A Mathematical Type for Physical Variables

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Abstract. In identifying the requirements of a markup language for describing the mathematical semantics of physics-based models, we pose the question: "Is there a mathematical type for physical variables?" While this question has no a priori answer, since physics is fundamentally empirical, it appears that a large body of physics may be described with a single mathematical type. Briefly stated, that type is formed as the mathematical product of a physical unit, such as *meter* or *second*, and an element of a Clifford algebra. We discuss some of the properties of this mathematical type and its use in documentation of physics-based models.

1 Introduction

We are interested in creating a markup language for the representation of physical models, i.e., a physics markup language. Our primary requirement for a physics markup language is to represent the models that physicists and engineers create and so, necessarily, the components with which they build those models. The principal reason for creating such a language is to improve the communication of the semantics of models of the physical world in order to support interoperability of physics-based models with each other, such as with multiphysics simulation, and interoperability with other non-physics-based models. Physics-based models are used extensively in modeling and simulation (M&S) frameworks to support a wide array of predictive and decision making applications of practical importance. An open and standard way of documenting the physical and mathematical semantics of physics-based models, such as a markup language might provide, would go a long way towards lowering the costs of model development and validation. Additionally, since models form the basis of the theoretical development of physics, communication of research results and physics education would also be favorably impacted.

In approaching these goals, we ask "What information is it necessary to specify in order to transmit knowledge of a physical model and to make the transmission unambiguous?" In particular, we are interested in identifying the specific mathematical concepts necessary for expressing the physical semantics since, once identified, they may be dealt with somewhat independently. We observe that the typical computer code representing a physics-based model follows from a mathematical model derived from the application of mathematically phrased

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14. ABSTRACT
In identifying the requirements of a markup language for describing the mathematical semantics of physics-based models, we pose the question: ?Is there a mathematical type for physical variables?? While this question has no a priori answer, since physics is fundamentally empirical, it appears that a large body of physics may be described with a single mathematical type. Briefly stated, that type is formed as the mathematical product of a physical unit, such as meter or second and an element of a Clifford algebra. We discuss some of the properties of this mathematical type and its use in documentation of physics-based models.

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physical laws to mathematical representations of physical objects. There is a rich array of mathematical concepts used in these mathematical representations. This raises the question as to how we approach the problem of representing all of these mathematical concepts. For example, we need to specify the dimensionality of the physical objects being modeled, their spatio-temporal extents, and the embedding space. We also note that: the physical quantities and corresponding units used to describe physical properties have a mathematical structure; the physical laws that are applied usually have a differential expression; and, invariance with respect to various transformations is a key concept. Each of these, while they carry physical semantics, must be mathematically expressed.

2 Mathematical Requirements of a Physics Markup Language

Physical semantics ultimately rests on mathematical phrasing. To be meaningful, scientific theories are required to provide predictions that are testable. In practice, this means we must be able to compare mathematically computed predictions to numerical measurement data. Accordingly, the first things we need to express in a physics markup language are the mathematical symbols that represent the properties of physical objects. To be useful, a physics-based model must represent a physical object with at least one of the object's measurable properties, which has physical dimension expressed in specified units. Very often these properties are modeled as variables and they are used to represent such things as the positions, velocities, and accelerations of a physical object, which typically vary as a function of time. A prediction results when, given the model, we can solve for a given variable. We refer to these variables as physical variables. We need to be able to express not only physical variables, but also the mathematical operations upon physical variables and the mathematical relationships between physical variables.

It is often said that most models in physics are ultimately partial differential equations with boundary conditions. In order to specify these relations between physical variables, we will require the ability to specify, in addition to the physical variables themselves, differential operators, such as gradient, divergence, and curl, acting on scalar and vector fields, as well as equality and inequality relationships. Note that this use of the term field is not the usual mathematical meaning as in, for example, "the real numbers form a field", but is specifically a physicists meaning. A physicists notion of a physical field (temperature field, gravitational field, etc.) is a scalar or vector quantity defined at each point in a space-time domain. Specifying a model in terms of differential equations is an implicit form of specification, since in order to express the variables as explicit functions of time we will require a solution to the equations.

