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Entropy Viscosity and L1-based Approximations of PDEs: Exploiting Sparsity

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14. ABSTRACT

Our goal is to develop robust numerical methods for solving mathematical models of nonlinear phenomena such as nonlinear conservation laws, advection-dominated multi-phase flows, and free-boundary problems, where shocks, fronts, and contact discontinuities are driving features and pose significant difficulties for traditional numerical methods. We have discovered that time-dependent nonlinear conservation equations can be stabilized by using the so-called entropy viscosity method and we proposed to to investigate this new technique. We explored in detail the approximation properties of the entropy viscosity method along the following directions. (i) New discretization methods including Discontinuous Galerkin and Lagrangian hydrodynamics; (ii) New fields of applications of the entropy viscosity concept, such as multiphase flows, using phase field techniques. This novel robust approximation technique for solving nonlinear problems developing shock or sharp interfaces will benefit every areas of science and engineering where controlling or dealing with this type of phenomenon is still an enormous challenge.

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Entropy Viscosity and L1-based Approximations of PDEs: Exploiting Sparsity

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1 Overview on the research program

The objective of this project is to develop and analyze efficient and robust numerical methods for simulation of mathematical models of nonlinear phenomena such as nonlinear conservation laws, advection-dominated flows, and other related problems where nonlinear effect such as shocks, fronts, and contact discontinuities pose significant difficulties for traditional numerical methods.

The main thrust of the research program for the period July 2012 to June 2015 was to explore and investigate the approximation properties of a technique that we call entropy viscosity method. The main stabilization mechanism in this method is a nonlinear dissipation proportional to the local size of an entropy production. We believe that the entropy viscosity idea is an important conceptual breakthrough. In the three year period of the current project we have made progresses in the following directions:

1. **Robust explicit viscous method.** We have constructed a first-order approximation technique for nonlinear scalar conservation equations that is

maximum principle satisfying on any mesh, any space dimension and for any flux. The approximation is explicit in time and done with continuous finite elements; the flux just needs to be Lipschitz continuous.

2. **Error estimates.** We have proved error estimates for the above first-order method. The methods converges strongly to the unique entropy solution with a convergence rate which is at least $\mathcal{O}(h^{\frac{1}{4}})$ irrespective of the space dimension.

3. **Second-order method.** We have proposed an accurate and maximum principle preserving method for nonlinear scalar conservation equations using the so-called flux correction technique (FCT). The method combines the first-order technique and the entropy-viscosity methodology.

4. **Viscous regularization of the Euler equations.** We have investigated viscous regularization techniques for the compressible Euler equations using fundamental results from thermodynamics. A general class of viscous regularization that is compatible with all known generalized entropy inequalities has been identified.

5. **Positivity in the Euler equations.** We have started to analyze the positivity properties of explicit viscous schemes using continuous finite elements for the compressible Euler equations. We have obtained an explicit positivity-preserving scheme and shown that positivity does not depend on the speed of sound. The technique does not involve any Riemann solvers (approximate or exact).

6. **Entropy-viscosity LES.** We have investigated the idea of using the notion of entropy-viscosity as a LES technique for the incompressible Navier-Stokes equations at high Reynolds numbers.

7. **High order schemes for other models:** We are developing a new second order methods for the so-called Mean Filed Games (MFG). The MFG model is a coupled nonlinear system which emerges in financial and crowd dynamics models.

2 Entropy viscosity

We review in this section some elementary facts about the entropy viscosity methodology. Nonlinear scalar conservation equations can all be put into the following

general form:

$$\partial_t u(\mathbf{x},t) + \nabla\cdot f(u(\mathbf{x},t)) = 0, \qquad \mathbf{x}\in\Omega,\ t>0, \tag{1}$$

with $u|_{t=0} = u_0$ and appropriate boundary conditions. The scalar initial boundary value problem has a unique entropy solution which satisfies an additional set of differential inequalities

$$\partial_t E(u) + \nabla\cdot F(u) \leq 0, \tag{2}$$

for any pairs $E(u)$ and $F(u)$ such that E is convex and $F'(u) = E'(u)f'(u)$. The function E is called entropy and F is the associated entropy flux. For convex fluxes (i.e., if f is convex) in one space dimension it is known that one entropy pair, for example the one generated by $E(u) = \frac{1}{2}u^2$, is enough to select the unique entropy solution. Physical systems have at least one entropy pair and the entropy inequality (2) is the mathematical form of the second principle of thermodynamics. It is expected that the auxiliary inequality (2) serves as a selection criteria and guarantees convergence of the numerical approximation to the correct *physical solution* of the nonlinear system. Therefore, it is desirable (and necessary) to somehow incorporate the entropy dissipation (2) in a numerical scheme.

