Reasoning Under Uncertainty: Variations of Subjective Logic Deduction

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***Abstract*—This work develops alternatives to the classical subjective logic deduction operator. Given antecedent and consequent propositions, the new operators form opinions of the consequent that match the variance of the consequent posterior distribution given opinions on the antecedent and the conditional rules connecting the antecedent with the consequent. As a result, the uncertainty of the consequent actually map to the spread for the probability projection of the opinion. Monte Carlo simulations demonstrate this connection for the new operators. Finally, the work uses Monte Carlo simulations to evaluate the quality of fusing opinions from multiple agents before and after deduction.**

I. Introduction

The world is not absolute and an ontological representation of the world should account for various shades of gray. Thus, traditional propositional logic is inadequate for general reasoning. Recently, probabilistic logics have emerged to account for the fact that propositions and even the rules that connect propositions are not always true [1]–[3]. At any given instance, if one observes a proposition or rule, it will either be true or false. Over the ensemble of all possible observations, a probabilistic model states that a proposition or rule can be observed to be true with a given ground truth probability. For many cases, the ground truth probabilities are not known and can only be inferred from the observations. Subjective logic (SL) was introduced as a means to represent the belief and uncertainty of these ground truth probabilities based upon the evidence observed by one or multiple agents [3]. In essence, SL is a form of evidential reasoning.

The main feature of SL is that an opinion about a proposition corresponds to a posterior distribution for the probability that the proposition is true given the evidence or observations thus far. Specifically, the posterior distribution is approximated as a Dirichlet distribution whose parameters have a one-to-one correspondence with the subjective opinion. A number of operators have been developed for SL that generalize the operators that exist in propositional and probabilistic logic [3]. These operators have been designed to match the mean of the distribution of the output proposition probabilities given that that the distributions for the input proposition probabilities are Dirichlet. Intuitively, the variance of the output proposition probability distribution represents the uncertainty, but most SL operators do not explicitly account for the variance. In some cases, the operators simply maximize uncertainty while matching the mean. As a result, the statistical meaning of uncertainty can be lost after such SL operators.

This work focuses on alternatives to the SL deduction operator [4], [5] that generalizes the notion of modus ponens from propositional logic, where opinions (or knowledge) about the antecedent and the implication rules leads to an opinion about the consequent. SL deduction represents the catalyst to formulate defeasible logics that incorporate uncertainty. Initial efforts to this end appear in [6]–[8]. In SL deduction, the consequent opinion does represent the mean posterior distribution for the consequent probabilities. However, the uncertainty does not relate to the statistics of this posterior distribution. This work develops two alternative deduction operators to determine more meaningful uncertainty values. Motivated from our previous work on expanding SL for partial observation updates [9], [10], the moment matching (MM) deduction method forms a consequent opinion that characterizes both the mean and variance of the consequent posterior distribution. Similarly, the mode/variance matching (MdVM) method characterizes the mode and variance of the consequent posterior distribution. The performance of these various deduction operators are evaluated using Monte Carlo simulations over known probabilistic ground truths. Furthermore, the simulations consider distributed agents where opinions are fused via the SL consensus operator before or

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after deduction. Note that the consensus operator perfectly account for uncertainty when the individual opinions are formed from independent observations.

This paper is organized as follows. Section II reviews SL including the deduction and consensus operators. Then, the alternative deduction operators are introduced in Section III. Section IV discusses some of the properties of the new deduction operators relative to SL deduction, for both single and multiagent scenarios, and Section V details the Monte Carlo evaluations. Finally, Section VI concludes the paper.

II. Subjective Logic

A. SL Basics

A full overview of SL is available in [3]. SL assumes a probabilistic world such that at any given instant of time, proposition x is true with probability p_x and it is false, i.e., its complement $\bar{x}$ is true, with probability $p_{\bar{x}} = 1 - p_x$. In general, SL can form beliefs over a mutually exclusive set of propositions X, i.e., a frame of discernment. Based upon observations that accumulate, a subjective opinion is formed that consists of the beliefs in the proposition in X and an uncertainty value that add to one. This opinion maps to a Dirichlet distribution representing the posterior distribution of the truthfulness of the propositions in X, i.e. p_x, given the observations. For simplicity of presentation, this paper focuses on a binary frame X consisting of proposition x and its complement where the posterior distribution for p_x given the observations is modeled by the beta distribution.

Specifically, SL represents the subjective[1] opinion x as a triple $\omega_x = (b_x, d_x, u_x)$ such that $b_x + d_x + u_x = 1$ where $b_x, d_x, u_x \geq 0$ are the belief, disbelief, and uncertainty in x due to the evidence. This triple is equivalent to a beta distribution for generating probabilities of x because there is a bijective mapping from ω_x to the beta distribution parameters $\boldsymbol{\alpha}_x$. This mapping assumes some base rate (prior) probability a_x that x is true along with a prior weight W that determines the sensitivity of uncertainty to evidence.The mapping is such that the probability projection (or mean) of the beta distribution forms the pignistic probabilities

$$m_x = E[p_x] = b_x + u_x a_x, \tag{1}$$
$$m_{\bar{x}} = E[p_{\bar{x}}] = d_x + u_x(1 - a_x). \tag{2}$$

Note that these pignistic probabilities form the estimates for $\hat{p}_x = E[p_x]$ and $\hat{p}_{\bar{x}} = E[p_{\bar{x}}]$ in light of the collected evidence and the base rates. For complete uncertainty (i.e., $u_x = 1$), the pignistic probabilities revert to the prior, and as uncertainty drops to zero, they are represented by the evidence in the belief and disbelief values.

The relationship between an opinion and the corresponding beta distribution is given by the following mapping from opinion ω_x to $\boldsymbol{\alpha}_x$:

$$\alpha_x = \frac{W}{u_x} b_x + W a_x, \quad \alpha_{\bar{x}} = \frac{W}{u_x} d_x + W(1 - a_x). \tag{3}$$

[1]Subjective in the sense that the opinion is formed by an agent's own observations.

