqrt{E_s / \rho_s (1 - \nu_s^2)} \\ \tilde{D} \frac{\partial^2 \psi_s}{\partial x^2} - \kappa G_s h_s \left( \psi_s + \frac{\partial w_s}{\partial x} \right) &= -\frac{\rho_s h_s^3}{12} \omega^2 \psi_s \\ \kappa G_s h_s \left( \frac{\partial^2 w_s}{\partial x^2} + \frac{\partial \psi_s}{\partial x} \right) &= -\rho_s h_s \omega^2 w_s, \quad \tilde{D} = \frac{E_s h_s^3}{12(1 - \nu_s^2)} \end{aligned} \quad (A1)$$

$u_s, w_s, \psi_s$  are displacements along local coordinates  $x$  and  $y$  (see Fig. 1c),  $E_s, G_s, \nu_s, \rho_s$  are extensional and shear moduli, Poisson ratio, and mass density,  $h_s$  is thickness and  $\kappa$  is shear constant. The corresponding constitutive relations are

$$f_x = \frac{E_s h_s}{(1-\nu_s^2)} \frac{\partial u_s}{\partial x}$$

$$q_x = \kappa G_s h_s \left( \frac{\partial w_s}{\partial x} + \psi_s \right), \quad m_x = \tilde{D} \frac{\partial \psi_s}{\partial x} \quad (\text{A2})$$

$f_x, q_x, m_x$  are axial force, shear force and moment (see Fig. 1c). The general solution of (A1) takes the form

$$\begin{Bmatrix} u_s(x) \\ w_s(x) \\ \psi_s(x) \end{Bmatrix} = \begin{bmatrix} e^{k_{se}x} & 0 & 0 \\ 0 & e^{\lambda_{s1}x} & e^{\lambda_{s2}x} \\ 0 & \delta_{s1} e^{\lambda_{s1}x} & \delta_{s2} e^{\lambda_{s2}x} \end{bmatrix} \begin{Bmatrix} C_{su} \\ C_{sw1} \\ C_{sw2} \end{Bmatrix}, \quad \delta_{sj} = -\frac{1}{\lambda_{sj}} (\lambda_{sj}^2 + k_{ss}^2) \quad (\text{A3})$$

$\lambda_{s1,2}$  satisfy the dispersion relation

$$(-\lambda_{sj}^2 + k_{se}^2)(-\lambda_{sj}^2 + k_{ss}^2) - \frac{12k_{se}^2}{h_s^2} = 0 \quad (\text{A4})$$

$$k_{ss} = \omega / c_{ss}, \quad c_{ss} = \sqrt{\kappa G_s / \rho_s}$$

Substituting (A4) in (A2) in terms of the state vector  $\mathbf{S}_s = \{\mathbf{f}_s, \mathbf{g}_s\}^T$

$$\mathbf{S}_s(x) = \mathbf{B}_s(x) \mathbf{C}_s$$

$$\mathbf{f}_s = \{f_x, q_x, m_x\}_s^T, \quad \mathbf{g}_s = \{u, w, \psi\}_s^T \quad (\text{A5})$$

$\mathbf{B}_s(x)$  is a  $3 \times 3$  matrix of exponentials and  $\mathbf{C}_s = \{C_{su}, C_{sw1}, C_{sw2}\}^T$ . Evaluating (A5) at the two ends of the strip  $(0, l_s)$  then eliminating  $\mathbf{C}_s$  yields the strip's transfer matrix

$$\mathbf{S}_s(l_s) = \mathbf{B}_s(l_s) \mathbf{B}_s^{-1}(0) \mathbf{S}_s(0) = \mathbf{T}_s(0 \rightarrow l_s) \mathbf{S}_s(0) \quad (\text{A6})$$

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	3-D model		2-D model		
plate	$p_0$	$p_1$	$\tilde{p}_0$	$\tilde{p}_1$	$p_2$
top	0.06	1.0	0.0223	0.371	3.5
bottom	0.05	0.1	0.0325	0.065	1.5
side	0.039	0	0.039	0	0

