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Searcher Speed for a Stationary Tight Crossover Barrier

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ABSTRACT

This technical note reports what appears to be a hitherto missing piece of barrier-patrol theory: the solution of the equation for a stationary (also known as 'symmetric') tight crossover barrier. It has long been known, at least implicitly, that the equation in question is cubic in the ratio of searcher speed to target speed. The exact solution is derived here; it is algebraically complicated. The key result of this note is the identification of a remarkably simple approximate solution that is remarkably accurate for barriers wider than a few times the effective sweep width of the searcher's sensor.

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Searcher Speed for a Stationary Tight Crossover Barrier

Executive Summary

This technical note reports what appears to be a hitherto missing piece of barrier-patrol theory: the solution of the equation for a stationary (also known as 'symmetric') tight crossover barrier. The crossover barrier is the path that a patrolling asset must follow if it wishes to ensure that all targets attempting to cross the barrier come within range of its sensors at some stage. Such a barrier is said to be 'tight', meaning that there are no coverage gaps through which a target may slip, whether deliberately or by chance. Crossover barriers are not tight in general, but can be designed to be so.

The equation for the stationary tight crossover barrier has long been known, but (it seems) not its solution, perhaps because it is cubic in the ratio of searcher speed v_s to target speed v_t. The exact solution is given in §3.3 below. It is algebraically complicated but it turns out to have a remarkably simple yet accurate approximation for wide barriers:

$$\frac{v_\mathrm{s}}{v_\mathrm{t}} \approx \frac{L}{W},$$

where L is the width of the barrier being patrolled and W is the effective sweep width of the searcher's sensor. This result is within 5% of the exact solution for $L/W > 2.8$ and within 1% for $L/W > 6.7$. If this should be insufficiently accurate for a particular application, then the analysis shows that

$$\frac{v_\mathrm{s}}{v_\mathrm{t}} \approx \frac{L}{W} - \frac{W}{2L} + \frac{1}{2}\left(\frac{W}{L}\right)^2 - \frac{5}{8}\left(\frac{W}{L}\right)^3.$$

This is accurate to within 5% for $L/W > 1.8$, to within 1% for $L/W > 2.4$ and to within 0.1% for $L/W > 3.9$.

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1. Introduction

The crossover barrier is the path that a searcher must follow in patrolling a barrier if it wishes to ensure that all targets attempting to cross the barrier come within range of its sensors at some stage. Such a barrier is said to be 'tight', meaning that there are no coverage gaps through which a target may slip. Crossover barriers are not tight in general, but can be designed to be so.† One of the inputs to that design is choice of the searcher's speed.

Equations for the tight crossover barrier have long been known: they appear in a famous text on search theory from the World War Two era [1] and in a more recent well known maritime operations-research text [2]. However, although implicitly deriving the equation for searcher speed, neither text solves it. This may be because the equation is cubic. The texts also avoid explicitly stating the equation, perhaps for the same reason.‡ The equation and its solution are presented here; the solution, though complicated, turns out to have a remarkably simple approximation that is remarkably accurate.

Although available in the cited references, the next section recapitulates the analysis of the crossover barrier, for completeness and to give an explicit statement of the cubic equation. Section 3 presents the solution, §4 gives a discussion and §5 a conclusion.

2. Setting the Problem Up

Koopman introduces the concept of the tight crossover barrier with the aid of Figure 1. When the searcher is at O, it is—obviously—directly over any target that happens to be at O at the same time. If the searcher steers a course at a lead angle α to the barrier, given by

$$\sin\alpha = v_t/v_s, \tag{1}$$

then it will pass directly over any target that is on the line OO′ in Figure 1 at the time that the searcher is at O. In Equation (1), v_s is the searcher's speed and v_t is the target's speed (called u by Koopman), both assumed to be constant.

† The question of whether and when barriers need to be tight is briefly discussed in §4.

‡ Alternatively, perhaps the equation was not stated by previous authors because they regarded the searcher's speed as an input quantity rather than a quantity to be determined. The present work was prompted by an application [3] that benefited from a determination of the searcher's speed.