A traditional way of selecting the entropy solution of (1) consists of adding viscous dissipation

$$\partial_t u^\epsilon + \nabla\cdot f(u^\epsilon) - \nabla\cdot(\epsilon\nabla u^\epsilon) = 0, \tag{3}$$

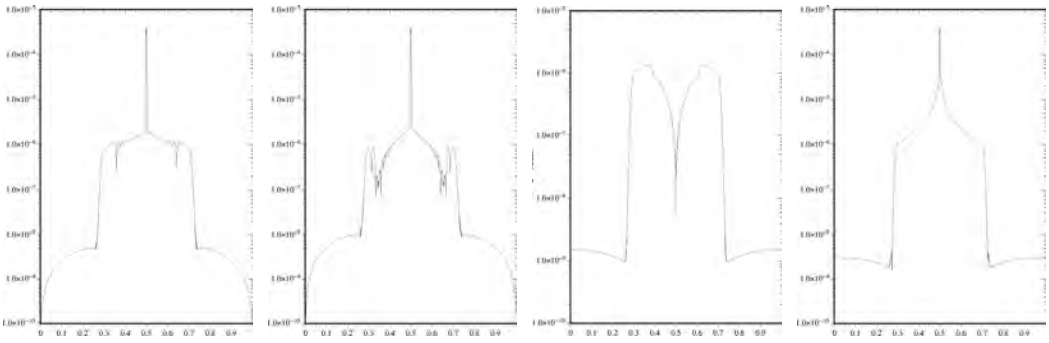
where $\epsilon > 0$ and it can be shown in general that $u^\epsilon \to u$ when $\epsilon \to 0$. The parameter ϵ is usually taken to be proportional to the local mesh size when constructing numerical approximation of (3) and this limits the convergence rate to first-order at most. The use of artificial viscosity to solve nonlinear conservation equations has been pioneered by Neumann and Richtmyer and popularized later by Smagorinsky for LES purposes and by Ladyženskaja for theoretical purposes in the analysis of the Navier-Stokes equations. However, the early versions of artificial viscosities were overly dissipative and the interest for development of such methods has faded over the years. The research direction shifted towards Discontinuous Galerkin and Finite Volume methods, where upwinding and limiters have been shown to be efficient and to yield high-order accuracy. Up to a few exceptions slope/flux limiting is a one-dimensional concept that does not easily generalize to unstructured meshes in higher dimensions. Moreover, the theoretical understanding of the stability and convergence of limiters is currently restricted to uniform grids and scalar equations in one space dimension. For the above reasons

and, among other things, the fact that artificial viscosities are easy to implement, the interest for artificial viscosity has lately been revived in the DG literature and in the Continuous Galerkin (CG) literature as well.

It is usually argued in the literature that good artificial viscosities can be computed from measures of the local regularity of the solution or from local residuals of the PDE. We propose to take a slightly different route by considering the local residual of an entropy equation to construct an artificial viscosity.

$$\epsilon \sim ch^2|\partial_t E(u) + \nabla \cdot F(u)|. \tag{4}$$

One immediate consequence of this choice is that the viscosity is proportional to the entropy production, which is known to be large in shocks and to be zero in contact discontinuities. As a result, this strategy makes an automatic distinction between shocks and contact discontinuities, and this subtle distinction cannot be made by any of the two classes of methods mentioned above. We also think that using the residual of the conservation equation may be less robust than using the entropy residual. This argument is based on the observation that consistency requires the residual of the PDE to converge to zero in the distribution sense as the mesh-size goes to zero, whereas the very nature of entropy implies that the entropy residual converges to a Dirac measure supported in the shocks. This implies that the entropy residual focuses far better in shocks than the PDE residual, and it is in this sense that we claim that the PDE residual is less reliable than the entropy residual. This can be better understood by considering the simple case of the one-



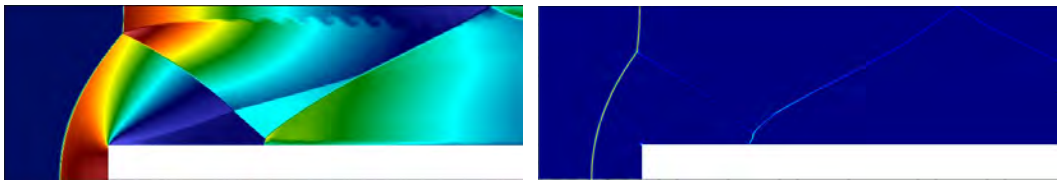
(a) $E(u) = \frac{1}{2}u^2$ (b) $E(u) = \frac{1}{2}u^2, c = 1$ (c) $E(u) = u, c = \frac{1}{2}$ (d) $E(u) = u, c = 1.$

Figure 1: Entropy viscosity in log scale for the Burgers equation at $t = 0.25$

dimensional Burgers equation $\partial_t u + \frac{1}{2}\partial_x u^2 = 0$ with initial data $u(x, 0) = 1$ if $x < 0$ and $u(x, 0) = 0$ otherwise. The entropy solution is $u(x, t) = 1 - H(x - \frac{1}{2}t)$

where H the Heaviside function. One observes that the residual of the equation is zero, whereas the entropy residual is an negative measure: $\frac{1}{2}\partial_t u^2 + \frac{1}{3}\partial u^3 = -\frac{1}{12}\delta(x - \frac{1}{2}t)$, where δ is the Dirac measure. This effect is very well illustrated on the one-dimensional Burgers equation over the interval $(0, 1)$ with initial data $\sin(\pi x)$, see Figure 1. We show in this figure the entropy viscosity computed by using either the quadratic entropy $E(u) = \frac{1}{2}u^2$ (first and second panel from the left) or $E(u) = u$ (third and fourth panels). Choosing $E(u) = u$ corresponds to using the residual of the PDE for the viscosity. These tests show that choosing $E(u) = \frac{1}{2}u^2$ leads to better focusing of the viscosity and is more robust with respect to the multiplier c (see definition (4)).