Table I. Parameters of pressure profile



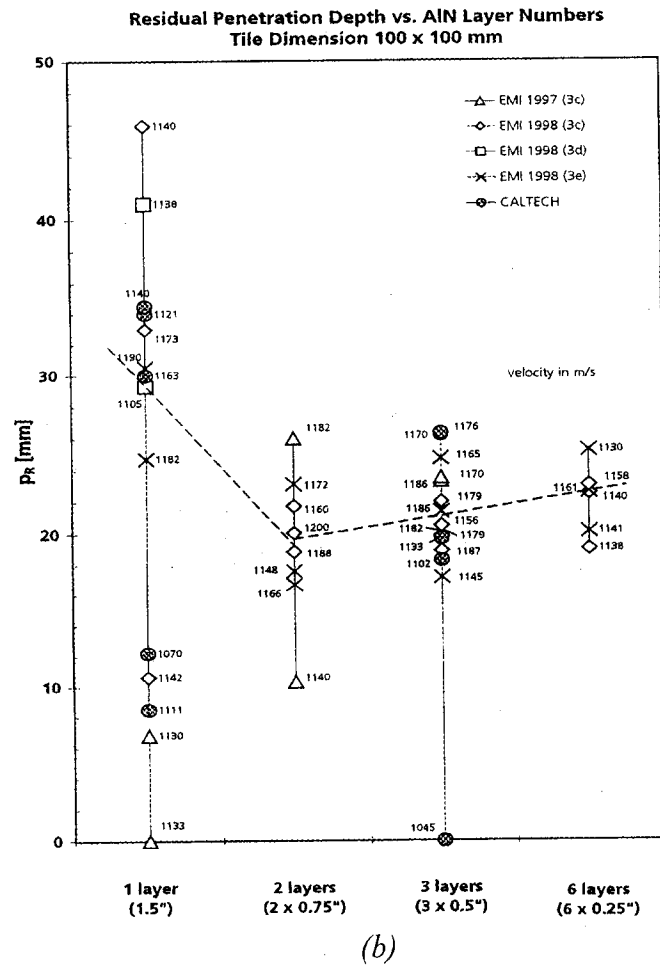
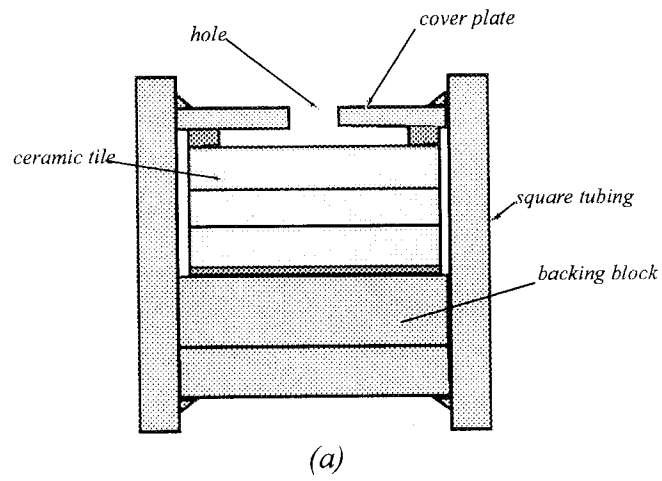