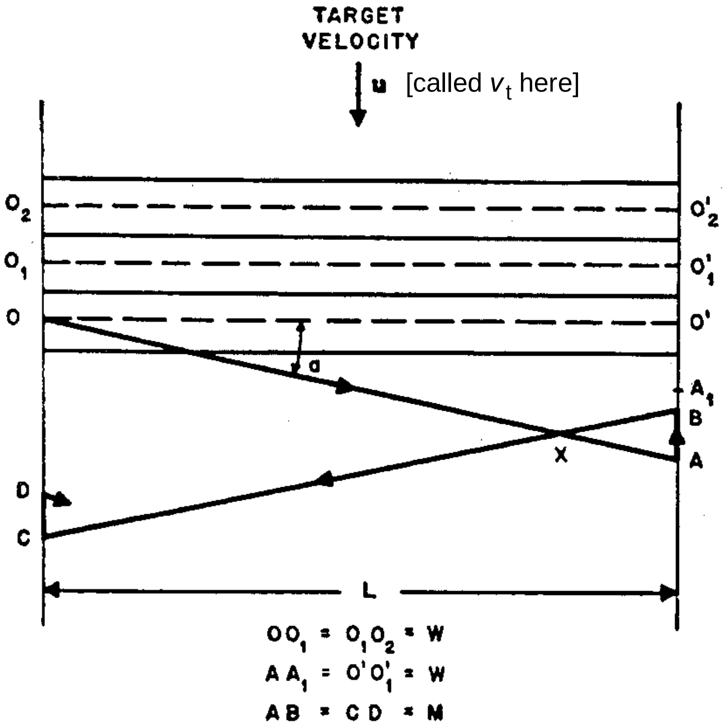

Figure 1: 'The crossover barrier patrol designed to be tight' (both figure and caption are due to Koopman [1(p.198)])

If the searcher's sensor has an effective sweep width of W, then, in traversing the path OA, the searcher has the opportunity to detect any target located in a band of width W centred on OO′ at the time that the searcher is at O. That is, the band centred on OO′ is cleared in the time that it takes the searcher to traverse the path OA. Having done this and arrived at A, the searcher next wishes to clear the band that was centred on O_1O_1' when the searcher was at O. Now, a target at O′ when the searcher is at O has moved to A_1 by the time that the searcher is at A. Because of the chosen lead angle (Eqn 1) and the fact that the distance OO_1 equals W, point A_1 is exactly a distance W upstream of A. Hence the searcher must manœuvre upstream—the 'upsweep'—to a point B, which is defined such that the time taken for the searcher to traverse AB equals the time taken for the target to traverse A_1B. In other words, both arrive at B at the same time, and the searcher is in the right position to do the return sweep BC across the barrier, during which it clears the band of width W that was centred on O_1O_1' at the time that the searcher was at O.

Continuing in this manner ensures that the barrier is tight: all targets attempting to cross the barrier must at some time come within a distance $W/2$ of the searcher. The case shown in Figure 1 leads to a 'retreating barrier', so-called because point D is downstream of point O. This and the other two cases are shown in Figure 2. Which case obtains in any given situation is determined by the length of the upsweep needed to maintain tightness. Koopman uses the term 'symmetric' to describe the case of present interest, but notes that the term 'stationary' is also used [1(p.200)].

A stationary barrier is obtained when the point B coincides with O′. It means that time taken for the searcher to traverse the path OAO′ must equal the time taken for a target to

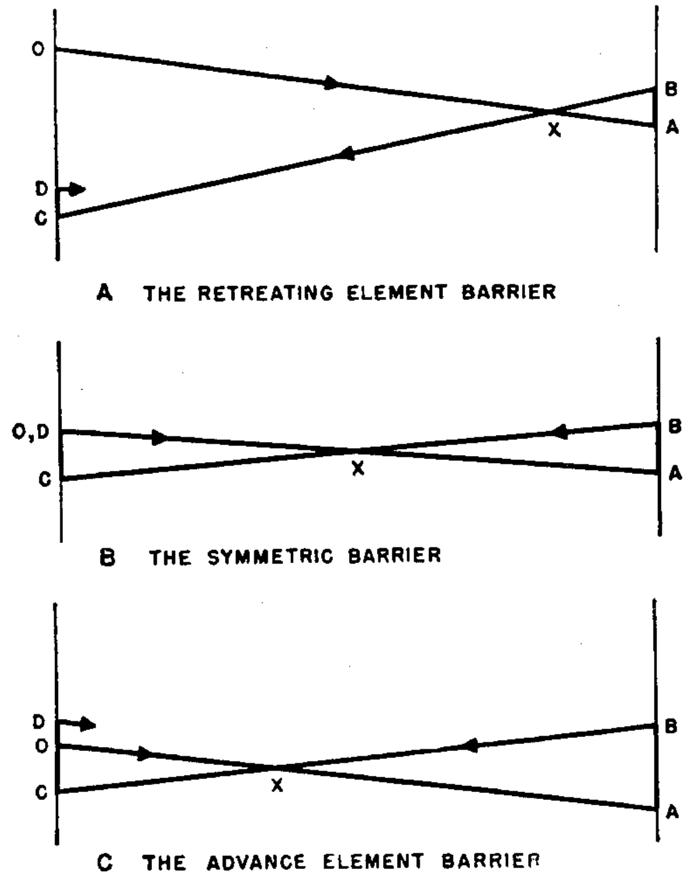

Figure 2: 'The three cases of crossover barrier patrol' [1(p.201)]

traverse the path $O_1'O'$. Expressed mathematically [1,2]:

$$\frac{L-W}{\sqrt{{v_s}^2 - {v_t}^2}} + \frac{W}{v_s + v_t} = \frac{W}{v_t}. \tag{2}$$

