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Transmitting Information by Propagation in an Ocean Waveguide: Computation of Acoustic Field Capacity

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14. ABSTRACT
The maximum rate at which information can be transferred without error by acoustic wave propagation between two spatially distinct regions in an ocean waveguide is investigated by applying information-theoretic principles. An expression for this upper bound on transmission rate (i.e., information capacity) is determined within an ocean environment comprised of a penetrable, absorptive bottom with spatially uniform sound speeds and densities for both the water column and bottom, in conjunction with a correlated noise field. The acoustic source is described by a continuous, spatially confined distribution and, together with a receive volume, can be viewed in the context of multiple input–multiple output (MIMO) communications between source and receive regions in the waveguide. In this scenario, the acoustic field is interpreted as a carrier of information in the information-theoretic sense, and an expression for the field capacity using a full wave formulation of propagation in this waveguide is derived. Using the compactness of the Green function operator and singular value decomposition, the capacity is written as a sum of link capacities identified with independent pseudo-channels, subject to constraints on the source power distribution among these channels and knowledge of the environment at the transmit region.

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TRANSMITTING INFORMATION BY PROPAGATION IN AN OCEAN WAVEGUIDE: COMPUTATION OF ACOUSTIC FIELD CAPACITY

1. INTRODUCTION

There is considerable interest in the development of underwater sensor networks to perform a variety of operations such as communications, detection, localization, and tracking of acoustic emitters. However, the fundamental limits of distributed processing within these networks is difficult to characterize and largely unknown, due in part to the large number of degrees of freedom associated with network topology, communication and signal processing parameters, and environmental variability. While it is clear that energy constraints must be addressed in an analysis of fundamental limits, "information" [1] is also an intrinsic quantity of interest, subject to its own constraints, and is the raison d'être for the network. "Information" is transferred (communicated) and transformed (processed) through various components of the system, defined here to include both the sensor network and its ocean environment. To address these limits of distributed processing, an information-theoretic point of view is considered in this report, and to make progress along this line of reasoning, it is helpful to decompose the problem into two parts. These components involve network *sensing* and network *processing* subsystems. A plausible initial approach to the problem should quantify how much information, constrained by the physics of the system, is potentially available for the network to process. This network sensing component involves quantification of the "information" transferred from the emitter(s) to a region of space that contains the sensor node locations. The amount of information sensed by the network, limited only by the physics of the system, then serves as an upper bound on what can be achieved by network processing. While the distributed network concept discussed above serves as motivation and context for the analysis presented in this report, network characteristics, sensor spatial configurations, and signal processing schemes are not addressed here.

Since information coded in or associated with the acoustic source signal is transferred to the sensors via wave propagation, it is important to develop a quantitative relationship between propagation and an objective measure of "information." A conventional description of propagation in underwater acoustics with well-separated source and receiver regions usually involves waves carrying energy and momentum along multiple paths. An alternative, complementary description is considered here whereby a wave field is interpreted as a carrier of information along multiple paths, storing and transferring it from a source to a receiver region via a noisy waveguide channel. While this statement may appear intuitively obvious, an objective description of "information" is a subject of considerable subtlety. This alternative interpretation will be explored here by applying Shannon's Information Theory [2] to examine relationships between wave propagation and information capacity in a simple waveguide environment. Though network issues are not specifically addressed here, the resulting analysis indicates how the acoustic field places a quantitative limit on the information available to distributed network nodes for further processing.

Information Theory provides a rigorous mathematical framework for developing inequalities that describe limitations (bounds) on information transfer imposed by the physical nature of the system. The capacity is the upper bound on the error-free information transmission rate, typically measured in bits/unit

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time/Hz, and represents here a fundamental limit imposed by propagation physics. We specifically consider the issue of how much information can be reliably associated (stored) within an acoustic field propagating in a noisy ocean waveguide, with the goal of obtaining an upper bound based on the physics of the problem. Note that this question is independent of the question of how this information is *processed* by the network. The sensors simply occupy a subset of the space volume containing the acoustic field and space is treated as a capacity-bearing structure, analogous to the role played by temporal diversity associated with spectral bandwidth [3]. The capacity is obtained from the mutual information (see Appendix), a functional of the *a priori* and *a posteriori* probabilities describing the information gain associated with transmission and reception of a message tranferred through a noisy channel. The message can be transmitted intentionally or unintentionally. In the former case, the source is controllable and one can maximize the mutual information over the possible source distributions to obtain the capacity. In the latter case, one cannot perform this operation since the source signal is a given quantity and not controllable. Since the emphasis here is on acoustic field capacity, the results are more closely related to communication issues than, for example, detection or localization. The latter objectives have recently been addressed in underwater acoustics using Information Theory [4, 5].

The modern origins of the research involving links between information and field structure are found in the electromagnetics and optics literature [6, 7]. Primary motivation for the resurgence of interest in the study of fundamental relationships between (classical) wave propagation and Information Theory is the potential for large capacity gains, initially associated with terrestrial wireless communications [8–10] using multiple input–multiple output (MIMO) system configurations and, more recently, (acoustic) underwater distributed networks [11–13]. Since the emphasis here is on the acoustic field capacity rather than the capacity associated with a specific MIMO architecture, some care is needed because a continuous field has an infinite number of degrees of freedom. It turns out, however, that the field can be represented exactly in the discrete form of a singular value decomposition (SVD) of the integral Green function operator $\mathbb{G}$, where the resulting series representation can be truncated with negligible error due to the properties of its singular values. This implies that the original infinite-dimensional matrix representation of $\mathbb{G}$ can be well-approximated by a finite number of degrees of freedom, and the information-theoretic analysis used for studying MIMO capacity can be applied to determine the capacity of the acoustic field. The degrees of freedom are associated with independent pseudo-channels through which information is transferred between source and receive regions.

An additional issue in this analysis is the description of the noise field for the underwater problem. In terrestrial communications theory, it is usually treated as white, uncorrelated Gaussian noise with a diagonal noise covariance matrix representation. Under certain oceanographic conditions, however, the ambient noise field can be spatially correlated in depth and off-diagonal components can be present. In other underwater acoustics situations, a diagonal covariance matrix may be more appropriate. A representation of the noise field in terms of a general noise covariance matrix suitable for including correlated noise will be associated with the waveguide environment and incorporated into the capacity computation. A recent effort to compute capacity in an underwater channel under a variety of conditions [14] considers the specific case of a Kuperman-Ingenito noise covariance model [15] that accounts for correlations in the noise field.

For the purpose of orientation, a description of acoustic propagation between discrete sources and receivers in terms of a MIMO framework is presented in Section 2. This is followed in Section 3 with a brief derivation of an equation for the source–receiver mutual information which is then applied to determine a general expression for information capacity in a stationary, correlated noise field. The result is obtained by solving a constrained optimization problem. In Section 4, propagation is treated from a different point of view, as a mapping between two Hilbert spaces associated with source and receive regions. This allows the continuous, infinte-dimensional Green function operator to be rewritten in terms of singular value decomposition and, after specifying a particular ocean environment, yields a result for the capacity using explicit expressions for the singular vectors and singular values. This rather abstract formulation is general enough to handle more complex situations. Numerical results are also presented in Section 4, given the special case of a spatially uniform ocean environment with an uncorrelated Gausian noise field. A summary and conclusions are given in Section 5. An appendix is included to briefly introduce and summarize some basic concepts of Information Theory used in this analysis.

2. REPRESENTATION OF THE ACOUSTIC FIELD IN A MIMO MODEL

In this section, propagation is interpreted within a communications framework and related to a MIMO communications model. Consider a spatially continuous source distribution referenced with cylindrical coordinates $\boldsymbol{r}' = (\rho', z', \phi')$, located in a transmit volume V_{T} with source strength $S(\rho', z', \phi')$. The pressure field at an arbitrary location in a volume V_{R} of the waveguide, spatially disjoint from V_{T} , can be obtained through linear superposition by integrating over the source distribution weighted by an appropriate Green function:

$$P(\rho, z, \phi; m) = \iiint\limits_{V_{\mathsf{T}}} G(\rho, \rho', z, z', \phi, \phi'; m) S(\rho', z', \phi'; m) \rho' d\rho' dz' d\phi'. \tag{1}$$

The source is assumed to be narrow-band and the frequency f is suppressed for convenience. This expression is valid within an arbitrary time window indexed by an integer m and centered at $(m-1/2)\Delta t$ where the window duration, Δt , is defined to be less than the smallest correlation time associated with environmental fluctuations in the waveguide medium. The Green function can then be viewed as time-invariant within that particular time window, though the channel properties (and therefore the Green function) may vary between time windows according to some probability distribution that describes the channel statistics. It is assumed here that the Green function is known at the transmitter. A MIMO model is usually written in the form of a finite-dimensional vector-matrix equation corresponding to a finite number of sources (transmitters) and receivers, and this is discussed below. However, in Section 4 a more rigorous analysis of Eq. (1), treated as a mapping between two Hilbert spaces associated with the source and receive regions, represents an appropriate generalization for dealing with continuous acoustic fields as described by Eq. (1) and yields additional insight into the relationship between information and propagation.

