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Modeling Uncertainty in Military Supply Chain Management Decisions

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Barry R. Cobb, Professor

Department of Economics and Business

Virginia Military Institute

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This report introduces a mixture distribution approach to modeling the probability density function for lead time demand (LTD) in problems where a continuous review inventory system is implemented. The method differs from the typical ?moment-matching? approach by focusing on building up an accurate, closed-form approximation to the LTD distribution from its components by using mixtures of truncated exponential (MTE) functions. First, construction of the lead time distribution is illustrated and the approach is compared to two other possible lead time distributions. This distribution is then utilized to determine optimal order policies in cases where a buyer makes its decisions alone, and later in a situation where members of a two-level supply chain coordinate their actions. Next, a mixture of polynomials (MOP) approach is introduced and utilized to model the LTD distribution for the case of discrete lead time and a daily demand distribution that assumes a standard form. Finally, the MOP model is extended to model an LTD distribution from empirical lead time and daily demand data.

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Abstract

This report introduces a mixture distribution approach to modeling the probability density function for lead time demand (LTD) in problems where a continuous review inventory system is implemented. The method differs from the typical "moment-matching" approach by focusing on building up an accurate, closed-form approximation to the LTD distribution from its components by using mixtures of truncated exponential (MTE) functions. First, construction of the lead time distribution is illustrated and the approach is compared to two other possible lead time distributions. This distribution is then utilized to determine optimal order policies in cases where a buyer makes its decisions alone, and later in a situation where members of a two-level supply chain coordinate their actions. Next, a mixture of polynomials (MOP) approach is introduced and utilized to model the LTD distribution for the case of discrete lead time and a daily demand distribution that assumes a standard form. Finally, the MOP model is extended to model an LTD distribution from empirical lead time and daily demand data.

Keywords: modeling, supply chain management, lead time demand

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About the Author

Barry Cobb is a professor in the Department of Economics and Business at the Virginia Military Institute (VMI) in Lexington. He teaches operations management, managerial economics, and management science. His research has been published in such journals as Decision Analysis, Decision Support Systems, INFORMS Transactions on Education, and the Journal of the Operational Research Society.

Barry Cobb
Department of Economics and Business
Virginia Military Institute
Lexington, VA 24450
Tel: (540) 464-7452
Fax: (540) 460-4059
E-mail: cobbbr@vmi.edu

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Modeling Uncertainty in Military Supply Chain Management Decisions

Numerous probability models have been suggested for representing uncertain demand during lead time in continuous-review inventory management systems when both lead time and demand per unit time are variable. A common approach to finding a distribution for lead time demand involves modeling lead time (LT) and demand per unit time (DPUT) with standard probability density functions (PDFs). Based on the distributions assigned, a compound probability distribution is determined for demand during lead time, or lead time demand (LTD). The latter distribution is used to determine reorder point and safety stock policies, and may be used to estimate inventory costs. In some cases, analytical formulas for optimal reorder point, safety stock, or stockout costs are available in terms of the compound distribution's parameters, while in other situations the values associated with certain percentiles of the compound LTD distribution are estimated to provide these values. While the problem of finding an appropriate LTD distribution has been well-studied, papers written in recent years have continued to pursue methods that overcome unrealistic distributional assumptions (Ruiz-Torres and Mahmoodi, 2010; Vernimmen et al, 2008).

This paper illustrates an approach for constructing a mixture distribution for LTD that allows the LT and DPUT distributions to be state-dependent. This method also allows input distributions that take any standard or empirical form. Use of the mixture distribution technique is first demonstrated in the context described by Cobb (2013), which is a single item continuous-review inventory model for one buyer. For single-firm operating in a continuous-review inventory system, the mixture distribution method for modeling the LTD distribuition differs from the typical "moment-matching" approach. The method focuses on building up an accurate, closed-form approximation to the LTD distribution from its components by using mixtures of truncated exponential (MTE) functions.

After the mixture distribution approach is described, a two-level supply chain model where the buyer operates under uncertain demand and utilizes a continuous review inventory system will be considered. In this two-echelon supply chain model, credit terms (Chaharsooghi and Heydari, 2010), quantity discounts (Li and Liu, 2006; Chaharsooghi et al., 2011), and rebates (Cobb and Johnson, 2014) have been suggested as coordinating incentives that allow the supply chain members to divide the cost savings resulting from coordinating their order quantity and reorder point decisions. In each of these cases, LTD is assumed to be normally distributed. This assumption is not always realistic, particularly when demand per unit time and lead

time are each random variables such that LTD has a compound probability distribution (Eppen and Martin, 1988; Lau and Lau, 2003; Lin, 2008). This paper will incorporate the previously described model (Cobb, 2013) into the two-echelon supply chain problem to show that this model can obviate the need to assume that demand for the entire lead time period is normally distributed.

Mixture of polynomials (MOP) models (Shenoy and West, 2011) are an alternative to the MTE model for approximating PDFs. These models are implemented in the two-level supply chain model in two situations. First, MOPs are used to fit LTD distributions given each possible lead time value when the PDFs have a standard functional form. Next, these distributions are approximated from historical data. In each case, the mixture distribution approach can be applied to calculate a closed-form approximation to the LTD distribution.

The next section describes lead time demand distributions and uses an example dataset to show how standard PDFs can be used as approximations to the LTD distribution. The mixture distribution method is also used for the example problem. Next, the different approximations to the LTD distribution are used to find optimal inventory order quantity and reorder point policies. This is followed by an illustration of how the mixture distribution approach can allow more complicated LTD distributions to be incorporated into such problems. The two-level supply chain model is then introduced, and the mixture distribution approach is used to model LTD in the context of decentralized, centralized, and coordinated supply chains. In the next two sections, the MOP approximations are described for the standard PDF case and the situation where the MOP distributions are estimated from historical data. The final section concludes the paper.

Lead Time Demand Distributions

LTD in a continuous-review inventory system is often assumed to follow a compound probability distribution. Suppose L is a random variable for lead time (LT) and D represents random demand per unit of time (DPUT). LTD is a random variable X determined as

$$X = D_1 + D_2 + \cdots + D_i + \cdots + D_L . \tag{1}$$

Therefore, X is a sum of random, independent and identically distributed (i.i.d.) instances of demand. The mean and variance of X can be calculated as

$$E(X) = E(L) \cdot E(D) \quad \text{and} \quad Var(X) = E(L) \cdot Var(D) + [E(D)]^2 \cdot Var(L) \tag{2}$$

Suppose the data in Table 1 is available to estimate a LTD distribution. This table contains 50 observations of daily demand for an inventory item and 10 observations for lead time on orders of the same item. The expected value of daily demand is $E(D)$=2.88 and the variance of this random variable is $Var(D)$=2.84. Lead

time has an expected value and variance of $E(L)=5.3$ and $Var(L)=6.9$, respectively. According to the formulas in (2), the expected value and variance of LTD are $E(X)=15.26$ and $Var(X)=72.3$, respectively.

The remainder of this section will illustrate three possible methods for approximating the LTD distribution underlying the data in Table 1.

Normal Approximation

The service level is defined as the percentage of replenishment order cycles where demand during lead time is satisfied. To determine the reorder point (R) required to achieve a desired service level, a typical textbook approach is to assume the LTD distribution is normal and use normal distribution tables or Excel formulas. For example, to find the R needed to achieve a 95% service level for the LTD distribution with expected value and variance described in Table 1, the Excel formula NORM.INV(0.95,15.26,72.3^0.5) can be used to find $R=29.25$.