The appearance of $L - W$ in the numerator of the first term is a refinement on the equations in references 1 and 2, reflecting the concept that the searcher need not transit the full width L of the barrier, but can turn $W/2$ from each side. Koopman [1] mentions this on pp. 202–3. It is included here because, as discussed in §3.2, it leads to the expected result for searcher speed—namely zero—when L equals W.†

We suppose that L, W and v_t are known and ask: what speed should the searcher make to give a stationary barrier? Rearranging Equation (2) in favour of v_s gives

$$V^3 - V^2 - pV - p = 0, \tag{3}$$

in which two new quantities are introduced to emphasise the simplicity of the equation:

$$V = \frac{v_s}{v_t}, \qquad p = \left(\frac{L}{W} - 1\right)^2. \tag{4}$$

† This is a side issue, since the focus here is on wide barriers (i.e. large L), but it is nice to have.

As simple as Equation (3) appears, it is nevertheless a cubic that does not factorise for arbitrary values of p, so one must to resort to the general solution [4(§3.8.2)]. This is the subject of the next section.

3. Solution

3.1 Cubic Determinant; Range of Parameters

The first step in obtaining the general numerical solution of a cubic equation is to compute the cubic determinant, called '$q^3 + r^2$' in reference 4. This plays a role analogous to that of the quantity commonly called '$B^2 - 4AC$' in the numerical solution of quadratic equations, in the sense that:

- when $q^3 + r^2 < 0$, the cubic has three distinct real solutions
- when $q^3 + r^2 = 0$, all solutions are real and at least two of them are equal
- when $q^3 + r^2 > 0$, one solution is real and the other two are a complex-conjugate pair.

The formulae in Abramowitz and Stegun [4(§3.8.2)] give the cubic determinant of Equation (3) as

$$q^3 + r^2 = \frac{-p}{27}\left(p^2 - 11p - 1\right). \tag{5}$$

This is zero at three values of the equation's parameter p:

$$p = 0, \qquad p = \tfrac{1}{2}\left(11 \pm 5\sqrt{5}\right). \tag{6}$$

The expression with the surd evaluates to 11.09017 and –0.09017.

Obviously we are only interested in real values of V. Equation (4) also makes it clear that solutions are unphysical if $p < 0$ or $V < 0$. However, it turns out to be instructive to ignore these restrictions initially and explore the full behaviour of the cubic. This is the topic of the next subsection, after which we resume constructing the general solution.

3.2 Behaviour of the Cubic

The interesting places to look are around the p values in Equation (6). Figure 3 shows plots of the cubic's left-hand side for nine p values from a little below $p = -0.09017$ to a little above $p = 0$. When $p < -0.09017$, the one real solution is negative—Figures 3(a, b). This is joined at $p = -0.09017$ by a positive double solution. The middle row of panels shows p progressing toward zero: the double solution becomes two distinct solutions and the middle and negative solutions move toward each other. They meet to form another double solution when $p = 0$ (Fig. 3f). For this value of p, the equation is easy to solve by inspection: the double solution is $V = 0$ and the single solution is $V = 1$.