Figure 1. Experiment : (a) setup, (b) residual penetration data

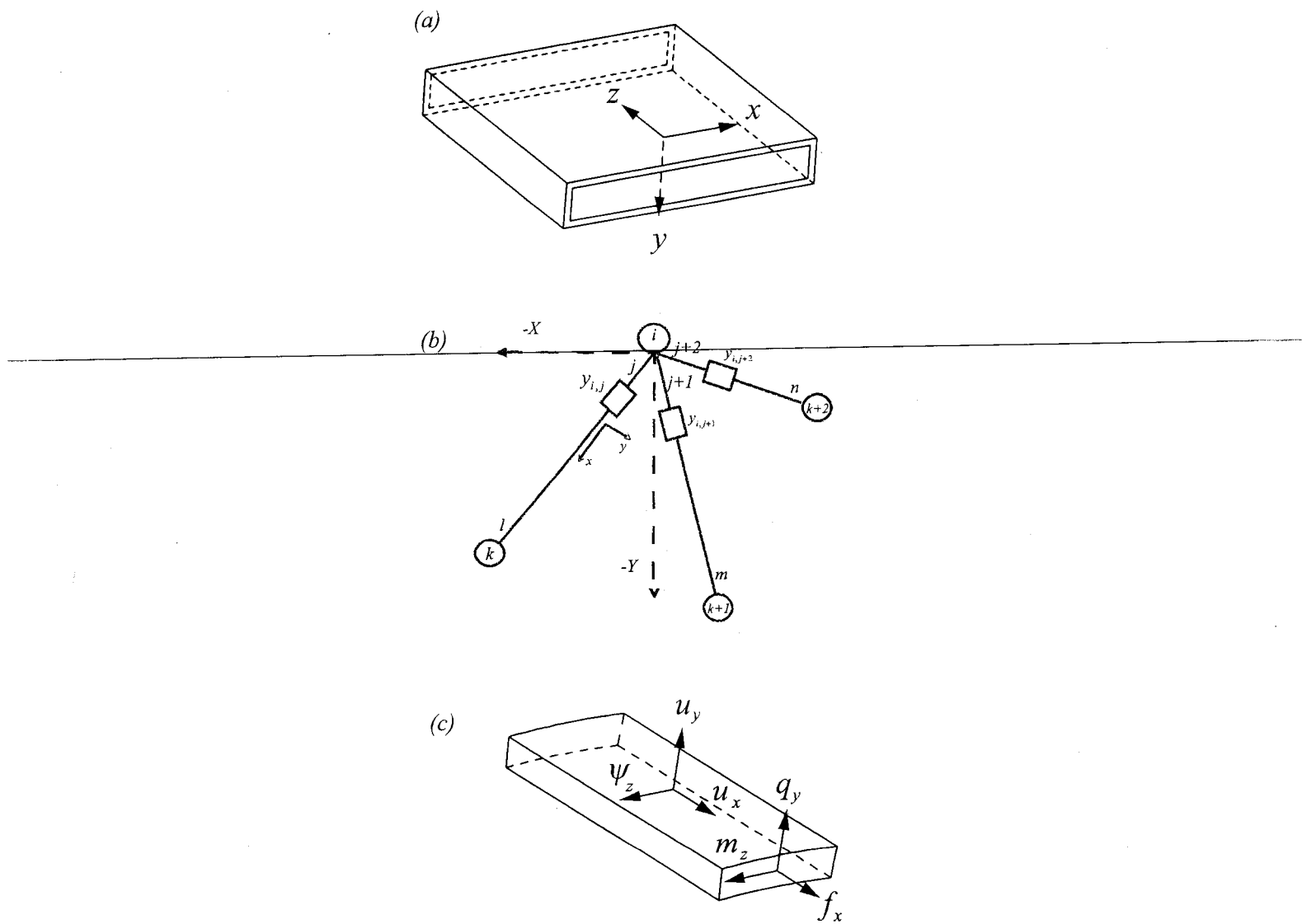


Figure 2. 2-D model : (a) geometry, (b) corner-branch system, (c) strip element

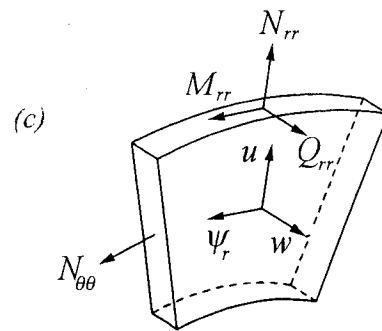
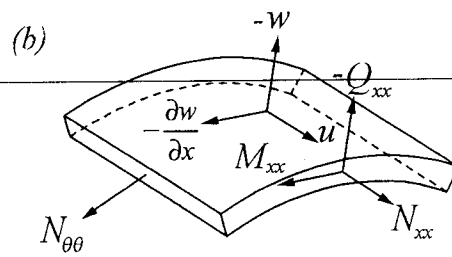
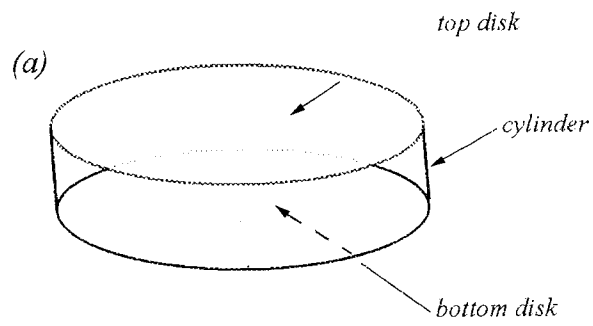