There are many more mathematical concepts that we need to express to represent models with physical variables. We know, for example, that: classical mechanics makes use of scalars, vectors, and tensors defined in space-time; these vectors have length, giving metric properties to objects defined in space-time;

quantum physics makes use of Dirac spinors and Hilbert space vectors (bra-ket notation); general relativity requires transformation between covariant and contra-variant forms using a non-trivial metric tensor; and, models of physical objects possess spatial extent and often have defined boundary surfaces. Differential equations need to be expressed over definite volumes, and boundary conditions need to be expressed on the bounding surfaces of those volumes. We often want to specify a preferred geometric basis for the expressed geometry, such as rectangular, cylindrical, or spherical coordinates. Until we can express the semantics of these many mathematical concepts, we will not be able to express a large body of physical models.

Statements of invariance are also important relationships between physical variables. While the equations that make up a model may implicitly obey some invariance, and additional statement of such invariance may seem redundant, specific statements of known invariance are useful in understanding a particular model and in performing computational evaluations using the model. Invariance is, in general, specified with respect to operations performed on physical variables by particular operators. Such operators include Euclidean transformations (spatial rotations, translations, and reflections) and Lorentz transformations (space-time rotations, boosts, and reflections).

While in order to make specific predictions it is common to consider models as providing unique solutions for all of its physical variables as functions on space-time, this is not always necessary. There is value in using models to express incomplete knowledge of as well, which may result in sets of multiple possible solutions. We may, for example, only only be able to specify that two variables within a model have a functional dependence, i.e., X is a function of Y, without knowing more detail. We may want to specify that a variable has exclusive dependence on another or that it is independent of another. We often need to state physical principles as inequalities, for example, for which there are many solutions. It may be that we want to develop reusable models that can be used to predict many different variables, but not necessarily simultaneously, where each variable may have distinct dependencies, or lack thereof, on given initial conditions. In building these models, we may need to develop a more clear definition of what constitutes a model and under what conditions a model permits solutions to be determined.

To summarize, a markup language for physics must support the following mathematical concepts:

a) A physical attribute which has physical dimension and may be represented in defined units. It may be represented with a scalar magnitude, or, if it is a more complex property, by a vector, a tensor, or other object with the necessary algebraic properties.
b) A physical object has spatial presence and extent, properties that are represented as point-like, 1-dimensional, or arbitrary dimensional attributes. These properties may be described within the space-time reference frame of the physical object itself, or within the space-time reference frame of another physical object.

c) The attributes of a physical object may satisfy a specified set of differential equations or other mathematical relations.

Finally, it cannot be supposed that this is a complete tally of useful mathematical semantics. For example, statements of general physical laws, such as Newtonian universal gravitation, will be aided by the use of mathematical quantifiers to specify, for example, that gravitational forces are present between *all pairs* of massive physical objects within a model. In general, it seems desirable, if not necessary, to be able to express a full range of mathematical relations between variables in physical models.

3 A Type for Physical Variables

The fundamental components that physicists use to build models are physical variables, parameters that represent the physical quantity attributes of physical objects. A physical quantity is an observable, measurable property of an object in the physical world. A principal difficulty we have in representing physical variables in a markup language is that physical variables do not generally have a well-defined type, where we use the term type much as a computer scientist or mathematician would, i.e., a class of objects with a well-defined set of allowed operations. Physicists and engineers typically act as applied math practitioners with a well-schooled intuition, and they are not always fussy about mathematical formalism. The types of physical variables are rarely declared as part of a problem statement or model definition, and it is common to find abrupt transitions in usage, from one implied type to another. While one might well consider attempting to capture the reasoning abilities of these applied math practitioners as an exercise in artificial intelligence, that is a separate research topic of its own. We are undertaking here the problem of capturing as precisely as possible the mathematical description of such models, and describing as concise a set of clearly defined types as possible. The reason for looking more carefully at the formal mathematical representations of physical variables is to determine what is a sufficient amount of information to require for a semantic representation of physical models.

So, we begin this inquiry into developing a physics markup language by posing the following primary question: "What is the type necessary for representing physical variables?" Upon reflection, we may question why we should expect there to be a single type for representing physical variables. We state, somewhat axiomatically, that the objective of physics is to describe physical interactions mathematically. One may dispute the underlying axiomatic assumption that physical interactions may be described mathematically, but, pragmatically, we are only interested in those interactions that may be so described, since that is what affords us the ability to make predictions.