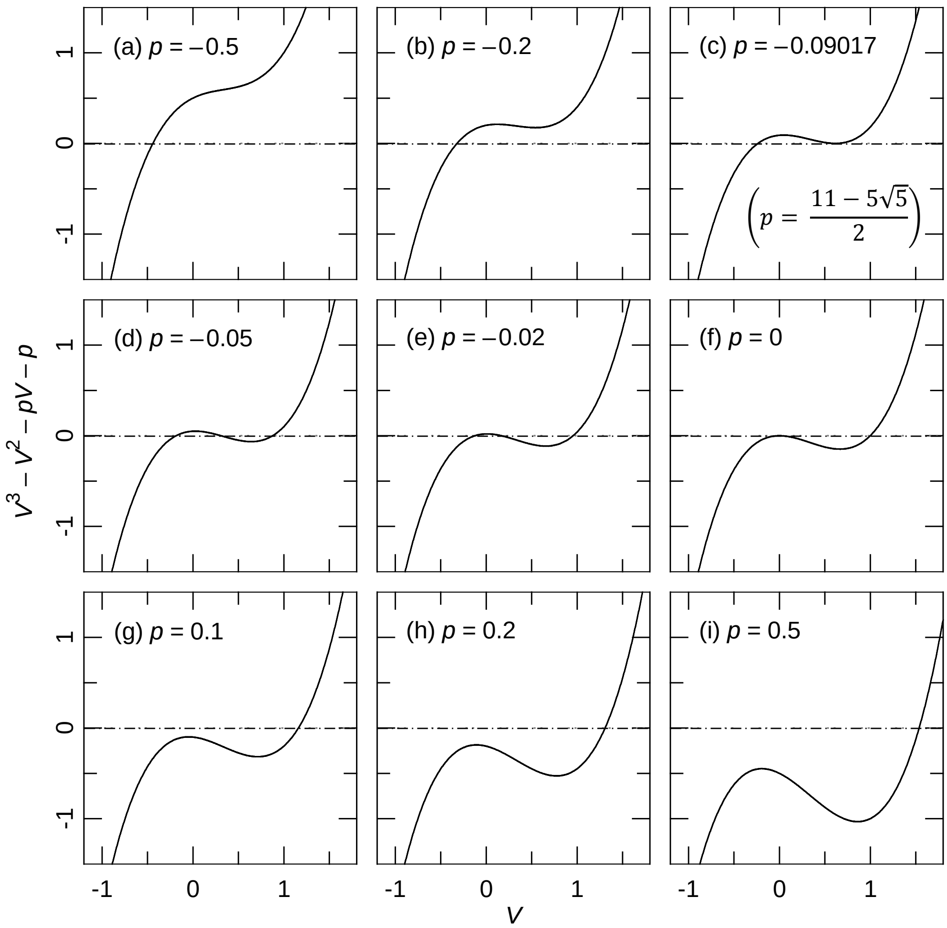

Figure 3: The cubic equation for a selection of values of its parameter p *in the vicinity of the two smaller critical values:* p = *–0.09071 (panel c) and* p = *0 (panel f)*

Figure 3(f) is an important panel because it marks the start of the region of physical solutions: when $p = 0$, the width L of the barrier just equals the sweep width W of the searcher's sensor. In this situation—and for narrower barriers, of course—the searcher should remain stationary in the middle of the barrier, so $V = 0$ is the appropriate solution.

In the bottom row of panels, we are back to a single real solution, which steadily increases in value as p increases. This row of panels shows that, as p approaches zero from above, the solution approaches unity from above but, as argued in the previous paragraph, the appropriate solution jumps to $V = 0$ at $p = 0$.

The third p value in Equation (6) is 11.09017. The top row of panels in Figure 4 continues the sequence of the bottom row of Figure 3 toward this value, showing the return of the double solution when $p = 11.09017$ (Fig. 4d).

Beyond $p = 11.09017$ there are again three real solutions, but only one is positive, so this must be the one of interest for present purposes. Numerical exploration using a spreadsheet suggests that the middle solution approaches $V = -1$ from below as $p \to \infty$. If so, it follows that there is only ever one positive solution. We return to this point in §3.4.2 to confirm the result by a power-series expansion of the general solution.

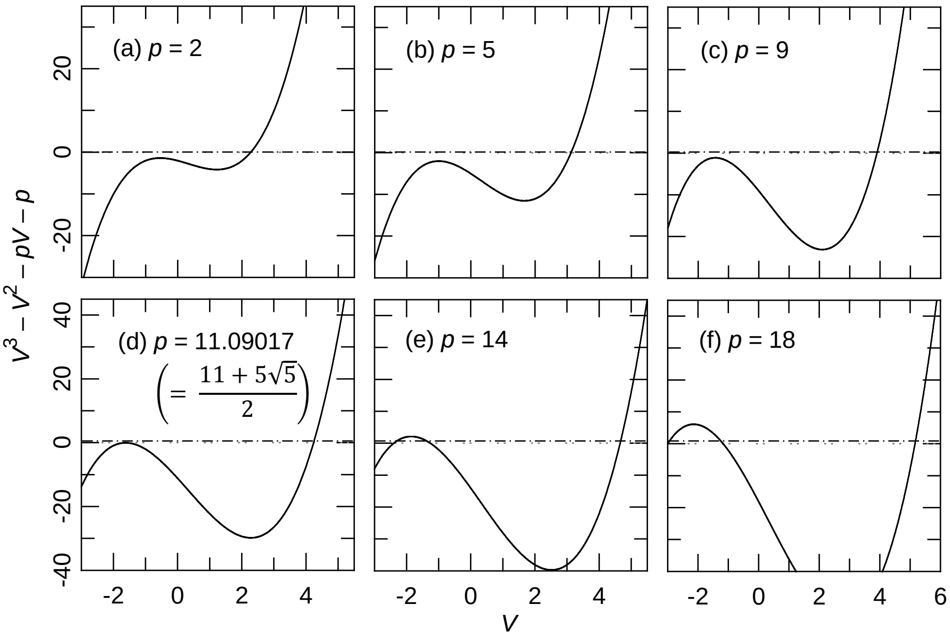

Figure 4: Continuing on from Figure 3 – the cubic equation for parameter values out to and beyond the largest critical value of p = *11.09017, which appears in panel (d)**