We now illustrate the entropy viscosity concept on the compressible Euler equations by using the physical entropy to construct the entropy viscosity. We show in Figure 2 the density field at $t = 2.86$ and the viscosity field at $t = 4$ for the classical wind tunnel problem with a forward facing step at Mach 3. We observe that the viscosity focuses very well and there is almost no viscosity in the contact discontinuity that develops at the top of the flow.

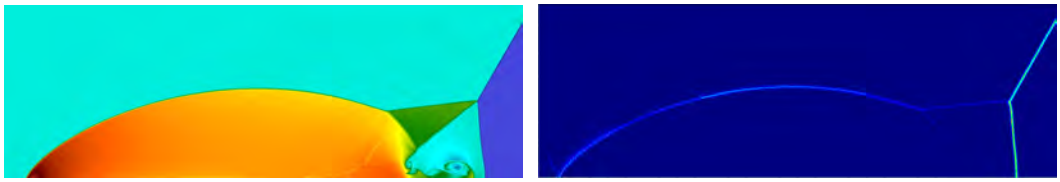


(a) Density at $t = 2.86$ (b) Entropy viscosity flux at $t = 4$

Figure 2: Wind tunnel with a step at Mach 3, $\mathbb{P}_1$ approximation.

We show in Figure 3 the density field at $t = 0.2$ for the double Mach reflection problem at Mach 10. We observe again that there is almost no viscosity in the contact discontinuity that develops from the first triple point; moreover the jet develops the standard instability at the bottom of the domain.

We have also developed two different techniques to enforces boundary conditions on curved boundaries, which is not an easy task for nonlinear conservation equations. We are also using the entropy viscosity framework to construct a goal oriented refinement strategy. The method is well developed by now. We have solved various benchmark problems with finite elements, spectral elements and Fourier. We show typical results in Figure 4.



(a) Density (b) Entropy viscosity

Figure 3: Double Mach reflection, $t = 0.2$, $\mathbb{P}_1$ approximation.

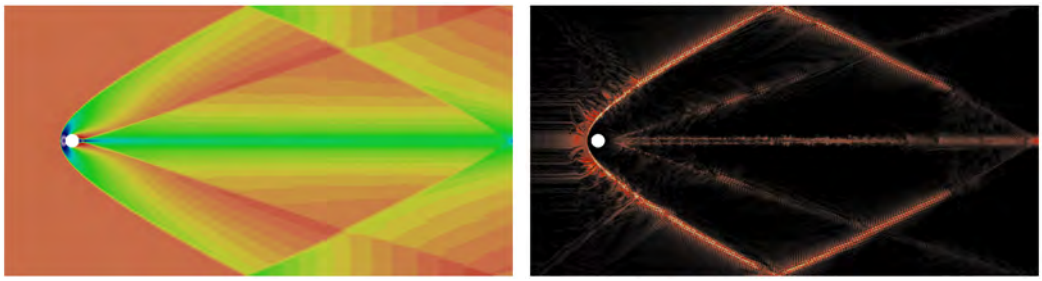


Figure 4: Supersonic around a cylinder in a tunnel. $\mathbb{P}_1$ approximation. Density field (left) and entropy viscosity (right).

3 Achievements in the period July 2012-June 2015

We describe in this section what we have achieved during the period July 2012-June 2015.

3.1 A robust viscous method

It is traditional to stabilize numerical methods for solving transport equations or scalar conservation equations by adding artificial dissipation. In the continuous finite element literature the viscosity is added by augmenting the Galerkin formulation with the weak form of the operator $-\nabla\cdot(\nu_h\nabla)$ where ν_h is scalar-valued and $\nu_h = c\beta h$. Here c is an adjustable constant, $\beta \geq 0$ is proportional to the local wave speed and h is a measure of the local mesh size. The first obstacle one runs into when using this approach is that of a proper definition of the meshsize h on non-uniform anisotropic meshes. Although many clever and reasonably well justified ideas have been proposed to address this non-trivial issue, to the best of our knowledge, none of them has yet lead to a provable maximum principle holding for every nonlinear flux and every mesh, assuming piecewise linear approximation