The reverse mapping from $\boldsymbol{\alpha}_x$ to ω_x is

$$(b_x, d_x, u_x) = \left(\frac{\alpha_x - W a_x}{s_x}, \frac{\alpha_{\bar{x}} - W(1 - a_x)}{s_x}, \frac{W}{s_x} \right), \tag{4}$$

where

$$s_x = \alpha_x + \alpha_{\bar{x}} = \frac{W}{u_x} \tag{5}$$

is the Dirichlet strength of the distribution, which is inversely proportional to the uncertainty.

The evidential nature of SL can be understood by realizing that the beta distribution is the conjugate prior for the binomial distribution. In other words, given an initial prior of $\boldsymbol{\alpha}_{x,0} = \begin{bmatrix} W a_x, & W a_{\bar{x}} \end{bmatrix}$, the posterior beta distribution of $\boldsymbol{\alpha}_x$ is the results of $s_x - W$ "coin flip" observations where $\alpha_x - W a_x$ observations were true and $\alpha_{\bar{x}} - W a_{\bar{x}}$ were false. In essence, the uncertainty is inversely proportional to the number of direct observations made about x.

B. SL Consensus

The SL consensus operation forms a composite opinion for the subjective opinion of multiple agents [11]. It assumes that the agents make independent observations to form their opinions. Given agents A and B having opinions ω_x^A and ω_x^B, the consensus opinion $\omega_x^{A \diamond B} = \omega_x^A \oplus \omega_x^B$ is obtained by converting these opinions into beta parameters via (3). Using the coin flip interpretation, it is easy see that the fused beta parameters are obtained by simply summing up the observations made by the individual agents and inserting the prior, i.e.,

$$\boldsymbol{\alpha}_x^{A \diamond B} = (\boldsymbol{\alpha}_x^A - W \boldsymbol{a}_x^A) + (\boldsymbol{\alpha}_x^B - W \boldsymbol{a}_x^B) + W \boldsymbol{a}_x^{A \diamond B}, \tag{6}$$

where the base rates for the various agents can be different. Finally, the fused beta parameters are converted in the fused opinion via (4). Clearly, the consensus operation is associative, and independent opinions can be fused in a sequential manner, e.g., fusing opinion $A \diamond B$ with C, fusing the result with D and so forth. Alternatively, the fusion of all agents can be accomplished by summing up all the evidences at once in the beta parameter space. This direct consensus of multiple agents can be expressed as $\omega_x^{\diamond_i A_i} = \bigoplus_i \omega_x^{A_i}$.

C. Subjective Logic Deduction

In deduction, one uses knowledge about an antecedent proposition x and knowledge about how the antecedent implies a consequent proposition y to formulate knowledge about the consequent. In probabilistic terms, the states of the x and y propositions occur via the joint probabilities p_{yx}, $p_{y\bar{x}}$, $p_{\bar{y}x}$, and $p_{\bar{y}\bar{x}}$. Complete knowledge of x entails knowing the marginal probability p_x. Likewise, complete knowledge of how x implies y means knowing the conditional probabilities $p_{y|x}$ and $p_{y|\bar{x}}$. Deduction is simply determining the marginal probability p_y from p_x, $p_{y|x}$ and $p_{y|\bar{x}}$. However, knowledge is usually not complete and is acquired through evidence such as from coin flip observations.

Subjective logic deduction uses the opinions ω_x, $\omega_{y|x}$, and $\omega_{y|\bar{x}}$ to determine the deduced opinion $\omega_{y||x}$. Note that the conditional opinions $\omega_{y|x}$, and $\omega_{y|\bar{x}}$ can be formed by direct coin flip observations of the truth or not of y when x is observed to be true or false, respectively. By the interpretation of $\omega_{y\|x}$ as a beta distribution, the mean of the distribution is equal to the true mean of the distribution p_y given that the distributions for p_x, $p_{y|x}$, and $p_{y|\bar{x}}$ are beta distributions parameterized according to the corresponding opinions and $p_y = p_{y|x}p_x + p_{y|\bar{x}}p_{\bar{x}}$. As shown in Section III, the mean of p_y is

$$m_y = m_{y|x}m_x + m_{y|\bar{x}}m_{\bar{x}}.$$

The deduction operator calculates the opinion on y from the three input opinions as

$$\omega_{y\|x} = b_x\omega_{y|x} + d_x\omega_{y|\bar{x}} + u_x\omega_{y|\hat{x}}, \tag{7}$$

where $\omega_{y|\hat{x}}$ is the opinion of y given a vacuous opinion about x, i.e., $u_x = 1$. The value of $\omega_{y|\hat{x}}$ has maximum uncertainty $u_{y|\hat{x}}$ such that belief and disbelief in y is bounded below by the conditional opinions, and the mean of the vacuous conditional opinion is consistent with the mean of p_y when x is vacuous so that $m_x = a_x$. Specifically,

$$\begin{aligned}
&b_{y|\hat{x}} \geq \min\{b_{y|x}, b_{y|\bar{x}}\},\\
&d_{y|\hat{x}} \geq \min\{d_{y|x}, d_{y|\bar{x}}\},\\
&b_{y|\hat{x}} + u_{y|\hat{x}}a_y = m_{y|x}a_x + m_{y|\bar{x}}(1-a_x).
\end{aligned} \tag{8}$$

Geometrically, the deduction operator assumes that the opinion of y is a point in a simplex determined by the opinion of x relative to the simplex vertices, which are the conditional opinions. While the geometric interpretation is appealing, it does not capture the statistics of the distribution of p_y beyond its mean. As shown later, the uncertainty $u_{y\|x}$ loses any correspondence to the variance of p_y.

D. Interpretation of Uncertainty

Intuitively, the bias and variance of m_x as an estimator for p_x should decrease to zero as uncertainty goes to zero. This subsection determines a relationship between the bias or variance and uncertainty by using the coin flip interpretation. First of all, m_x is a biased estimate of the ground truth p_x that is only unbiased asymptotically as the influence of the base rates disappear. An unbiased estimate considers only the observed evidence, not the prior, to estimate p_x. In the beta parameter space, the unbiased estimate for p_x is given by

$$\nu_x = \frac{\alpha_x - Wa_x}{s_x - W}, \tag{9}$$

and via (3), this unbiased estimate can be expressed as

$$\nu_x = \frac{b_x}{1-u_x}. \tag{10}$$

Unlike the regularized[2] mean m_x, the unbiased estimate is undefined when uncertainty is one. When the prior weight $W = 2$ and $a_x = 0.5$, the unbiased estimate corresponds to the mode of the beta distribution describing the posterior for p_y. This observation motivates the use of the mode/variance matching deduction method in the next section. Estimating the bias of m_x as the difference between m_x and ν_x leads to the following relationship between bias and uncertainty

$$\text{bias} \approx \frac{u_x}{1-u_x}(a_x - m_x). \tag{11}$$

Clearly, the bias goes to zero as uncertainty goes to zero.