A single-user, narrow-band MIMO system architecture with T transmitters and R receivers located within a waveguide is given by the following model. The transmitted signal is represented in a particular time interval m by a T-dimensional column vector $\boldsymbol{s} = \left[S_1(\boldsymbol{r}_1'; m) \ldots S_T(\boldsymbol{r}_T'; m)\right]^t$ where a superscript t denotes vector transpose. Signal vector components are treated here as uncorrelated random variables whose values can change in time, so that $\boldsymbol{s}$ describes a random vector process. From an information-theoretic viewpoint, each input node transmits a phasor that codes a symbol in the time window indexed by m. The set of phasors obtained from all the nodes can be interpreted as representing a transmitted message in this time block. While the MIMO architecture assumes a set of spatially distributed point transmitters (sources), it should be noted that this source representation can also be viewed as a discretized version of the continuous source distribution in Eq. (1) that consists of a sum of weighted delta-functions. The complex pressure field, $\boldsymbol{p}$, at the R point receivers is written in terms of an (unprimed) R - dimensional column vector

$\boldsymbol{p} = [P_1(\boldsymbol{r}_1;m)\ldots P_R(\boldsymbol{r}_R;m)]^t$. It is coupled to the sources through an R x T complex matrix $\boldsymbol{G}$ whose elements are Green functions. Including the presence of additive noise $\boldsymbol{n} = [n_1(\boldsymbol{r}_1;m)\ldots n_R(\boldsymbol{r}_R;m)]^t$ at each receiver location during each time interval, the MIMO model for an arbitrary time window can be written in terms of a linear transformation of a complex random vector process

$$\boldsymbol{p} = \boldsymbol{G}\boldsymbol{s} + \boldsymbol{n}, \tag{2}$$

where the Green function matrix is given by

$$\boldsymbol{G} = \begin{pmatrix} G_{11}(\boldsymbol{r}_1,\boldsymbol{r}_1^{'};m) & G_{12}(\boldsymbol{r}_1,\boldsymbol{r}_2^{'};m) & . & . & G_{1T}(\boldsymbol{r}_1,\boldsymbol{r}_T^{'};m) \\ G_{21}(\boldsymbol{r}_2,\boldsymbol{r}_1^{'};m) & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ G_{R1}(\boldsymbol{r}_R,\boldsymbol{r}_1^{'};m) & . & . & . & G_{RT}(\boldsymbol{r}_R,\boldsymbol{r}_T^{'};m) \end{pmatrix}. \tag{3}$$

For the rest of the paper, m will be suppressed for notational convenience, and we confine the discussion to a fixed, though arbitrary, time window unless otherwise noted. Each matrix element in Eq. (3), from an information-theoretic viewpoint, represents a path through which information can be communicated between locations $\boldsymbol{r}_j^{\prime}$ and $\boldsymbol{r}_k$ for the jk^{th} source–receiver link. The matrix elements clearly depend on the geometry of the channel as well as the environmental parameters. The rank of $\boldsymbol{G}$ determines the number of independent communication modes (degrees of freedom) between transmitters and receivers. Note that the rank is finite for the MIMO system model.

In the next section, it is seen that the noise field enters into the computation of acoustic field capacity in terms of its spatial covariance. When surface wave-generated ambient noise is present, the noise covariance exhibits off-diagonal components [15]. For calm seas, a diagonal noise covariance structure may be a more appropriate description. In the following section, the capacity in a correlated noise field is first developed and, as a special case, an analytical solution for a diagonal noise field is obtained.

3. CAPACITY OF THE ACOUSTIC FIELD IN A CORRELATED NOISE FIELD

To orient the reader, some basic comments concerning Information Theory and notation used in this section are presented in the Appendix. The field capacity is determined here through singular value decomposition of matrices associated with the Green functions and noise distribution, in conjunction with a constrained optimization performed on the mutual information. The latter quantity is obtained using the model described by Eq. (2). The constraints are associated with source power, and the optimization connects the notions of both information and energy with the acoustic field.

For clarity, a well-known expression for the mutual information, Eq. (9), is first developed; it is independent of a particular form for the matrix elements in Eq. (3), and obtained following a standard approach [2]. The derivation of the acoustic field capacity is then obtained from this expression for the case of spatially correlated ambient noise, subject to constraints on the source power. For continuous probability density functions q , the conditional mutual information associated with the model defined by Eq. (2) is defined by

$$I(\boldsymbol{s};\boldsymbol{p}|\boldsymbol{G}) = \int\int_{\mathscr{S}} q(\boldsymbol{s},\boldsymbol{p}|\boldsymbol{G}) \log_2 \left\{ \frac{q(\boldsymbol{s},\boldsymbol{p}|\boldsymbol{G})}{q(\boldsymbol{p}|\boldsymbol{G})q(\boldsymbol{s})} \right\} d\boldsymbol{s}d\boldsymbol{p} \tag{4}$$

where $\mathscr{S}$ represents the support set of the random vectors $\boldsymbol{s}, \boldsymbol{p}$. The density functions are conditioned on the Green function matrix $\boldsymbol{G}$ since it is assumed to be known, corresponding to a fixed realization of $\boldsymbol{G}$ in a particular time window. If the matrix $\boldsymbol{G}$ is stochastic (uncertain), the average (ergodic) capacity [16] can be determined by an expectation of the maximized mutual information over all realizations of $\boldsymbol{G}$. To make progress, Eq. (4) is evaluated in terms of the differential entropy h. The integrals can be identified as differential entropy terms by expanding the log term and the mutual information can be expressed as

$$I(\boldsymbol{s};\boldsymbol{p}|\boldsymbol{G}) = h(\boldsymbol{p}|\boldsymbol{G}) - h(\boldsymbol{p}|\boldsymbol{s},\boldsymbol{G}). \tag{5}$$

Substituting Eq. (2) into the second term on the right, $h(\boldsymbol{Gs}+\boldsymbol{n}|\boldsymbol{s},\boldsymbol{G}) = h(\boldsymbol{n})$ since only the noise makes the acoustic field uncertain given a transmitted signal vector $\boldsymbol{s}$ and known Green function matrix. Therefore, the mutual information for this MIMO transmission model can be written as

$$I(\boldsymbol{s};\boldsymbol{p}|\boldsymbol{G}) = h(\boldsymbol{p}|\boldsymbol{G}) - h(\boldsymbol{n}). \tag{6}$$

This is a standard result [2, 17], though here the conditioning on $\boldsymbol{G}$ is made explicit. The maximization of Eq. (6) with respect to $q(\boldsymbol{s})$, the distribution on the source function, coupled with any constraints on the source function yields the capacity of the transmitted field. In other words, "probing" over all possible source distributions subject to the constraints and picking the one that maximizes I yields the maximum rate of transmission of information by wave propagation in a channel subject to additive noise. Because $h(\boldsymbol{n})$ is determined only by the noise distribution, independent of any channel input, maximizing $I(\boldsymbol{s};\boldsymbol{p}|\boldsymbol{G})$ is equivalent to maximizing $h(\boldsymbol{p}|\boldsymbol{G})$. It is common to assume a zero mean, circularly symmetric, complex Gaussian (ZMCSCG) noise field and that representation is chosen here. Under this assumption, $\boldsymbol{n}$ is a proper Gaussian random vector whose density function is given by the standard multidimensional form [9, 18]. For all random vectors $\boldsymbol{p}$ with a given covariance matrix, the entropy of $\boldsymbol{p}$ is maximized when $\boldsymbol{p}$ is ZMCSCG since a normal distribution maximizes the entropy over all distributions with the same covariance [9, 18], implying that this is the optimal distribution on $\boldsymbol{s}$ as well. In addition, of all the additive noise processes with fixed variance, it can be shown that Gaussian noise processes result in the smallest channel capacity [17]. Therefore, an estimate of the capacity for this distribution will yield a conservative bound on the information transfer rate. For Gaussian distributions, the mutual information can be computed explicitly from the covariances of both the received acoustic field and the noise. Assuming the signal and noise are statistically independent, the received covariance obtained from Eq. (2) is given by $E\left[\boldsymbol{pp}^\dagger\right] = E\left[\boldsymbol{Gss}^\dagger\boldsymbol{G}^\dagger\right] + E\left[\boldsymbol{nn}^\dagger\right]$ where † defines the conjugate (Hermitian) transpose. This expression for the received field covariance, $\boldsymbol{K}_p = E\left[\boldsymbol{pp}^\dagger\right]$, can be written in terms of covariance matrices involving the received signal and noise vectors:

$$\boldsymbol{K_p} = \boldsymbol{G}E\left[\boldsymbol{ss}^\dagger\right]\boldsymbol{G}^\dagger + \boldsymbol{K_n} = \boldsymbol{G\Theta G}^\dagger + \boldsymbol{K_n} \tag{7}$$

or $\boldsymbol{K_p} = \boldsymbol{K_s} + \boldsymbol{K_n}$, where $\boldsymbol{\Theta} = E\left[\boldsymbol{ss}^\dagger\right]$ and $\boldsymbol{K}_s = \boldsymbol{G\Theta G}^\dagger$. For $\boldsymbol{p}$ having the above mentioned distribution, the entropy is given by $h(\boldsymbol{p}|\boldsymbol{G}) = \log_2\det(\pi e\boldsymbol{K}_p)$ where $\det(\cdot)$ denotes the determinant of the quantity in parentheses [9, 18]. Using this result and an analogous entropy expression for $\boldsymbol{K_n}$, as well as Eqs. (6) and (7), one can write the capacity as a maximization of the mutual information over all source covariance matrices subject to the constraints that the total source power, Π, is finite and that each transmitter contribution to the source power is positive semi-definite:

$$C = \max_{\boldsymbol{\Theta}:\mathrm{tr}(\boldsymbol{\Theta})\leq\Pi,(\boldsymbol{\Theta})_{ii}\geqq 0}\left[\log_2\det\left(\pi e\left(\boldsymbol{K}_s+\boldsymbol{K}_n\right)\right) - \log_2\det\left(\pi e\boldsymbol{K}_n\right)\right]. \tag{8}$$

Here, $\mathrm{tr}(\boldsymbol{\Theta})$ represents the trace of the source covariance matrix. Note that the maximization over all input distributions is accounted for in the above expression through the discussion following Eq. (6). Making use of the determinant relations [9] $\det(c\boldsymbol{A}) = c^b \det(\boldsymbol{A})$ for a square matrix of dimension b with c a scalar, $\det(\boldsymbol{A}^{-1}) = 1/\det(\boldsymbol{A})$, $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$ and Sylvesters determinant theorem, $\det(\boldsymbol{I}_M + \boldsymbol{AB}) = \det(\boldsymbol{I}_N + \boldsymbol{BA})$, where $\boldsymbol{A}$ is a MxN matrix and $\boldsymbol{B}$ is a NxM matrix yields the result

$$C = \max_{\boldsymbol{\Theta}:\mathrm{tr}(\boldsymbol{\Theta})\leq\Pi,(\boldsymbol{\Theta})_{ii}\geq 0} \left[\log_2 \det\left(\boldsymbol{I}_R + \left(\boldsymbol{G}^\dagger \boldsymbol{K}_n^{-1} \boldsymbol{G}\right)\boldsymbol{\Theta}\right)\right]. \tag{9}$$

This is a general form that will be used to compute the capacity of the system described by Eq. (2) subject to the constraints on transmitted power. Note that the second term in parentheses is analogous to a signal-to-noise-ratio (SNR), a form that can be seen more clearly by writing the determinant term as $\det(\boldsymbol{I}_T + (\boldsymbol{G\Theta G}^\dagger)\boldsymbol{K}_n^{-1})$ and invoking the special case of independent, identically distributed Gaussian noise $\boldsymbol{K}_n^{-1} = (\beta^2)^{-1}\boldsymbol{I}$ with β^2 the noise variance (power). The result gives $\det(\boldsymbol{I}_T + \frac{(\boldsymbol{G\Theta G}^\dagger)}{\beta^2})$, where the ratio represents a received signal-to-noise-power.