of course. Once a meshsize h has been chosen, one then runs into the problem of choosing the constant to multiply βh to form the viscosity. Although $\frac{1}{2}$ seems to be a reasonable choice justified by the one-dimensional analysis on uniform grids, we do not know of any rational for tuning this constant in two and three space dimensions on arbitrary grids besides heuristic arguments and trial and errors tests. Even-though it is possible to establish well-founded heuristic arguments to tune the constant, again we do not know of any argument yielding a provable maximum principle for every nonlinear flux and every mesh. The last argument that finally lead us to revisit the first-order viscosity theory is that it is not robust with respect to the shape of the cells. When trying to reproduce the one-dimensional argument in arbitrary space dimension with continuous finite elements one observes that the convex combination argument can work only if $\int \nabla\varphi_i \cdot \nabla\varphi_j \, \mathrm{d}x < 0$ for all pairs of shape functions, φ_i, φ_j, with common support of nonzero measure. This is the well-known acute angle condition assumption. This condition is easy to violate, in particular in 3D, and makes any method based on the bilinear form $\int \nu_h \nabla u \cdot \nabla v \, \mathrm{d}x$ not robust.

Starting from the above observations, in [4] we have revisited the first-order viscosity theory from top to bottom and introduced a dissipation mechanism based on a graph Laplacian. In this theory the viscosity is defined cell-wise to ensure that the solution is maximum principle preserving. The method is explicit, works with continuous finite elements, does not involve any adjustable constants, can be defined on arbitrary meshes in any space dimension, and is independent of the local anisotropy of the mesh.

3.2 Error estimates

We have analyzed the convergence properties of the first-order Lagrange finite element technique introduced in [4] for scalar nonlinear conservation laws. We have proved in [9] that the error in the $L^\infty_t(L^1_x)$-norm is at most $\mathcal{O}(h^{\frac{1}{4}})$ under the appropriate CFL condition in any space dimension and for any shape-regular mesh family; the mesh may be composed of an arbitrary combination of simplices, prisms, cuboids, etc. The estimate is established by using the technique of the doubling of the variables introduced by Kruskov (1970) and first used by Kuznecov (1976) to prove error estimates. This approach completely bypasses the compensated compactness argument. To the best our knowledge, this is the first time that a priori error estimates have been established for an explicit method using continuous Lagrange finite elements to approximate nonlinear scalar conservation equations on nonuniform meshes. Similar results have been established

by Cockburn and Gremaud (1996), but the error estimate therein is $\mathcal{O}(h^{\frac{1}{8}})$ and the algorithm is a shock capturing streamline diffusion method using implicit time stepping and an artificial viscosity scaling like $h^{\frac{3}{4}}$ in the shocks.

We have shown in [9] that the error in the L^1-norm is directly controlled by entropy residuals, i.e., the key to obtain an error bound is to estimate the entropy production for every Kružkov entropy. Using a graph-Laplacian-based viscous dissipation was a major breakthrough in this enterprise. This result strengthens our point of view that entropy residuals are important objects.

3.3 Convergence of high-order schemes

In [1], we establish the L^2-stability of an entropy viscosity technique applied to nonlinear scalar conservation equations. First- and second-order explicit time-stepping techniques using continuous finite elements in space are considered. The method is shown to be stable independently of the polynomial degree of the space approximation under the standard CFL condition. This technique is extended to the Navier-Stokes equations and is implemented as LES model in SFEMaNS.

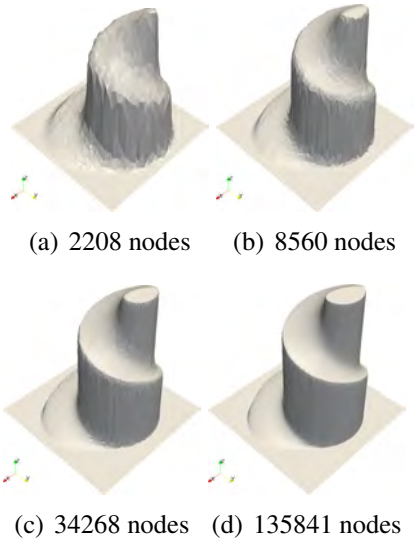
In a sequence of papers [13, 12], we have addressed the long standing question about convergence of well known high-order central schemes towards the entropy solution of a scalar conservation law. The goal was to find tools which will allow us to analyze such methods and prove their convergence. We proved in [13, 12] a new maximum principle for central second order schemes based on nonlinear piecewise linear reconstruction. The novelty is that the maximum principle proof is valid for general k-monotone fluxes and convergence to the entropy solution is proven if the flux is strictly convex. Moreover, the second order reconstruction is not reduced to first order in the regions of local extrema. It has been an open problem how to prove stability and error estimates for such schemes which work in practice but their local reconstruction violates a regular maximum principle. In our most recent paper, together with the former student Orhan Mehmetoglu, we were able to prove strong convergence of the above mentioned numerical methods towards the unique entropy solution in the case of a conservation law with just bounded initial data [14]. This is the largest possible class of initial conditions considered in practice and the results for first order schemes were classical in this setup. However, this was not known any of the second order schemes based on minmod-type limiters.