Similarly, one can relate the uncertainty value to the variance of the estimator. Given N_x coin flip observations, it is well known that the Cramer-Rao lower bound (CRLB) is given by [12]

$$\text{CRLB} = \frac{p_x p_{\bar{x}}}{N_x}. \tag{12}$$

Since $N_x = s_x - W$ and $m_x \approx p_x$, the approximate variance for m_x or ν_x can be related to uncertainty via

$$\text{var} \approx \frac{u_x}{1-u_x}\frac{m_x m_{\bar{x}}}{W}. \tag{13}$$

Again, as uncertainty goes to zero, so does the variance. Overall, the two expression for bias and variance can be computed for any subjective opinion. These expressions approximate the true bias and spread for estimates over the ensemble of possible observations that could have occurred from the ground truth.

III. Deduction Alternatives

The proposed deductive methods fit a beta distribution to the actual marginal distribution for p_y when given the opinions ω_x, $\omega_{y|x}$ and $\omega_{y|\bar{x}}$. Since these opinions are generated from independent observations, the joint distribution for p_x, $p_{y|x}$ and $p_{y|\bar{x}}$ is the product of the individual beta distributions so that

$$\begin{aligned}
f_{X,Y|X}(p_x, p_{y|x}, p_{y|\bar{x}}) &= \frac{{p_x}^{\alpha_x-1}(1-p_x)^{\alpha_{\bar{x}}-1}}{B_2(\alpha_x, \alpha_{\bar{x}})}\cdot\\
&\cdot\frac{{p_{y|x}}^{\alpha_{y|x}-1}(1-p_{y|x})^{\alpha_{\bar{y}|x}-1}}{B_2(\alpha_{y|x}, \alpha_{\bar{y}|x})}\frac{{p_{y|\bar{x}}}^{\alpha_{y|\bar{x}}-1}(1-p_{y|\bar{x}})^{\alpha_{\bar{y}|\bar{x}}-1}}{B_2(\alpha_{y|\bar{x}}, \alpha_{\bar{y}|\bar{x}})},
\end{aligned} \tag{14}$$

where the parameters $\boldsymbol{\alpha}_x$, $\boldsymbol{\alpha}_{y|x}$ and $\boldsymbol{\alpha}_{y|\bar{x}}$ are determined from the corresponding opinions via (3), and $B_2(\cdot,\cdot)$ is the beta function

$$B_2(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

By making the following substitution of variables

$$\begin{aligned}
p_y &= p_{y|x}p_x + p_{y|\bar{x}}(1-p_x),\\
p_{x|y} &= \frac{p_{y|x}p_x}{p_{y|x}p_x + p_{y|\bar{x}}(1-p_x)},\\
p_{x|\bar{y}} &= \frac{(1-p_{y|x})p_x}{(1-p_{y|x})p_x + (1-p_{y|\bar{x}})(1-p_x)},
\end{aligned} \tag{15}$$

the joint distribution with respect to p_y, $p_{x|y}$, and $p_{x|\bar{y}}$ is expressed as

$$f_{Y,X|Y}(p_y, p_{x|y}, p_{x|\bar{y}}) = g_x g_y g_{x|y} g_{x|\bar{y}}, \tag{16}$$

where

$$g_x = \frac{{p_x}^{\beta_x}(1-p_x)^{\beta_{\bar{x}}}}{B_2(\alpha_x, \alpha_{\bar{x}})},$$

[2]The prior regularizes the mean.

$$g_y = p_y^{\alpha_{y|x}+\alpha_{y|\bar{x}}-1}(1-p_y)^{\alpha_{\bar{y}|x}+\alpha_{\bar{y}|\bar{x}}-1},$$

$$g_{x|y} = \frac{{p_{x|y}}^{\alpha_{y|x}-1}(1-p_{x|y})^{\alpha_{y|\bar{x}}-1}}{B_2(\alpha_{y|x},\alpha_{\bar{y}|x})},$$

$$g_{x|\bar{y}} = \frac{{p_{x|\bar{y}}}^{\alpha_{\bar{y}|x}-1}(1-p_{x|\bar{y}})^{\alpha_{\bar{y}|\bar{x}}-1}}{B_2(\alpha_{y|\bar{x}},\alpha_{\bar{y}|\bar{x}})},$$

$p_x = p_{x|y}p_y + p_{x|\bar{y}}(1-p_y)$, $\beta_x = \alpha_x - \alpha_{y|x} - \alpha_{\bar{y}|x}$ and $\beta_{\bar{x}} = \alpha_{\bar{x}} - \alpha_{y|\bar{x}} - \alpha_{\bar{y}|\bar{x}}$.

The marginal distribution is obtained from (16) via

$$f_Y(p_y) = \int_0^1 \int_0^1 f_{Y,X|Y}(p_y, p_{x|y}, p_{x|\bar{y}})dp_{x|\bar{y}}dp_{x|y} \quad (17)$$

In general, this marginal is not a beta distribution, and a simple analytical expression for the distribution appears unobtainable. It is possible to determine this distribution using numerical integration techniques. Nevertheless, statistics of this distributions can be obtained directly from joint distribution of p_x, $p_{y|x}$ and $p_{y|\bar{x}}$ in (14). The new deduction methods use these statistics to approximate the marginal distribution f_Y by a beta distribution.