Table 1. Observations for Daily Demand and Lead Time

Daily demand (DPUT)	1	2	2	1	4	1	1	1	1	1
	3	5	3	2	5	4	2	2	3	2
	2	3	3	3	1	3	6	3	6	2
	5	1	5	3	2	6	1	2	4	1
	3	2	2	2	6	5	5	1	3	7
Lead time (LT)	3	5	3	4	4	5	5	10	5	10

The normal approximation to the LTD distribution and the reorder point $R=29.25$ are illustrated graphically in Figure 1. By implementing this policy, we would expect to stockout on 5% of replenishment order cycles.
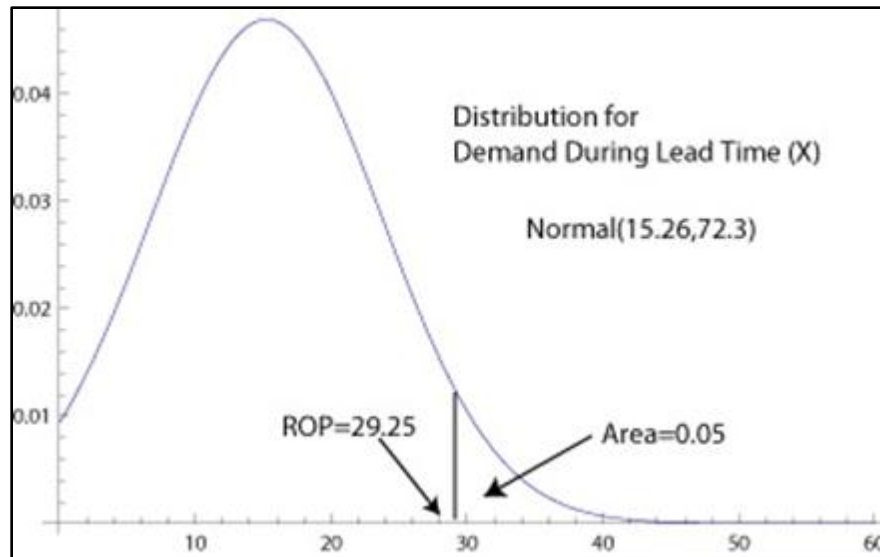
Figure 1. LTD Distribution and Reorder Point.

Negative Binomial Approximation

While the normal approximation to the LTD distribution is popular, there are numerous other approximations that have been suggested in the literature. For example, Taylor (1961) suggests using the negative binomial (NB) distribution for the case where the Poisson distribution is a good fit for DPUT and LT has a gamma distribution. Denote the approximate LTD distribution by $\hat{f}$. Here we assume the NB(r,p) distribution for LTD is

$$\hat{f}(x; r, p) = \frac{\Gamma(x+r)}{x! \cdot \Gamma(r)} (1 - p)^r p^x \qquad x = 0, 1, 2, \dots \qquad (3)$$

where $\Gamma(\cdot)$ is the gamma function. Given this formulation, $E(X) = rp/(1 - p)$ and $Var(X) = E(X)/(1 - p)$. There are two ways of finding a reorder point that will provide an appropriate service level with this NB formulation. Taylor (1961) provides a formula to calculate stockout probabilities as a function of the underlying Poisson and Gamma distributions. These can be calculated for possible reorder point values until a suitable value that meets the service level objective is found. Excel can also be used to enumerate the probabilities of achieving a certain service level with various possible values of *R*. Unfortunately, the built-in NEGBINOM.DIST function only accepts integer values of the *r* parameter, so these probabilities must be calculated using the formula in (3) and the GAMMALN function.

For the data in Table 1, we can use the empirical expected value and variance to solve two equations and two unknowns and obtain *r*=4.08 and *p*=0.79. This NB distribution is shown in Figure 2. The value of *R* that provides approximately a 95% service level is *R*=31.
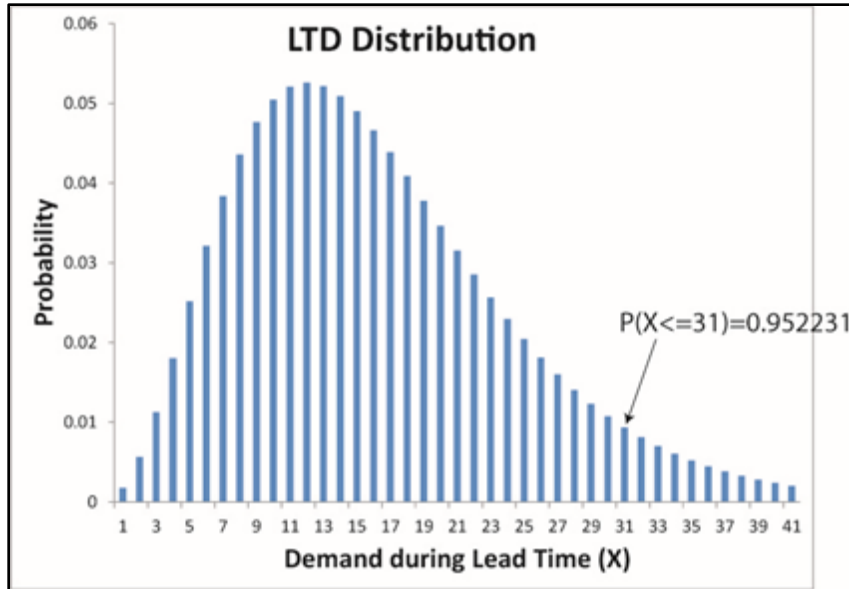
Figure 2.   Negative Binomial Distribution for LTD.

This solution is essentially the same as the one found using Taylor's (1961) analytical formulas. In this case, the Poisson daily demand assumption may be reasonable, because *E*(*D*) and *Var*(*D*) are very similar, a feature of the Poisson distribution.

Mixtures of Truncated Exponentials (MTE) Approximation

The functional form of some PDFs, such as the negative binomial PDF in (3) do not permit integration in closed-form. The means the result of an expected value calculation with such a PDF does not have a functional form that can be used for further computation. These calculations could include, for example, building a cost function to perform nonlinear optimization to find optimal inventory policies. One approach suggested to overcome this limitation is the MTE model (Moral et al., 2001).

An example of a 4-piece, 2-term (ignoring the constant) MTE function that can be used to model LTD given a lead time of *L*=3 for the problem in the previous section is:

$$\hat{f}_{X|\{L=3\}}(x) =$$

$$\begin{cases} -0.7148 + 0.6681 \exp\{0.0325x\} + 0.000048 \exp\{0.989x\} & \text{if } 2.5 \leq x < 5 \\ -96.721 - 318.54 \exp\{-1.945x\} + 96.76 \exp\{0.000128x\} & \text{if } 5 \leq x < 8 \\ 0.1383 - (1.63E - 06) \exp\{x\} + (2.89E - 09)\exp\{1.5x\} & \text{if } 8 \leq x < 11.5 \\ -0.0252 + 0.9786 \exp\{-0.205x\} & \text{if } 11.5 \leq x \leq 17.5 \end{cases} \quad (3)$$

