



Geometric Structure-Preserving Discretization Schemes for Nonlinear Elasticity

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Final Report for GEOMETRIC STRUCTURE-PRESERVING DISCRETIZATION SCHEMES FOR ELASTICITY

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1 Summary

We introduced a smooth complex for nonlinear elasticity that can be considered as the tensorial analogue of the standard grad-curl-div complex. This mathematical structure simultaneously describes the kinematics and the kinetics of large deformations. The relation between this complex and the de Rham complex allows one to readily derive the necessary and sufficient conditions for the compatibility of displacement gradient and the existence of stress functions on non-contractible bodies. The main application of the nonlinear elasticity complex is in developing mixed finite element methods for large deformations, which will be pursued in a future project. To this end, the smooth complex should be extended to also include less smooth tensors. We introduced this extension by using the so-called partly Sobolev spaces. The result is a Hilbert complex involving second-order tensors on flat compact manifolds with boundary. We then used the general framework of Hilbert complexes to write Hodge-type and Helmholtz-type orthogonal decompositions for second-order tensors. As some applications of these decompositions in continuum mechanics, one can study the strain compatibility equations of nonlinear elasticity in the presence of Dirichlet boundary conditions.

For developing finite element methods for nonlinear elasticity using the above Hilbert complex, we first discretize this complex by means of appropriate finite element spaces and derive stable mixed finite element methods for the associated Laplacian. We use the general theory for approximation of Hilbert complexes and the finite element exterior calculus and introduce some stable mixed finite element methods for the Laplacian of second-order tensors with appropriate mixed boundary conditions. To this end, we introduce new finite elements for second-order tensors that are the tensorial analogues of some standard finite elements for vector fields. One important feature of the finite element spaces generated by these finite elements is that they respect the global topology of the underlying domains in the sense that they reproduce some topological properties of the domains regardless of the refinement level of meshes.

2 Nonlinear Elasticity Complex

Let $\mathcal{B} \subset \mathbb{R}^3$ be an open subset and suppose $\{X^I\}$ is the Cartesian coordinates on \mathcal{B} . We equip \mathcal{B} with metric \mathbf{G} , which is the Euclidean metric of \mathbb{R}^3 . The gradient of vector fields and the curl and the

divergence of $\binom{2}{0}$ -tensors are defined as

$$\begin{aligned} \mathbf{grad} : \mathfrak{X}(\mathcal{B}) &\rightarrow \Gamma(\otimes^2 T\mathcal{B}), & (\mathbf{grad} \mathbf{Y})^{IJ} &= Y^I{}_{,J}, \\ \mathbf{curl} : \Gamma(\otimes^2 T\mathcal{B}) &\rightarrow \Gamma(\otimes^2 T\mathcal{B}), & (\mathbf{curl} \mathbf{T})^{IJ} &= \varepsilon_{IKL} T^{JL}{}_{,K}, \\ \mathbf{div} : \Gamma(\otimes^2 T\mathcal{B}) &\rightarrow \mathfrak{X}(\mathcal{B}), & (\mathbf{div} \mathbf{T})^I &= T^{IJ}{}_{,J}, \end{aligned}$$

where “ $,J$ ” indicates $\partial/\partial X^J$. We also define the operator

$$\mathbf{curl}^\top : \Gamma(\otimes^2 T\mathcal{B}) \rightarrow \Gamma(\otimes^2 T\mathcal{B}), \quad (\mathbf{curl}^\top \mathbf{T})^{IJ} = (\mathbf{curl} \mathbf{T})^{JI}.$$

It is straightforward to show that $\mathbf{curl}^\top \circ \mathbf{grad} = 0$, and $\mathbf{div} \circ \mathbf{curl}^\top = 0$. Thus, we obtain the following complex

$$0 \longrightarrow \mathfrak{X}(\mathcal{B}) \xrightarrow{\mathbf{grad}} \Gamma(\otimes^2 T\mathcal{B}) \xrightarrow{\mathbf{curl}^\top} \Gamma(\otimes^2 T\mathcal{B}) \xrightarrow{\mathbf{div}} \mathfrak{X}(\mathcal{B}) \longrightarrow 0, \quad (2.1)$$

that, due to its resemblance with the **gcd** complex (i.e. the standard grad-curl-div complex of vector fields), is called the **gcd** complex. Interestingly, similar to the **gcd** complex, useful properties of the **gcd** complex also follow from the de Rham complex. This can be described via the \mathbb{R}^3 -valued de Rham complex as follows. Let $d : \Omega^k(\mathcal{B}) \rightarrow \Omega^{k+1}(\mathcal{B})$ be the standard exterior derivative given by

$$(d\beta)_{I_0 \dots I_k} = \sum_{i=0}^k (-1)^i \beta_{I_0 \dots \widehat{I}_i \dots I_k, I_i},$$

where the hat over an index implies the elimination of that index. Any $\alpha \in \Omega^k(\mathcal{B}; \mathbb{R}^3)$ can be considered as $\alpha = (\alpha^1, \alpha^2, \alpha^3)$, with $\alpha^i \in \Omega^k(\mathcal{B})$, $i = 1, 2, 3$. One can define the exterior derivative $d : \Omega^k(\mathcal{B}; \mathbb{R}^3) \rightarrow \Omega^{k+1}(\mathcal{B}; \mathbb{R}^3)$ by $d\alpha = (d\alpha^1, d\alpha^2, d\alpha^3)$. Since $d \circ d = 0$, we also conclude that $d \circ d = 0$, which leads to the \mathbb{R}^3 -valued de Rham complex $(\Omega(\mathcal{B}; \mathbb{R}^3), d)$. Given $\alpha \in \Omega^k(\mathcal{B}; \mathbb{R}^3)$, let $[\alpha]_{I_1 \dots I_k}^i$ denote the components of $\alpha^i \in \Omega^k(\mathcal{B})$. By using the global orthonormal coordinate system $\{X^I\}$, one can define the following isomorphisms