A. Moment Matching

The moment matching (MM) deduction method approximates f_Y by a beta distribution that matches the mean and variance of f_Y. The idea of moment matching for SL originated in [9] where a new method was introduced to update subjective opinions based upon partial observations of the coin flip (or dice roll) when only likelihoods of the possible results are known. This notion was further expanded in [10] to account for uncertainty in the likelihood calculations. For the deduction operation, the first step is to determine the mean and variance of f_Y. The mean is derived via

$$\begin{aligned}
m_y &= E[p_y] \\
&= E[p_{y|x}]E[p_x] + E[p_{y|\bar{x}}]E[1-p_x] \\
&= \int_0^1 \frac{{p_x}^{\alpha_x}{p_{\bar{x}}}^{\alpha_{\bar{x}}-1}}{B_2(\alpha_x,\alpha_{\bar{x}})}dp_x \int_0^1 \frac{{p_{y|x}}^{\alpha_{y|x}}{p_{\bar{y}|x}}^{\alpha_{\bar{y}|x}-1}}{B_2(\alpha_{y|x},\alpha_{\bar{y}|x})}dp_{y|x} \\
&+ \int_0^1 \frac{{p_x}^{\alpha_x-1}{p_{\bar{x}}}^{\alpha_{\bar{x}}}}{B_2(\alpha_x,\alpha_{\bar{x}})}dp_x \int_0^1 \frac{{p_{y|\bar{x}}}^{\alpha_{y|\bar{x}}}{p_{\bar{y}|\bar{x}}}^{\alpha_{\bar{y}|\bar{x}}-1}}{B_2(\alpha_{y|\bar{x}},\alpha_{\bar{y}|\bar{x}})}dp_{y|\bar{x}} \\
&= \frac{\alpha_x}{\alpha_x+\alpha_{\bar{x}}}\frac{\alpha_{y|x}}{\alpha_{y|x}+\alpha_{\bar{y}|x}} + \frac{\alpha_{\bar{x}}}{\alpha_x+\alpha_{\bar{x}}}\frac{\alpha_{y|\bar{x}}}{\alpha_{y|\bar{x}}+\alpha_{\bar{y}|\bar{x}}},
\end{aligned}$$

which leads to

$$m_y = m_{y|x}m_x + m_{y|\bar{x}}(1-m_x). \quad (18)$$

The second order moment is determined via

$$\begin{aligned}
E[p_y^2] &= \frac{\int_0^1 {p_x}^{\alpha_x+1}{p_{\bar{x}}}^{\alpha_{\bar{x}}-1}dp_x}{B_2(\alpha_x,\alpha_{\bar{x}})}\frac{\int_0^1 {p_{y|x}}^{\alpha_{y|x}+1}{p_{\bar{y}|x}}^{\alpha_{\bar{y}|x}-1}dp_{y|x}}{B_2(\alpha_{y|x},\alpha_{\bar{y}|x})} \\
&+ \int_0^1 \frac{{p_x}^{\alpha_x-1}{p_{\bar{x}}}^{\alpha_{\bar{x}}+1}}{B_2(\alpha_x,\alpha_{\bar{x}})}dp_x \int_0^1 \frac{{p_{y|\bar{x}}}^{\alpha_{y|\bar{x}}+1}{p_{\bar{y}|\bar{x}}}^{\alpha_{\bar{y}|\bar{x}}-1}}{B_2(\alpha_{y|\bar{x}},\alpha_{\bar{y}|\bar{x}})}dp_{y|\bar{x}} \\
&+ 2\int_0^1 \frac{{p_x}^{\alpha_x}{p_{\bar{x}}}^{\alpha_{\bar{x}}}}{B_2(\alpha_x,\alpha_{\bar{x}})}dp_x \int_0^1 \frac{{p_{y|x}}^{\alpha_{y|x}}{p_{\bar{y}|x}}^{\alpha_{\bar{y}|x}-1}}{B_2(\alpha_{y|x},\alpha_{\bar{y}|x})}dp_{y|x} \\
&\cdot \int_0^1 \frac{{p_{y|\bar{x}}}^{\alpha_{y|\bar{x}}}{p_{\bar{y}|\bar{x}}}^{\alpha_{\bar{y}|\bar{x}}-1}}{B_2(\alpha_{y|\bar{x}},\alpha_{\bar{y}|\bar{x}})}dp_{y|\bar{x}}.
\end{aligned}$$

After some simplifying steps

$$\begin{aligned}
E[p_y^2] &= m_x m_{y|x}\frac{\alpha_x+1}{s_x+1}\frac{\alpha_{y|x}+1}{s_{y|x}+1} \\
&+ (1-m_x)m_{y|\bar{x}}\frac{\alpha_{\bar{x}}+1}{s_x+1}\frac{\alpha_{y|\bar{x}}+1}{s_{y|x}+1} \\
&+ 2m_x(1-m_x)m_{y|x}m_{y|\bar{x}}\frac{s_x}{s_x+1}
\end{aligned}$$

Then the variance is computed as

$$\begin{aligned}
\sigma_y^2 &= E[p_y^2] - (E[p_y])^2, \\
&= m_{y|x}m_x\left(\frac{\alpha_{y|x}+1}{s_{y|x}+1}\frac{\alpha_x+1}{s_x+1} - m_x m_{y|x}\right) \\
&+ m_{y|\bar{x}}(1-m_x)\left(\frac{\alpha_{y|\bar{x}}+1}{s_{y|\bar{x}}+1}\frac{\alpha_{\bar{x}}+1}{s_x+1} - (1-m_x)m_{y|\bar{x}}\right) \\
&- 2\frac{m_x(1-m_x)m_{y|x}m_{y|\bar{x}}}{s_x+1}.
\end{aligned}$$

Final simplification leads to

$$\begin{aligned}
\sigma_y^2 &= \frac{m_x m_{y|x}(1-m_{y_x})}{s_{y|x}+1}\frac{m_x s_x+1}{s_x+1} \\
&+ \frac{(1-m_x)m_{y|\bar{x}}(1-m_{y|\bar{x}})}{s_{y|\bar{x}}+1}\frac{(1-m_x)s_x+1}{s_x+1} \\
&+ \frac{m_x(1-m_x)(m_{y|x}-m_{y|\bar{x}})^2}{s_x+1}.
\end{aligned} \quad (19)$$

The next step is to determine the parameters of the beta distribution $\boldsymbol{\alpha}_y$ whose mean and variance are given by (18) and (19), respectively. It is well known that the mean and variance of the beta distribution are determined by $\boldsymbol{\alpha}_y$ via [13]

$$m_y = \frac{\alpha_y}{s_y}, \quad (20)$$

$$\sigma_y^2 = \frac{m_y(1-m_y)}{s_y+1}. \quad (21)$$

In other words, the parameters are determined from the moments via

$$\begin{aligned}
s_y &= \frac{m_y(1-m_y)}{\sigma_y^2} - 1, \\
\alpha_y &= m_y s_y, \quad \alpha_{\bar{y}} = (1-m_y)s_y.
\end{aligned} \quad (22)$$