Continuing with the more general case of correlated noise, define $\boldsymbol{W} \equiv \boldsymbol{K}_n^{-1/2}\boldsymbol{G}$ and substitute into Eq. (9) to obtain

$$C = \max_{\boldsymbol{\Theta}:\mathrm{tr}(\boldsymbol{\Theta})\leq\Pi,(\boldsymbol{\Theta})_{ii}\geq 0} \left[\log_2 \det\left(\boldsymbol{I}_R + \left(\boldsymbol{W}^\dagger \boldsymbol{W}\right)\boldsymbol{\Theta}\right)\right]. \tag{10}$$

The matrix $\boldsymbol{W}$ can be thought of as a projection of the Green function (channel) matrix onto the noise covariance, or as a pre-whitening filter. Performing a singular value decomposition on this matrix, $\boldsymbol{W} = \boldsymbol{K}_n^{-1/2}\boldsymbol{G} = \boldsymbol{E}\boldsymbol{\Lambda}_W\boldsymbol{F}^\dagger$, where $\boldsymbol{E}$ and $\boldsymbol{F}$ contain right and left singular vectors, respectively, and $\boldsymbol{\Lambda_W}$ is a diagonal matrix containing the singular values. Using the identity $\det(\boldsymbol{I}+\boldsymbol{AB}) = \det(\boldsymbol{I}+\boldsymbol{BA})$, the capacity can be expressed in the form

$$C = \max_{\boldsymbol{\Theta}:\mathrm{tr}(\boldsymbol{\Theta})\leq\Pi,(\boldsymbol{\Theta})_{ii}\geq 0} \left[\log_2 \det\left(\boldsymbol{I}_R + \boldsymbol{\Lambda}_W^2 \boldsymbol{F}^\dagger \boldsymbol{\Theta}\boldsymbol{F}\right)\right]. \tag{11}$$

Applying Hadamard's inequality [2], $\det\boldsymbol{A} \leq \prod_{i=1}^n (\boldsymbol{A})_{ii}$, to Eq. (11) yields an upper bound on the capacity in a correlated noise field:

$$C \leq \max_{\boldsymbol{\Theta}:\mathrm{tr}(\boldsymbol{\Theta})\leq\Pi,(\boldsymbol{\Theta})_{ii}\geq 0} \left[\log_2 \prod_i \left(1 + \left(\boldsymbol{\Lambda}_W^2 \boldsymbol{F}^\dagger \boldsymbol{\Theta}\boldsymbol{F}\right)_{ii}\right)\right]. \tag{12}$$

The equality is obtained (the bound is achievable) when $\boldsymbol{F}^\dagger\boldsymbol{\Theta}\boldsymbol{F}$ is a diagonal matrix since $\boldsymbol{\Lambda_W}$ is already diagonal, comprised of singular values of $\boldsymbol{K_n}^{-1/2}\boldsymbol{G}$. This condition can be satisfied in the following manner. Defining $\boldsymbol{B} \equiv F^\dagger\boldsymbol{\Theta}F$, choose pseudo-sources $\boldsymbol{b} \equiv \boldsymbol{F}^\dagger\boldsymbol{s}$ so that $\boldsymbol{B} = E[\boldsymbol{bb}^\dagger]$ is diagonal. Equality is then obtained in Eq. (12) by choosing the elements of $\boldsymbol{b}$ as zero-mean independent Gaussian random variables. Noting that the left singular vectors $\boldsymbol{F}$ are unitary, the total power constraint can be rewritten in terms of a trace condition on $\boldsymbol{B}$. This follows from using the result that the trace of a matrix product is invariant under cyclic permutation [19], $tr(\boldsymbol{B}) = tr\left(\boldsymbol{F}^\dagger\boldsymbol{\Theta}F\right) = tr\left(\boldsymbol{FF}^\dagger\boldsymbol{\Theta}\right) = tr(\boldsymbol{\Theta}) = \Pi$. Eq. (12) can now be written as

$$C = \max_{\boldsymbol{B}:\mathrm{tr}\boldsymbol{B}\leq\Pi;(\boldsymbol{B})_{ii}\geq 0} \sum_i \log_2\left[1 + \left(\boldsymbol{\Lambda}_W^2\boldsymbol{B}\right)_{ii}\right]. \tag{13}$$

Observing that the sum represents a linear combination of concave functions, the Kuhn-Tucker conditions [20] for constrained maximization of a concave function imply that a Lagrange multiplier λ can be introduced to maximize Eq. (13) with respect to the input power distribution. Converting the logarithm from base 2 to the Napierian base, the Lagrangian, L, can be expressed in the form

$$L = \frac{1}{\ln 2}\sum_i \ln\left[1 + (\boldsymbol{\Lambda}_W^2)_{ii}(\boldsymbol{B})_{ii}\right] + \lambda \sum_i ((\boldsymbol{B})_{ii} - \Pi) \tag{14}$$

with the last term representing the total energy constraint. Now evaluate $\partial L/\partial B_{ii} = 0$, since the maximization is with respect to the distribution of source power. Differentiation yields the intermediate result $\left[(\ln 2)\left(1 + (\boldsymbol{\Lambda}_W^2)_{ii}(\boldsymbol{B})_{ii}\right)\right]^{-1}(\boldsymbol{\Lambda}_W^2)_{ii} + \lambda = 0$ and solving for the diagonal components of $\boldsymbol{B}$ gives

$$(\boldsymbol{B})_{ii} = \varepsilon - \frac{1}{\left(\boldsymbol{\Lambda}_W^2\right)_{ii}} \tag{15}$$

where $\varepsilon \equiv -1/(\lambda \ln 2)$ is the scaled Lagrange multiplier. The second constraint is included by writing Eq. (15) in the form

$$(\boldsymbol{B})_{ii} = \left[\varepsilon - \frac{1}{\left(\boldsymbol{\Lambda}_W^2\right)_{ii}}\right]^+ \equiv \max\left\{0, \varepsilon - \frac{1}{\left(\boldsymbol{\Lambda}_W^2\right)_{ii}}\right\}. \tag{16}$$

The final step in the capacity calculation is to determine the (scaled) Lagrange multiplier ε; this parameter is obtained from Eq. (16) using the total energy constraint $\sum_i (\boldsymbol{B})_{ii} = \Pi$, yielding

$$\Pi = \sum_{r=1}^{J}\left[\varepsilon - \frac{1}{\left(\boldsymbol{\Lambda}_W^2\right)_{rr}}\right]^+ = J\varepsilon - \sum_{r=1}^{J}\frac{1}{\left(\boldsymbol{\Lambda}_W^2\right)_{rr}}. \tag{17}$$

The singular values in this expression are ordered from largest to smallest and the summation is over all relevant degrees of freedom J, where J is determined by waterfilling [21]. Solving for ε, one obtains

$$\varepsilon = \frac{1}{J}\left[\Pi + \sum_{r=1}^{J}\frac{1}{\left(\boldsymbol{\Lambda}_W^2\right)_{rr}}\right]. \tag{18}$$

Substitution of Eqs. (18) and (15) yields

$$(\boldsymbol{B})_{ii} = \frac{1}{J}\left[\Pi + \sum_{r=1}^{J}\frac{1}{\left(\boldsymbol{\Lambda}_W^2\right)_{rr}}\right] - \frac{1}{\left(\boldsymbol{\Lambda}_W^2\right)_{ii}} \tag{19}$$

and the final expression for the capacity of the acoustic field in a correlated surface noise field is given by combining Eq. (13) and Eq. (19):

$$C = \sum_{i=1}^{J} \log_2 \left(\frac{\left(\mathbf{\Lambda}_W^2\right)_{ii}}{J} \left[\Pi + \sum_{r=1}^{J} \frac{1}{\left(\mathbf{\Lambda}_W^2\right)_{rr}} \right] \right). \tag{20}$$

Recalling that $\boldsymbol{K}_n^{-1/2}\boldsymbol{G} = \boldsymbol{E}\mathbf{\Lambda}_W\boldsymbol{F}^\dagger$, the SVD of the Green function, $\boldsymbol{U}\mathbf{\Lambda}\boldsymbol{V}^\dagger$, can be substituted in this expression and then the diagonal matrix $\mathbf{\Lambda}_W$ is obtained through left and right multiplication by the unitary matrices $\boldsymbol{E}^\dagger$ and $\boldsymbol{F}$ respectively, giving the singular values

$$\mathbf{\Lambda}_W = \boldsymbol{E}^\dagger\boldsymbol{K}_n^{-1/2}\boldsymbol{U}\mathbf{\Lambda}\boldsymbol{V}^\dagger\boldsymbol{F}. \tag{21}$$

Therefore, singular vectors and singular values associated with the channel matrix and noise covariance completely specify the field capacity for an arbitrary time window whose duration is less than the shortest correlation time related to environmental variability in the waveguide.

4. EXAMPLE: FIELD CAPACITY IN UNCORRELATED NOISE

In the previous section, a rather general expression for the field capacity was obtained in terms of singular vectors and singular values associated with the Green function and noise covariance matrices. Additional insight into the relationship between information capacity and the acoustic field can be obtained by giving an explicit example for a special case where an analytical solution can be constructed. Such a calculation is performed here for a shallow water waveguide with a constant sound speed and penetrable, absorptive bottom. In this example, the noise field is chosen to be uncorrelated with equal noise power β^2 distributed along the diagonal: $\boldsymbol{K}_n = \beta^2\boldsymbol{I}$. Substituting $\boldsymbol{K}_{\boldsymbol{n}}^{-1/2}$ into Eq. (21) yields $\boldsymbol{E}\mathbf{\Lambda}_W\boldsymbol{F}^\dagger = \boldsymbol{U}\frac{\Lambda}{\beta}\boldsymbol{V}^\dagger$ where it is clear that, in this case, $\mathbf{\Lambda}_W$ reduces to the diagonal matrix containing the singular values, σ_i, of the Green function matrix scaled by β^{-1}. Substituting this result into the general expression for the capacity, Eq. (20) gives

$$C = \sum_{i=1}^{J} \log_2 \left(\frac{\sigma_i^2}{\beta^2 J} \left[\Pi + \beta^2 \sum_{r=1}^{J} \frac{1}{\sigma_r^2} \right] \right). \tag{22}$$

In this expression, the singular values are ordered from largest to smallest. The value of J is determined by a waterfilling procedure [21] which distributes the source power depending on the noise level on each channel. Given the source power and noise power, the acoustic field capacity is determined once the singular values of the Green function matrix are obtained.