To answer the question as to why we should expect a single type for physical variables, consider the following. If for each interaction of two physical variables we were to be given a physical variable of a new type, it would not take very long for the resulting type proliferation to make it difficult, if not impossible, to describe the physical universe. Describing the physical universe is certainly easier if there is a countable, closed system of defined types, and easier still if there are but a finite number of defined types. More importantly, we should expect that if the physical universe is closed, so too in our mathematical description of the physical universe should the set of objects that represent physical variables be closed under those operations that represent physical interactions. The requirement of closure merely reflects the idea that physical interactions should be a function of the physical quantities of the interacting objects and should result in physical effects, where the effects may also be represented using physical variables. Without a requirement of closure for physical variables, we would allow non-physical results from the interaction of physical objects or we would allow physical effects to result from non-physical interactions. We therefore require the definition, from a formal perspective, of a type for representing physical variables, which has a mathematical description, being essentially a set that is closed under defined operations.

We also undertake this inquiry with the understanding that a practical solution today may well be improved upon later since it is impossible to anticipate all of the future developments of theoretical physics. This reality should not deter us, however, from attempting to answer our primary question, since there is significant challenge and great utility in handling only those representations of physical variables that have been described to date.

In summary, we need to represent the idea of physical variables, the mathematical symbols used to represent specific physical properties of physical objects. The physical variable may be thought of as having all of the mathematical properties that the applicable physical theories indicate that they should have, and also be capable of holding the corresponding measurable values. The values may be arrived at by measurement of the corresponding physical objects attributes, or by prediction arrived at by applying physical laws, e.g., equations, to other measured attributes of a system of physical objects.

4 The Physical Dimensional Properties of Physical Variables

The term "physical quantity" is a fundamental one in physics, narrowly defined by the International System of Units (SI) as the measurable properties of physical objects. Common usage often substitutes the phrase physical dimension for the SI defined phrase physical quantity, and uses the term physical quantity more loosely. A "physical dimension" in this sense should not be confused with the separate notion of spatial dimensions, e.g., those defined by three spatial basis vectors.

The SI has also defined base quantities: they are length, mass, time, electric current, thermodynamic temperature, amount of substance, and luminous intensity with corresponding dimensions represented by the symbols L, M, T, I, Θ, N, and J [1]. Derived quantities may be created by taking products, ratios, and powers of the base quantities. A measurement generally returns a positive, definite quantity and a zero value implies an immeasurably small amount of the quantity. The result of a simple (scalar) measurement of a physical quantity is represented as the product of a scalar real number and a physical unit, where the physical unit is a scale factor for a physical quantity or physical dimension. While there is debate within the physics research community as to how many physical dimensions are truly fundamental, standard practice is to use the seven SI base quantities mentioned above. The SI also provides corresponding standard base units for the seven base quantities: meter; second; kilogram; ampere; Kelvin; mole; and candela. Within the SI standard, many other units, called derived units, are defined in terms of these base units.

While the SI system is commonly used, it is not used exclusively. Other systems may have a different set of fundamental dimensions, base units, or both. A simple way to characterize the system used for a given model is to specify, for n fundamental dimensions, an n-tuple of defined units. This explicitly specifies the base units while implicitly specifying the base dimensions and supports the expression of a model for any set of defined absolute quantities.

In its most comprehensible form, then, a physical variable represents a quantity, like a length, which is generally measured as a finite precision, real number of units, where the units are some reference or standard units. While an individual measurement is most easily thought of as a scalar quantity, physical variables may have multiple components which are more suitably represented as vectors or tensors. Measurement of these more complex objects is correspondingly complex.