The strength parameter s_y can be computed directly from the mean and strength values from the opinions ω_x, $\omega_{y|x}$ and $\omega_{y|\bar{x}}$. By inserting (18) and (19) into (22),

$$s_y = \frac{A+B+C}{\frac{A}{s_x+1} + \frac{B}{s_{y|x}+1}\frac{m_x s_x+1}{s_x+1} + \frac{C}{s_{y|\bar{x}}+1}\frac{(1-m_x)s_x+1}{s_x+1}} - 1. \quad (23)$$

where

$$A = m_x(1-m_x)(m_{y|x}-m_{y|\bar{x}})^2, \quad (24)$$

$$B = m_x m_{y|x}(1-m_{y|x}), \quad (25)$$

$$C = m_{\bar{x}} m_{y|\bar{x}}(1-m_{y|\bar{x}}). \quad (26)$$

Overall, MM deduction approach takes the steps given in Algorithm 1.

Algorithm 1 Moment Matching Deduction

Input: ω_x, $\omega_{y|x}$ and $\omega_{y|\bar{x}}$

Output: $\omega_{y\|x}$

1) Calculate the $m_x, m_{y|x}, m_{y|\bar{x}}$ and $s_x, s_{y|x}, s_{y|\bar{x}}$ via (1) and (5), respectively.
2) Calculate m_y and s_y^* using (18) and (23), respectively.
3) Let $s_y = \max\left\{s_y^*, \frac{Wa_y}{m_y}, \frac{W(1-a_y)}{1-m_y}\right\}$.
4) Set $\omega_{y\|x} = \left(m_y - \frac{Wa_y}{s_y}, 1 - m_y - \frac{W(1-a_y)}{s_y}, \frac{W}{s_y}\right)$.

Note that Step 3 is included to ensure that the belief and disbelief value can never become negative after the deduction update. Changing the Dirichlet strength by this step has no effect on the mean value. In essence, it lowers the variance of the beta fit to f_Y while maintaining the mean of f_Y so that belief masses are not negative. Because this approach retains the true mean of f_Y it fits within the framework of a subjective logic operator. Unlike the traditional subjective logic operator as reviewed in Section II-C, the MM deduction operator better characterizes the spread of the posterior f_Y. As Section V will show, the Dirichlet strength for the MM deduction operator provides a better approximation for the effective number of coin flips of y than deduction provides.

B. Mode and Variance Matching

As discussed earlier, when the beta distribution represents the posterior for p_x due to the coin flip experiment, the mean of the density represents a biased estimator for ground truth p_x. On the other hand, the mode of the posterior is the unbiased estimator of p_x. Asymptotically, the mean and mode are equal and both estimators become unbiased. It can be argued that it is as important (if not more) for the mode of the approximate beta distribution for p_y to match the mode of the marginal distribution given by (17). At this point, we do not have a closed formed or efficient means to determine the mode of this distribution, i.e., the marginal of (16). As a surrogate, we use the unbiased estimator of p_y derived from the unbiased estimates of p_x, $p_{y|x}$ and $p_{y|\bar{x}}$ (see (10)), i.e.,

$$\nu_y = \frac{b_{y|x}b_x}{(1-u_{y|x})(1-u_x)} + \frac{b_{y|\bar{x}}d_x}{(1-u_{y|\bar{x}})(1-u_x)}. \tag{27}$$

This surrogate is only available when neither opinions ω_x, $\omega_{y|x}$, $\omega_{y|\bar{x}}$ are vacuous, i.e., $u_x, u_{y|x}, u_{y|\bar{x}} < 1$. To determine the beta parameters, we note that the mode of the beta distribution [13] relates to the parameters via[3]

$$\nu_y = \frac{\alpha_y - 1}{s_y - 2}. \tag{28}$$

Then using (20), (21), and (28), it can be shown that the Dirichlet strength that matches the mode and variance of f_Y is the largest root of the following cubic polynomial

$$\sigma_y^2 s_y^3 + (\sigma_y^2 - \nu_y(1-\nu_y))s_y^2 - (1-2\nu_y)^2 s_y + (1-2\nu_y)^2 = 0. \tag{29}$$

[3]The expression for the mode assumes that $\alpha_y, \alpha_{\bar{y}} > 1$, which also implies $s_y > 2$.

It can also be shown that a real solution $s_y \geq 2$ always exist for (29) when $W = 2$. Then the mean of the beta distribution whose strength s_y solves (29) and whose mode is given by (28) has a mean given by

$$m_y = \frac{\nu_y(s_y - 2) + 1}{s_y}. \tag{30}$$

Overall, the mode/variance matching (MdVM) deduction approach is described by Algorithm 2.

Algorithm 2 Mode/Variance Matching Deduction

Input: ω_x, $\omega_{y|x}$ and $\omega_{y|\bar{x}}$

Output: $\omega_{y\|x}$

1) Calculate the $m_x, m_{y|x}, m_{y|\bar{x}}$ and $s_x, s_{y|x}, s_{y|\bar{x}}$ via (1) and (5), respectively.
2) Calculate ν_y using (27).
3) Calculate s_y as the largest root of (29).
4) Calculate m_y using (30).
5) Set $\omega_Y = \left(m_y - \frac{Wa_y}{s_y}, 1 - m_y - \frac{W(1-a_y)}{s_y}, \frac{W}{s_y}\right)$.

Unlike MM Deduction, the MdVM does not include the step that possibly increases the Dirichlet strength in order to ensure that the belief and disbelief values are positive. As long as $W = 2$ and $a_y = 0.5$, which represent reasonable prior values, b_y and d_y are guaranteed to be positive due to (30). In short, the MdVM is only designed to work for a uniform baseline rate with a prior weight of $W = 2$.