$$\begin{aligned} \mathfrak{v}_0 : \mathfrak{X}(\mathcal{B}) &\rightarrow \Omega^0(\mathcal{B}; \mathbb{R}^3), & [\mathfrak{v}_0(\mathbf{Y})]^i &= \delta_{iI} Y^I, \\ \mathfrak{v}_1 : \Gamma(\otimes^2 T\mathcal{B}) &\rightarrow \Omega^1(\mathcal{B}; \mathbb{R}^3), & [\mathfrak{v}_1(\mathbf{T})]^i{}_J &= \delta_{iI} T^{IJ}, \\ \mathfrak{v}_2 : \Gamma(\otimes^2 T\mathcal{B}) &\rightarrow \Omega^2(\mathcal{B}; \mathbb{R}^3), & [\mathfrak{v}_2(\mathbf{T})]^i{}_{JK} &= \delta_{iI} \varepsilon_{JKL} T^{IL}, \\ \mathfrak{v}_3 : \mathfrak{X}(\mathcal{B}) &\rightarrow \Omega^3(\mathcal{B}; \mathbb{R}^3), & [\mathfrak{v}_3(\mathbf{Y})]^i{}_{123} &= \delta_{iI} Y^I, \end{aligned}$$

where δ_{iI} is the Kronecker delta. Let \mathbf{T}^\top be the transpose of \mathbf{T} , i.e. $(\mathbf{T}^\top)^{IJ} = T^{JI}$, and let $\{\mathbf{E}_I\}$ be the standard basis of \mathbb{R}^3 . For $\mathbf{T} \in \Gamma(\otimes^2 T\mathcal{B})$, we define $\vec{\mathbf{T}}_{\mathbf{N}}$ to be the traction of \mathbf{T}^\top in the direction of unit vector $\mathbf{N} = N^I \mathbf{E}_I \in S^2$, where $S^2 \subset \mathbb{R}^3$ is the unit 2-sphere. Thus, $\vec{\mathbf{T}}_{\mathbf{N}} = N^I T^{IJ} \mathbf{E}_J$. One can write

$$\mathfrak{v}_k(\mathbf{T}) = \left(\mathfrak{v}_k(\vec{\mathbf{T}}_{\mathbf{E}_1}), \mathfrak{v}_k(\vec{\mathbf{T}}_{\mathbf{E}_2}), \mathfrak{v}_k(\vec{\mathbf{T}}_{\mathbf{E}_3}) \right), \quad k = 1, 2. \quad (2.2)$$

It is easy to show that

$$\mathfrak{v}_1 \circ \mathbf{grad} = d \circ \mathfrak{v}_0, \quad \mathfrak{v}_2 \circ \mathbf{curl}^\top = d \circ \mathfrak{v}_1, \quad \mathfrak{v}_3 \circ \mathbf{div} = d \circ \mathfrak{v}_2.$$

Therefore, the following diagram commutes for the **gcd** complex.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathfrak{X}(\mathcal{B}) & \xrightarrow{\text{grad}} & \Gamma(\otimes^2 T\mathcal{B}) & \xrightarrow{\text{curl}^\top} & \Gamma(\otimes^2 T\mathcal{B}) & \xrightarrow{\text{div}} & \mathfrak{X}(\mathcal{B}) & \longrightarrow & 0 \\
 & & \downarrow \mathfrak{z}_0 & & \downarrow \mathfrak{z}_1 & & \downarrow \mathfrak{z}_2 & & \downarrow \mathfrak{z}_3 & & \\
 0 & \longrightarrow & \Omega^0(\mathcal{B}; \mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathcal{B}; \mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathcal{B}; \mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathcal{B}; \mathbb{R}^3) & \longrightarrow & 0
 \end{array} \tag{2.3}$$

The contraction $\langle \mathbf{T}, \mathbf{Y} \rangle$ of $\mathbf{T} \in \Gamma(\otimes^2 T\mathcal{B})$ and $\mathbf{Y} \in \mathfrak{X}(\mathcal{B})$ is a vector field that in the orthonormal coordinate system $\{X^I\}$ reads $\langle \mathbf{T}, \mathbf{Y} \rangle = T^{IJ} Y^J \mathbf{E}_I$. Clearly, if $\mathbf{N}_\mathcal{C}$ is the unit outward normal vector field of a closed surface $\mathcal{C} \subset \mathcal{B}$, then $\langle \mathbf{T}, \mathbf{N}_\mathcal{C} \rangle$ is the traction of \mathbf{T} on \mathcal{C} . Suppose $\mathbb{H}_{\text{gcd}}^k(\mathcal{B})$ is the k -th cohomology group of the **gcd** complex. Diagram (2.3) implies that \mathfrak{z}_k also induces the isomorphism $\mathbb{H}_{\text{gcd}}^k(\mathcal{B}) \approx \oplus_{i=1}^3 \mathbb{H}_{dR}^k(\mathcal{B})$ between the cohomology groups. Using this fact, one can prove the following theorem.

Theorem 2.1. *An arbitrary tensor $\mathbf{T} \in \Gamma(\otimes^2 T\mathcal{B})$ is the gradient of a vector field if and only if*

$$\text{curl}^\top \mathbf{T} = 0, \quad \text{and} \quad \int_\ell \langle \mathbf{T}, \mathbf{t}_\ell \rangle dS = 0, \quad \forall \ell \subset \mathcal{B}, \tag{2.4}$$

where ℓ is an arbitrary closed curve in \mathcal{B} and \mathbf{t}_ℓ is the unit tangent vector field along ℓ .

Similarly, one can derive the necessary and sufficient conditions for the existence of a potential for \mathbf{T} induced by curl^\top . The upshot is the following theorem.

Theorem 2.2. *Given $\mathbf{T} \in \Gamma(\otimes^2 T\mathcal{B})$, there exists $\mathbf{W} \in \Gamma(\otimes^2 T\mathcal{B})$ such that $\mathbf{T} = \text{curl}^\top \mathbf{W}$, if and only if*

$$\text{div} \mathbf{T} = 0, \quad \text{and} \quad \int_{\mathcal{C}} \langle \mathbf{T}, \mathbf{N}_\mathcal{C} \rangle dA = 0, \quad \forall \mathcal{C} \subset \mathcal{B}, \tag{2.5}$$

where \mathcal{C} is an arbitrary closed surface in \mathcal{B} and $\mathbf{N}_\mathcal{C}$ is its unit outward normal vector field.