4.1 Representation of the Green Function in Terms of Singular Vectors and Singular Values

To obtain an explicit expression for the singular values in Eq. (22), the pressure field in Eq. (1) is expressed in terms of a singular value decomposition and a comparison is made between two equivalent expressions for the acoustic field. The received field in V_T is considered in terms of a mapping between two Hilbert spaces, and the Green function operator is written as a singular value decomposition. To proceed, a useful approach is to interpret Eq. (1) as an operator equation $P = [\mathbb{G}S](\boldsymbol{r})$ where $\mathbb{G} \equiv \int\int\int G$ is an integral operator with a Green function kernel, mapping acoustic source functions in V_{T} into functions representing the pressure field P in V_{R} at arbitrary field points $\boldsymbol{r} = (\rho, z, \phi)$. Mathematically, Eq. (1) is viewed as a linear mapping between two infinite-dimensional Hilbert spaces $\mathscr{H}_S, \mathscr{H}_P$ whose elements (vectors) comprise the set of source and field functions, respectively. The source (transmitting) region and the field (receiving) region are chosen as disjoint in order to avoid singularities in the Green function; under this condition, the Green function is analytic [3, 22]. For this analytic kernel, the operator is bounded and compact [23, 24]. It is known that a bounded, infinite-dimensional linear operator $\mathbb{G}\colon \mathscr{H}_S \to \mathscr{H}_P$ can always be represented by a unique infinite-dimensional matrix with respect to a prescribed basis [24]. For each of these distinct spaces $\mathscr{H}_S, \mathscr{H}_P$, one can define eigen-bases denoted as $\{\Phi_1, \Phi_2 \cdots\}$ and $\{\Psi_1, \Psi_2 \cdots\}$, respectively. Therefore, one can express the pressure and source distributions through the basis expansions $P = \sum_{i=1}^{\infty} < P, \Psi_i > \Psi_i$ and $S = \sum_{i=1}^{\infty} < S, \Phi_i > \Phi_i$ where the inner products $<,>$ denote generalized Fourier coefficients defined by $\langle f, g \rangle \equiv \iiint f g d^3\boldsymbol{r}$. Forming an inner product of both sides of the operator equation with respect to arbitrary orthogonal basis vectors Ψ_j for P, one can write the inner product as $< P, \Psi_j > = < \mathbb{G}\left[\sum_i < S, \Phi_i > \Phi_i\right], \Psi_j > = \sum_i < S, \Phi_i >< \mathbb{G}\Phi_i, \Psi_j >$ where S is expressed in arbitrary orthogonal basis Φ_i. The result can be written as an infinite-dimensional matrix equation for the source and field spectral coefficients [23, 24]:

$$\begin{pmatrix} < P, \Psi_1 > \\ < P, \Psi_2 > \\ \vdots \end{pmatrix} = \begin{pmatrix} < \mathbb{G}\Phi_1, \Psi_1 > & < \mathbb{G}\Phi_2, \Psi_1 > & \dots \\ < \mathbb{G}\Phi_1, \Psi_2 > & < \mathbb{G}\Phi_2, \Psi_2 > & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} < S, \Phi_1 > \\ < S, \Phi_2 > \\ \vdots \end{pmatrix} \tag{23}$$

where the $(l, m)th$ matrix element is given by integrals over the source and receive volumes,

$$< \mathbb{G}\Phi_l, \Psi_m > = \int\int\int \Psi_m^*(\boldsymbol{r}) G(\boldsymbol{r}, \boldsymbol{r}') \Phi_l(\boldsymbol{r}') d^3\boldsymbol{r} d^3\boldsymbol{r}'. \tag{24}$$

Note that Eq. (23) can be interpreted in the communication framework as well, since knowledge of the spectral coefficients is equivalent to specifying the acoustic field. The matrix elements then link source and field expansion coefficients in arbitrary bases, determining how much information in the source is transmitted to the receive region by propagation [22, 23].

The matrix in Eq. (23) can be diagonalized by specifying an appropriate basis set. This is achieved here by choosing as bases the right and left orthonormal singular vectors ($\Phi_i \to \nu_i, \Psi_i \to \psi_i$) of $\mathbb{G}$. Using orthogonality and the fact that [24] $\mathbb{G}\nu_i = \sigma_i \psi_i$, the matrix then becomes diagonal since $< \sigma_i \psi_i, \psi_j > = \sigma_i < \psi_i, \psi_j > = \sigma_i \delta_{ij}$. For this choice of bases, the expansion coefficients are related by

$$< P, \psi_i > = \sigma_i < S, \nu_i > \tag{25}$$

where σ_i is the ith diagonal element of $\boldsymbol{\Lambda}$. This set of uncoupled equations can be written in vector form as

$$\hat{\boldsymbol{p}} = \boldsymbol{\Lambda}\hat{\boldsymbol{s}} + \hat{\boldsymbol{n}}. \tag{26}$$

Comparing this expression with Eq. (2), it is seen that they have the same form, but the diagonalization of the Green function leads to an infinite dimensional set of uncoupled equations. Furthermore, using Eq. (25)

we find that the acoustic field can be written in terms of a singular value decomposition of $\mathbb{G}$:

$$P(\rho,z,\phi) = [\mathbb{G}S](\boldsymbol{r}) = \sum_{i=1}^{\infty} \sigma_i < S(\rho',z',\phi'), \nu_i(\rho',z',\phi') > \psi_i(\rho,z,\phi). \tag{27}$$

Therefore, the field is expressed as a linear combination of left singular functions weighted by projections of the source distribution onto the right singular vectors. It will sometimes be helpful to modify this result slightly by introducing a multiple index format rather than the single index scheme used in Eq. (27). This modification is just the opposite of "vectorizing" a matrix, where vectorization involves writing a matrix in a vector format by concatenating column vectors comprising the matrix. One can then express Eq. (27) equivalently as

$$P(\rho,z,\phi) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} \sigma_{m,n} \left\langle S(\rho',z',\phi'), \nu_{m,n}(\rho',z',\phi') \right\rangle \psi_{m,n}(\rho,z,\phi). \tag{28}$$

This form facilitates the explicit identification of the singular values and singular vectors for the choice of waveguide environment; it is seen below that the multiple indices refer to depth and azimuthal eigenmodes associated with the propagated field. To explicitly identify each of the terms in this equation, a free-space solution method [22] is extended here to a bounded medium (here, an ocean waveguide). The starting point in this computation is to obtain a solution of the inhomogeneous Helmholtz equation in cylindrical coordinates. The Green function satisfies the differential equation

$$\left[\nabla^2 + k^2(z)\right] G(\rho,\rho',z,z',\phi,\phi') = \frac{\delta(\rho-\rho')}{\rho'} \delta(z-z')\delta(\phi-\phi') \tag{29}$$

and in cylindrical coordinates can be written in the form

$$G(\rho,\rho',z,z',\phi,\phi') = \sum_{n=1}^{\infty} u_n(z')u_n(z)H_0^{(1)}(k_n|\boldsymbol{\rho}-\boldsymbol{\rho'}|) \tag{30}$$

where $\boldsymbol{\rho} \equiv (\rho,\phi)$ and $|\boldsymbol{\rho}-\boldsymbol{\rho'}| = \sqrt{\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')}$ by the law of cosines. The depth-dependent eigenfunctions are $u_n(z)$, and $H_0^{(1)}$ is a zeroth order Hankel function of the first kind. Applying Graf's theorem [25] (the addition theorem for Hankel functions), the Green function for a point source not located at the coordinate origin can be referenced with respect to the coordinate origin and expressed as

$$G(\rho,\rho',z,z',\phi,\phi') = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} u_n(z')u_n(z)J_m(k_n\rho')H_m^{(1)}(k_n\rho)e^{im(\phi-\phi')} \tag{31}$$

where J_m is the m^{th} order Bessel function and $H_m^{(1)}$ is the m^{th} order Hankel function of the first kind. After substituting Eq. (31) into Eq. (1) and rearranging terms, one obtains

$$P(\rho,z,\phi) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \left[\iiint_T S(\rho',z',\phi')u_n(z')J_m(k_n\rho')e^{-im\phi'}\rho' d\rho' dz' d\phi' \right] \cdot \left[u_n(z)H_m^{(1)}(k_n\rho)e^{im\phi}\right] \tag{32}$$

where the depth-dependent normal modes $u_n(z)$ need to be determined explicitly for a particular sound speed distribution and boundary conditions to allow identification of the singular values by comparison with Eq. (28). Since the right and left singular vectors, $(\nu_{m,n}, \psi_{m,n})$, are ortho-normal, Eq. (32) must be equivalent to Eq. (28) once the functions in Eq. (32) are ortho-normalized. The singular vectors satisfy $\int\int\int |\nu_{m,n}|^2 d^3\boldsymbol{r}'$ =1 and $\int\int\int |\psi_{m,n}|^2 d^3\boldsymbol{r}$ =1. They are given by

$$\nu_{m,n}(\rho',z',\phi') = \frac{u_n(z')J_m(k_n\rho')e^{-im\phi'}}{\left[\int_{\rho'=0}^{R_s}\int_{z'=0}^{\infty}\int_{\phi'=0}^{2\pi} u_n^2(z')\left[J_m\left(k_n\rho'\right)\right]^2\rho' d\rho' dz' d\phi'\right]^{1/2}} \tag{33}$$

and

$$\psi_{m,n}(\rho,z,\phi) = \frac{u_n(z)H_m^{(1)}(k_n\rho)e^{im\phi}}{\left[\int_{\rho=R_s}^{\infty}\int_{z=0}^{\infty}\int_{\phi=0}^{2\pi} u_n^2(z)\left|H_m^{(1)}(k_n\rho)\right|^2\delta\left(\rho-R_{cyl}\right)\rho d\rho dz d\phi\right]^{1/2}}. \tag{34}$$