IV. Properties of Deduction

The traditional SL deduction calculates the consequent opinion as a linear combination of conditional opinions as given by (7). As a result, the uncertainty for y is computed as

$$u_y = b_x u_{y|x} + d_x u_{y|\bar{x}} + u_x u_{y|\hat{x}}. \tag{31}$$

The linear relationship does exist for the new deduction approaches. In fact, the uncertainty given by SL deduction appears to always be larger than that given by the two other deduction operations. Figure 1 plots the uncertainty for y over all possible opinions of x when $\omega_{y|x} = (.8, .1, .1)$ and $\omega_{y|\bar{x}} = (.1, .6, .3)$ for the three deduction methods. The figure reveals that the uncertainty for the new deductions methods is lower than that of SL deduction. Unlike SL deduction, the relationship between uncertainty and ω_x is nonlinear for the new methods. All three methods provide the same uncertainty for dogmatic $\omega_x = (1, 0, 0)$ or $\omega_x = (0, 1, 0)$. Certainly, SL deduction should have higher uncertainty for the vacuous $\omega_x = (0, 0, 1)$ because it maximizes uncertainty for this case.

In general, it can be shown that for the dogmatic opinion that x is true (or false) leads to $\omega_{y\|x} = \omega_{y|x}$ (or $\omega_{y\|x} = \omega_{y|\bar{x}}$) for all three methods.[4] As a result, when the rules are known to be true with full certainty, all three deduction methods become equivalent to modus ponens in standard propositional logic. For the slightly more general case that the propositions and

[4]Note that MdVM assumes $W = 2$ and $a_y = 0.5$.

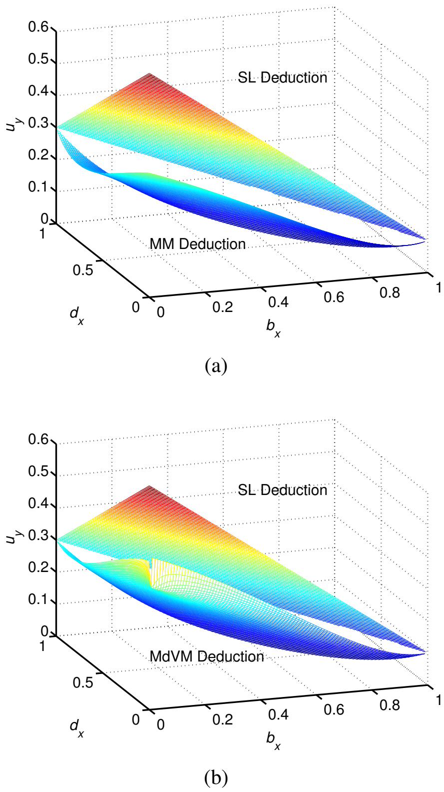

Figure 1. Uncertainty of y as a function of ω_x when $\omega_{y|x} = (.8, .1, .1)$ and $\omega_{y|\bar{x}} = (.1, .6, .3)$: (a) SL and MM Deduction and (b) SL and MdVM Deduction.

rules are not completely true or false, but the opinions are still dogmatic, all three methods still provide the same results as they simplify to probabilistic reasoning.

The observations from Figure 1 shows that the SL deduction leads to an opinion whose uncertainty is larger than or equal to the uncertainty of the other two deduction methods appear to hold in general. However, aside from some special cases, we do not have a proof to determine if indeed this trend always holds. It should be emphasized that SL and MM deduction lead to the same probability projection m_y. They only differ in that MM provides a lower uncertainty value that relates to a confidence interval for m_y to be near the ground truth p_y. On the other hand, MdVM deduction provides a different m_y value than SL and MM deduction, but the uncertainty value it provides also relates to the confidence interval. Given that the SL framework requires an accurate representation of the mean of the posterior and only an approximation for the variance, MM deduction can be viewed as another SL operator, but not MdVM deduction.

In a multiagent scenario, different agents can form different opinions about x and the conditional rules based on their observations. In this paper, we assume each agent is able to properly observe the coin flip experiments and each agent is truthful. These assumptions lead to the fact that the proper processing is to perform consensus for each of the propositions and conditional rules first followed by deduction. This *consensus before deduction* (CBD) approach can be thought of as a centralized fusion architecture where each agent sends their opinion to a central agent who performs fusion. The approach, while proper, can lead to heavy bandwidth usage by sending many opinions to the central agent.

An alternative approach is for each agent to perform deduction using their local opinions, and send a smaller set of inferred opinions to the central agent who simply performs consensus. This distributed *deduction before consensus* (DBC) approach is attractive from a networking perspective. However, it must be noted that DBC can lead to vastly different opinions than CBD. The next section will investigate how each deduction method fares under the DBC architecture. Nevertheless, much research is needed to understand the error bounds associated to DBC so that fusion architectures with guaranteed performance can be designed.

In some extreme cases, DBC can perform very poorly. For example, consider a case where Agent A has opinions $\omega_x^A = (1, 0, 0)$, $\omega_{y|x}^A = (0, 0, 1)$, and $\omega_{y|\bar{x}}^A = (0, 0, 1)$, and that Agent B has opinions $\omega_x^B = (0, 0, 1)$, $\omega_{y|x}^B = (1, 0, 0)$, and $\omega_{y|\bar{x}}^B = (0, 1, 0)$. Agent A has complete evidence about x but has collected no evidence about the rules. Conversely, Agent B has collected independent evidence to obtain full knowledge of the rules but no evidence for x. Although the evidences collected by Agents A and B are independent, CBD and DBC give vastly different opinions. For any of the three methods, the proper CBD approach leads to $\omega_{y\|x}^{A\diamond B} = (1, 0, 0)$, but the DBC approach leads to $\omega_{y\|x}^{A\diamond B} = (0, 0, 1)$. The problem is that opinions on x and on the rules enhance each other in determining the inferred opinion. In this extreme case, both agents can only deduce a vacuous opinion because they either had no evidence for the proposition or for the rules. Because the opinions enhance each other, we expect that for independent observations, CBD provides opinions with lower uncertainty. In short, deduction creates a dependency between deduced opinions that SL consensus does not account for.

For another case, consider that the number of agents is growing without bound, all agents use the same opinions for the conditional rules, and the antecedent x is always true, i.e., $p_x = 1$. In the CBD approach the fusion of the opinions on x by the agents will converge to the dogmatic opinion $\omega_x^\diamond = (1, 0, 0)$, and the uncertainty of the deduced opinion will converge to $u_{y|x}$. On the other hand, DBC allows for the fused uncertainty to converge to zero. Because the agents are utilizing the same opinions about the rules, the deduced opinions are not statistically independent, and consensus is overly optimistic about the quality of the fused opinion.