3.4 Construction of a second-order method

Imposing both the maximum principle and high-order accuracy in space is a difficult task for numerical methods approximating scalar conservation equations. The Godunov theorem even asserts that it is impossible for one-step linear approximation methods to be at least second-order accurate in space and be monotonicity preserving. Satisfying the maximum principle and having high-order accuracy on arbitrary meshes in any space dimension are very desirable properties in all applications, but these are two contradicting requirements: (i) on one hand one needs to add artificial viscosity to guarantee the maximum principle; (ii) on the other hand one needs to stay as close as possible to the Galerkin discretization of the problem to be high-order accurate in space.

In [5], we have extended the first-order technique introduced in [4] to an explicit second-order maximum principle preserving numerical method that works on arbitrary meshes in any space dimension with any Lipschitz flux using continuous Lagrange finite elements. The four key ingredients are the first-order technique of [4], a novel treatment of the consistent mass matrix by [6], a high-order technique (entropy-viscosity method of [7]) and the Boris-Book-Zalesak flux correction technique. The main characteristics of the new method are: (i) it is maximum principle preserving; (ii) it preserves the accuracy of the entropy viscosity method; (iii) the dispersion errors induced by mass lumping are corrected in the flux limiting step. As an illustration of the method we show the solution of the two-dimensional scalar conservation equation $\partial_t u + \nabla \cdot \mathbf{f}(u) = 0$ with the non-convex flux $\mathbf{f}(u) = (\sin u, \cos u)$ and initial data $u(\mathbf{x}, 0) = \frac{14\pi}{4}$, if $\sqrt{x^2 + y^2} \leq 1$, $u(\mathbf{x}, 0) = \frac{\pi}{4}$, otherwise. It is a challenging test case for many high-order numerical schemes because the solution has a two-dimensional composite wave structure. For example, it is known that some central-upwind schemes based on WENO5, Minmod 2, or SuperBee reconstructions converge to non-entropic solutions. The computational domain $[-2, 2]\times[-2.5, 1.5]$ is triangulated using nonuniform meshes and the solution is approximated up to $t = 1$ using $\mathbb{P}_1$ finite elements. The graph of the limited solu-

Figure 5: KPP solution

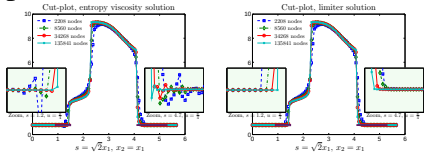


(a) 2208 nodes (b) 8560 nodes

(c) 34268 nodes (d) 135841 nodes

tion on four meshes (2208, 8560, 34268, and 135841 $\mathbb{P}_1$ nodes, respectively) are shown in Figure 5. The helicoidal composite wave is clearly visible.

Figure 6: Cross section, KPP problem



To better evaluate the amount of maximum principle violation by the entropy viscosity solution and visualize the action of the flux correction technique, we show in Figure 6 a cross section along the line $x_2 = x_1$ of the graph of the entropy viscosity and limited solutions at $t = 1$ on four different meshes (2208, 8560, 34268, and 135841 $\mathbb{P}_1$ nodes, respectively). The cross section for the entropy solution is shown in the left panel and that for the limited solution is shown in the right panel. Two regions are magnified to emphasize the slight violations of the local maximum principle by the entropy viscosity solution. Using the curvilinear abscissa $s = \sqrt{2}x_1$, the first region is the rectangle $(s, u) \in [1.2 \pm 0.3]\times[\frac{\pi}{4} \pm 5\times10^{-2}]$, the second region is the rectangle $(s, u) \in [4.7 \pm 0.6]\times[\frac{\pi}{4} \pm 1\times10^{-2}]$. Observe that in both cases the over- and under-shoots of the entropy solution are barely noticeable and seem to diminish as the meshes are refined. Note again that the limited solution satisfies the maximum principle, and the convergence of the limited solution to the exact solution seems to be monotone.

3.5 Viscous regularization for Euler

In our original implementation of the entropy viscosity method [7], we used the Navier-Stokes system as a viscous regularization of the Euler equations and we built a high-order method from there. Although the numerical results looked good on standard benchmark tests (see Figure 2, Figure 3), extensive numerical experiments revealed that the Navier-Stokes system is not a robust regularization in general. Two key reasons for that are: (i) the Navier-Stokes regularization has no mechanism that can keep the density positive; (ii) the Navier-Stokes regularization is known to violate the minimum entropy principle if the thermal diffusivity is set to zero. Therefore, the viscous model has oscillations that are not present in the inviscid limit. These observations have lead us to investigate this problem carefully. In [10] we have identified all viscous regularizations of the Euler system that are compatible with the minimum entropy principle and with some generalized entropy inequalities. Moreover, we have identified one single family of regularizations that preserves the positivity of the density, satisfies a minimum entropy principle, and is compatible with all the generalized entropies inequalities

as introduced in Harten et al. (1998).