The denominators contain iterated integrals that can be determined separately. The radial distance containing the source distribution is R_s . Referring to Fig. 1, the left singular vectors are evaluated on the surface of a cylinder exterior to the source region at a range $\rho = R_{cyl}$ so that in this example, the field is evaluated on a surface rather than within a volume. The angular integral contribution yields just a factor of $\sqrt{2\pi}$ because of assumed horizontal isotropy in the sound speed field. Now consider the integral over depth-dependent eigenfunctions appearing in both normalizations. For a uniform sound speed field in the water column and a penetrable bottom comprised of a half space (see Fig. 1), the un-normalized eigenfunctions in the two layers (water column and bottom) are given by [26]

$$\begin{aligned} u_{n1}(z') &= \sin\left(k_{zn}z'\right) \quad &; z' \leq H, \\ u_{n2}(z') &= \sin(k_{zn}H)e^{-\chi_n(z'-H)} \quad &; z' \geq H, \end{aligned} \tag{35}$$

where $(\frac{\omega}{c_1})^2 - k_n^2 = k_{zn}^2$ and $(\frac{\omega}{c_2})^2 - k_n^2 = -\chi_n^2$ and k_n is the horizontal wavenumber. The normalization integral over depth can be written in the form

$$D = \left[\int_{z'=0}^{H} u_{n1}^2(z')dz' + \left(\frac{\mu_1}{\mu_2}\right)\int_{z'=H}^{\infty} u_{n2}^2(z')dz'\right] \tag{36}$$

and by straightforward evaluation of the integrals using Eq. (35) and the boundary conditions at the bottom interface, $u_{n1}(H) = u_{n2}(H)$, $\frac{1}{\mu_1}\left(\frac{du_{n1}}{dz}\right)_{z=H} = \frac{1}{\mu_2}\left(\frac{du_{n2}}{dz}\right)_{z=H}$ to eliminate χ_n yields

$$D = \frac{k_{zn}H - \sin(k_{zn}H)\cos(k_{zn}H) - \left(\frac{\mu_1}{\mu_2}\right)^2\sin^2(k_{zn}H)\tan\left(k_{zn}H\right)}{2k_{zn}}. \tag{37}$$

Next, consider the integral over range involving the Bessel function in Eq. (33). This integral is given by [27]

$$\int_{\rho'=0}^{R_s}\left[J_m\left(k_n\rho'\right)\right]^2\rho' d\rho' = \left[\frac{(\rho')^2}{2}\left[\left[J_m(k_n\rho')\right]^2 - J_{m-1}\left(k_n\rho'\right)J_{m+1}\left(k_n\rho'\right)\right]\right]_{\rho'=0}^{\rho'=R_s}. \tag{38}$$

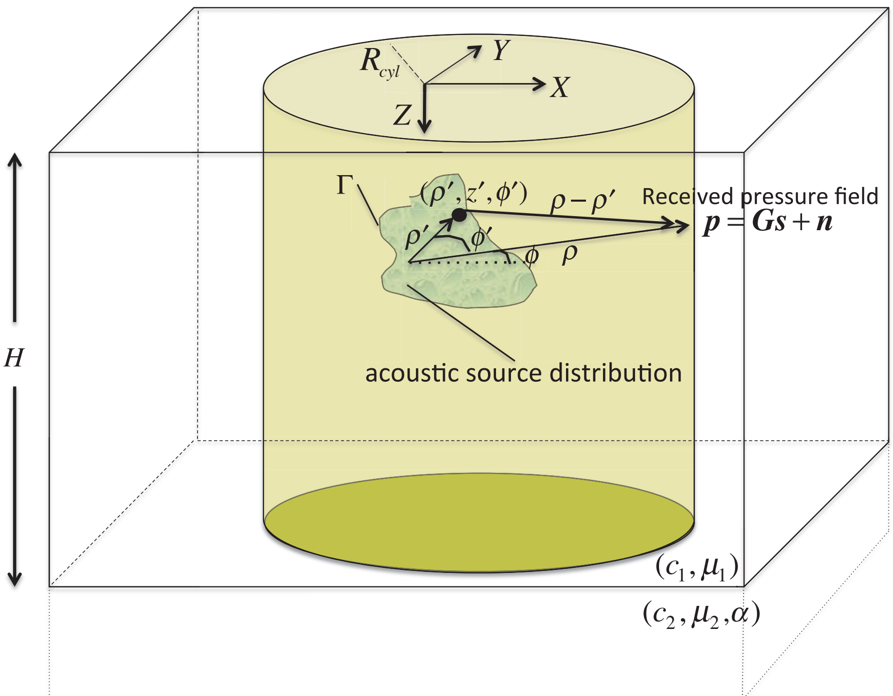

Fig. 1 — Waveguide configuration and cylindrical coordinate system used to evaluate the acoustic field and capacity on a cylindrical surface exterior to an arbitrary source distribution

Combining these results, the normalization coefficients, $a_{m,n}$, for the right singular vector $v_{m,n}$ in Eq. (33) can be written as

$$a_{m,n} = \begin{array}{l} \left[\dfrac{2k_{zn}}{k_{zn}H - \sin(k_{zn}H)\cos(k_{zn}H) - \left(\frac{\mu_1}{\mu_2}\right)^2 \sin^2(k_{zn}H)\tan(k_{zn}H)} \right]^{1/2} \\ \cdot \frac{1}{\sqrt{\pi}R_s} \left[[J_m(k_n R_s)]^2 - J_{m-1}(k_n R_s) J_{m+1}(k_n R_s) \right]^{-1/2} \end{array} . \tag{39}$$

To evaluate this expression explicitly, one also needs the eigenvalues k_{zn} and radial wave numbers k_n ; these are determined by solving the following transcendental equation resulting from the second boundary condition

$$\tan(k_{zn}H) = -\frac{\mu_2}{\mu_1} \frac{k_{zn}}{\sqrt{\left(\frac{\omega}{c_1}\right)^2 - \left(\frac{\omega}{c_2}\right)^2 - k_{zn}^2}} \tag{40}$$

and substituting the roots from its numerical solution into the expression for the horizontal wave numbers

$$k_n = \sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_{zn}^2}. \tag{41}$$

Therefore, the right singular vectors are given by

$$v_{m,n}(\rho', z', \phi') = a_{m,n} u_n(z') J_m(k_n \rho') e^{-im\phi'} \tag{42}$$

with m,n corresponding to azimuthal and depth eigenvalue indices.

Finally, consider the left singular vectors $\psi_{m,n}$, where the only new iterated integral in the normalization factor is given by $\int \left|H_m^{(1)}(k_n\rho)\right|^2 \delta\left(\rho - R_{cyl}\right)\rho d\rho$ involving the Hankel function, evaluated on the lateral surface of the cylinder where the field is received external to the source region. The normalization factors for the left singular vector are then given by

$$b_{m,n} = \left[\frac{2k_{zn}}{k_{zn}H - \sin(k_{zn}H)\cos(k_{zn}H) - \left(\frac{\mu_1}{\mu_2}\right)^2 \sin^2(k_{zn}H)\tan(k_{zn}H)}\right]^{1/2} \cdot \frac{1}{\sqrt{2\pi}\left|H_m^{(1)}(k_n R_{cyl})\right|} \tag{43}$$

giving the left singular vectors explicitly as

$$\psi_{m,n}(\rho, z, \phi) = b_{m,n} u_n(z) H_m^{(1)}(k_n\rho) e^{im\phi}. \tag{44}$$

Using the normalization factors in Eq. (39) and Eq. (43) and comparing the equivalent expressions in Eqs. (28) and (32), the singular values $\sigma_{m,n}$ are given by $(a_{m,n}b_{m,n})^{-1}$ as

$$\begin{aligned}\sigma_{m,n} = \;& \frac{\pi R_s}{\sqrt{2}k_{zn}}\left[k_{zn}H - \sin(k_{zn}H)\cos(k_{zn}H) - \left(\frac{\mu_1}{\mu_2}\right)^2 \sin^2(k_{zn}H)\tan(k_{zn}H)\right] \\ & \cdot \left|H_m^{(1)}(k_n R_{cyl})\right| \left[\left[J_m(k_n R_s)\right]^2 - J_{m-1}(k_n R_s) J_{m+1}(k_n R_s)\right]^{1/2}.\end{aligned} \tag{45}$$

This result is now extended to include the effect of bottom absorption by generalizing the wave vector in the bottom, $k_2 = 2\pi f/c_2$, to represent a complex variable with absorption coefficient [28] defined by α. One can then write $k_2 = \frac{\omega}{c_2} \to k_2\left[1 + i\frac{\alpha}{2}\right] = \frac{2\pi f}{c_2}\left[1 + \frac{i\alpha}{2}\right]$. To proceed, it is convenient to slightly rewrite the transcendental Eq. (40) using $\left(\frac{\omega}{c_1}\right)^2 = k_1^2 = k_n^2 + k_{zn}^2$, letting $k_n = \xi$ and keeping only linear terms in α to obtain the dispersion relation

$$\tan\left(H\sqrt{k_1^2 - \xi^2}\right) = \frac{-\mu_2}{\mu_1}\frac{\sqrt{k_1^2 - \xi^2}}{\sqrt{\xi^2 - k_2^2(1 + i\alpha)}}. \tag{46}$$

Because of absorption, ξ is also complex and is defined here as $\xi \equiv q + i\gamma/2$ where γ is the modal attenuation coefficient and $\gamma_n \ll \xi_n$ for all roots ξ_n of Eq. (46). This dispersion relation is now expanded in a Taylor series in α and γ , keeping only first order terms in each variable. After equating the real parts of the expansion one obtains [28]

$$\tan\left(H\sqrt{k_1^2 - q^2}\right) = \frac{-\mu_2}{\mu_1}\frac{\sqrt{k_1^2 - q^2}}{\sqrt{q^2 - k_2^2}} \tag{47}$$

where the eigenvalues q_n determined by solving this equation (the dispersion relationship for a Pekeris waveguide without absorption, Eq. (40)) are constrained by $k_2 \leq q_n \leq k_1$. The modal attenuation coefficients, γ_n , are then obtained [28] by equating the imaginary parts of the expansion, giving

$$\gamma_n = \frac{\alpha k_2^2 \frac{\mu_2}{\mu_1}\left(k_1^2 - q_n^2\right)}{2q_n\sqrt{q_n^2 - k_2^2}\left[q_n^2 - k_2^2 + \left(\frac{\mu_2}{\mu_1}\right)^2\left(k_1^2 - q_n^2\right)\right]\left[\frac{H}{2} + \frac{\left(k_1^2 - k_2^2\right)\frac{\mu_2}{\mu_1}}{2\sqrt{q_n^2 - k_2^2}\left[q_n^2 - k_2^2 + \left(\frac{\mu_2}{\mu_1}\right)^2\left(k_1^2 - q_n^2\right)\right]}\right]}. \tag{48}$$

This expression, which is frequency dependent, is evaluated in terms of the solution of Eq. (47) for the eigenvalues $q = q_n$ of the unattenuated case.