V. Simulations

The three deduction methods were evaluated by simulating opinions formed by actual coin flip observations. The ground truth for proposition x varied from $p_x = 0$ up to $p_x = 1$, and the conditional probabilities were set to $p_{y|x} = 0.8$ and $p_{y|\bar{x}} = 0.1$. All agents formed opinions for the antecedent ω_x using $N_x = 10$ observations and formed opinions for the two conditionals $\omega_{y|x}$ and $\omega_{y|\bar{x}}$ using $N_{y|x} = N_{y|\bar{x}} = 100$ observations. For a given value of p_x, the deduction methods were evaluated over 100 Monte Carlo simulations of the formed opinions. In all simulations, $W = 2$ and $a_x = a_y = 0.5$.

Figure 2 plots the error performance for the three methods in terms of average bias, standard deviation, and root mean squared (RMS) error average over the 100 Monte Carlo simulation per a given p_x. The solid color curves are the computed errors using the ground truth $p_y = 0.8p_x + 0.1(1 - p_x)$ for the three methods, and the corresponding color dashed curves are the average predicted error derived from the uncertainty u_y via (11) and (13), which has no knowledge of the ground truth. The actual errors for SL and MM deduction are the same because they produce the same m_y value. However, the high uncertainty for SL deduction overestimates the standard deviation. On the other hand, the uncertainty for MM and MdVM deduction is able to accurately portray the error spread. The MdVM method exhibits the smallest bias, and the uncertainty for MdVM also accurately portrays its bias. The predicted bias for the MM is similar to that of the MdVM despite the higher bias for the MM method.

Figure 3 provides the error performance results for 100 agents forming independent opinions, transmitting their deduced y to a central agents who computes a consensus opinion for y, i.e., the DBC fusion approach. As mentioned in the previous section, DBC is not the optimal approach. Furthermore, SL and MM deduction now provide different results. Overall, the uncertainty value for the MM and MdVM methods can still predict the standard deviation, but they predict much less bias. The uncertainty for the SL method still overestimates the standard deviation. It is interesting to note that SL deduction usually achieves less RMS error than MM deduction when the base rate $a_y = 0.5$ does not match the ground truth p_y. On the other hand, the MdVM method is able to predict the spread and achieve better RMS error because of its lower bias.

Figure 4 shows the error performance results for the 100 agents that transmit their intrinsic opinions to the central agent that performs CBD. For this case, the overall biases are significantly smaller than the DBC case. As in the single agent case, the error performance for the SL and MM methods are the same, but the MM method is able to predict the actual errors from its uncertainty value. In effect, this CBD is equivalent to a single agent with $N_x = 1,000$ and $N_{y|x} = N_{y|\bar{x}} = 10,000$ observations. With so many observations, the mean and mode are almost equal, and the performances of the MM and MdVM methods are almost identical. Interestingly, the standard deviations due to the CBD approaches are similar to that of the DBC approaches. This is probably due to the fact that the agents collected independent evidences for both the antecedent and rules, and the amount of evidence was balanced across the agents. In other words, the simulated scenario enables DBC to be viable unlike the conditions discussed in previous section. Some preliminary results with unbalanced observations over agents have shown much lower uncertainty for DBC than CBD as one would expect from independent agents since observations of propositions and rules enhance each other in the deduction operation. These results also indicate that MdVM is usually better than SL, which in turn is usually better than MM.

VI. Conclusions

This work introduces two new deduction methods that operate over subjective opinions referred to as moment and mode/variance matching (MM and MdVM, respectively). Unlike the traditional SL deduction method, the uncertainty from the computed opinions for the consequent are able to predict the confidence bounds for the mean projection (or estimate) in the two new methods. However, the MM deduction method is unable to predict its bias. We believe that this is why the performance of the MdVM is usually better than MM for DBC type fusion of opinions.

SL combines probabilistic logic with uncertainty. It is computationally appealing by representing uncertainty for each proposition (or rule) as a single parameter that with the belief values map to a beta (or more generally a Dirichlet) distribution for the possible ground truth generating probabilities given the evidence. After many logical operations such as deduction, this representation of uncertainty is an approximation which comes with a cost. As seen in the simulations, the uncertainty does not necessarily predict both the bias and variance inherent in the posterior distribution of the consequent. Furthermore, it is unclear how to control the approximation error when combining deduction with fusion in a distributed manner. For instance, the bias in DBC fusion initially goes down when combining more agents, but it eventually saturates. Future work should investigate the conditions that causes DBC to provide diminishing return and determine the maximum number agents that can be fused before saturation occurs. Furthermore, the implication on DBC for multiple agents using common or dependent opinions as well as variations in the amount of evidences to form opinions need to be better understood. This understanding will hopefully lead to principled methods to design distributed reasoning engines that incorporate uncertainty.

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References

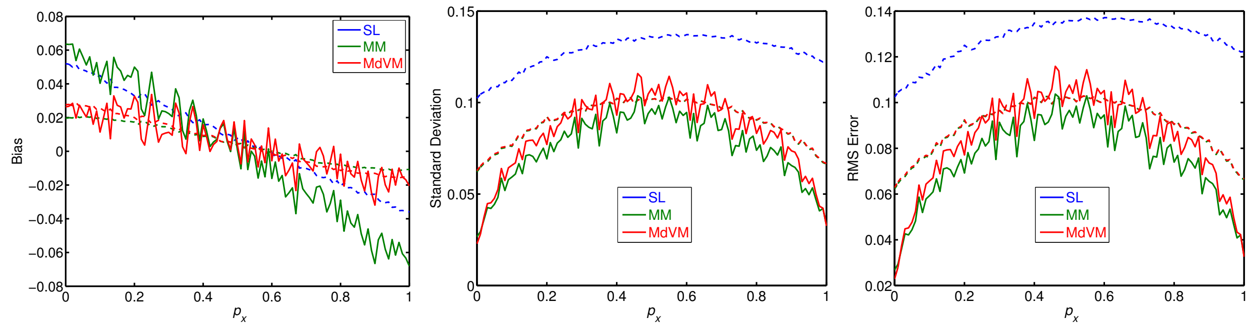
[1] N. J. Nilsson, "Probabilistic logic," *Artificial Intelligence*, vol. 28, no. 1, pp. 71–87, Feb. 1986.