3.6 Positivity for the compressible Euler equation

In [8] we have started to apply the graph Laplacian strategy to the Euler equations and we have been able to construct an explicit first-order method with continuous finite elements that is capable of preserving the positivity of the density and the internal energy. Quite surprisingly, the theorem that we have established shows that the amount of dissipation that is needed for this property to hold does not invoke the speed of sound; the viscosity is directly proportional to the material velocity only. The result holds for every reasonable equation of state. This result goes against a few received ideas in the field. This work just started during Summer 2014, but it promises to be very fruitful and has already open doors on barely explored territories.

3.7 LES and entropy-viscosity

Simulating turbulence is computationally demanding since large gradients and eddy-phenomena exist in general at scales that cannot be correctly represented by the mesh. The small scales of turbulent flows produce still smaller ones through the coupling of wave modes via the action of the nonlinear term; this induces an accumulation of energy at the grid scale. The consequence of this cascade phenomenon is that turbulent flows can be considered as having singular behaviors at the mesh scale. A possible remedy to this phenomenon that we proposed in [7, 2] consists of monitoring the local kinetic energy balance and introducing a localized dissipation in these regions that is proportional to the violation of this balance. The deviation from the local energy balance, which we call the entropy residual,

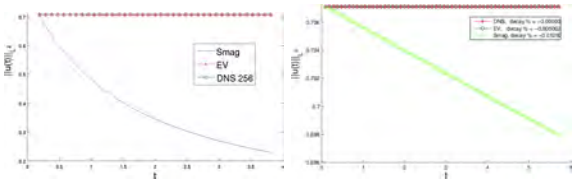
$$D_h(\mathbf{x},t) := \partial_t(\tfrac{1}{2}\mathbf{u}_h^2) + \nabla\cdot((\tfrac{1}{2}\mathbf{u}_h^2 + p_h)\mathbf{u}_h) - \tfrac{1}{Re}\Delta(\tfrac{1}{2}\mathbf{u}_h^2) + \tfrac{1}{Re}\nabla\mathbf{u}_h{:}\nabla\mathbf{u}_h - \mathbf{f}\cdot\mathbf{u}_h$$

can be thought of as an indicator for local entropy production in analogy with entropy production for scalar conservation laws. (Here $(\mathbf{u}_h, p_h)$ are the approximate velocity and pressure and h denotes the grid scale.) In a resolved flow $D_h(\mathbf{x},t)$ should be on the order of the consistency error of the method; a small value. In [8] we construct a viscosity proportional to $|D_h(\mathbf{x},t)|$. This so-called entropy-viscosity is defined by $\nu_E(\mathbf{x},t) := \min\left(c_{\max}h(\mathbf{x})\|\mathbf{u}_h(\mathbf{x},t)\|,\, c_E h^2(\mathbf{x})\frac{|D_h(\mathbf{x},t)|}{\|\mathbf{u}^2\|_{L^\infty(\Omega)}}\right)$. The momentum equation is then

modified by adding the term $-\nabla\cdot(\nu_E(\mathbf{x},t)\nabla\mathbf{u})$. In the definition of $\nu_E(\mathbf{x},t)$, the constants $c_{\max}$ and c_E are tunable parameters which depend only on the numerical method and the geometry of the mesh. This ensures that the viscosity never exceeds the first-order upwind viscosity. When the local grid size is small enough so that all the scales are resolved, the residual $|D_h(\mathbf{x},t)|$ and $\nu_E(\mathbf{x},t)$ are of the order of the consistency error and the consistency error times $h(\mathbf{x})^2$, respectively. The entropy-viscosity is therefore consistent, i.e., vanishes when all of the features of the flow are properly resolved at the grid scale.

We have done series of validation tests on the above model in the periodic cube using a Fourier code developed at LANL. These tests are reported in [3] To evaluate the consistency of the method we have tested an inviscid flow with the following two-dimensional initial data: $u = \cos(8\pi x)\sin(8\pi y)$, $v = -\sin(8\pi x)\cos(8\pi y)$, $w = 0$. By construction, the flow should remain two-dimensional and laminar and the total kinetic energy should be constant for some time, until the accumulation of numerical round-off errors trigger three-dimensional instabilities. The Euler solution is computed up to $t = 4$ on a 32^3 grid and up to $t = 6$ on a 512^3 grid using the entropy-viscosity technique and the Smagorinsky model in the Fourier DNS code. Insets (a) and (b) of Figure 7 show the time evolution of the kinetic energy for the entropy-viscosity solution and the Smagorinsky solution. It is striking that the Smagorinsky solution loses energy fast even though the flow is laminar, whereas the entropy-viscosity solution tracks the DNS solution rather closely. The conclusion of these tests is that the entropy-viscosity model outperforms the Smagorinsky model since it is consistent whereas the Smagorinsky model is not.

Figure 7: Consistency of EV-LES model.



(a) 32^3 grid. (b) 512^3 grid.