Figure 3. 3-D axisymmetric model

(a) geometry, (b) cylinder element, (c) disk element

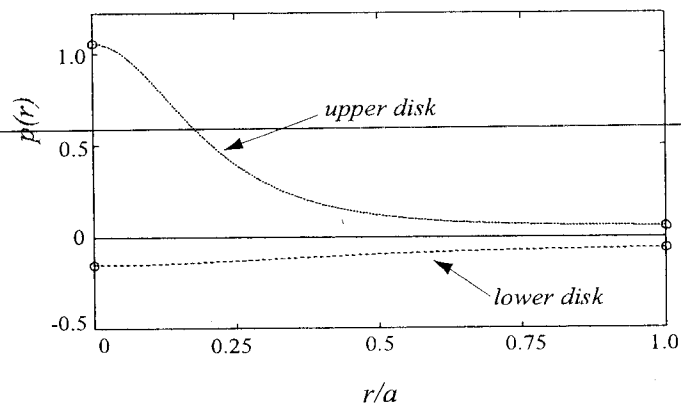


Figure 4. Radial distribution of internal pressure

$$p(r) = p_0 + p_1 \operatorname{sech}(p_2 r)$$

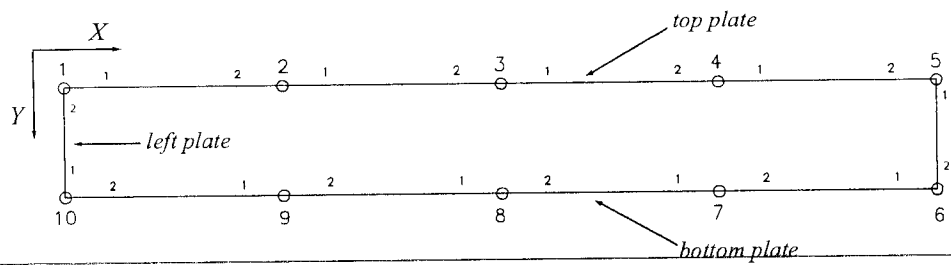


Figure 5. Cross-section of 2-D capsule with corner and branch numbers

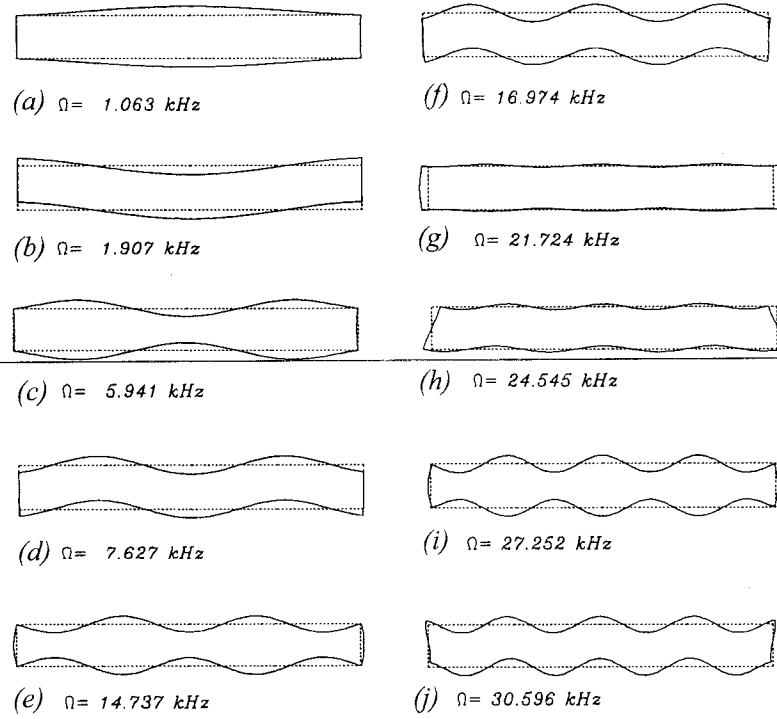


Figure 6. Resonant frequencies and modes of 2-D capsule