[2] M. Richardson and P. Domingos, "Markov logic networks," *Machine Learning*, vol. 62, no. 1–2, pp. 107–136, 2006.

[3] A. Jøsang, "Subjective logic," Jul. 2011, draft book in preparation.

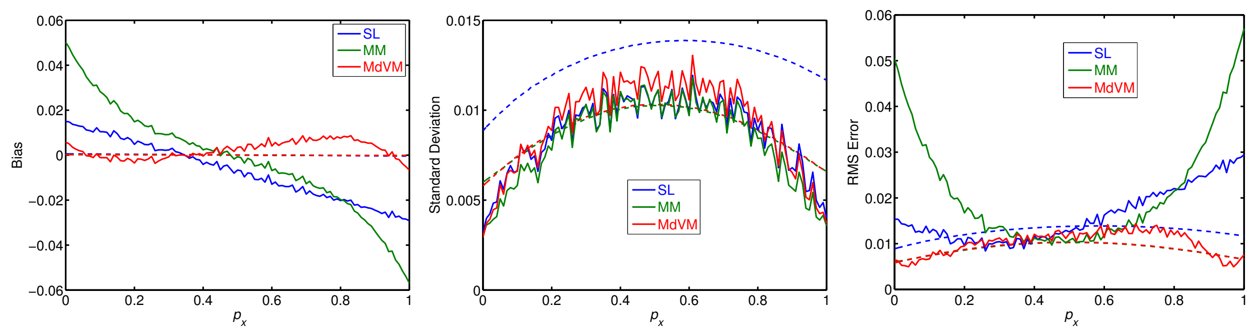
[4] A. Jøsang, S. Pope, and M. Daniel, "Conditional deduction under uncertainty," in *Proceedings of the 8th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2005)*, 2005.

[5] A. Jøsang, "Conditional reasoning with subjective logic," in *Proceedings of the 8th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2005)*, Barcelona, Spain, Jul. 2005.



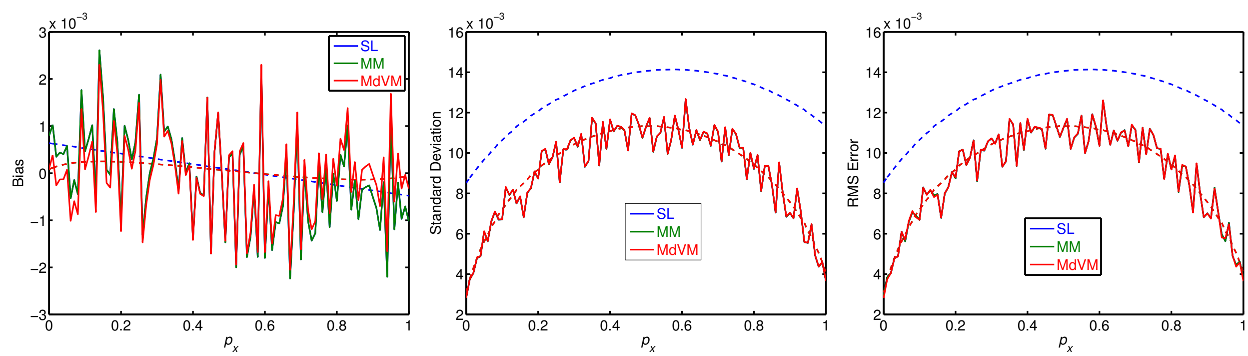
(a) (b) (c)

Figure 2. Quality of the deduced opinion versus ground truth for a single agent: (a) Bias, (b) standard deviation and (c) RMS error. Note that the actual error curves for SL and MM overlap.



(a) (b) (c)

Figure 3. Quality of the deduced opinion versus ground truth from 100 agents fused via DBC: (a) Bias, (b) standard deviation and (c) RMS error.



(a) (b) (c)

Figure 4. Quality of the deduced opinion versus ground truth from 100 agents fused via CBD: (a) Bias, (b) standard deviation and (c) RMS error. Note that actual error curves for SL and MM overlap and exhibit almost similar performance as the MdVM curves.

[6] C. Chesnevar, G. Simari, T. Alsinet, and L. Godo, "A logic programming framework for possibilistic argumentation with vague knowledge," in *Proc. of the 20th Conference on Uncertainty in Artificial Intelligence*, Banff, Canada, Jul. 2004, pp. 76–84.

[7] M. Şensoy, A. Fokoue, J. Z. Pan, T. J. Norman, Y. Tang, N. Oren, and K. Sycara, "Reasoning about uncertain information and conflict resolution through trust revision," in *Proc. of the Autonomous Agents and Multiagent Systems (AAMAS) Conference*, St. Paul, MN, May 2013.

[8] S. Han, B. Koo, A. Hutter, and W. Stechele, "Forensic reasoning upon pre-obtained surveillance metadata using uncertain spatio-temporal rules and subjective logic," in *Analysis, Retrieval and Delivery of Multimedia Content*, N. Adami, A. Cavallaro, R. Leonardi, and P. Migliorati, Eds. Springer, 2013, pp. 125–157.

[9] L. M. Kaplan, S. Chakraborty, and C. Bisdikian, "Fusion of classifiers: A subjective logic perspective," in *Proc. of the IEEE Aerospace Conference*, Big Sky, MT, Mar. 2012.

[10] ——, "Subjective logic with uncertain partial observations," in *Proc. of the International Conference on Information Fusion*, Singapore, Jul. 2012.

[11] A. Jøsang, "The consensus operator for combining beliefs," *Artificial Intelligence Journal*, vol. 142, no. 1–2, pp. 157–170, Oct. 2002.

[12] K. Miura, "An introduction to maximum likelihood estimation and information geometry," *Interdisciplinary Information Sciences*, vol. 17, no. 3, pp. 155–174, 2011.

[13] A. K. Gupta and S. Nadarajah, Eds., *Handbook of Beta Distribution and Its Applications*. CRC Press, 2004.