Figure 8 shows a table displaying the relative kinetic energy loss for the DNS, entropy-viscosity, and the Smagorinsky solutions at time $t = 4$ on four grids 32^3, 64^3, 128^3, 256^3. The DNS does not lose any energy at all resolutions and the entropy-viscosity solution does not lose any significant amount of energy, even at low resolution. The Smagorinsky solution on the other hand has lost 68% of the energy by time $t = 4$ on the 32^3 grid and 3.3% on the 256^3 grid. This test

Figure 8: $\frac{\|\mathbf{u}-\mathbf{u}_0\|_{L^2}}{\|\mathbf{u}_0\|_{L^2}}$ *at* $t = 4$

	DNS	EV	Smag.
32	4.0e-13	2.1e-06	6.8e-1
64	1.6e-16	1.1e-06	3.6e-1
128	6.5e-14	5.6e-10	1.2e-1
256	1.1e-13	8.7e-11	3.3e-2

confirms that contrary to the Smagorinsky method, the entropy-viscosity method does not dissipate energy in the laminar regions of the flow. In Figure 9, we show the energy spectra of simulation runs at $Re \approx 6500$ for various resolutions. All runs are compared against a resolved DNS run at resolution 256^3 (called "No Model"). Each under-resolved simulation is done using the entropy-viscosity model and the Smagorinsky model. Note that the unregularized DNS fail to capture the correct spectra as expected, while the flows regularized with entropy-viscosity perform significantly better. Note also that the entropy viscosity model is always closer to the DNS spectrum than the Smagorinsky model.

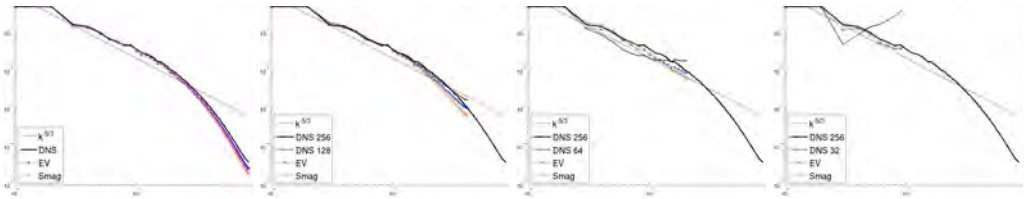


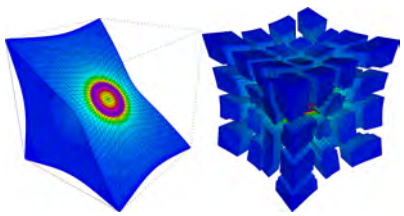
Figure 9: *Energy spectra. Entropy-Viscosity (EV) vs. Smagorinsky (Smag).*

The above work has been done before we developed the graph Laplacian viscosity theory in [4] and got rid of $c_{\max}$, $\|\mathbf{u}_h(\mathbf{x},t)\|$ and $h(\mathbf{x})$ in the definition of the first-order viscosity. An obvious task ahead of us is to extend the scalar theory to the incompressible Navier-Stokes equations and develop a LES model that is parameter free.

3.8 Lagrangian hydrodynamics

Through collaborations with colleagues at Lawrence Livermore National Laboratory, we have extended the entropy viscosity methodology to Lagrangian hydrodynamics, [11]. Our experiments have shown that the method extends naturally to this setting without any particular difficulty, thereby proving again that the methodology that we propose is very flexible. We show here some computations done by our student Vladimir Tomov who has been be partially supported by the grant (V. Tomov has currently postdoctoral position at Lawrence Livermore National Laboratory). Figure 10 shows the 3D Noh implosion test. The field

Figure 10: 3D Noh problem.



(a) Density (b) 64 MPI tasks

shown is the left panel is density. The mesh subdivision among the 64 parallel tasks that were used for this computation is shown in the right panel. The method is stabilized by using the entropy viscosity technique.

3.9 High-order schemes for other models

Together with Vladimir Tomov we are developing a new second order scheme for approximating nonlinear systems called Mean Field Games (MFG). This type of models have been emerging recently in financial applications and modeling crowd dynamics. The MFG system couples a forward Hamilton-Jacobi equation and a backward transport equation. Currently, there are now known second order numerical approximations for such models. Together with Tomov, we have been working on the design of a second order approximation of the inviscid MFG model and we are preparing a paper on that [15].

3.10 Mentoring

The work described above has been done in collaboration with various graduate students and post-doctoral researchers. More specifically the work done on the graph-Laplacian viscosity has been done with the visiting assistant professor Murtazo Nazarov. The contribution of M. Nazarov to the work described in §3.1-§3.4 was important. M. Nazarov visited TAMU between 2011 and 2014. He now holds a tenure-track position in Uppsala, Sweden. The work described in §3.4 was also done with the help of the graduate student Yong Yang (expected to graduate in 2016). The work on LES was done in collaboration with Adam Larios (visiting assistant professor (2011-2014)) and former graduate student Travis Thompson. Adam Larios has now a tenure-track position at the university of Nebraska. Travis Thompson (graduated in 2013) has been research assistant at the university of Tennessee Joint Institute for Computational Sciences at Oak Ridge National Laboratory; he is now research assistant at Rice University. Additional work on the use the entropy viscosity in Lagrangian hydrodynamics has been done by the former graduate student Vladimir Tomov. He graduated in 2014 and now is research assistant at Lawrence Livermore National Laboratory. Some of the funds from the 2011/2015 AFOSR grant were used to support these post-doctoral researchers and graduate students.