4.1 The Mathematics of Units and Dimensions

As asserted earlier, the semantics of physics is largely contained within the mathematical properties of the components with which we describe physical models. We now begin to examine the mathematical properties of physical variables. The operation of taking the physical dimension of a physical variable, X, is usually written with square brackets, as $[X]$. This operation, which is idempotent, i.e., $[[X]] = [X]$, is like a projection, where the information about magnitude, units, and spatial directionality of the physical variable is all lost. We can enumerate some of the properties of physical variables under the physical dimension bracket operation:

All physical variables have physical dimension composed of the fundamental dimensions:

$$[X] = \mathsf{L}^{\alpha}\mathsf{M}^{\beta}\mathsf{T}^{\gamma}\mathsf{I}^{\delta}\Theta^{\epsilon}\mathsf{N}^{\zeta}\mathsf{J}^{\eta}, \tag{1}$$

where the exponents, α, β, γ, δ, ϵ, ζ, and η, are rational numbers.

Physical variables may be added if they are of the same dimension:

$$\text{If } [X] = [Y], \text{ then } [X + Y] = [X] = [Y]; \tag{2}$$

The physical dimension of a product of physical variables is the same as the commutative and associative product of the physical dimensions of the factor variables:

$$[X * Y] = [X] * [Y] = [Y] * [X]\ ; \tag{3}$$

$$[X * Y * Z] = [X * Y] * [Z] = [X] * [Y * Z] \quad (4)$$

The physical dimension of the reciprocal of a physical variable is the reciprocal of the physical dimension of the variable:

$$[X^{-1}] = [X]^{-1} \quad (5)$$

The physical dimension of a real number is defined to be 1. Formally, the physical dimensions of physical variables form a commutative, or abelian group. This group may be written multiplicatively, which corresponds to the usual way in which dimensional quantities are manipulated in most physical applications. Written multiplicatively, the group elements are the identity, 1, and, in the case of SI, $n = 7$ base quantities, L, M, T, I, Θ, N, and J, along with their powers and their products. Being abelian, this group may also be written additively, where the group element representation is as an n-tuple of exponents for the n base quantities. The additive representation of the group is useful in performing dimensional analysis. The exponents of the dimensions are often integers, although for convenience in some applications the exponents are extended to the rational numbers. When written additively, the physical dimensions of physical variables may be seen to form a vector space, where vector addition corresponds to multiplication of the underlying physical variables and scalar multiplication of the n-tuple of exponents corresponds to raising the physical variables to various powers.

By taking the physical dimension of a physical variable we have lost some essential pieces of information, which we now seek to recover. In particular, for what is commonly thought of as a scalar physical variable, we need to represent the combination of the units and magnitude of the physical property. In order to do so, we here introduce the following notation: $X = \{X\}_u * u$, where a physical variable is factored into two parts: the first part is $\{X\}_u$, while the second part is the unit, u, that the physical dimension is expressed in, i.e., $[X] = [u]$, The first part, $\{X\}_u$, which is properly scaled with respect to the unit, u, is the non-physically-dimensioned part of the physical variable, i.e., $[\{X\}_u] = 1$. We will call this part of the physical variable, $\{X\}_u$, the spatial part.

Units provide a scale factor for each of the base dimensions, giving a base unit for each base dimension. A unit is either a base unit or a unit derived by (commutative) products and ratios of base units. For example, a product of two units of length results in a unit having physical dimension $\mathsf{L} * \mathsf{L} = \mathsf{L}^2$, a unit for area. A product or ratio of different units may be reduced if they have fundamental dimensions in common. A ratio of two different units having the same physical dimension, when reduced, results in a dimensionless real number called a *conversion factor*. For example, $[foot] = [meter] = \mathsf{L}$ so $meter/foot \approx 3.28$.

We finally note that we can represent the logarithm of the physical variable as the formal sum

$$ln(X) = ln(\{X\}_u) + \alpha * ln(u_1) + ...\eta * ln(u_7) \quad (6)$$

where $u = \prod_{i=1}^{7} u_i$ in the case of seven base units. We can more simply represent this as

$$(z, \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta) \tag{7}$$

where $z = ln(\{X\}_u)$, representing the measured quantity in units derived from base units. In this representation of the physical variable the operation of taking the physical dimension is seen to be a true projection operator, i.e.,

$$[(z, \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)] = (0, \alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta). \tag{8}$$

where the result is an element of the additive representation of the group of physical dimensions. The space of fundamental and derived physical dimensions so represented comprises a vector space, where the *vector addition* operation corresponds to multiplication of physically dimensioned quantities and the *scalar multiplication* operation corresponds to raising physical quantities to powers. A change of units is represented as a translation operation in the first (dimensionless) component of the $(1 + n)$-tuple that represents the physical variable when that component is an element of a scalar field.