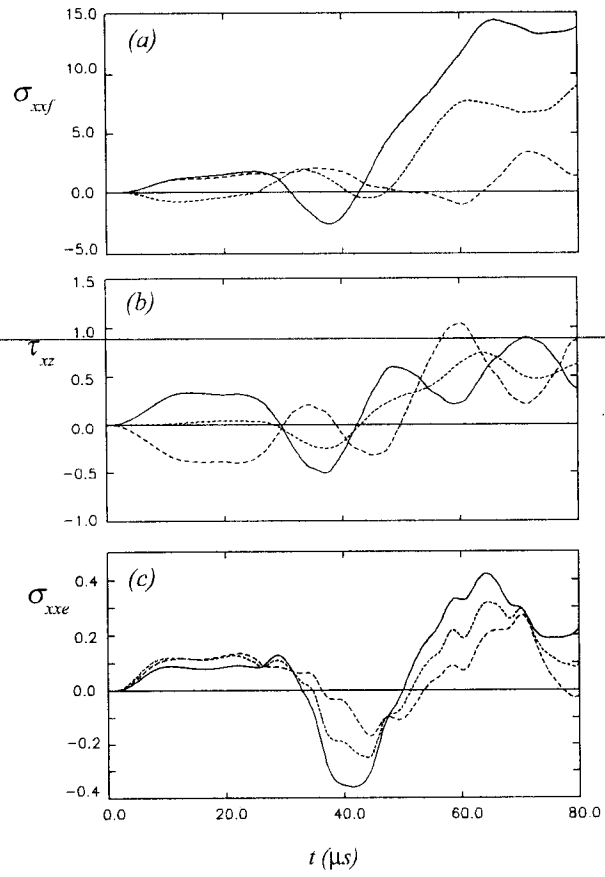


Figure 7. 2-D stress histories on left plate: (a)  $\sigma_{xxf}$ , (b)  $\tau_{xz}$ , (c)  $\sigma_{xze}$

—  $Y=0$ , ---  $Y=w_y/2$ , .....  $Y=w_y$

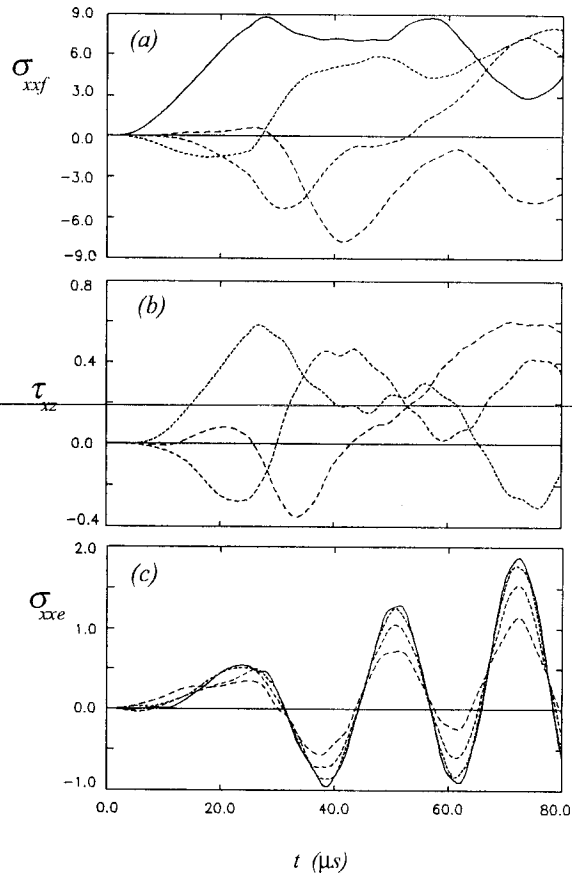


Figure 8. 2-D stress histories on top plate: (a)  $\sigma_{xxf}$ , (b)  $\tau_{xz}$ , (c)  $\sigma_{xze}$

—  $X = 0.5 l_x$ , — — —  $X = 0.375 l_x$ , - - - -  $X = 0.25 l_x$ , - . - .  $X = 0.125 l_x$