**Panel (f) shows that* V = *–3 is a solution of the equation when* p = *18. That is, the equation factorises for this particular parameter value. It is easy to find other examples by rearranging Equation* (3) *in favour of* p:

$$p = \frac{V^3 - V^2}{V + 1} .$$

One can now insert any desired value of V *to obtain the value of* p *for which the equation has that* V *value as a solution. So, for example,* V = *–2 is a solution when* p = *12,* V = *2 is a solution when* p = *4/3,* V = *3 is a solution when* p = *9/2, and so on.*

3.3 General Solution

The physically valid region is $p \geq 0$. When there are three real solutions in this region, the numerical survey of the last subsection shows that the solution of interest is always the largest one (except for the isolated special case of $p = 0$). This is the solution called z_1 by Abramowitz and Stegun [4(§3.8.2)]. In the following, it is identified simply as V.

For $p < 11.09017$, when $q^3 + r^2 > 0$, this solution is

$$V = \frac{1}{3}\left(1 + \sqrt[3]{18p + 1 + 3Q} + \sqrt[3]{18p + 1 - 3Q}\right), \tag{7}$$

where

$$Q = \sqrt{3p\left(1 + 11p - p^2\right)} . \tag{8}$$

For the critical value of p equalling 11.09017 (or, more accurately, $(11 + 5\sqrt{5})/2$), Q equals zero and Equation (7) simplifies in the obvious manner.

That leaves the third and most interesting region of p > 11.09017, where $q^3 + r^2 < 0$ (interesting because it corresponds to wide barriers). Although the solution is real, deriving an expression for it requires an excursion into the complex plane. The result is [4(§3.8.2)]

$$V = \frac{1}{3} + \frac{2}{3}\cos\left(\frac{1}{3}\arctan\frac{\sqrt{3p\left(p^2 - 11p - 1\right)}}{6p + 1/3}\right)\sqrt[6]{27p^3 + 27p^2 + 9p + 1}\,. \tag{9}$$

Equations (7) and (9) look very different, but they pass seamlessly from one to the other at p = 11.09017; for there the argument of the arctan function is zero—so the value of the cos function is unity—and number inside the sixth root in Equation (9) equals the square of the number inside either cube root in Equation (7).

The complication in Equations (7) and (9) hides a remarkable approximate simplicity. This, the key result of this note, is revealed by displaying V as a function of L/W rather than of p, as Figure 5 shows. It is clear that, for large L/W (and hence large p),

$$V = \frac{v_\mathrm{s}}{v_\mathrm{t}} \approx \frac{L}{W}\,. \tag{10}$$

The approximation is accurate to within 5% for $L/W > 2.8$ and to within 1% for $L/W > 6.7$.

Equation (10) is all the more surprising because naive insertion of $V \approx \sqrt{p}$ into Equation (3) would clearly seem rule out it as a solution. An algebraic demonstration that it is indeed a good approximate solution can be obtained by application of power-series expansions. This is detailed in §3.4.1. A quadratic approximation to Equation (3) is possible, as described in §3.5. It turns out to lead to Equation (10) also.

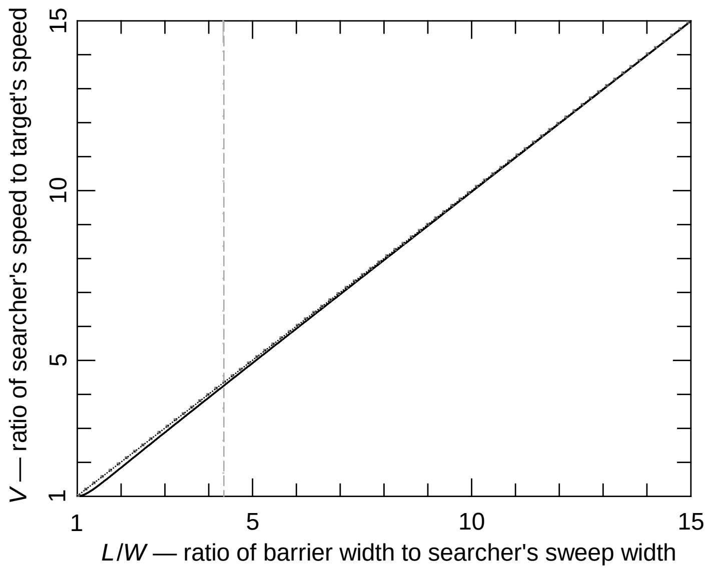

Figure 5: Solution of Equation (3) displayed as a function of the ratio L/W. *The grey broken line marks the value of* L/W *corresponding to* p = *11.09017; that is, the plot is of Equation (7) to the left of the broken line and of Equation (9) to the right. The dotted line is a plot of* V = L/W, *showing how close this is to the exact solution.*