This function was found by simulating 500 series of three observations for daily demand from values in Table 1 using a bootstrapping approach. The

constants—coefficients on the exponential terms and coefficients on the variable *X*—were determined by fitting a function to the simulated histogram. There is an established literature on fitting MTE functions to historical data; in this case, the method suggested by Moral et al. (2002) was utilized. A graphical view of the MTE function overlaid on the simulated histogram is shown in Figure 3.
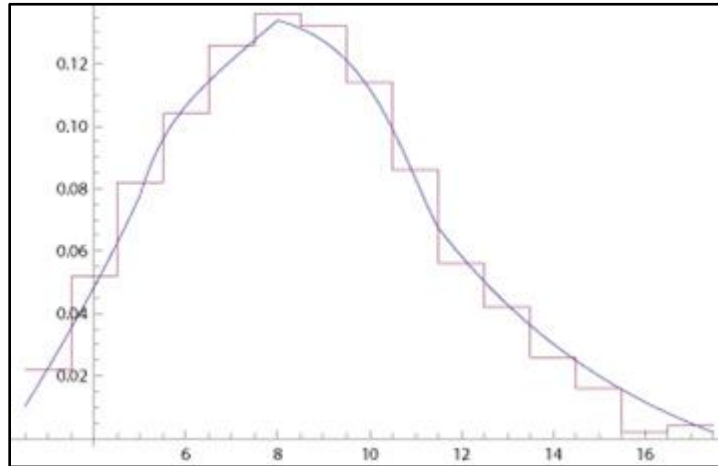
Figure 3.   MTE Distribution for LTD Given a Lead Time of 3 days.

Similar functions $\hat{f}_{X|\{L=l\}}$ can be constructed for the other possible lead time values, *L*=4, 5, and 10. From the data on lead time observations in Table 1, we can estimate *P*(*L*=3)= *P*(*L*=4) = *P*(*L*=10) = 0.2 and *P*(*L*=5)=0.4. A mixture distribution approach (Cobb, 2013) can be employed to find the LTD distribution. Here, the LTD distribution is determined as

$$\hat{f}_X(x) = P(L = 3) \cdot \hat{f}_{X|\{L=3\}}(x) + P(L = 4) \cdot \hat{f}_{X|\{L=4\}}(x) + P(L = 5) \cdot \hat{f}_{X|\{L=5\}}(x) \quad (4)$$

$$+P(L = 10) \cdot \hat{f}_{X|\{L=10\}}(x).$$

The MTE function is shown in Figure 4 overlaid on the previously described NB distribution. This MTE function has 17 pieces and up to six terms in each piece. For illustrative purposes, a continuous NB parameterization is displayed. Since the class of MTE functions is closed under addition, multiplication, and integration (Moral et al., 2001), the mixture distribution resulting from the calculation above is also an MTE function. Thus, it retains the same desirable mathematical properties.

We can perform closed-form integrations of the MTE LTD distribution to find a reorder point that achieves a desired service level. In this case,

$$\int_0^{33.3} \hat{f}_X(x)\, dx \approx 0.95 \,,$$

so we can set *R*=33.3 to obtain a 95% service level.
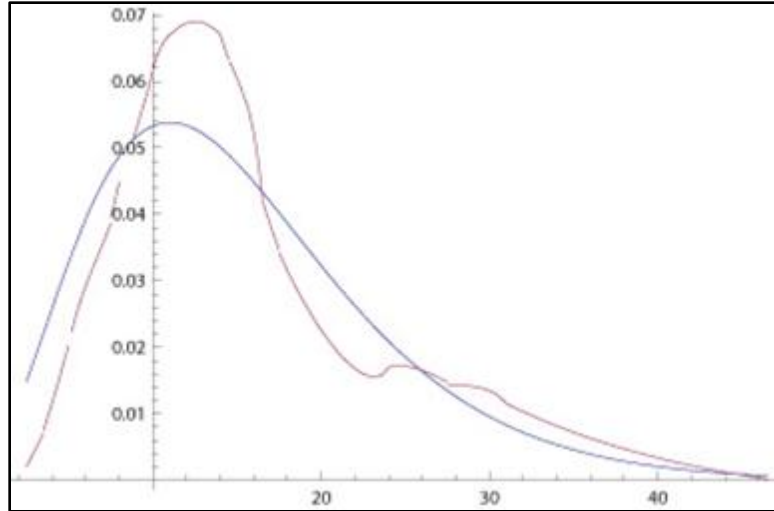
Figure 4. MTE LTD Distribution Overlaid on a NB Approximation.

The next section discusses the use of the MTE function for finding inventory policies in a continuous-review inventory system.

Calculating Inventory Policies

Suppose that we want to determine an optimal order quantity and reorder point in a continuous-review inventory system (a "(*Q,R*)" policy). We will consider four models that could be used to find the best policy given the data available (see Table 1): 1) a normal approximation to the LTD distribution; 2) the NB approximation to the LTD distribution; 3) the MTE mixture distribution; and 4) a simulation-optimization model that simulates lead time and demand values from the empirical distributions developed from Table 1. We term the latter model the "actual" solution.

A simple cost function with no backordering allowed (Johnson and Montgomery, 1974) for this problem is

$$TC(Q,R) = K \cdot \frac{Y}{Q} + \frac{\pi \cdot Y \cdot S_R}{Q} + h \cdot (0.5Q + R - E(X)) \,. \tag{5}$$

In this equation, *K* is the fixed cost per order, *Y* is the expected annual demand, *h* is the holding cost per unit per year, and π is the stockout cost per unit. The average inventory includes safety stock of *R-E(X)*. The shape of the distribution for LTD determines the expected shortage per cycle, S_R. For a given reorder point,

$$S_R = \int_R^\infty (x - R) \cdot \hat{f}_X(x) \, dx. \tag{6}$$

Suppose *Y=E(D)* · 250 working days = 720, *K*=30, *h*=4, and π=5. The key to finding an optimal (*Q,R*) combination is to evaluate S_R as part of constructing the total cost function in (5). With the MTE function, the calculation in (6) can be performed in closed-form, and the result substituted into (5) to obtain a closed-form total cost

function. The expected shortage per cycle as a function of R is an 8-piece expression, with selected terms shown below:

$$\hat{S}_R(r)$$
$$= \begin{cases} \begin{aligned} &-3876.5 + 4.66\exp\{-0.205r\} + 6.31\exp\{-0.172r\} \\ &+3888.1\exp\{0.005r\} - 21.82r - 0.04r^2 \end{aligned} & \text{if } 16.15 \le r < 16.5 \\ \begin{aligned} &-3890.6 + 4.66\exp\{-0.205r\} + 6.31\exp\{-0.172r\} \\ &+3888.1\exp\{0.005r\} + 20.6\exp\{-0.140r\} - 20.64r - 0.07r^2 \end{aligned} & \text{if } 16.5 \le r < 17.5 \\ \begin{aligned} &-3889.2 + 6.31\exp\{-0.172r\} + 3888.1\exp\{0.005r\} \\ &+20.6\exp\{-0.140r\} - 20.76r - 0.07r^2 \end{aligned} & \text{if } 17.5 \le r < 23.5 \\ \quad\vdots \\ -8.74 + 29.87\exp\{-0.78r\} + 0.28r - 0.002r^2 & \text{if } 31 \le r < 46.5\,. \end{cases}$$

Optimization over the resulting cost function is fast. The example here was solved in Mathematica 9.0 by using the ArgMin function. The results obtained using the four methods under consideration are shown in Table 2. An iterative approach (Hadley and Whitin, 1961) in combination with numerical integration was implemented to find the solutions using the normal or NB approximations. The table shows the values Q^* and R^* which—when implemented simultaneously—minimize annual total cost. The computing (CPU) times required to obtain the solutions are also shown. The simulation-optimization solution was simply stopped after running for several hours, and the values obtained were assumed to be the best possible solution.