While physicists routinely perform legitimate mathematical manipulations of physical dimensions, they do this intuitively and the formal mathematical structure of physical dimensions is rarely expressed.

5 A Type for the Spatial Part of Physical Variables

The spatial part of physical variables, i.e., $\{X\}_u$, has the following properties: we can multiply it by a scalar; we can add more than one together; we can multiply more than one together. The first two of these properties indicate that they form a vector space. The third property, multiplication of physical variables, is trivial when the spatial part of a physical variable is a scalar. After scalars, the most common object representing the spatial part of physical variables are *vectors*. When a physicist or engineer refers to a "vector", they usually mean a rank-1 tensor. Physicists and engineers also use higher rank tensors, most commonly rank-2 tensors.

Typically used vector multiplications are: the scalar, inner, or dot product; and, the vector cross product, or Gibbs' vector product. Well known, though less commonly used, is the dyadic, outer, or tensor product, where higher rank tensors may be constructed from lower rank tensors. Usually a metric is tacitly assumed, typically Euclidean. Other metrics are required for special and general relativistic mechanics.

The manner in which these vectors and tensors are manipulated by physicists is largely ad hoc, rather than uniform, and is usually derived from the work of prior physical scientists. Maxwell popularized Hamilton's quaternions, using them to express electrodynamics. Quaternions were superseded by the vector analysis of Gibbs [2] and Heaviside, which survives to this day, largely unaltered except by addition of new concepts, objects and operations. The mathematics used in quantum mechanics today follows the style of usage originated by the physicists that originally employed it. While we do not mean to suggest incorrectness in their treatment, much of the mathematics used by physicists is taught by physicists. A mathematician might find an absence of definition and uniformity in the mathematical properties of physical variables as they are most commonly used.

Considering these issues, a principle question that that we raise is: "What is the type of the spatial part of physical variables?" By asking this question we mean to proceed to understand the formal mathematical structure of these objects. Because physics is at root empirical, the best answer that can be provided is to propose a type of object that appears to meet the criteria of matching the known objects used by physicists as physical variables. Each time a new concept, object, or operation is added, it would be helpful to formally extend an axiomatic mathematical framework to incorporate the new in with the old. The purpose for doing this was stated previously: closure in the world of physical quantities and interactions should be reflected by mathematical closure in the physical variables used to represent the physical world. Happily, this question has been constructively considered and the best answer to date appears to be that the spatial part of physical variables may be described by Clifford algebras [3].

As usually encountered in the education of a physicist, physical variables, specifically the spatial part of physical variables, appear to consist of several types. Most commonly encountered are real scalars or vectors. Complex scalars and vectors are also commonly used in representing physical variables. Minkowski four-vector notation is well-known to students of physics to be a better notation than Gibbs' vector notation for electrodynamics, particularly the "Electrodynamics of Moving Bodies", [4] i.e., special relativity. General relativity introduces multi-ranked tensors; elementary quantum mechanics introduces Hilbert spaces and the non-commutative spinors. Finally, modern quantum particle theories make liberal use of elements of various Lie algebras. To the casual observer, there appears to be a multiplicity of types.

Clifford algebras are not commonly used by most physicists, though they are heavily used in some forefront research areas of theoretical physics. While there is currently some effort [5] to change this state of affairs, one may reasonably ask why we should introduce into a discussion of standards a construct that is not commonly used. The answer is based on two requirements. First, there is the important problem of being able to translate or otherwise relate models expressed in different notations. If there is one notational representation that can capture the semantics of a catchall of individual notations, then it is useful to have it present at least as an underlying representation, even if it is infrequently expressed explicitly in the specification of models. That is, since it represents the current understanding of the fundamental underlying mathematics for most, if not all, physical models, representing Clifford algebras is sufficient to represent physical variables in most known models. Secondly, since many models explicitly reference Clifford algebras, it is necessary to represent Clifford algebras in order to represent the semantics of those models.