3.4 Power-Series Expansion for Wide Barriers

This subsection has two aims, first to justify the claims above concerning Equation (10) and secondly to show that the middle solution in Figures 4(e, f) is not positive for any value of $p > 11.09017$ and therefore is never a candidate for the solution of interest.†

3.4.1 Positive Solution

Starting with the sixth root in Equation (9), its binomial-theorem expansion for large p is

$$\sqrt[6]{27p^3+27p^2+9p+1} \approx \sqrt{3p}\left(1+\frac{1}{6p}-\frac{1}{72p^2}\right) \tag{11}$$

to order $1/p^2$. To the same order, the numerator of the argument of the arctan function is

$$\sqrt{3p\left(p^2-11p-1\right)} \approx \sqrt{3p^3}\left(1-\frac{11}{2p}-\frac{125}{8p^2}\right), \tag{12}$$

and so the whole argument is

$$\frac{\sqrt{3p\left(p^2-11p-1\right)}}{6p+1/3} \approx \sqrt{\frac{p}{12}}\left(1-\frac{50}{9p}-\frac{9925}{648p^2}\right) \quad . \tag{13}$$

The inverse tangent function is expanded using the result [5(¶505.2)]

$$\arctan x = \frac{\pi}{2}-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}+\ldots, \tag{14}$$

which gives

$$\frac{1}{3}\arctan\frac{\sqrt{3p\left(p^2-11p-1\right)}}{6p+1/3} \approx \frac{\pi}{6}-\sqrt{\frac{4}{3p}}\left(1+\frac{14}{9p}+\frac{2993}{360p^2}\right) \tag{15}$$

to order $1/p^2$. To expand the cos function, we use ¶415.08 of Dwight [5] in the form

$$\cos(\theta-x) = \cos\theta + x\sin\theta - \frac{x^2\cos\theta}{2} - \frac{x^3\sin\theta}{6} + \frac{x^4\cos\theta}{24} + \ldots, \tag{16}$$

to obtain

$$\cos\left(\frac{1}{3}\arctan\frac{\sqrt{3p\left(p^2-11p-1\right)}}{6p+1/3}\right) \approx \frac{\sqrt{3}}{2}+\sqrt{\frac{1}{3p}}\left(1-\sqrt{\frac{1}{p}}+\frac{4}{3p}-\frac{3}{p}\sqrt{\frac{1}{p}}+\frac{7859}{1080p^2}\right). \tag{17}$$

Finally, putting Equations (11) and (17) together leads to

---

†If the middle solution is never positive, then obviously the smallest solution is never positive either.

$$V \approx \sqrt{p} + 1 - \frac{1}{2}\sqrt{\frac{1}{p}} + \frac{1}{p} - \frac{17}{8p}\sqrt{\frac{1}{p}} + \frac{2021}{405p^2} \tag{18}$$

to order $1/p^2$.

Taking a hint from Figure 5, we convert Equation (18) from a series in p to a series in L/W. For readability's sake, it is helpful to give the ratio a name by writing $R_\mathrm{L} = L/W$. Then (cf. Eqn 4):

$$p = R_\mathrm{L}{}^2 \left(1 - \frac{2}{R_\mathrm{L}} + \frac{1}{R_\mathrm{L}{}^2} \right), \tag{19}$$

so that Equation (18) becomes

$$V \approx R_\mathrm{L} \left(1 - \frac{1}{2R_\mathrm{L}{}^2} + \frac{1}{2R_\mathrm{L}{}^3} - \frac{5}{8R_\mathrm{L}{}^4} + \frac{3613}{3240R_\mathrm{L}{}^5} \right). \tag{20}$$

Note the absence of a term in $1/R_\mathrm{L}$. It may seem excessive for present purposes to carry Equation (20) to order $1/R_\mathrm{L}{}^5$, but this turns out to be the lowest-order term with a non-zero coefficient when Equation (20) is substituted into Equation (3), a fact that emphasises the extent to which $V \approx R_\mathrm{L}$ (i.e. Eqn 10) is a good approximation. To see this in detail, we go term by term through Equation (3):

$$V^2 \approx R_\mathrm{L}{}^2 \left(1 - \frac{1}{R_\mathrm{L}{}^2} + \frac{1}{R_\mathrm{L}{}^3} - \frac{1}{R_\mathrm{L}{}^4} + \frac{2803}{1620R_\mathrm{L}{}^5} \right), \tag{21}$$