Table 2. Results for Inventory Policies Determined Using Four Approaches.

Method	Q^*	R^*	TC	CPU (sec.)
Normal Approximation	108	25	482.99	3.57
NB Approximation	110	25	482.89	3.76
MTE Mixture Distribution	110	27	481.10	1.26
Simulation-Optimization	108	27	480.82	∞

Table 2 shows that the MTE mixture distribution works equally as well as the other approaches when implemented to obtain an optimal (Q,R) policy. The next section illustrates that the mixture distribution approach can be used to model more complicated LTD distributions.

State-Dependent Variables

The advantage of the mixture distribution approach (Cobb, 2013) in inventory management problems is that more complex LTD distributions can be constructed by building the model from its components while still maintaining a closed-form representation. In some cases, expert knowledge can be used to assign state-dependent distributions for DPUT and/or LT.

As an illustration, suppose the first row of 10 observations in Table 1 can be associated with replenishment orders where a significant number of missions were

canceled due to weather, creating reduced demand. This reduced demand is assumed to occur on 20% of replenishment orders; thus, demand can be considered to have two states: regular (with 80% probability) and low (20% of the time).

To demonstrate another approach to finding MTE approximations, the dataset in Table 1 will be used in this example to first determine a standard PDF that best fits the empirical data for each demand state. In this case, the log-normal distribution with $\mu = 1.03$ and $\sigma^2 = 0.31$ is selected for the regular state and the $N(0.27, 0.19)$ is chosen for state 2. The demand in each state for a given lead time period is then a sum of i.i.d. log-normal random variables. This sum has no known distribution, but approximations for the PDF of a sum of log-normal random variables exist. Following Cobb et al. (2013), the Fenton-Wilkinson approximation (Fenton, 1960) is implemented and MTE distributions are fit to these approximations for each state and each possible lead time value. For state 1 and state 2, these functions are denoted by $\hat{f}^{(1)}_{X|\{L=l\}}$ and $\hat{f}^{(2)}_{X|\{L=l\}}$, respectively. The conditional PDF for LTD given $L = l$ is then calculated as

$$\hat{f}_{X|\{L=l\}}(x) = 0.8 \cdot \hat{f}^{(1)}_{X|\{L=l\}}(x) + 0.2 \cdot \hat{f}^{(2)}_{X|\{L=l\}}(x).$$

The PDF for LTD is constructed as in equation (4). The new LTD distribution is bi-model, as shown in Figure 5.
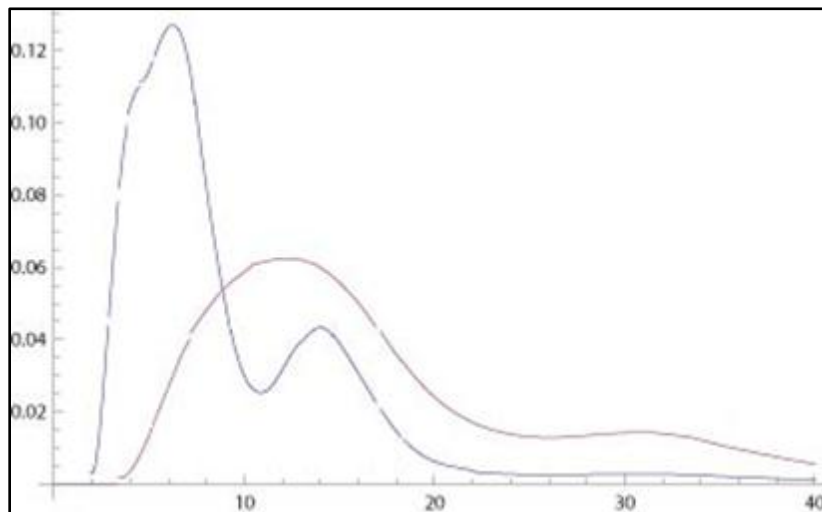
Figure 5.   Mixture Distribution for LTD With State-dependent Demand

Suppose the state-dependent, bi-modal distribution shown in Figure 5 is the correct PDF for LTD. Using this distribution as part of the total cost function to find the optimal (Q,R) policy results in a 21% savings when compared to implementing the policies found earlier using the MTE distribution shown in Figure 4 (or one of the other approximations). The mixture distribution approach still yields a closed-form function for S_R and the optimization is still fast.

Coordinated Supply Chains

In this section, we consider a two-echelon supply chain as depicted in Figure 6. A buyer experiencing random demand places its orders for inventory with the supplier.
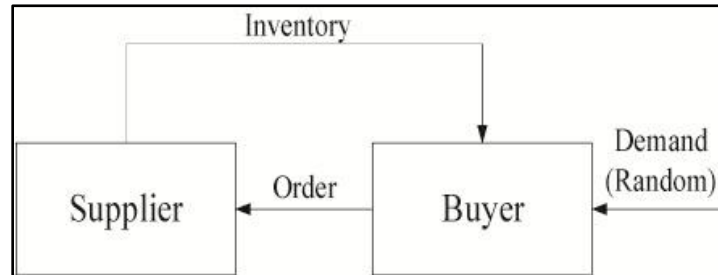
Figure 6. The Cost Function for the Buyer in This Problem is as Follows
(Hadley and Whitin, 1963; Johnson and Montgomery, 1974):

$$TC_b(Q, R, V) = (K_b - V) \cdot \frac{Y}{Q} + \frac{\pi \cdot Y \cdot S_R}{Q} + h_s \cdot (0.5Q + R - E(X)). \tag{7}$$

Most of the notation is the same as for the cost function defined in equation (5). The subscript *b* has been added to the fixed cost per order, annual unit holding cost, and total cost to identify this amount with the buyer. The subscript *s* will similarly represent the seller. The quantity *V* is a rebate provided by the seller to the buyer on a per order basis as an incentive for the buyer to adopt policies that benefit both parties (Cobb and Johnson, 2014). As discussed in the introduction, credit options and price discounts have also been considered in this two-level supply chain as coordination incentives (Chaharsooghi and Heydari, 2010; Chaharsooghi et al., 2011; Li and Liu, 2006).

The cost function for the supplier in this problem is:

$$TC_s(Q, N, V) = \left(\frac{K_s}{N} + V\right) \cdot \frac{Y}{Q} + h_s(N - 1)0.5Q. \tag{8}$$

In this two-level supply chain model, the buyer selects an order quantity and reorder point. The supplier receives orders of size *Q* from the buyer and purchases inventory from its vendors in a quantity that is an integer multiple *N* of the buyer's order size.

The supply chain can operate in one of three modes. First, the buyer can select Q_d and R_d without considering the effect of its selection on the supplier's costs. In response, the supplier selects N_d to minimize its own costs. This is referred to as the *decentralized* mode and since there is no coordination, the rebate amount is *V*=0. Total costs in the supply chain are TC^d = $TC_b(Q_d, R_d, 0) + TC_s(Q_d, N_d, 0)$. Second, the buyer and supplier can agree on values for Q_c, R_c, and N_c that minimize the sum of the cost functions in equations (7) and (8).

Since the members cooperate fully and are *centralized*, there is again no requirement for the supplier to provide a coordination incentive and $V=0$. Total costs in this mode are denoted by $TC^c = TC_b(Q_c,R_c,0)+TC_s(Q_c,N_c,0)$.