We can also write an analogue of the **gcd** complex for two-point tensors. Let $\mathcal{S} = \mathbb{R}^3$ with coordinate system $\{x^i\}$, which is the Cartesian coordinates of \mathbb{R}^3 . Suppose $\varphi : \mathcal{B} \rightarrow \mathcal{S}$ is a smooth mapping and let $T_X \varphi(\mathcal{B}) := T_{\varphi(X)} \mathcal{S}$. Note that although φ is not necessarily an embedding, the dimension of $T_X \varphi(\mathcal{B})$ is always equal to $\dim \mathcal{S}$. We can define the following operators for two-point tensors that belong to $\Gamma(T\varphi(\mathcal{B}))$ and $\Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B})$:

$$\begin{array}{ll}
 \mathbf{Grad} : \Gamma(T\varphi(\mathcal{B})) \rightarrow \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}), & (\mathbf{Grad} \mathbf{U})^{iI} = U^i{}_{,I}, \\
 \mathbf{Curl}^\top : \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \rightarrow \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}), & (\mathbf{Curl}^\top \mathbf{F})^{iI} = \varepsilon_{IKL} F^{iL}{}_{,K}, \\
 \mathbf{Div} : \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \rightarrow \Gamma(T\varphi(\mathcal{B})), & (\mathbf{Div} \mathbf{F})^i = F^{iI}{}_{,I}.
 \end{array}$$

We have $\mathbf{Curl}^\top \circ \mathbf{Grad} = 0$, and $\mathbf{Div} \circ \mathbf{Curl}^\top = 0$. Thus, the **GCD** complex, which we also call the nonlinear elasticity complex, can be written as:

$$0 \longrightarrow \Gamma(T\varphi(\mathcal{B})) \xrightarrow{\mathbf{Grad}} \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \xrightarrow{\mathbf{Curl}^\top} \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \xrightarrow{\mathbf{Div}} \Gamma(T\varphi(\mathcal{B})) \longrightarrow 0.$$

By using the following isomorphisms

$$\begin{aligned}
 \mathbf{I}_0 &: \Gamma(T\varphi(\mathcal{B})) \rightarrow \Omega^0(\mathcal{B}; \mathbb{R}^3), & [\mathbf{I}_0(\mathbf{U})]^i &= U^i, \\
 \mathbf{I}_1 &: \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \rightarrow \Omega^1(\mathcal{B}; \mathbb{R}^3), & [\mathbf{I}_1(\mathbf{F})]^i{}_J &= F^{iJ}, \\
 \mathbf{I}_2 &: \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) \rightarrow \Omega^2(\mathcal{B}; \mathbb{R}^3), & [\mathbf{I}_2(\mathbf{F})]^i{}_{JK} &= \varepsilon_{JKL} F^{iL}, \\
 \mathbf{I}_3 &: \Gamma(T\varphi(\mathcal{B})) \rightarrow \Omega^3(\mathcal{B}; \mathbb{R}^3), & [\mathbf{I}_3(\mathbf{U})]^i{}_{123} &= U^i,
 \end{aligned}$$

one concludes that the following diagram commutes.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Gamma(T\varphi(\mathcal{B})) & \xrightarrow{\mathbf{Grad}} & \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) & \xrightarrow{\mathbf{Curl}^\top} & \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B}) & \xrightarrow{\mathbf{Div}} & \Gamma(T\varphi(\mathcal{B})) & \longrightarrow & 0 \\
 & & \downarrow \mathbf{I}_0 & & \downarrow \mathbf{I}_1 & & \downarrow \mathbf{I}_2 & & \downarrow \mathbf{I}_3 & & \\
 0 & \longrightarrow & \Omega^0(\mathcal{B}; \mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathcal{B}; \mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathcal{B}; \mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathcal{B}; \mathbb{R}^3) & \longrightarrow & 0
 \end{array}$$

The above isomorphisms also induce an isomorphism $H_{\mathbf{GCD}}^k(\mathcal{B}) \approx \oplus_{i=1}^3 H_{dR}^k(\mathcal{B})$, where $H_{\mathbf{GCD}}^k(\mathcal{B})$ is the k -th cohomology group of the **GCD** complex. Let $\{\mathbf{E}_I\}$ and $\{\mathbf{e}_i\}$ be two copies of the standard basis of \mathbb{R}^3 . For $\mathbf{F} \in \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B})$, and $\mathbf{n} = n^i \mathbf{e}_i \in S^2$, let $\vec{\mathbf{F}}_{\mathbf{n}} = n^i F^{iJ} \mathbf{E}_J \in \mathfrak{X}(\mathcal{B})$. Then, one can write

$$\mathbf{I}_k(\mathbf{F}) = \left(\iota_k(\vec{\mathbf{F}}_{\mathbf{e}_1}), \iota_k(\vec{\mathbf{F}}_{\mathbf{e}_2}), \iota_k(\vec{\mathbf{F}}_{\mathbf{e}_3}) \right), \quad k = 1, 2.$$

Let $\langle \mathbf{F}, \mathbf{Y} \rangle := F^{iI} Y^I \mathbf{e}_i$. The above relations for the **GCD** complex allow us to obtain the following results that can be proved similarly to Theorems 2.1 and 2.2.

Theorem 2.3. *Given $\mathbf{F} \in \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B})$, there exists $\mathbf{U} \in \Gamma(T\varphi(\mathcal{B}))$ such that $\mathbf{F} = \mathbf{Grad} \mathbf{U}$, if and only if*

$$\mathbf{Curl}^\top \mathbf{F} = 0, \quad \text{and} \quad \int_\ell \langle \mathbf{F}, \mathbf{t}_\ell \rangle dS = 0, \quad \forall \ell \subset \mathcal{B}.$$

Moreover, there exists $\Psi \in \Gamma(T\varphi(\mathcal{B}) \otimes T\mathcal{B})$ such that $\mathbf{F} = \mathbf{Curl}^\top \Psi$, if and only if

$$\mathbf{Div} \mathbf{F} = 0, \quad \text{and} \quad \int_{\mathcal{C}} \langle \mathbf{F}, \mathbf{N}_{\mathcal{C}} \rangle dA = 0, \quad \forall \mathcal{C} \subset \mathcal{B}.$$