$$V^3 \approx R_\mathrm{L}{}^3 \left(1 - \frac{3}{2R_\mathrm{L}{}^2} + \frac{3}{2R_\mathrm{L}{}^3} - \frac{9}{8R_\mathrm{L}{}^4} + \frac{1993}{1080R_\mathrm{L}{}^5} \right), \tag{22}$$

$$pV \approx R_\mathrm{L}{}^3 \left(1 - \frac{2}{R_\mathrm{L}} + \frac{1}{2R_\mathrm{L}{}^2} + \frac{3}{2R_\mathrm{L}{}^3} - \frac{17}{8R_\mathrm{L}{}^4} + \frac{9283}{3240R_\mathrm{L}{}^5} \right), \tag{23}$$

leading, with Equation (19), to the final result

$$V^3 - V^2 - pV - p \approx \frac{-8}{405R_\mathrm{L}{}^2}, \tag{24}$$

rather than zero. This highlights just how good an approximation Equation (20) is: the 4 to 5 leading terms in Equations (21)–(23) sum to zero and the next set of terms almost so (in the sense that 8/405 is a small number and dividing by $R_\mathrm{L}{}^2$ makes it even smaller).

3.4.2 Middle Solution

The purpose of this subsection is to show that the middle solution is never positive. It is clearly not positive in Figure 4(e,f), so we explore the behaviour as $p \to \infty$. For Equation (3) with $p > 11.09017$, the middle solution is the one called z_3 by Abramowitz and Stegun [4(§3.8.2)]. Here, it is called V_m. It is convenient to define a variable for the argument of the cos function in Equation (9):

$$A = \frac{1}{3}\arctan\frac{\sqrt{3p\left(p^2 - 11p - 1\right)}}{6p + 1/3}. \tag{25}$$

In terms of this, the middle solution of Equation (3) is [4(§3.8.2)]

$$V_{\mathrm{m}} = \frac{1}{3}\left(1 - \cos A + \sqrt{3}\sin A\right)\sqrt[6]{27p^3 + 27p^2 + 9p + 1}. \tag{26}$$

Expansions of the sixth root and $\cos A$ have already been obtained (Eqns 11 and 17). The result for $\sin A$ is

$$\sin A \approx \frac{1}{2} - \sqrt{\frac{1}{p}}\left(1 + \frac{1}{3}\sqrt{\frac{1}{p}} + \frac{4}{3p} + \frac{1}{p}\sqrt{\frac{1}{p}} + \frac{7859}{1080p^2}\right)^{\dagger}. \tag{27}$$

Collecting it all up gives

$$V_{\mathrm{m}} \approx -1 - \frac{2}{p} - \frac{4042}{405p^2}, \tag{28}$$

which demonstrates that the middle solution approaches –1 from below as $p \to \infty$; that is, it is never positive.

3.5 Quadratic Approximation to Equation (3)

Equation (2) can be rewritten using the lead angle α (Eqn 1) in place of the two speeds. The result is

$$\frac{L - W}{\cos\alpha} + \frac{W}{1 + \sin\alpha} = \frac{W}{\sin\alpha}. \tag{29}$$

The approximation $\cos \alpha \approx 1$—appropriate for small values of α—reduces the equation from a cubic to a quadratic. In terms of the variables V and p (Eqn 4), it becomes

$$V^2 - \sqrt{p}V - \sqrt{p} = 0. \tag{30}$$

This has the solutions

$$V^2 = \frac{\sqrt{p}}{2}\left(1 \pm \sqrt{1 + \frac{4}{\sqrt{p}}}\right), \tag{31}$$

of which only the one with the upper sign is real. The power-series expansion of this expression is

$$V \approx \sqrt{p} + 1 - \sqrt{\frac{1}{p}} + \frac{2}{p} - \frac{5}{p}\sqrt{\frac{1}{p}} + \frac{14}{p^2}, \tag{32}$$

which may be compared with Equation (18). The first two terms are identical, but thereafter the two series differ, with the coefficients in Equation (32) being larger than those in Equation (18).

† Yes, the term of order $1/p^2$ is indeed identical to the term of the same order in $\cos A$ (Eqn 17), but note the extra factor of $1/\sqrt{3}$ multiplying the parentheses in Eqn (17).

The agreement between the first two terms of Equations (18) and (32) means that both equations give the same linear approximation, namely Equation (10). The quadratic equation is much the easier to solve, but its applicability becomes apparent only in hindsight, once the solution to the cubic equation is in hand.