If the parties are not centralized but can coordinate their policies, the potential exists to divide cost savings of $TC^+ = TC^d-TC^c$. An interval $[V_{min}, V_{max}]$ can be calculated (Cobb and Johnson, 2013) such that any value for the rebate V in the interval reduces the total costs in the supply chain to centralized levels. The smallest value of the rebate the buyer will accept can be found by solving $TC_b(Q_c,R_c,V) = TC_b(Q_d,R_d,0)$ for V. This value is denoted by V_{min}. The largest value of the rebate the seller will accept can be found by solving $TC_s(Q_c,N_c,V) = TC_s(Q_d,N_d,0)$ for V. This value is denoted by V_{max}. For the example in this paper, we will assume that if the parties agree to coordinate their policies (and implement Q_c, R_c, and N_c), the value of the rebate they select is $\bar{V}=(V_{min}+V_{max})/2$.

All of the two-echelon supply chain models referenced previously assume that demand for the entire lead time period is normally distributed. For the case where both Q and R are selected to minimize total costs, Charharsooghi and Heydari (2010) derive expressions that state the optimal value for Q (in either the decentralized or centralized mode) as a function of the optimal value for R (and vice versa) and the standard normal cumulative density function. The optimal values can be found by iterating between these two expressions. The supplier selects the integer value for N that minimizes its costs subject to the choices of the buyer.

By implementing the mixture distribution approach, we can develop closed-form expressions for the cost functions in (7) and (8) and find optimal solutions in the same manner as the solutions presented earlier in the paper for the (Q,R) inventory model. For illustration, assume $Y=E(D) \cdot 250$ working days = 720, $K_s=K_b=30$, $h_s=h_b=4$, and $\pi=5$. These parameters are the same as used in the earlier example and the supplier has the same cost structure as the buyer (obviously this may not always be true in practice).

For the previous example, employing the MTE mixture distribution in Figure 4 gives the same results in Table 2 for the decentralized case—$Q_d=110$ and $R_d=27$. In this mode, the supplier selects the multiple of the buyer's order quantity that minimizes its costs. Since $TC_s(110,1,0)=197$ and $TC_s(110,2,0)=316$, the supplier selects $N_d=1$. Total supply chain costs in the decentralized mode are $TC^d = 678$.

In the centralized mode, we find the optimal order quantity and reorder point that minimizes $TC_b(Q,R,0)+TC_s(Q,N,0)$ for several possible values of N, then choose the optimal values that give the lowest combined supply chain cost. Again, using the MTE mixture distribution allows the construction of a closed-form total cost function, and optimization over this function in Mathematica is fast. Using the MTE mixture distribution, we find that $Q_c=154$, $R_c=24$, and $N_c=1$. Total supply chain costs in the

centralized mode are TC^d = 648. Table 3 summarizes the optimal values for the decision variables in each mode and the total costs for each party and the supply chain. The answers obtained with the mixture distribution approach are compared with those obtained by using the solutions shown by Chaharsooghi and Heydari (2010).

Table 3. **Optimal Solutions and Total Costs for the Supply Chain in Three Modes of Operation**

Normal	Q	R	N	V	TC_b	TC_s	TC
Decentralized	108	25	1	0	483	200	683
Centralized	151	23	1	0	506	143	649
Coordinated	151	23	1	8.53	466	183	649

MTE Mixture	Q	R	N	V	TC_b	TC_s	TC
Decentralized	110	27	1	0	481	197	678
Centralized	154	24	1	0	507	141	648
Coordinated	154	24	1	8.51	467	181	648

A comparison of the solutions in the decentralized and centralized models shows that the costs in the entire supply chain can be reduced by $TC^+ = TC^d - TC^c$ = 30 if the centralized order quantity and reorder point are implemented. However, these policies increase costs for the buyer by 507-481=26. By using the solutions in Cobb and Johnson (2013) to find the value $\bar{V}$ that divides the cost savings of operating in the centralized mode between the buyer and the seller, the buyer is adequately compensated for increasing its order quantity. The rebate amount for this problem is 8.51 per order cycle. Both members experience costs that are lower than in the decentralized mode.

Alternative Approach

This section introduces an alternative approach to modeling the LTD distribution, the mixture of polynomials (MOP) model.

To illustrate the formation of the LTD distribution, we will utilize the following example from McClain and Thomas (1985) that has also been used by Eppen and Martin (1988). Demand in each time period is normally distributed with mean μ_D=40 and variance σ_D^2=30. Lead time (in periods of one day) may take on the values 7, 12, 14, 15, 16, and 25, and each value has a probability of 1/6.

Normal Approximation

Because the possible values for LT are dispersed over the range from 7 to 25, the distribution for LTD will be multi-modal. As such, there is no one standard PDF that is a good fit. The typical "textbook" approach to modeling the LTD distribution in this case is a normal approximation, and the normal distribution has

been used exclusively in the two-stage supply chain model under continuous review assumptions that will be presented later in the paper.

The normal approximation to the compound LTD distribution has a mean and variance as defined in equation (2). In the example under consideration, $E(L)$ = 14.83 days and LT has a variance of $Var(L)$=29.14. The formulas in (2) are used to determine that $E(X)$=593.33 and $Var(X)$=47047.2. If we want to find the reorder point (R) associated with a certain service level, say 95%, we can use the Excel function NORM.INV(0.95,593.33,47047.2^0.5) to find R=950. The service level is the probability that all customer orders are filled in given order cycle.

Eppen and Martin (1988) demonstrate that for this example, implementing R=950 will actually lead to very different service level than 95%. This is because the true distribution of LTD is a mixture of normal distributions. This is discussed in the next section.

Mixture of Normal Distributions

In this section and for the remainder of the paper, the distribution of LTD is denoted by f_X. The distribution of LTD conditional on a specific value $L = l$ for lead time is denoted by $f_{X|\{L=7\}}$. Similarly, the cumulative distribution function (CDF) for lead time demand is denoted by F_X, while the CDF conditional on a specific lead time $L = l$ is denoted by $F_{X|\{L=7\}}$.

In the example problem, if lead time is $L = 7$ days, the distribution $f_{X|\{L=7\}}$ is a normal PDF with mean 7·40=280 and variance 7·30=210. The means and variances of all the conditional LTD distributions can be similarly calculated. The marginal distribution for LTD is the mixture of normal distributions calculated as

$$f_X(x) = \frac{1}{6}\left(f_{X|\{L=7\}}(x) + f_{X|\{L=12\}}(x) + f_{X|\{L=14\}}(x) + f_{X|\{L=15\}}(x) + f_{X|\{L=16\}}(x) + f_{X|\{L=25\}}(x)\right).$$

The mixture of normal distributions for LTD is shown in Figure 7 overlaid on the normal approximation with mean 593.33 and variance 47047.2.
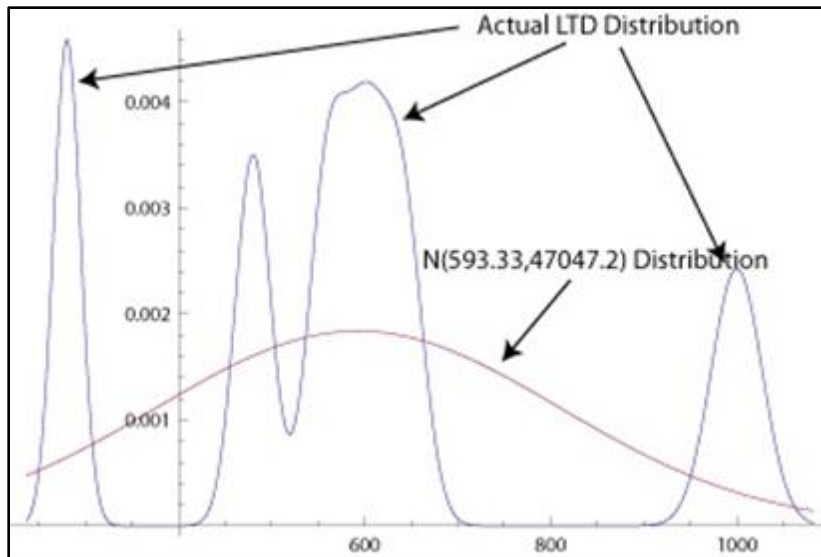
Figure 7.   LTD Distribution and Normal Approximation