3.11 Personnel Supported During Duration of Grant

1. Jean-Luc Guermond, Professor, Texas A&M University.
2. Bojan Popov, Professor, Texas A&M University.
3. Murtazo Nazarov, Visiting Assistant Professor, Texas A&M University. Now tenure-track at Uppsalla Univ.
4. Vladimir Tomov, Graduate student, Texas A&M University. Now post-doc at LLNL.
5. Yong Yang, Graduate student, Texas A&M University.
6. Manuel Quezada , Graduate student, Texas A&M University.

References

[1] A. Bonito, J.-L. Guermond, and B. Popov. Stability analysis of explicit entropy viscosity methods for non-linear scalar conservation equations. *Math. Comp.*, 83(287):1039–1062, 2014.

[2] J.-L. Guermond. On the use of the notion of suitable weak solutions in CFD. *Int. J. Nmer. Methods Fluids*, 57:1153–1170, 2008.

[3] J.-L. Guermond, A. Larios, and T. Thompson. Validation of an entropy-viscosity model for large eddy simulation. In *Direct and Large Eddy-Eddy Simulation IX*, volume 20, pages 43–48. ECOFTAC Series, 2015.

[4] J.-L. Guermond and M. Nazarov. A maximum-principle preserving C^0 finite element method for scalar conservation equations. *Comput. Methods Appl. Mech. Engrg.*, 272:198–213, 2014.

[5] J.-L. Guermond, M. Nazarov, B. Popov, and Y. Yang. A second-order maximum principle preserving Lagrange finite element technique for nonlinear scalar conservation equations. *SIAM J. Numer. Anal.*, 52(4):2163–2182, 2014.

[6] J.-L. Guermond and R. Pasquetti. A correction technique for the dispersive effects of mass lumping for transport problems. *Comput. Methods Appl. Mech. Engrg.*, 253:186–198, 2013.

[7] J.-L. Guermond, R. Pasquetti, and B. Popov. Entropy viscosity method for nonlinear conservation laws. *J. Comput. Phys.*, 230(11):4248–4267, 2011.

[8] J.-L. Guermond, R. Pasquetti, and B. Popov. From suitable weak solutions to entropy-viscosity. *J. Sci. Comput.*, 49(1):35–50, October 2011.

[9] J.-L. Guermond and B. Popov. Error estimates of a first-order lagrange finite element technique for nonlinear scalar conservation equations. *SIAM J. Numer. Anal.*, 2014. submitted.

[10] J.-L. Guermond and B. Popov. Viscous regularization of the Euler equations and entropy principles. *SIAM J. Appl. Math.*, 74(2):284–305, 2014.

[11] J.-L. Guermond, B. Popov, and V. Tomov. Entropy viscosity method for euler equations in lagrangian frame. 2015. In preparation.

[12] O. Mehmetoglu and B. Popov. Maximum principle of central schemes with k-monotone fluxes. In *HYP2010, Series in Contemporary Applied Mathematics CAM 17*, pages 227–237. Vol. 1, ISBN 978-7-04-034535-3, 2012.

[13] O. Mehmetoglu and B. Popov. On maximum principle and convergence of central schemes based on slope limiters. *Math. Comp.*, pages 219–231, 2012.

[14] O. Mehmetoglu and B. Popov. Convergence of the nt scheme for bounded initial data. 2013, preprint.

[15] B. Popov and V. Tomov. A second order scheme for mean field games. 2015. in press.

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Principal Investigator Name

The full name of the principal investigator on the grant or contract.

Jean-Luc Guermond

Program Manager

The AFOSR Program Manager currently assigned to the award

Fariba Fahroo

Reporting Period Start Date

07/01/2012

Reporting Period End Date

06/30/2015

Abstract

Our goal is to develop robust numerical methods for solving mathematical models of nonlinear phenomena such as nonlinear conservation laws, advection-dominated multi-phase flows, and free-boundary problems, where shocks, fronts, and contact discontinuities are driving features and pose significant difficulties for traditional numerical methods. We have discovered that time-dependent nonlinear conservation equations can be stabilized by using the so-called entropy viscosity method and we proposed to to investigate this new technique. We explored in detail the approximation properties of the entropy viscosity method along the following directions. (i) New discretization methods including Discontinuous Galerkin and Lagrangian hydrodynamics; (ii) New fields of applications of the entropy viscosity concept, such as multiphase flows, using phase field techniques. This novel robust approximation technique for solving nonlinear problems developing shock or sharp interfaces will benefit every areas of science and engineering where controlling or dealing with this type of phenomenon is still an

enormous challenge.

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Funding Summary by Cost Category (by FY, $K)

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Total			

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