3 Hilbert Complexes and Orthogonal Decompositions

Consider the following linear subspaces of $\Gamma(T\varphi(\bar{\mathcal{B}}))$ and $\Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})$:

$$\begin{aligned}
 \Gamma_j(T\varphi(\bar{\mathcal{B}})) &:= \left\{ \mathbf{U} \in \Gamma(T\varphi(\bar{\mathcal{B}})) : \mathbf{U}|_{\partial_j \bar{\mathcal{B}}} = 0 \right\}, \\
 \Gamma_{n_j}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) &:= \left\{ \mathbf{F} \in \Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) : \vec{\mathbf{F}}_{\mathbf{e}_i} \perp \partial_j \bar{\mathcal{B}}, \quad i = 1, \dots, n \right\}, \\
 \Gamma_{t_j}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) &:= \left\{ \mathbf{F} \in \Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) : \vec{\mathbf{F}}_{\mathbf{e}_i} \parallel \partial_j \bar{\mathcal{B}}, \quad i = 1, \dots, n \right\}.
 \end{aligned}$$

The operators \mathbf{Grad} , \mathbf{Curl}^Γ , and \mathbf{Div} can be restricted to the above subspaces which allows one to impose boundary conditions on the **GCD** complex. The upshot is the following commutative diagrams.

$$\begin{array}{ccccccc}
 0 \longrightarrow & \Gamma_j(T\varphi(\bar{\mathcal{B}})) & \xrightarrow{\mathbf{Grad}_j} & \Gamma_{n_j}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{Curl}_j^\Gamma} & \Gamma_{t_j}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{Div}_j} & \Gamma(T\varphi(\bar{\mathcal{B}})) \longrightarrow 0 \\
 & \downarrow I_0 & & \downarrow I_1 & & \downarrow I_2 & & \downarrow I_3 \\
 0 \longrightarrow & \Omega_{n_j}^0(\bar{\mathcal{B}}; \mathbb{R}^3) & \xrightarrow{d_{n_j}} & \Omega_{n_j}^1(\bar{\mathcal{B}}; \mathbb{R}^3) & \xrightarrow{d_{n_j}} & \Omega_{n_j}^2(\bar{\mathcal{B}}; \mathbb{R}^3) & \xrightarrow{d_{n_j}} & \Omega_{n_j}^3(\bar{\mathcal{B}}; \mathbb{R}^3) \longrightarrow 0 \\
 \\
 0 \longleftarrow & \Gamma(T\varphi(\bar{\mathcal{B}})) & \xleftarrow{\mathbf{Div}_j} & \Gamma_{t_j}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) & \xleftarrow{\mathbf{Curl}_j^\Gamma} & \Gamma_{n_j}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) & \xleftarrow{\mathbf{Grad}_j} & \Gamma_j(T\varphi(\bar{\mathcal{B}})) \longleftarrow 0 \\
 & \downarrow -I_0 & & \downarrow I_1 & & \downarrow I_2 & & \downarrow -I_3 \\
 0 \longleftarrow & \Omega_{t_j}^0(\bar{\mathcal{B}}; \mathbb{R}^3) & \xleftarrow{\delta_{t_j}} & \Omega_{t_j}^1(\bar{\mathcal{B}}; \mathbb{R}^3) & \xleftarrow{\delta_{t_j}} & \Omega_{t_j}^2(\bar{\mathcal{B}}; \mathbb{R}^3) & \xleftarrow{\delta_{t_j}} & \Omega_{t_j}^3(\bar{\mathcal{B}}; \mathbb{R}^3) \longleftarrow 0
 \end{array}$$

The isomorphisms I_0, \dots, I_3 are L^2 -isometries. The Hilbert spaces $L^2\Gamma(T\varphi(\bar{\mathcal{B}}))$, $H^1\Gamma_j(T\varphi(\bar{\mathcal{B}}))$, $H^C\Gamma_{n_j}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})$, and $H^D\Gamma_{t_j}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})$ are the completions of $(\Gamma(T\varphi(\bar{\mathcal{B}})), \langle \cdot, \cdot \rangle_{L^2})$, $(\Gamma_j(T\varphi(\bar{\mathcal{B}})), \langle \cdot, \cdot \rangle_{H^1})$, $(\Gamma_{n_j}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}), \langle \cdot, \cdot \rangle_{H^C})$, and $(\Gamma_{t_j}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}), \langle \cdot, \cdot \rangle_{H^D})$, respectively. These Hilbert spaces allow one to write the following Hilbert complex for two-point tensors:

$$\begin{array}{ccc}
 0 \longrightarrow & H^1\Gamma_1(T\varphi(\bar{\mathcal{B}})) & \xrightarrow{\mathbf{Grad}_1} & H^C\Gamma_{n_1}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{Curl}_1^\Gamma} & \\
 & & & & & \\
 & & & H^D\Gamma_{t_1}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) & \xrightarrow{\mathbf{Div}_1} & L^2\Gamma(T\varphi(\bar{\mathcal{B}})) \longrightarrow 0
 \end{array} \tag{3.1}$$

The dual of this Hilbert complex reads:

$$\begin{array}{ccc}
 & & \xleftarrow{\mathbf{Curl}_2^\Gamma} & H^C\Gamma_{n_2}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) & \xleftarrow{\mathbf{Grad}_2} & H^1\Gamma_2(T\varphi(\bar{\mathcal{B}})) \longleftarrow 0 \\
 0 \longleftarrow & L^2\Gamma(T\varphi(\bar{\mathcal{B}})) & \xleftarrow{\mathbf{Div}_2} & H^D\Gamma_{t_2}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) & &
 \end{array} \tag{3.2}$$

The complex (3.1) is isomorphic to $(H^d\Omega_{n_1}(\bar{\mathcal{B}}), d_{n_1})$, and hence, it is Fredholm with $H_{\mathbf{GCD}_1}^k(\bar{\mathcal{B}}) \approx H_{\mathbf{GCD}_1}^k(\bar{\mathcal{B}}) \approx \oplus_{i=1}^3 H_{dR}^k(\bar{\mathcal{B}}, \partial_1\bar{\mathcal{B}})$, where $H_{\mathbf{GCD}_1}^k(\bar{\mathcal{B}})$ and $H_{\mathbf{GCD}_1}^k(\bar{\mathcal{B}})$ are the k -th cohomologies of the smooth **GCD** complex (with boundary conditions on $\partial_1\bar{\mathcal{B}}$) and the Hilbert complex (3.1), respectively. Let $\mathcal{H}_{n_1, t_2}^\varphi(\bar{\mathcal{B}}) := \ker \mathbf{Curl}_1^\Gamma \cap \ker \mathbf{Div}_2$ be the kernel of the Laplacian L_φ associated to (3.1) and (3.2). Then, $\mathcal{H}_{n_1, t_2}^\varphi(\bar{\mathcal{B}})$ only consists of smooth harmonic two-point tensors and $\mathcal{H}_{n_1, t_2}^\varphi(\bar{\mathcal{B}}) \approx H_{\mathbf{GCD}_1}^1(\bar{\mathcal{B}})$. One can show that:

Theorem 3.1. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^3$ be a smooth, compact 3-manifold with boundary and suppose $\varphi : \bar{\mathcal{B}} \rightarrow \mathbb{R}^3$ is a smooth mapping. The Hilbert complex (3.1) induces the following L^2 -orthogonal decompositions: The Hodge decomposition*

$$\begin{aligned}
 L^2\Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) = \\
 \mathbf{Grad}(H^1\Gamma_1(T\varphi(\bar{\mathcal{B}}))) \oplus \mathcal{H}_{n_1, t_2}^\varphi(\bar{\mathcal{B}}) \oplus \mathbf{Curl}^\Gamma(H^C\Gamma_{n_2}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})),
 \end{aligned}$$

and, equivalently, the Helmholtz decompositions

$$\begin{aligned}
 L^2\Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}) &= \mathbf{Grad}(H^1\Gamma_1(T\varphi(\bar{\mathcal{B}}))) \oplus \ker \mathbf{Div}_2 \\
 &= \ker \mathbf{Curl}_1^\Gamma \oplus \mathbf{Curl}^\Gamma(H^C\Gamma_{n_2}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})),
 \end{aligned}$$

where

$$\begin{aligned}\ker \mathbf{Div}_2 &= \mathbf{Curl}^\top(H^C\Gamma_{n_2}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})) \oplus \mathcal{H}_{n_1, t_2}^\varphi(\bar{\mathcal{B}}), \\ \ker \mathbf{Curl}_1^\top &= \mathbf{Grad}(H^1\Gamma_1(T\varphi(\bar{\mathcal{B}}))) \oplus \mathcal{H}_{n_1, t_2}^\varphi(\bar{\mathcal{B}}).\end{aligned}$$

If in addition a two-point tensor is of class $C^{r, \mu}$ (C^∞), then its components in the above decompositions are of class $C^{r, \mu}$ (C^∞) as well.

Corollary 3.2. *Let $\bar{\mathcal{B}} \subset \mathbb{R}^3$ be a smooth, compact 3-manifold with boundary and suppose $\varphi : \bar{\mathcal{B}} \rightarrow \mathbb{R}^3$ is a smooth mapping. The necessary and sufficient conditions for the existence of a \mathbf{Grad}_1 -potential for $\mathbf{F} \in L^2\Gamma(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}})$ are*

$$\mathbf{F} \in H^C\Gamma_{n_1}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}), \quad \mathbf{Curl}^\top \mathbf{F} = 0, \quad \langle \mathbf{F}, \mathbf{K} \rangle_{L^2} = 0, \quad \forall \mathbf{K} \in \mathcal{H}_{n_1, t_2}^\varphi(\bar{\mathcal{B}}).$$

Similarly, the necessary and sufficient conditions for the existence of a \mathbf{Curl}_2^\top -potential for \mathbf{F} are

$$\mathbf{F} \in H^D\Gamma_{t_2}(T\varphi(\bar{\mathcal{B}}) \otimes T\bar{\mathcal{B}}), \quad \mathbf{Div} \mathbf{F} = 0, \quad \langle \mathbf{F}, \mathbf{K} \rangle_{L^2} = 0, \quad \forall \mathbf{K} \in \mathcal{H}_{n_1, t_2}^\varphi(\bar{\mathcal{B}}).$$

Similar results are also valid for 2-manifolds in \mathbb{R}^2 .

4 Mixed formulations of the tensor Laplacian

By the (weak) 3D Laplacian for second-order tensors (or simply the 3D tensor Laplacian) with mixed boundary conditions, we mean the following boundary-value problem: Given $\mathbf{Q} \in L^2(\otimes^2 T\mathcal{B})$, find $\mathbf{T} \in L^2(\otimes^2 T\mathcal{B})$ such that

$$\begin{aligned}\mathcal{L}(\mathbf{T}) &= \mathbf{Q}, \\ \mathbf{T} \perp \mathcal{S}_1, \quad (\mathbf{div} \mathbf{T})|_{\mathcal{S}_1} &= 0, \quad \mathbf{T}|_{\mathcal{S}_2}, \quad (\mathbf{curl}^\top \mathbf{T}) \perp \mathcal{S}_2,\end{aligned}\tag{4.1}$$

where $\mathcal{L} := \mathbf{curl}^\top \circ \mathbf{curl}^\top - \mathbf{grad} \circ \mathbf{div}$. The space of solutions of (4.1) for $\mathbf{Q} = 0$, is the space of *harmonic tensors* defined as

$$\mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) := \ker \mathcal{L}_1 = \ker \mathbf{curl}_{\mathcal{S}_1}^\top \cap \ker \mathbf{div}_{\mathcal{S}_2}.$$

One can write

$$\dim \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) = \dim \mathcal{H}^1(\mathcal{B}, \mathcal{S}_1) = 3b_1(\bar{\mathcal{B}}, \mathcal{S}_1).\tag{4.2}$$

Note that $\mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$ measures the non-uniqueness of solutions of (4.1) in the sense that if \mathbf{T} is a solution of (4.1), then so is $\mathbf{T} + \hat{\mathbf{T}}$, $\forall \hat{\mathbf{T}} \in \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$. The integer $b_1(\bar{\mathcal{B}}, \mathcal{S}_1)$ depends on the topological properties of both \mathcal{B} and \mathcal{S}_1 . For example, if \mathcal{B} is a 3D ball with a spherical hole, then $b_1(\bar{\mathcal{B}}, \emptyset) = 0$, and $b_1(\bar{\mathcal{B}}, \partial\mathcal{B}) = 1$. Therefore, (4.2) provides a connection between solutions of (4.1) and topological properties of \mathcal{B} and \mathcal{S}_1 .