Consider the reorder point R=950. We can find the service level (SL) associated with this reorder point by evaluating the conditional CDFs $F_{X|\{L=l\}}$ at 950 and weighting the results (Eppen and Martin, 1988). This is done as follows

$$SL(950) = \frac{1}{6}\left(F_{X|\{L=7\}}(950) + F_{X|\{L=12\}}(950) + F_{X|\{L=14\}}(950) + F_{X|\{L=15\}}(950)\right.$$
$$\left. + F_{X|\{L=16\}}(950) + F_{X|\{L=25\}}(950)\right),$$
$$SL(950) = \frac{1}{6}(1 + 1 + 1 + 1 + 1 + 0.034) = 83.8\%.$$

The conditional values for SL given a certain LT are calculated using the NORM.DIST formula in Excel; for example, the SL given L=25 is NORM.DIST(950,7·40,(7·30)^0.5,1). Calculation of the reorder point associated with a desired service level cannot be done directly with the exact LTD distribution, but a function such as Goal Seek in Excel can be implemented to find that R=1014 provides a 95% SL.

Mixture of Polynomials Approximation

If the functional form of f_X permits closed-form integration, the SL associated with a given reorder point, R, can be determined as

$$SL(R) = \int_0^R f_X(x)\,dx.$$

Since the functional form of the mixture of normal distributions for the example problem cannot be integrated in this way, built-in Excel functions for the normal CDF were used to calculate the service level. This required weighting the results from the conditional distributions for each possible lead time value.

One method for obtaining a closed-form distribution for LTD is the mixture of polynomials (MOP) model (Shenoy and West, 2011). The MOP model can be used to approximate PDFs by piecewise polynomials defined on hypercubes. MOP approximations of standard PDFs, such as the normal distribution, can be developed by using Lagrange interpolating polynomials with Chebyshev points (Shenoy, 2012). This method was used to define a 2-piece, 4th-degree MOP function that approximates the standard normal PDF as

$$g(z) = \begin{cases} 0.398 - 0.038z - 0.322z^2 - 0.148z^3 - 0.020z^4 & \text{if} - 3 \leq z < 0 \\ 0.398 + 0.038z - 0.322z^2 + 0.148z^3 - 0.020z^4 & \text{if } 0 \leq z \leq 3. \end{cases}$$

All piecewise functions in this paper are assumed to equal zero in undefined regions. Using this approximation, the PDF for lead time demand conditional on $L = l$ can be determined as

$$\hat{f}_{X|\{L=l\}}(x) = \frac{1}{\sqrt{l \cdot \sigma_D^2}} g\left(\frac{x - l \cdot \mu_D}{\sqrt{l \cdot \sigma_D^2}}\right).$$

The MOP function $\hat{f}_X$ that approximates the PDF f_X for LTD is determined as

$$\hat{f}_X(x) = \sum_{i=1}^{k} P(L = l_i) \cdot \hat{f}_{X|\{L=l\}}(x).$$

The index i has been added to the k possible values for lead time. This method can be used when the DPUT distribution is normal, or at least in any situation where we are willing to approximate the DPUT distribution with a normal distribution. Notice, this would be very different (and more accurate) than approximating the distribution for demand over the entire lead time with a normal distribution.

For the example problem, $\hat{f}_X$ is calculated as

$$\hat{f}_X(x) = \frac{1}{6} \cdot \left(\begin{array}{c} \hat{f}_{X|\{L=7\}}(x) + \hat{f}_{X|\{L=12\}}(x) + \hat{f}_{X|\{L=14\}}(x) + \hat{f}_{X|\{L=15\}}(x) \\ + \hat{f}_{X|\{L=16\}}(x) + \hat{f}_{X|\{L=25\}}(x) \end{array} \right).$$

The MOP approximation to the LTD distribution is a relatively compact 15-piece, 4th-degree polynomial defined as

$$\hat{f}_X(x)$$
$$= \begin{cases} -20.93 + 0.333x - 0.002x^2 + 5.24 \times 10^{-6}x^3 - 5.18 \times 10^{-9}x^4 & \text{if } 236.53 \leq x < 280 \\ -45.50 + 0.596x - 0.003x^2 + 6.36 \times 10^{-6}x^3 - 5.18 \times 10^{-9}x^4 & 280.00 \leq x < 323.47 \\ \vdots & \vdots \\ -261.34 + 0.996x - 0.001x^2 + 9.03 \times 10^{-7}x^3 - 2.15 \times 10^{-10}x^4 & \text{if } 1000 \leq x \leq 1082.16. \end{cases}$$

This closed-form function for the lead time demand distribution is easy to manipulate. It can be easily integrated to find a closed-form function for the CDF of lead time demand as follows:

$\hat{F}_X(x)$

$$= \begin{cases} 1059.39 - 20.93x + 0.167x^2 - 0.0007x^3 - 1.31 \times 10^{-6}x^4 - 1.036 \times 10^{-9}x^5 & \text{if } 236.53 \le x < 280 \\ 2770.91 - 45.50x + 0.298x^2 - 0.0010x^3 - 1.59 \times 10^{-6}x^4 - 1.036 \times 10^{-9}x^5 & 280.00 \ \le x < 323.47 \\ 0.1667 & 323.47 \ \le x < 423.01 \\ 4621.62 - 52.22x + 0.236x^2 - 0.0005x^3 - 5.99 \times 10^{-7}x^4 - 2.692 \times 10^{-10}x^5 & 423.01 \ \le x < 480 \\ \vdots & \vdots \\ 0.8333 & 705.73 \le x < 917.84 \\ \vdots & \vdots \\ 54808 - 261.34x + 0.498x^2 - 0.0005x^3 + 2.26 \times 10^{-7}x^4 - 4.30 \times 10^{-11}x^5 & \text{if } 1000 \le x \le 1082.16 \\ 1 & x > 1082.16. \end{cases}$$

Using this CDF to find the service level for a reorder point of 950 gives
$$\hat{F}_X(950) = \widehat{SL}(950) = 83.9\%.$$

Evaluating 950 gives $\hat{F}_X$ at each possible reorder point value between *E(X)* and the first value for *R* that provides a 95% service level gives *R*=1015 and this calculation requires 0.05 seconds of computing time.

In summary, the LTD distribution can be modeled using one normal distribution as an approximation over the entire lead time period. This method leads to poor results when calculating the service level for a given lead time and for finding a reorder point that achieves a targeted service level. The actual distribution for the example problem is a mixture of normal distributions, and Excel formulas and built-in functions can be utilized to find service levels and reorder points, albeit indirectly. The MOP model offers an alternative to constructing a closed-form LTD distribution that can be directly integrated and evaluated to find a CDF for lead time demand, service levels, and reorder points. As discussed in the remainder of the paper, this distribution can be utilized to find optimal inventory policies in a two-level supply chain under uncertain demand and continuous review assumptions.