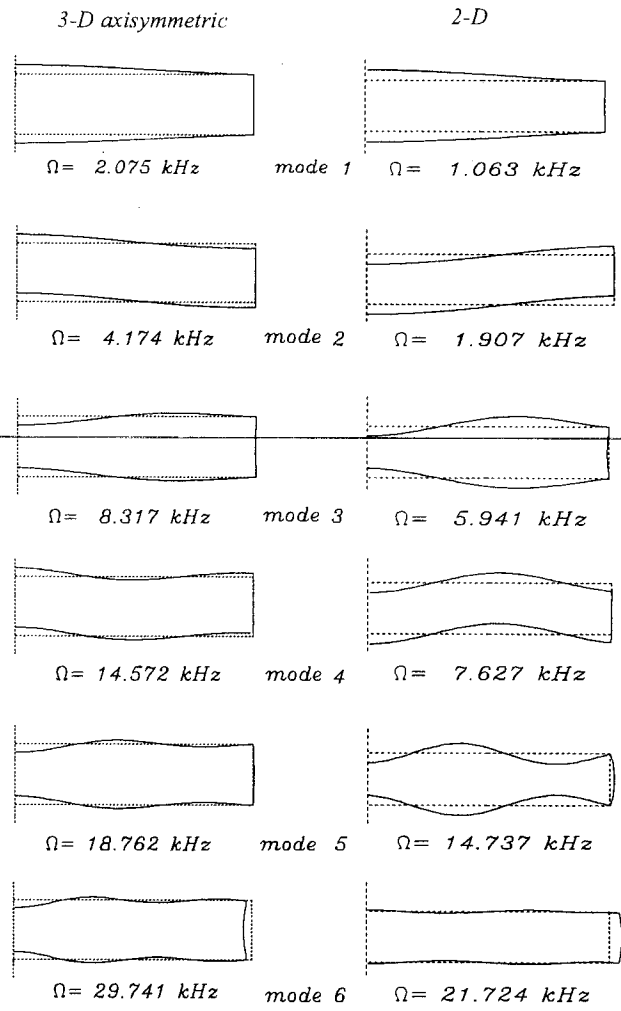


Figure 9. Comparison of modes from 3-D axisymmetric and 2-D models

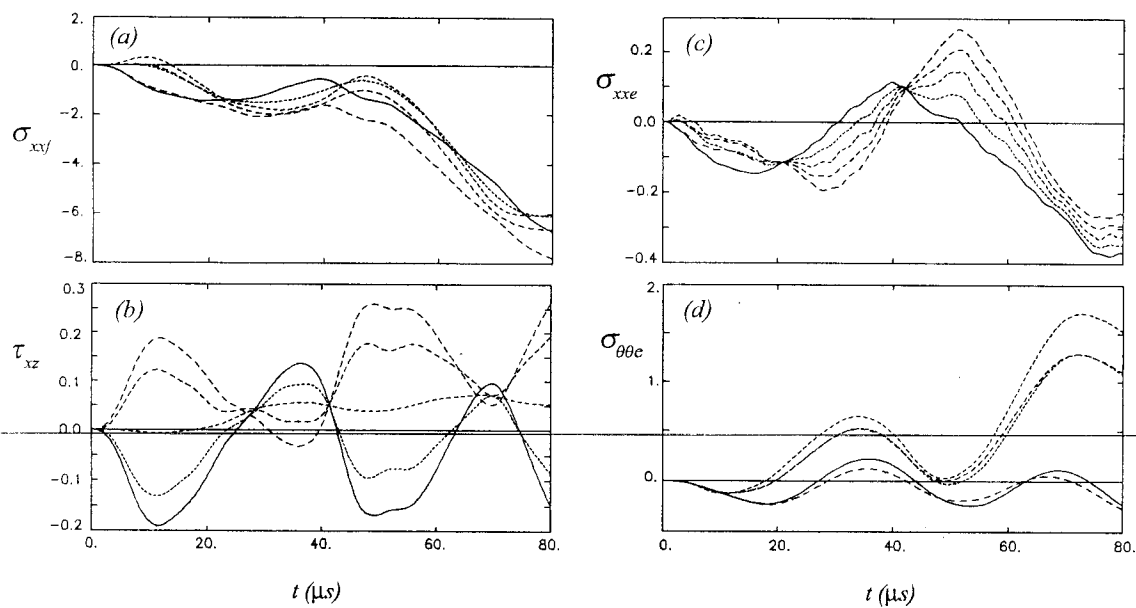


Figure 10. 3-D stress histories on cylinder: (a)  $\sigma_{xy}$ , (b)  $\tau_{xz}$ , (c)  $\sigma_{xze}$ , (d)  $\sigma_{\theta\theta e}$

—  $Y = 0$ ,    - - -  $Y = 0.25 w_y$ ,    . . .  $Y = 0.5 w_y$ ,  
- . -  $Y = 0.75 w_y$ ,    - - -  $Y = w_y$