4.2 Numerical Results

The acoustic field capacity in uncorrelated noise can now be obtained from Eqs. (22), (45), (47), and (48). It is instructive to start with a discussion of the singular values $\sigma_{m,n}$ given by Eq. (45) whose squared values appear in the capacity equation. As a baseline reference, a case without bottom absorption is considered first. The corresponding (squared) spectrum is illustrated in Fig. 2 for a frequency of $f = 600$ Hz and water depth $H = 200$ m as a function of vertical and azimuthal mode numbers n and m, respectively, with $R_s = 50$ m and $R_{cyl} = 15$ km. The uniform sound speed in the water column and sediment are 1500 m/s and 1677 m/s, respectively, and the sediment density is 1.83 kg/m^3. To numerically compute the spectrum, a second-order expansion [29] of the Hankel function was used for large z. Exterior to the source region the singular values in Eq. (45) fall off rather sharply as a function of azimuthal mode number [22], a result well known in the inverse scattering literature [30]. The "transition" occurs in the vicinity of the azimuthal mode number $m_{critical} = k_n R_s \approx 2\pi R_s/\lambda$, representing the integer number of acoustic wavelengths that can fit around the perimeter of the cylinder enclosing the source distribution. It is an expression of the fact that the number of azimuthal modes supported by the waveguide is finite and does not depend on the vertical mode structure. This transition, observed in numerous cases (not shown here) involving different values of R_s , implies that the number of degrees of freedom, J, involved in determining the capacity is finite. To further examine this point, consider the difference between the exact pressure field P_{exact} and a representation of the field P_{rep} with respect to some norm:

$$||P_{exact} - P_{rep}|| \leq \varepsilon. \tag{49}$$

For example, the representation could be a basis expansion fit to experimental data. Real measurements are affected by ambient noise, environmental variability, and other factors, so that it is reasonable to consider two fields as indistinguishable [3] if their difference, in a suitable norm, is below resolution ε for any spatial location where the field is measured. This statement also implies that the difference in either "information"

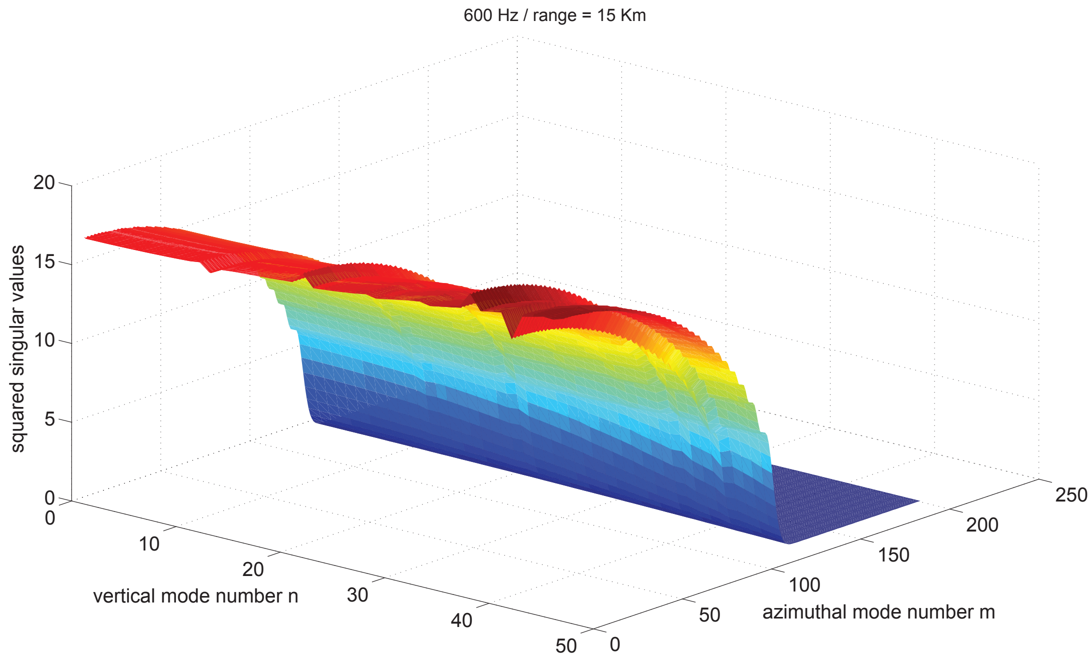

Fig. 2 — Squared singular value spectrum evaluated at a frequency of 600 Hz, water depth 200 m, and range of 15 km as a function of azimuthal and vertical mode numbers for the case of no bottom attenuation

content or capacity between P_{exact} and P_{rep} is indistinguishable below ε. Because of the cutoff of the singular values, only a finite number of them are necessary to represent both the infinite-dimensional operator $\mathbb{G}$ as a singular value decomposition and the capacity of the field. Consequently, the number of terms in the capacity expression given by either Eq. (20) or Eq. (22) is finite. It also follows that only a finite number of basis functions are required to specify the information capacity to resolution ε. Turning to the case when bottom absorption is included, modal attenuation alters the spectrum as a function of vertical wavenumber, but does not qualitatively affect the above conclusion concerning the representation of either the field or information by a finite number of degrees of freedom. The spectrum corresponding to the absorbing bottom case is illustrated in Fig. 3, where the parameters used in the computation of the attenuation coefficient in Eq. (48) are for a bottom composed of silt [31] (loss tangent 0.036) and the other parameters are the same as for the baseline case when bottom absorption is absent. The transition still occurs near azimuthal mode number $m_{critical}$ but the relative importance of the singular values now depends strongly on vertical mode number with the singular values decreasing as vertical mode number increases, as expected.

The procedure for determining the optimal value of J and capacity involves the following steps and is illustrated in Fig. 4 where, for generality, the noise power is allowed to vary between channels. Given the normalized noise power on the ith channel, β_i^2/σ_i^2, sequentually distribute the source power over the channels so that the power approaches its maximum value on any channel. If the normalized noise level is too high (e.g., channel 3 in Fig. 4), then no power is transmitted through that particular channel. Repeat this power allocation procedure until the total source power, Π, has been distributed across the useable

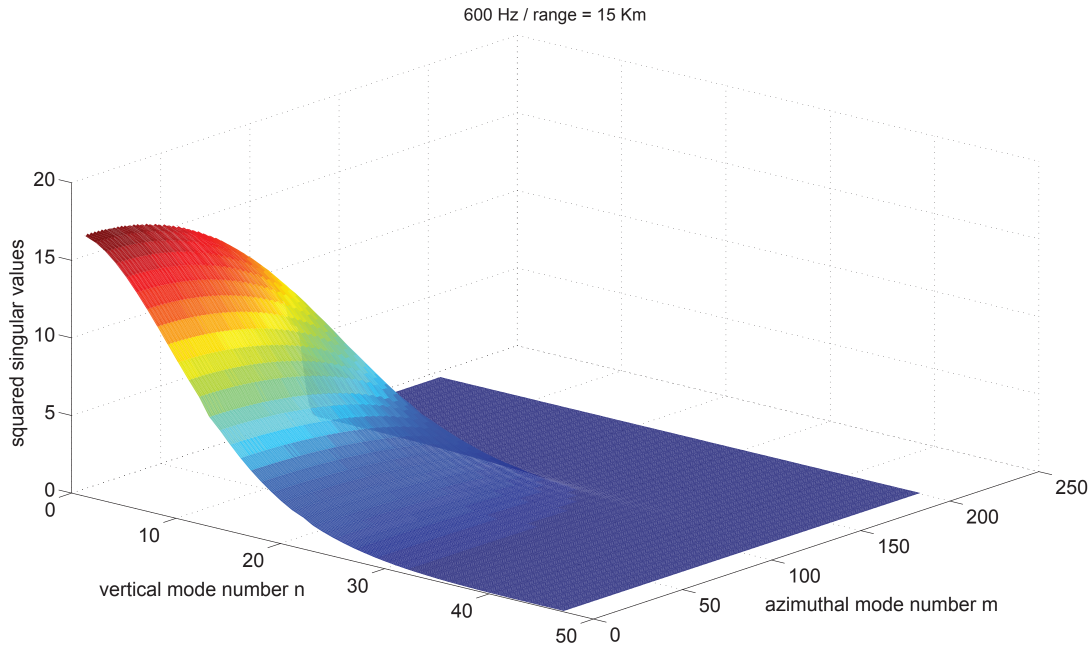

Fig. 3 — Squared singular value spectrum evaluated at a frequency of 600 Hz, water depth 200 m, and range of 15 km as a function of azimuthal and vertical mode numbers when bottom attenuation is present

channels. Using this waterfilling approach and assuming the same noise level on each channel, Eq. (22), capacity results are presented in Figs. 5 and 6 at selected frequencies between 200 and 800 Hz for two water depths, 100 m and 200 m. The results inFig. 5 correspond to a high SNR (source power 180 dB, noise power 60 dB) and in Fig. 6 to a lower SNR (source power 80 dB and noise power 60 dB). It is clear from Fig. 5 that capacity values at the four ranges considered increase with frequency and water depth. The number of azimuthal and depth modes increases with frequency as well, implying an increase in the number of degrees of freedom available for information transmisision and these extra degrees of freedom are interpreted as the cause of this trend. There is a trend toward decreasing capacity as a function of range, though at 200 Hz the capacity is approximately constant over range. For a given frequency of transmission, the gap between capacity values for the two water depths decreases as frequency decreases. For the low SNR case, the trend toward decreased capacity as a function of range is stronger but, interestingly, there is no longer a systematic drop in capacity from high to low frequency. An alternative way of viewing the results can be obtained through normalizing the capacity by the (range-dependent) area of the receiving surface, resulting in an average capacity per unit area at a particular frequency. These results are shown in Figs. 7 and 8 for the same SNR values.