To implement the MOP mixture distribution approach to find an optimal order quantity/reorder point combination, we first develop a closed-form expression for the expected shortage per cycle in (6) using the previously defined PDF $\hat{f}_X$. This function is an 8-piece, 6th-degree polynomial defined as

$\hat{S}_R(R)$

$$= \begin{cases} -3.16 \times 10^{-6} + 32362.4R - 138.1R^2 + 0.314R^3 \\ \qquad -0.0004R^4 + 2.75 \times 10^{-7}R^5 - 7.81 \times 10^{-11}R^6 & \text{if } 593.33 \le R < 600 \\ -4.11 \times 10^{-6} + 40257.0R - 164.4R^2 + 0.358R^3 \\ \qquad -0.0004R^4 + 2.87 \times 10^{-7}R^5 - 7.81 \times 10^{-11}R^6 & \text{if } 600 \le R < 621.48 \\ \vdots & \vdots \\ -9.57 \times 10^{-6} + 54807.6R - 130.7R^2 + 0.166R^3 \\ \qquad -0.0001R^4 + 4.52 \times 10^{-8}R^5 - 7.16 \times 10^{-12}R^6 & \text{if } 1000 \le R \le 1082.16. \end{cases}$$

Decentralized Solution

This function for S_R shown above can be substituted into equation (5) to create a piecewise cost function for the buyer. In this example, we will assume

$K_b = 50, h_b = 5$, and $\pi = 6$. Expected annual demand is based on 150 working days and equals $Y = 150 \cdot \mu D = 150 \cdot 40 = 6000$. This cost function is displayed as a function of Q for three values of R in Figure 8. By inspection, we can see that the optimal order quantity is lower for smaller values of R. In other words, we can better control costs by simultaneously selecting the order quantity and reorder point.
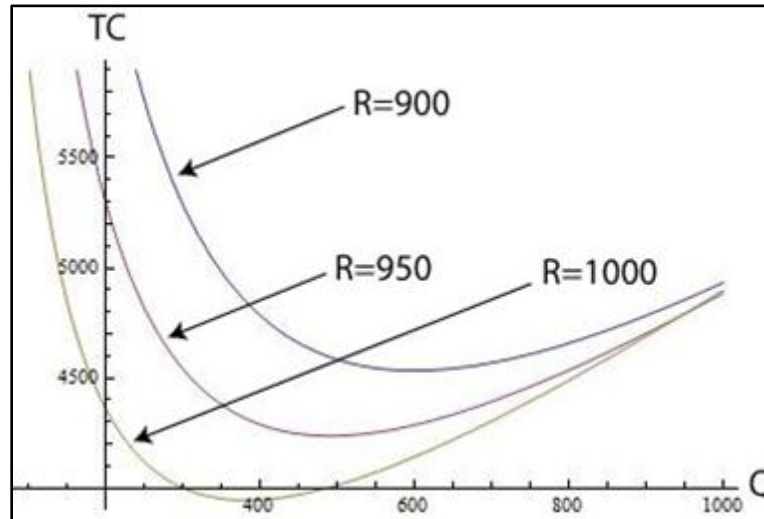
Figure 8.   Buyers' Cost as a Function of Order Quantity for Three Values of the Reorder Point

Optimization over the cost function developed using the MOP distribution for LTD is fast. Notice that the function for expected shortage per cycle is a MOP. When this expression is inserted in the cost function in (5), the result is a function with polynomial terms and some terms with Q in the denominator. The example here was solved using Mathematica 9.0 by using the ArgMin function. The resulting solutions are Q_d=364 and R_d=1014 with $TC_b(Q_d, R_d, 0)$=3924. The supplier's best response is to set N_d=1 and incur costs of $TC_s(Q_d, N_d, 0)$=2472, and total costs in the supply chain are TC^d=6396. The computing time expended is less than one second.

An iterative approach (Hadley and Whitin, 1961) in combination with numerical integration was implemented to find the solutions using the normal approximation to the LTD distribution using the partial solution provided by Chaharsooghi and Heydari (2010). The solutions are $Q_N^d = 447$ and $R_N^d = 925$. If these solutions are inserted in the "actual" cost function (the one developed with the MOP distribution for LTD), the result is $TC_b(Q_N^d, R_N^d, 0) = 4454$. Using the MOP mixture distribution yields an improvement in costs of 4454 – 3924 = 530 or 12%.

Centralized Solution

The closed-form function S_R for expected shortage per cycle developed using the MOP distribution for LTD can also be used to derive a cost function for the entire

supply chain in the centralized case. In this example we assume, K_s=150 and h_s=12.5. This function $TC^c(Q,R,N)$ is used to find the optimal combination (Q_c, R_c) for several possible values of the supplier's decision variable N. The value of N producing the lowest total cost once the corresponding optimal values for order quantity and reorder point are selected is deemed the best supplier policy. Typically, solving for the optimal (Q_c, R_c) with N=1 then checking to see if N=2 or N=3 produces a better solution is adequate.

The best order quantity in the centralized model for a given reorder point is higher than the optimal order quantity in the decentralized case. This is illustrated in Figure 9, where the total costs are graphed as a function of Q for the decentralized and centralized cases assuming a reorder point of R=1000. Visually, the centralized cost function appears to reach a minimum at a larger value of Q.
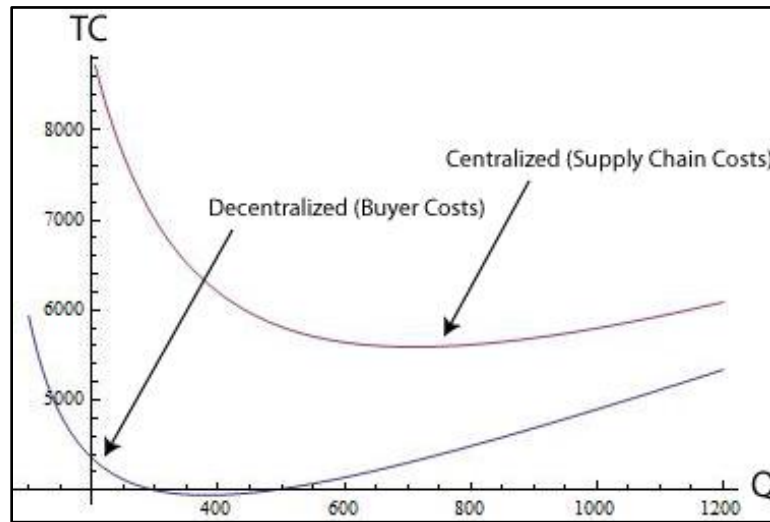
Figure 9.   Decentralized and Centralized Costs as a Function of Order Quantity for a Reorder Point of 1000.

The closed-form centralized cost function can again be easily utilized to find the optimal policy of (Q_c, R_c, N_c) = (718,993,1). The costs for the parties at the optimal solutions are as follows: $TC_b(Q_c, N_c, 0)$=4333; $TC_s(Q_c, N_c, V)$=1254; TC^c=5587. The solution again takes around one second of computing time to obtain.

The buyer incurs higher costs by 4333-3924=409 in the decentralized mode as compared to the centralized mode, where the supplier's costs are reduced by 2472-1254=1218. Total costs in the supply chain are lower than in the decentralized mode by 6396-5587=809.