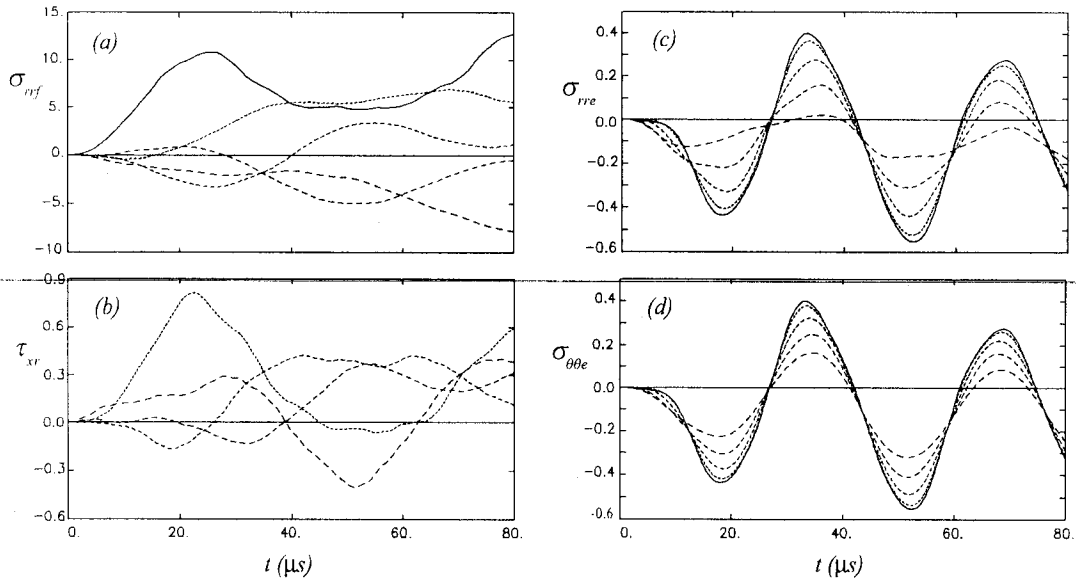


Figure 11. 3-D stress histories on top disk: (a)  $\sigma_{rrf}$ , (b)  $\tau_{xr}$ , (c)  $\sigma_{rre}$ , (d)  $\sigma_{\theta\theta e}$

—  $X = 0.5 l_x$ , - - -  $X = 0.375 l_x$ , ...  $X = 0.25 l_x$ ,  
- . -  $X = 0.125 l_x$ , - - -  $X = 0$

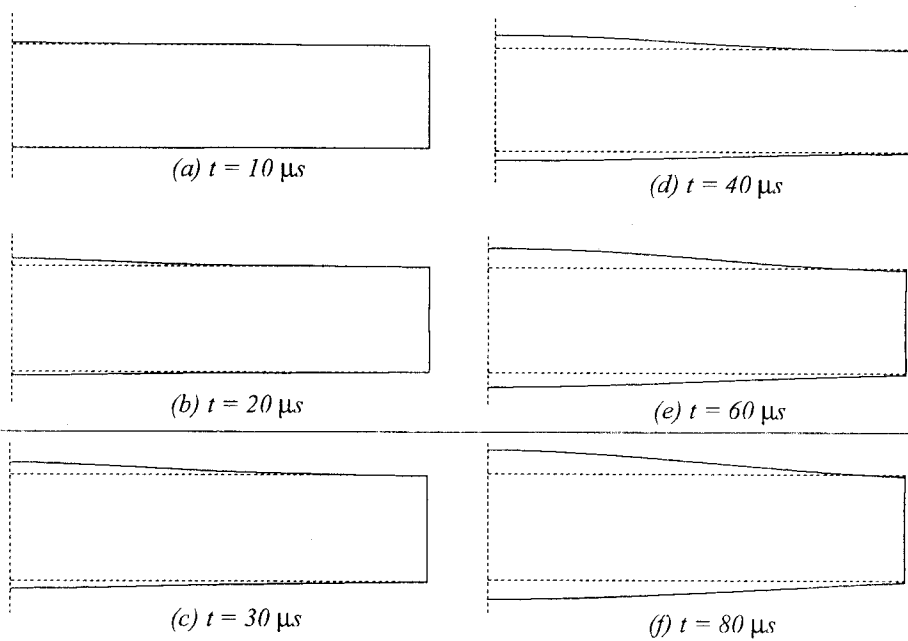


Figure 12. Time snapshots of 3-D capsule