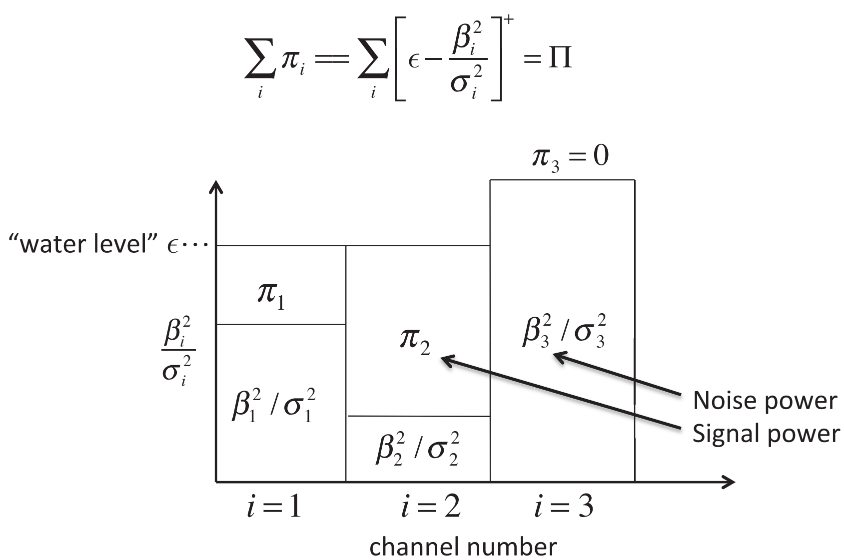

Fig. 4 — Illustration of the waterfilling procedure used to evaluate Eq. (22)

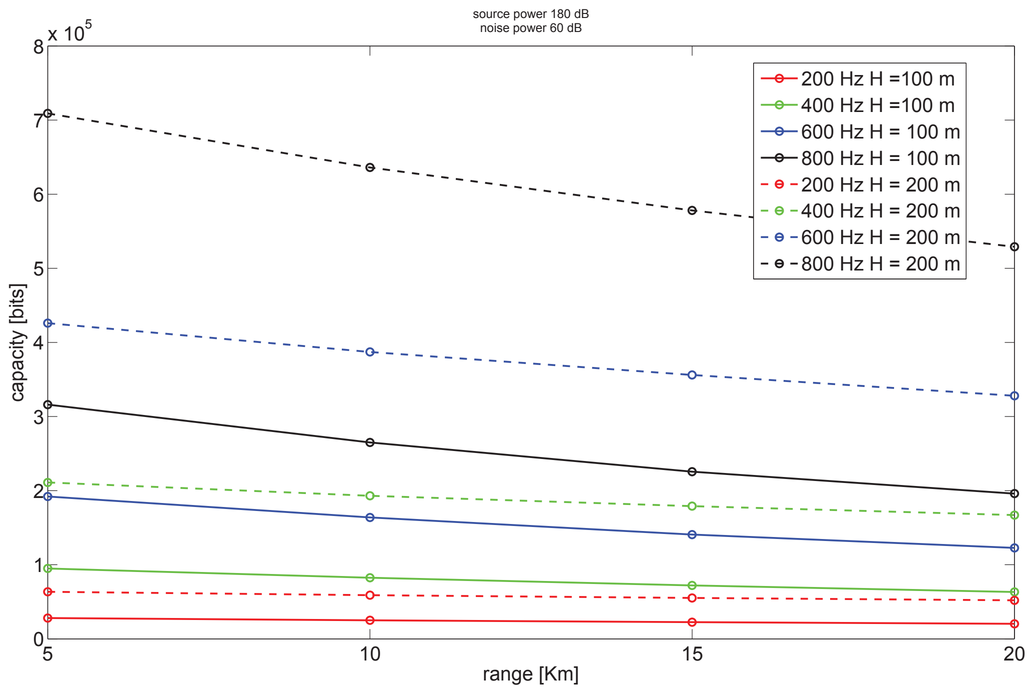

Fig. 5 — Field capacity on the cylindrical surfaces obtained from Eq. (22) in bits per channel use for source power 180 dB, noise power 60 dB at acoustic frequencies of 200, 400, 600, and 800 Hz for water depths of 100 m (solid lines) and 200 m (dotted lines)

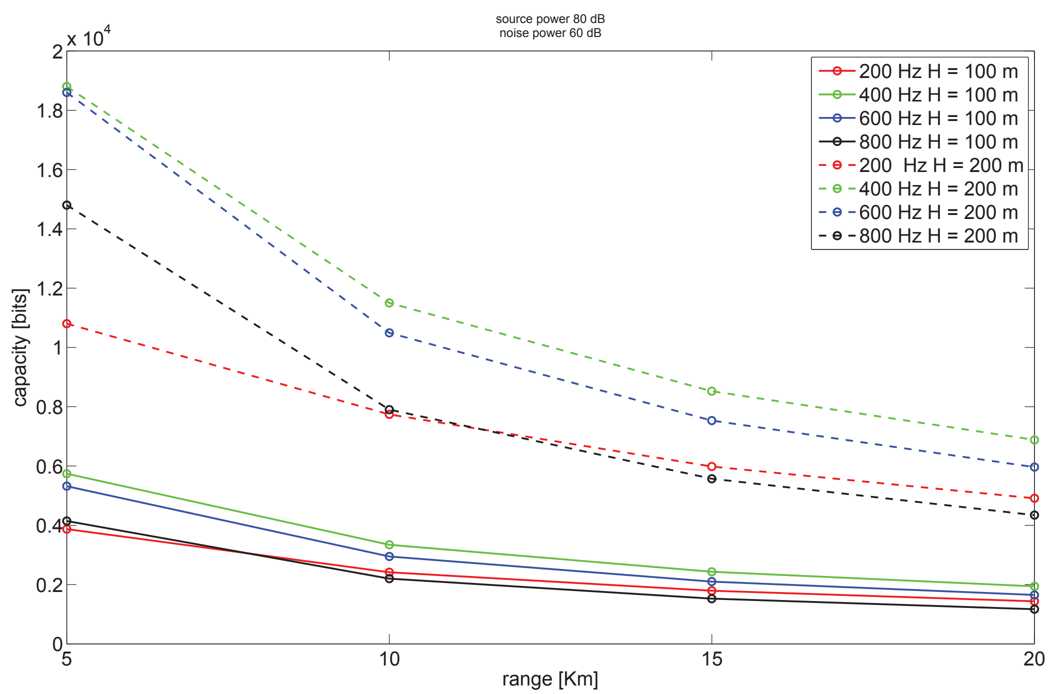

Fig. 6 — Field capacity on the cylindrical surfaces obtained from Eq. (22) in bits per channel use for source power 80 dB, noise power 60 dB at acoustic frequencies of 200, 400, 600, and 800 Hz for water depths of 100 m (solid lines) and 200 m (dotted lines)

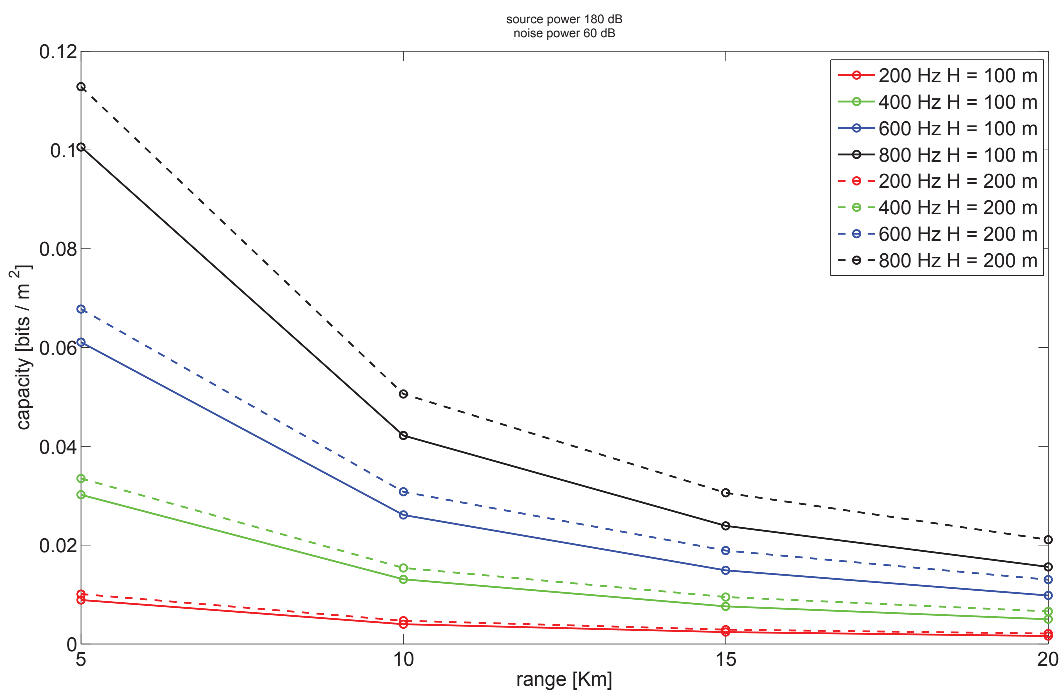

Fig. 7 — Normalized capacity for source power 180 dB, noise power 60 dB at acoustic frequencies of 200, 400, 600, and 800 Hz for water depths of 100 m (solid lines) and 200 m (dotted lines)