4. Discussion — Practical Applicability of the Analysis

It is probably the case that most barrier-patrol operations carried out in real life are not tight, as a matter of practical necessity. Most discussions of barrier-patrol effectiveness—e.g. [1,2,6(§5.2),7(§1.3)]—concentrate on non-tight barriers. Universally such analyses take the detection probability p_d to be the fraction of the total area actually swept by the patrolling asset. It is clear, however, that this p_d value will be achieved only if targets attempting to cross the barrier are unaware of the location and motion of the patrolling asset(s); for otherwise targets can time their crossing to ensure that it occurs in a gap,† thereby reducing to zero the detection effectiveness actually achieved by the patrol. This highlights the reason for preferring a tight barrier whenever practicable: a target whose speed is always less than the design speed v_t of the barrier cannot avoid coming within sensor range of the patrol at some time.

A barrier can be tight but not stationary, as Figure 2 illustrates. This may be acceptable if the barrier is temporary. If, on the other hand, it must be in place for a long time in a choke point, then the finite dimensions of the choke point place a limit on the rate at which the barrier can advance or retire.

To use Equation (10) to find the appropriate searcher speed for a stationary tight barrier, one needs to know values for the barrier width, sensor sweep width and target speed. The first two are determined by environmental factors and can be expected to be known in a given situation. The speed with which a target chooses to cross the barrier is not known by the searcher prior to the target being detected and tracked. In practice a patrolling asset would navigate the path of a stationary barrier at a speed v_s determined by other factors, such as minimising the rate of fuel consumption. Equation (10) then gives the target speed $v_\mathrm{t,stb}$ for which the stationary barrier is tight. The target may of course choose to cross at a different speed:

- If the target's actual speed is less than $v_\mathrm{t,stb}$, then neighbouring sweeps across the barrier—e.g. OA and BC in Figure 1—yield overlapping searched swaths. This is nevertheless still a tight barrier. It may even be an efficient mode of operation if the speed v_s is close to the search platform's maximum-efficiency speed.
- If, on the other hand, the target's actual speed is greater than $v_\mathrm{t,stb}$, then tightness is not achieved: there will be coverage gaps through which a target can slip. To exploit this, a target wishing to cross the barrier must be able to detect and track the

† Evasive targets, a fact of life in ASW, are outside the scope of two otherwise mathematically comprehensive treatments of search theory [8,9].

patrolling asset(s) without, of course, itself being detected. The effect may be partly counteracted in the case of search by passive sonar if the target's source level increases with its speed, for this may lead to an increase in the searcher's sweep width.

Every search platform has an upper limit to its speed, owing either to its top speed or to the maximum speed consistent with effective operation of its sensor. Equation (10) then implies an upper limit, for a given assumed target speed, on the width of barrier that it can patrol. Stationary tight barriers of greater width require multiple search assets [1,2].

There is another practical point in the use of crossover barriers: the searcher must decide in which direction it expects targets to attempt a crossing. The use of the lead angle (Eqn 1 and Fig. 1) means that barriers tight for targets crossing in one direction are not—and cannot be made to be—tight for targets crossing in the opposite direction. To my knowledge, the only way of achieving a barrier that is tight for both transit directions is to station multiple sensors, spaced by W or less, across the width of the barrier.

As a final practical point, it may be noted that the equations assume a constant sweep width W at all locations across the barrier. If W should vary, tightness could be ensured by designing the barrier using the minimum value of W encountered.

5. Conclusion

This note reports what appears to be a hitherto missing piece of barrier-patrol theory: the solution to the equation for a stationary, or symmetric, tight crossover barrier. It is presumed that this situation arose because the equation is cubic in the ratio of searcher speed v_s to target speed v_t. The exact solution is given in Equations (7) and (9). It turns out to have a remarkably simple yet accurate approximation for wide barriers:

$$\frac{v_s}{v_t} \approx \frac{L}{W}, \tag{33}$$

where L is the width of the barrier being patrolled and W is the effective sweep width of the searcher's sensor. Equation (33) is accurate to within 5% for $L/W > 2.8$ and to within 1% for $L/W > 6.7$.

To provide an algebraic understanding of the origin of the accuracy of Equation (33), the exact solution was expanded in a power series in W/L. This gives the next few terms in the approximation as

$$\frac{v_s}{v_t} \approx \frac{L}{W} - \frac{W}{2L} + \frac{1}{2}\left(\frac{W}{L}\right)^2 - \frac{5}{8}\left(\frac{W}{L}\right)^3, \tag{34}$$

which is accurate to within 5% for $L/W > 1.8$, to within 1% for $L/W > 2.4$ and to within 0.1% for $L/W > 3.9$.

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19. ABSTRACT

This technical note reports what appears to be a hitherto missing piece of barrier-patrol theory: the solution of the equation for a stationary (also known as 'symmetric') tight cross-over barrier. It has long been known, at least implicitly, that the equation in question is cubic in the ratio of searcher speed to target speed. The exact solution is derived here; it is algebraically complicated. The key result of this note is the identification of a remarkably simple approximate solution that is remarkably accurate for barriers wider than a few times the effective sweep width of the searcher's sensor.