Since $\mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$ is finite-dimensional, it is a closed subspace of $L^2(\otimes^2 T\mathcal{B})$, and therefore, using the orthogonal projection theorem, one can write $L^2(\otimes^2 T\mathcal{B}) = \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) \oplus \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^\perp$, where

$\mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^\perp$ is the orthogonal complement of $\mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$, i.e.

$$\mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^\perp := \{ \mathbf{T} \in L^2(\otimes^2 T\mathcal{B}) : \langle \mathbf{T}, \mathbf{S} \rangle_{L^2 \otimes^2} = 0, \forall \mathbf{S} \in \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) \}.$$

One can show that:

Theorem 4.1. *The boundary-value problem (4.1) admits a solution if and only if $\mathbf{Q} \in \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^\perp$.*

For approximating solutions of the tensor Laplacians with mixed boundary conditions by means of Galerkin methods, we need suitable weak formulations for these boundary-value problems. The boundary-value problem (4.1) can be considered as a special case of the *abstract Hodge Laplacian* and therefore, one can define the following mixed formulation for (4.1): Given $\mathbf{Q} \in L^2(\otimes^2 T\mathcal{B})$, find $(\mathbf{U}, \mathbf{T}, \mathbf{P}) \in H^1(T\mathcal{B}, \mathcal{S}_1) \times H^c(\otimes^2 T\mathcal{B}, \mathcal{S}_1) \times \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$ such that

$$\begin{aligned} \langle \mathbf{U}, \boldsymbol{\Upsilon} \rangle_{L^2 T} - \langle \mathbf{T}, \mathbf{grad} \boldsymbol{\Upsilon} \rangle_{L^2 \otimes^2} &= 0, & \forall \boldsymbol{\Upsilon} \in H^1(T\mathcal{B}, \mathcal{S}_1), \\ \langle \mathbf{grad} \mathbf{U}, \boldsymbol{\Theta} \rangle_{L^2 \otimes^2} + \langle \mathbf{curl}^\top \mathbf{T}, \mathbf{curl}^\top \boldsymbol{\Theta} \rangle_{L^2 \otimes^2} + \langle \mathbf{P}, \boldsymbol{\Theta} \rangle_{L^2 \otimes^2} &= \langle \mathbf{Q}, \boldsymbol{\Theta} \rangle_{L^2 \otimes^2}, & \forall \boldsymbol{\Theta} \in H^c(\otimes^2 T\mathcal{B}, \mathcal{S}_1), \\ \langle \mathbf{T}, \boldsymbol{\Pi} \rangle_{L^2 \otimes^2} &= 0, & \forall \boldsymbol{\Pi} \in \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2). \end{aligned} \quad (4.3)$$

The above equations are the Euler-Lagrange equations corresponding to a saddle point of the functional $\mathcal{J} : H^1(T\mathcal{B}, \mathcal{S}_1) \times H^c(\otimes^2 T\mathcal{B}, \mathcal{S}_1) \times \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2) \rightarrow \mathbb{R}$ given by

$$\begin{aligned} \mathcal{J}(\boldsymbol{\Upsilon}, \boldsymbol{\Theta}, \boldsymbol{\Pi}) &= \frac{1}{2} \langle \boldsymbol{\Upsilon}, \boldsymbol{\Upsilon} \rangle_{L^2 T} - \langle \mathbf{grad} \boldsymbol{\Upsilon}, \boldsymbol{\Theta} \rangle_{L^2 \otimes^2} - \frac{1}{2} \langle \mathbf{curl}^\top \boldsymbol{\Theta}, \mathbf{curl}^\top \boldsymbol{\Theta} \rangle_{L^2 \otimes^2} \\ &\quad - \langle \boldsymbol{\Pi}, \boldsymbol{\Theta} \rangle_{L^2 \otimes^2} + \langle \mathbf{Q}, \boldsymbol{\Theta} \rangle_{L^2 \otimes^2}. \end{aligned}$$

The first equation in (4.3) implies that $\mathbf{U} = \mathbf{div}_{\mathcal{S}_2} \mathbf{T}$. The second equation says that \mathbf{T} solves the equation $\mathcal{L}(\mathbf{T}) = \mathbf{Q} - \mathbf{P}$, where the harmonic tensor \mathbf{P} is the harmonic part of \mathbf{Q} provided by a Hodge-type decomposition. Hence $\mathbf{Q} - \mathbf{P} \in \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^\perp$, which is the necessary and sufficient condition for the existence of a solution due to Theorem 4.1. Finally, the third equation eliminates the degree of freedom for choosing a solution \mathbf{T} by requiring that $\mathbf{T} \in \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)^\perp$. The well-posedness of the mixed formulation (4.3) follows from the Fredholm property of the associated Hilbert complex. The upshot is the following theorem.

Theorem 4.2. *The mixed formulation (4.3) is well-posed. Thus, for any $\mathbf{Q} \in L^2(\otimes^2 T\mathcal{B})$, the problem (4.3) admits a unique solution $(\mathbf{U}, \mathbf{T}, \mathbf{P}) \in H^1(T\mathcal{B}, \mathcal{S}_1) \times H^c(\otimes^2 T\mathcal{B}, \mathcal{S}_1) \times \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$, and there is a constant $c > 0$ such that*

$$\| \mathbf{U} \|_{H^1 T} + \| \mathbf{T} \|_{H^c} + \| \mathbf{P} \|_{L^2 \otimes^2} \leq c \| \mathbf{Q} \|_{L^2 \otimes^2}.$$

In the mixed formulation (4.3), the boundary conditions on \mathcal{S}_1 and \mathcal{S}_2 are the essential and the natural boundary conditions, respectively, i.e. the boundary conditions on \mathcal{S}_1 are explicitly imposed in the solution spaces while the boundary conditions on \mathcal{S}_2 are imposed by the mixed formulation. Alternatively, one can write another mixed formulation of (4.1) as follows: Given $\mathbf{Q} \in L^2(\otimes^2 T\mathcal{B})$, find