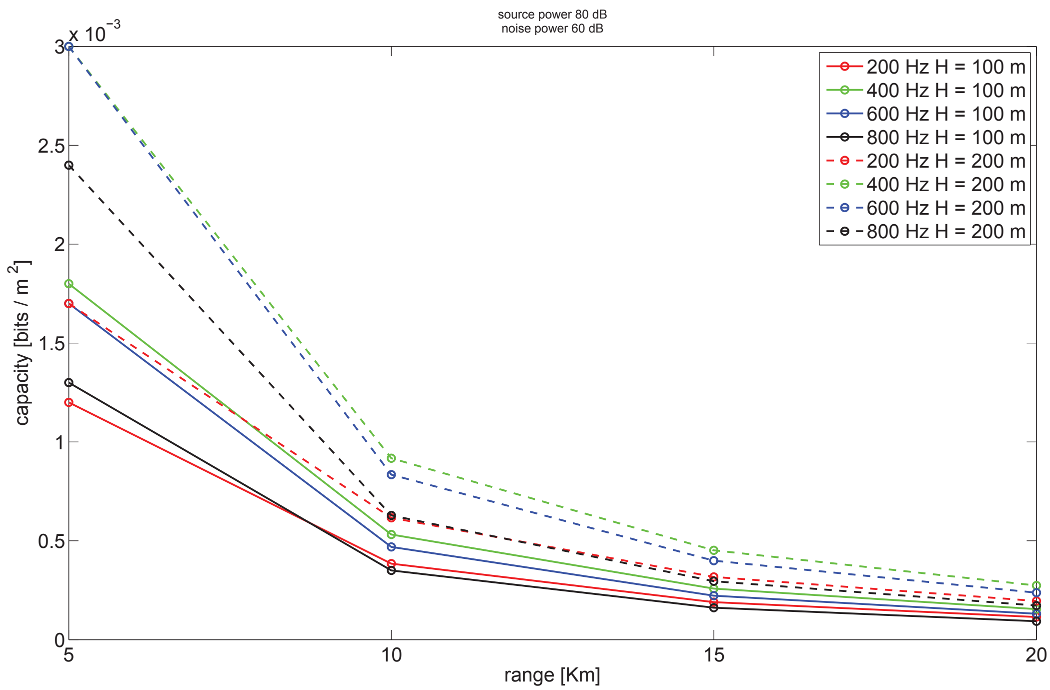

Fig. 8 — Normalized capacity for source power 80 dB, noise power 60 dB at acoustic frequencies of 200, 400, 600, and 800 Hz for water depths of 100 m (solid lines) and 200 m (dotted lines)

5. SUMMARY AND CONCLUSIONS

In this report, a discussion of the rationale for obtaining bounds on the performance of distributed underwater sensor networks led naturally to an information-theoretic decomposition of this basic problem into two parts. We specifically considered the first part in detail by addressing the question of how much information, in the Shannon sense, is potentially available or can be transferred to a region of space by an acoustic field that was propagated from an arbitrary controllable source distribution through a noisy ocean waveguide environment. From an information-theoretic viewpoint, any spatial configuration of sensors placed in this region would be subject to this bound, which was derived from constraints imposed by the physics of propagation.

The solution to this problem involved computation of the acoustic field capacity in a bounded (waveguide) environment, where the capacity represents the maximum rate of information transfer through a noisy channel with asymptotically negligible probability of error. The capacity of the field was computed on a cylindrical surface exterior to the source region, so that placement of any sensor nodes would be constrained to that surface. This constraint is not a limitation of the methodology and any arbitrary source–receiver volume combination could be considered, albeit at the cost of additional geometric complexity. It was argued that a MIMO model was sufficient to provide an adequate framework for computing the acoustic field capacity in this problem and the information-theoretic approach applied here reflects this view. A rather general solution was obtained within a time window during which the decorrelation time associated with environmental variability was sufficiently long that the Green functions were fixed. These functions were also assumed known at the transmitter (source) so that power could be distributed in an optimal fashion for this channel. As a specific example, an explicit solution was constructed for the special case of a Pekeris-type waveguide with uniform sound speed and density in both the water column and absorbing bottom with the frequency dependent modal attenuation coeffficients developed perturbatively. This solution was obtained by singular value decomposition of the Green function operator with the resulting capacity on the cylinder surface written as a finite sum of link capacities corresponding to a finite number of degrees of freedom associated with the acoustic field. The finite number of degrees of freedom was seen to be a consequence of the structure of the Green function operator, as illustrated by its singular value spectrum, which showed a characteristic transition-like behavior to negligible values as a function of the source radius. The link capacity terms were logarithmic functions of the squared singular values of the Green function operator, the source power and noise power in that link, thus relating the quantification of information directly to the energy and spatial structure of the propagated field.

Field capacity results were obtained by a waterfilling procedure for several acoustic frequencies and water depths, where the effect of bottom absorption and frequency were readily apparent. Note that the capacity of the field, as illustrated in Figs. 3 and 4, is larger than the corresponding sum of the link capacities between source(s) and nodes forming a network, as any nodes would occupy some fraction of the surface area of the cylinder surrounding the source distribution. Therefore the capacity bound associated with the field is not a supremum or least upper bound for a network of sensors covering a surface area less than that of the cylinder. A particular source distribution was not specified here; instead, it was kept arbitrary for the purpose of generality. However, for a particular choice of sensor network, the maximum rate of information transfer from the source region to the nodes would depend on the specific receiver node distribution on the surface as well as the source configuration itself.

A number of extensions of this work to develop more realistic upper bounds could be pursued. A practical upper bound would involve allowance for a small, non-zero probability of error in transmission of information through the noisy channel, an issue beyond the scope of this discussion but accessible through more sophisticated information-theoretic arguments than are presented here [2]. In the analysis presented in this report, it was assumed that the Green functions were known. This is rarely the case in practice, and imperfect channel state information leads to degradation of the channel capacity, as the source power distribution cannot readily be adapted to the channel characteristics. For example, internal gravity waves as well as incomplete knowledge of time and space varying surface boundaries, contribute to uncertainty in the Green functions and hence to uncertainty in the calculation of capacity. These effects can be mitigated to some extent by considering the ergodic or outage capacity, or by treating the uncertainty in the channel explicitly through a stochastic formulation of the Green functions. The general (correlated noise) result, Eqs. (20) and (21), needs an additional model of the specifics of the noise covariance including, for example, ambient surface noise and local ship noise in order to treat realistic noise fields. Range dependence of the sound speed field leading to mode coupling is another issue that will affect the computation of capacity. Finally, the capacity was determined in the space domain, ignoring the role of temporal diversity though such diversity allows higher transmission rates by way of spectral bandwidth. While the emphasis here was on the spatial domain, both spatial and temporal diversity should be included in future work to determine more realistic upper bounds induced by wave propagation.

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1. For clarity, the term "information," placed in quotes, is used in the text with respect to its colloquial meaning. In contrast, when the word is not placed in quotes, information is interpreted with respect to its technical use as a quantitative measure of the amount of knowledge or uncertainty represented by a random variable or sequence of random variables without any reference to its content or meaning.

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Appendix A

APPENDIX: GENERAL COMMENTS ON INFORMATION THEORY

Since Shannon's Information Theory is not typically encountered in the context of underwater acoustics, this appendix briefly discusses a few general concepts associated with this subject so that the analysis in the text is somewhat self-contained. The literature on Information Theory is large, and the reader is referred to standard references for details [A1–A3]. It should be emphasized that this theory does not define what "information" represents; rather, it treats only the amount of information and purposely excludes any discussion of its meaning or content. Information is treated in a probabilistic manner. In the discrete form of the theory, it is associated with a discrete random variable D or a sequence of random variables comprising a random vector $\boldsymbol{D} = (D_1 ... D_i)$, with corresponding probability mass functions $q(D = d)$ or $q(D_1 = d_1 \ldots D_i = d_i)$ for the discrete random sequence. For a single random variable, the basic quantitative measure of information, $H(D)$, is a functional of the probability mass function and defined as the entropy of D: $H(D) \equiv -\sum_d q(d) \log_2 q(d)$. It is measured in bits and the sum is over the possible outcomes/events associated with D. An analogous definition can be written for a random vector. Since entropy is a function only of the probability distribution, it does not depend on the possible values taken by the random variable. Entropy can be viewed as a representation of the average uncertainty associated with the random variable or sequence, or interpreted as the number of bits on the average necessary to represent the random variable. In the context of the issue addressed here, entropy can also be viewed as the average amount of information in a probabilistically generated message linked to a random variable or sequence. As a reasonable quantitative measure of "information," entropy can be justified mathematically from a small set of plausible axioms, but agrees only in part with the colloquial meaning of the word.

For the purposes of this paper, the values taken by the elements of $\boldsymbol{D}$ can be interpreted as a random sequence of symbols physically encoded in the form of amplitude and phase variations of the acoustic field emitted by a source during a particular time interval, and correspond to a transmitted signal. The symbols encode a "message," though as mentioned above, the meaning or content of the message is irrelevant from the perspective of Information Theory. The choice of code is not of interest here, we simply assume that it has been specified. The message could be transmitted purposefully, i.e., designed and coded by an engineer, or it could be sent unwillingly, i.e., the message is simply the radiated pressure field from the source, propagating through the channel, and representing (coding) the location of the source itself [A4]. The goal here is to establish an upper bound on what transmission rate is achievable due to constraints imposed by propagation, without reference to a particular coding scheme. Transmission of information is assumed to occur through a noisy channel where, before reception, $q(D = d)$ represents the *a priori* probability of transmitting the symbol d. After transmission, a symbol d' is received and associated with *a posteriori* probability $q(d|d')$, the probability that d was transmitted, given the received symbol d'. Notationally, $q(d)$ and $q(d')$ will refer to two different random variables, D and D' with different probability mass functions. The difference between the entropies of the *a priori* and *a posteriori* probabilities is defined as the mutual information, $I(D;D')$, and measures the gain in information due to the reception of $D' = d'$. Mutual information can be expressed as a measure of the dependence between two random variables, $I(D;D') = \sum_d \sum_{d'} q(d,d') \log_2 \frac{q(d,d')}{q(d)q(d')}$, where $q(d,d')$ represents a joint distribution between d and d'. It can also be interpreted as the reduction in the uncertainty of d due to the knowledge of d'. For a communication channel in which the output D'

depends probabilistically on the input D, the maximum of $I(D;D')$ taken over all possible distributions of D is defined as the capacity, C, given by $C = \max_{q(D)} I(D,D')$. The problem of obtaining the capacity is often expressed as a constrained optimization problem. This is the case discussed in the text, where the constraints are related to acoustic source power. Shannon's second theorem proved that the link capacity represents the maximum rate (upper bound) at which information can be transmitted over a noisy channel and recovered with negligible probability of error [A1–A3]. The theorem's validity involves an existence proof and does not, by itself, indicate how to construct a code that satisfies the theorem. However, it provides motivation to seek such codes and many have been developed consistent with the Shannon (asymptotic) limit on information transfer rates. The general theory can be extended to continuous random variables and distributions; that is the form used in the analysis, replacing sums by integrals over the distributions. For example, the continuous form for the entropy, called differential entropy h, can be defined as $h(D) = \int q(d) \log_2 q(d)$ where the integral is over the support of the random variable. In the report, the source and the noise are treated stochastically as required by Information Theory, while the environment is assumed to be deterministic and known within a time window. The capacity of the field can be determined under these conditions once the channel properties are defined explicitly.

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