5.1 Features of a Clifford Algebra

The objects of Clifford algebras are vectors, although they may not always seem as recognizable to physicists as the usual vectors that come from the Vector Analysis of Gibbs and Heaviside. The vectors of Clifford algebras are also referred to as *multivectors* and represent a richer set of objects than those in Gibbs' Vector Analysis. Some multivectors are the usual vectors of Gibbs' Vector Analysis, some are scalars and some are higher ranked tensors. Some of these multivectors represent formal sums of the usual scalars, vectors, and tensors. Some of these multivectors may be used to represent subspaces. Some of these multivectors are used to represent rotations, translations, spinors and other objects normally described by Lie groups. In summary, the principle mathematical objects of interest to physicists are all elements of Clifford algebras.

A key element of Clifford algebras is the Clifford product, an associative vector product with an inverse. The other vector products previously mentioned here do not have these properties. Since Clifford algebras also have an identity element and closure holds for the Clifford product, there is a resulting group structure for the vectors in a Clifford algebra. Of particular interest, Lie algebras are sub-algebras of Clifford algebras. A complete description of Clifford algebras is well beyond the scope of this paper and is well described elsewhere [3].

One may well ask "If Clifford algebras are as powerful as advertised, why did physicists ever commit to the standard Vector Analysis?" There may be several speculative answers possible [6]. Certainly the work of Grassman, which gave rise to Clifford algebras, may not have been as well publicized among physicists as Gibbs' work was, though Gibbs was certainly aware of it. Additionally, the standard Vector Analysis serves quite well for much of classical physics, so its continued use is a reasonable satisficing strategy. How, then, is Gibbs' Vector Analysis not the best fit for physics? It begins to be less comfortably used when vector objects of rank greater than one, i.e., tensors, are required, but, most certainly, spinors appear to be foreign objects within Vector Analysis. Perhaps one of the sorest points is that vector cross-product defined by Gibbs only exists in three dimensions. Modern physicists like to stretch well beyond three-dimensions. In Clifford algebras the cross product has been defined for spaces of any dimension. Outside of three dimensional space it is not a simple vector, and does not appear to be describable within Vector Analysis.

We note the following several points that may be of particular interest to mathematicians. Hestenes narrows the range of Clifford algebras of interest to physicists to *geometric algebras*. Geometric algebras are the subset of Clifford algebras defined over the reals and possessing a non-singular quadratic form [3] [7], so, from the mathematical perspective, expressing elements and operations of a Clifford algebra are sufficient for doing the same for elements of a geometric algebra. A concise axiomatic development of geometric algebra and its differential calculus, called geometric calculus, are provided by Hestenes [3]. Geometric calculus claims greater generality than Cartan's calculus of differential forms [3].

Of particular interest to physicists and other intuitive mathematicians, geometric algebras have natural and well developed geometric interpretations [11] which, interestingly, have been exploited in computer graphics rendering using the coordinate-free representations of rotations and translations. The work of reformulating physics in this coherent notation, not overwhelming, but no small task, has been underway for many years [8] [9] with the result that it appears to have great potential for unifying the mathematics of physics. Geometric calculus has even been successfully applied to gauge theory gravity [10], one of the more esoteric research frontiers in physics.

We are left to conclude that the standard Gibbs' Vector Analysis is by comparison just a convenient shorthand, derived from the ideas of Grassman which have reached a fuller and richer expression in Clifford algebras. In sum, Clifford algebras generally, and geometric algebra in particular, provide a coherent algebraic method for representing the spatial part of physical variables for most of classical and modern physics. It is certainly not the most commonly used notation, but other notations may be readily translated into it.

6 Summary

Our purpose here has been to sketch the essential mathematical properties of physical variables. One reason for doing this is to help clarify the mathematical semantics as separate from, though necessary to, the expression of physical semantics. Having made this separation, experts in representing mathematical semantics are now enabled to aid in the development of a physics markup language by independently expanding mathematical semantic representations. In particular, semantic representations of the mathematical properties of physical dimensions and units, and of Clifford algebras, which include geometric algebras, will greatly enable the expression of the physical semantics of physics-based models. We believe that the expression of Clifford algebras in this way will be significantly more straightforward from a mathematical perspective because it is mathematically better defined than the collection of notations used for different sub-theories within the physics community.

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