$(\mathbf{S}, \mathbf{T}, \mathbf{P}) \in H^c(\otimes^2 T\mathcal{B}, \mathcal{S}_2) \times H^d(\otimes^2 T\mathcal{B}, \mathcal{S}_2) \times \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$ such that

$$\begin{aligned} \langle\langle \mathbf{S}, \boldsymbol{\Sigma} \rangle\rangle_{L^2 \otimes^2} - \langle\langle \mathbf{T}, \mathbf{curl}^\top \boldsymbol{\Sigma} \rangle\rangle_{L^2 \otimes^2} &= 0, & \forall \boldsymbol{\Sigma} \in H^c(\otimes^2 T\mathcal{B}, \mathcal{S}_2), \\ \langle\langle \mathbf{curl}^\top \mathbf{S}, \boldsymbol{\Theta} \rangle\rangle_{L^2 \otimes^2} + \langle\langle \mathbf{div} \mathbf{T}, \mathbf{div} \boldsymbol{\Theta} \rangle\rangle_{L^2 T} + \langle\langle \mathbf{P}, \boldsymbol{\Theta} \rangle\rangle_{L^2 \otimes^2} &= \langle\langle \mathbf{Q}, \boldsymbol{\Theta} \rangle\rangle_{L^2 \otimes^2}, & \forall \boldsymbol{\Theta} \in H^d(\otimes^2 T\mathcal{B}, \mathcal{S}_2), \\ \langle\langle \mathbf{T}, \boldsymbol{\Pi} \rangle\rangle_{L^2 \otimes^2} &= 0, & \forall \boldsymbol{\Pi} \in \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2). \end{aligned} \quad (4.4)$$

In this mixed formulation, in contrary to (4.3), the boundary conditions on \mathcal{S}_2 are the essential boundary conditions and those on \mathcal{S}_1 are the natural boundary conditions.

5 Discrete Hilbert Complexes

It is possible to discretize the Hilbert complex mentioned earlier using some appropriate finite element spaces. Let \mathcal{B}_h be a 3D triangulation. Then, one can write the following polynomial complexes for second-order tensors:

$$0 \longrightarrow \mathcal{P}_r^H(T\mathcal{B}_h, \mathcal{S}_1) \xrightarrow{\text{grad}} \mathcal{P}_{r-1}^c(\otimes^2 T\mathcal{B}_h, \mathcal{S}_1) \xrightarrow{\mathbf{curl}^\top} \mathcal{P}_{r-2}^d(\otimes^2 T\mathcal{B}_h, \mathcal{S}_1) \xrightarrow{\mathbf{div}} \mathcal{P}_{r-3}(T\mathcal{B}_h) \longrightarrow 0, \quad (5.1a)$$

$$0 \longrightarrow \mathcal{P}_r^H(T\mathcal{B}_h, \mathcal{S}_1) \xrightarrow{\text{grad}} \mathcal{P}_{r-1}^c(\otimes^2 T\mathcal{B}_h, \mathcal{S}_1) \xrightarrow{\mathbf{curl}^\top} \mathcal{P}_{r-1}^{d-}(\otimes^2 T\mathcal{B}_h, \mathcal{S}_1) \xrightarrow{\mathbf{div}} \mathcal{P}_{r-2}(T\mathcal{B}_h) \longrightarrow 0, \quad (5.1b)$$

$$0 \longrightarrow \mathcal{P}_r^H(T\mathcal{B}_h, \mathcal{S}_1) \xrightarrow{\text{grad}} \mathcal{P}_r^{c-}(\otimes^2 T\mathcal{B}_h, \mathcal{S}_1) \xrightarrow{\mathbf{curl}^\top} \mathcal{P}_{r-1}^d(\otimes^2 T\mathcal{B}_h, \mathcal{S}_1) \xrightarrow{\mathbf{div}} \mathcal{P}_{r-2}(T\mathcal{B}_h) \longrightarrow 0, \quad (5.1c)$$

$$0 \longrightarrow \mathcal{P}_r^H(T\mathcal{B}_h, \mathcal{S}_1) \xrightarrow{\text{grad}} \mathcal{P}_r^{c-}(\otimes^2 T\mathcal{B}_h, \mathcal{S}_1) \xrightarrow{\mathbf{curl}^\top} \mathcal{P}_r^{d-}(\otimes^2 T\mathcal{B}_h, \mathcal{S}_1) \xrightarrow{\mathbf{div}} \mathcal{P}_{r-1}(T\mathcal{B}_h) \longrightarrow 0. \quad (5.1d)$$

Note that (5.1a) is a subcomplex of (5.1b), (5.1b) is a subcomplex of (5.1c), and (5.1c) is a subcomplex of (5.1d). These discrete complexes are Hilbert complexes. Regardless of the refinement level of the mesh \mathcal{B}_h , the cohomology groups of the above discrete Hilbert complexes are the same as those of the nonlinear elasticity complex.

6 Stable Mixed Finite Element Methods for the Tensor Laplacian

The bounded cochain projections between the nonlinear elasticity complex and the discrete complexes introduced earlier allow one to use the general theory for approximation of Hilbert complexes for approximating the nonlinear elasticity complexes. Consider the following mixed formulation, which is associated to the first Laplacians of the discrete Hilbert complexes (5.1): Given $\mathbf{Q} \in L^2(\otimes^2 T\mathcal{B})$, find $(\mathbf{U}_h, \mathbf{T}_h, \mathbf{P}_h) \in V_h^H \times V_h^c \times \mathcal{H}_h$ such that

$$\begin{aligned} \langle\langle \mathbf{U}_h, \boldsymbol{\Upsilon} \rangle\rangle_{L^2 T} - \langle\langle \mathbf{T}_h, \mathbf{grad} \boldsymbol{\Upsilon} \rangle\rangle_{L^2 \otimes^2} &= 0, & \forall \boldsymbol{\Upsilon} \in V_h^H, \\ \langle\langle \mathbf{grad} \mathbf{U}_h, \boldsymbol{\Theta} \rangle\rangle_{L^2 \otimes^2} + \langle\langle \mathbf{curl}^\top \mathbf{T}_h, \mathbf{curl}^\top \boldsymbol{\Theta} \rangle\rangle_{L^2 \otimes^2} + \langle\langle \mathbf{P}_h, \boldsymbol{\Theta} \rangle\rangle_{L^2 \otimes^2} &= \langle\langle \mathbf{Q}, \boldsymbol{\Theta} \rangle\rangle_{L^2 \otimes^2}, & \forall \boldsymbol{\Theta} \in V_h^c, \\ \langle\langle \mathbf{T}_h, \boldsymbol{\Pi} \rangle\rangle_{L^2 \otimes^2} &= 0, & \forall \boldsymbol{\Pi} \in \mathcal{H}_h, \end{aligned} \quad (6.1)$$

where the pair (V_h^H, V_h^c) can be either

$$(\mathcal{P}_{r+1}^H(T\mathcal{B}_h, \mathcal{S}_1), \mathcal{P}_r^c(\otimes^2 T\mathcal{B}_h, \mathcal{S}_1)) \text{ or } (\mathcal{P}_r^H(T\mathcal{B}_h, \mathcal{S}_1), \mathcal{P}_r^{c-}(\otimes^2 T\mathcal{B}_h, \mathcal{S}_1)),$$

with $r \geq 1$. Also depending on the choice of (V_h^H, V_h^c) , $\mathcal{H}_h := \ker \mathbf{curl}_{\mathcal{S}_1}^\top \cap \ker \mathbf{grad}_h^*$, is the space of discrete harmonic tensors associated to one of the discrete Hilbert complexes (5.1). Clearly, we have $V_h^H \subset H^1(T\mathcal{B}, \mathcal{S}_1)$ and $V_h^c \subset H^c(\otimes^2 T\mathcal{B}, \mathcal{S}_1)$. However, $\mathcal{H}_h \not\subset \mathcal{H}(\mathcal{B}, \mathcal{S}_1, \mathcal{S}_2)$, in general. Therefore, the discrete mixed formulation (6.1) is a generalized (or non-conformal) Galerkin method for the mixed formulation (4.3).

The Fredholm property of the original Hilbert complex and the existence of the bounded cochain projections imply that for a shape regular family of triangulations $\{\mathcal{B}_h\}$, the mixed finite element methods based on (6.1) are stable and convergent. The rate of convergence is determined by the smoothness of the data \mathbf{Q} , the smoothness of the domain, and the degrees of polynomials that generate the finite element spaces. One can also show that the error is the optimal order allowed by discrete solution spaces if there is sufficient elliptic regularity. For example, let $(V_h^H, V_h^c) = (\mathcal{P}_{r+1}^H(T\mathcal{B}_h, \mathcal{S}_1), \mathcal{P}_r^c(\otimes^2 T\mathcal{B}_h, \mathcal{S}_1))$, and suppose that the solutions \mathbf{T} and \mathbf{P} are of Sobolev class H^2 . Then, one can write

$$\begin{aligned} & \| \mathbf{U} - \mathbf{U}_h \|_{L^2 T} + h \| \mathbf{grad}(\mathbf{U} - \mathbf{U}_h) \|_{L^2 \otimes^2} + h \| \mathbf{T} - \mathbf{T}_h \|_{L^2 \otimes^2} \\ & + h^2 \| \mathbf{curl}^\top(\mathbf{T} - \mathbf{T}_h) \|_{L^2 \otimes^2} + h \| \mathbf{P} - \mathbf{P}_h \|_{L^2 \otimes^2} = O(h^{r+2}). \end{aligned}$$

Therefore, regarding the degree of the polynomial approximation, all components converge with the optimal order.

For implementing (6.1), one needs to calculate the space of discrete harmonic tensors \mathcal{H}_h . To this end, consider the following homogeneous problem: Find $(\mathbf{U}_h, \mathbf{T}_h) \in V_h^H \times V_h^c$ such that

$$\begin{aligned} & \langle \langle \mathbf{U}_h, \mathbf{\Upsilon} \rangle \rangle_{L^2 T} - \langle \langle \mathbf{T}_h, \mathbf{grad} \mathbf{\Upsilon} \rangle \rangle_{L^2 \otimes^2} = 0, \quad \forall \mathbf{\Upsilon} \in V_h^H, \\ & \langle \langle \mathbf{grad} \mathbf{U}_h, \mathbf{\Theta} \rangle \rangle_{L^2 \otimes^2} + \langle \langle \mathbf{curl}^\top \mathbf{T}_h, \mathbf{curl}^\top \mathbf{\Theta} \rangle \rangle_{L^2 \otimes^2} = 0, \quad \forall \mathbf{\Theta} \in V_h^c. \end{aligned}$$

Then, $(\mathbf{U}_h, \mathbf{T}_h)$ is a solution if and only if $\mathbf{U}_h = 0$, and $\mathbf{T}_h \in \mathcal{H}_h$.

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9 Publications

- 1) A. Angoshtari and A. Yavari, Stable mixed finite element methods for the tensor Laplacian with mixed boundary conditions, *In preparation*.
- 2) A. Angoshtari and A. Yavari, On the compatibility equations of nonlinear and linear elasticity in the presence of boundary conditions, *Submitted to ZAMP*.
- 3) A. Angoshtari and A. Yavari, The weak compatibility equations of nonlinear elasticity and the insufficiency of the Hadamard jump condition for non-simply connected bodies, *Submitted to Continuum Mechanics and Thermodynamics*.
- 4) A. Angoshtari and A. Yavari, Hilbert complexes, orthogonal decompositions, and potentials for nonlinear continua, *Submitted to Journal of Mathematical Physics*.
- 5) A. Angoshtari and A. Yavari, Differential complexes in continuum mechanics, *Archive for Rational Mechanics and Analysis*, **216**, 193-220, 2015.
- 6) A. Angoshtari and A. Yavari, A geometric structure-preserving discretization scheme for incompressible linearized elasticity, *Computer Methods in Applied Mechanics and Engineering*, **259**, 130-153, 2013.

10 Honors and Awards Received

None

11 AFRL Point of Contact

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12 Transitions

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13 New Discoveries

None

1.

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Abstract

We introduced a smooth complex for non-near elasticity that can be considered as the tensor analogue of the standard grad-curl-div complex. This mathematical structure simultaneously describes the kinematics and the kinetics of large deformations. The relation between this complex and the de Rham complex allows one to readily derive the necessary and sufficient conditions for the compatibility of displacement gradient and the existence of stress functions on non-contractible bodies. The main application of the non-near elasticity complex is in developing mixed finite element methods for large deformations, which will be pursued in a future project. To this end, the smooth complex should be extended to a so-called essential smooth tensors. We introduced this extension by using the so-called partly Sobolev spaces. The result is a Hilbert complex involving second-order tensors on flat compact manifolds with boundary. We then used the general framework of Hilbert complexes to write Hodge-type and Helmholtz-type orthogonal decompositions for second-order tensors. As some applications of these decompositions in continuum mechanics, one can study the strain compatibility equations of non-near elasticity in the presence of Dirichlet boundary conditions.

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A. Angoshtar and A. Yavar, A geometric structure-preserving discretization scheme for incompressible near incompressible elasticity, Computer Methods in Applied Mechanics and Engineering, 259, 130-153, 2013.

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Program Officer

Research Objectives

Technical Summary

Funding Summary by Cost Category (by FY, \$K)

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