The corresponding centralized solutions found using the normal approximation are $Q_N^c = 802$ and $R_N^c = 857$. If these solutions are inserted in the "actual" cost function for the supply chain (the one developed with the MOP

distribution for LTD), the result is TC_N^c = 5891. Using the MOP mixture distribution yields an improvement in costs of 5891 – 5587 = 304 or 5% in the centralized mode.

Coordinated Solution

While the buyer would prefer that the supply chain operate in decentralized mode and the supplier wants a centralized solution, both parties can potentially compromise and coordinate to divide the centralized costs savings. The closed-form cost functions developed using the MOP method again provide an approach to determine a supply chain coordination mechanism to make this work.

The buyer will accept a per order rebate as low as V_{min}, which can be found by solving $TC_b(718,993, V)$ = 3924, or 4333-8.356v=3924. The solution is V_{min}=49. The supplier will accept a per order rebate as high as V_{max}, which can be found by solving $TC_s(718,993, V)$ = 2472, or 1254+8.356v=2472. The solution is V_{max}=146.

In this example, at the centralized optimal order quantity, there are Y/Q_c=6000/718=8.356 order cycles per year, so the minimum incentive entails rebates of 8.356·49=409 and the maximum incentive entails rebates of 8.356·146=1218. One solution is to implement $\bar{V}=(V_{min}+V_{max})/2$=97.5 and require the supplier to provide 815 in rebates to the buyer. This brings the buyer's total costs to 3518, the supplier's total costs to 2069, and supply chain costs to 5587, which is the centralized level.

Empirical MOP Distributions

In previous sections of the report, we have seen the MTE approach implemented with empirical data, and the MOP approach implemented when the underlying LT and DPUT distributions were discrete and represented by a standard continuous PDF, respectively. This section will illustrate an approach to estimating an MOP function to approximate the LTD distribution when empirical data is available.

We suppose a modest amount of historical data is available for daily demand and lead times. In this example, we use a dataset **D** of **N** = 500 observations for daily demand with sample mean 39.66 and sample variance 30.64. These values are a random sample from the N(40,30) distribution. Fifty observations (N_L=50) of historical lead time values are available in dataset D_L with sample mean 14.94 and sample variance 37.04. There are values in the dataset for each possible lead time value. Empirical histograms are displayed for this sample data in Figure 10.
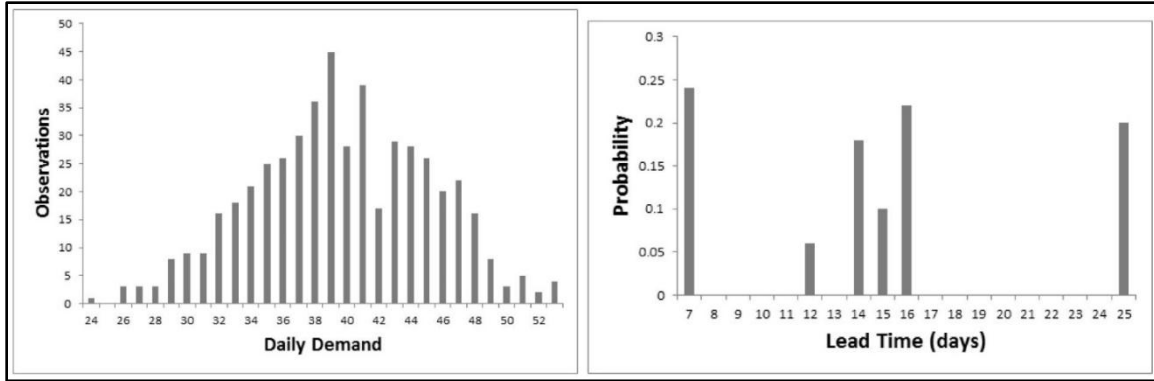
Figure 10. Histograms for the Daily Demand and Lead Time Data.

The datasets depicted in Figure 10 will be used to develop LTD distributions given each possible empirical lead time value. We assume that the observations of daily demand are i.i.d. Since this is the case, we use the dataset of **N** = 500 values to create six smaller datasets for daily demand given the possible values for *L*. For example, the first seven values for daily demand are 40, 37, 47, 47, 45, 34, and 31. These are summed to 281 to determine the first sample value in the dataset for LTD demand given a lead time of 7 days. The next seven consecutive values in the dataset sum to 263, so this is the second value in the *L*=7 dataset, and so on. This smaller dataset has 71 observations.

B-Spline Estimation of MOPs

Lopez-Cruz et al. (2014) suggest using a linear combination of B-spline functions to construct MOP approximations from datasets where the parametric form of the underlying probability distribution is unknown. B-spline functions are piecewise polynomial functions defined by the number of control points, *n*+1, and the degree of the polynomial, *d*. The control points define a knot vector **t** = {$t_0, t_1, t_2, \ldots, t_n$}.

B-spline functions (Zong and Lam, 1998) have two definitions, one when *d*=1 and another when *d*>1. When *d*=1, the functions are defined as

$$B_{j,1}(x) = \begin{cases} 1 & t_j \le x < t_{j+1} \\ 0 & otherwise \, . \end{cases}$$

For *d* >1, the functions are calculated as

$$B_{j,k}(x) = \left(\frac{x - t_j}{t_{j+k-1} - t_j} \right) \cdot B_{j,k-1}(x) + \left(\frac{t_{j+k} - x}{t_{j+k} - t_{j+1}} \right) \cdot B_{j+1,k-1}(x).$$

The control points are indexed by *j*=0,…,*n* and the degree of the functions are indexed by *k*=1,…,*d*. For this example, we assume t_0 is the smallest value in the dataset, t_n is the largest value in the dataset, and that the intervals between all of the

knots are the same distance. The resulting functions $Bj_{,k}$ are referred to as uniform B-splines. The B-spline functions are used to form a *n*-piece MOP density function

$$\hat{f}_{X|L=l}(x) = \sum_{i=1}^{m} \alpha_i \cdot B_{i,d}(x) \tag{9}$$

by selecting mixing coefficients α_i, i=1,…,m, where $m=n+d$-1. Thus, the PDF for LTD given a lead time $L=l$ will be a mixture of the m B-splines of order d.

Suppose a dataset $\mathbf{D} = \{x_1, …, x_{N_l}\}$ of observations of LTD, X, given a specific lead time $L = l$ is available. Zong (2006) suggests using the following iterative formula for determining the maximum likelihood estimators, $\{\hat{\alpha}_1, …, \hat{\alpha}_{1m}\}$, for the mixing coefficients in (9):

$$\hat{\alpha}_i^{(q)} = \frac{d}{N_l \cdot (t_i - t_{i-d})} \sum_{x \epsilon D} \frac{\hat{\alpha}_i^{(q-1)} B_{i,d}(x)}{\hat{f}_{X|L=l}\left(x; \hat{\alpha}_i^{(q-1)}\right)}. \tag{10}$$

Beginning with equivalent values for each α_i, the expression in (10) is used iteratively for i=1,…,m until $\left|(L^{(q)} - L^{(q-1)})/L^{(q)}\right| < \varepsilon$ where $L^{(q)}$ is the log-likelihood of $\mathbf{D}$ given $\hat{f}_{X|L=l}\left(x; \hat{\alpha}_i^{(q)}\right)$ at iteration q in the optimization process. Using $\varepsilon=10^{-6}$ appears to be adequate for most applications (López-Cruz et al., 2014).

The goal is to develop PDFs $\hat{f}_{X|L=l}$ for LTD given each possible value for L that are reasonably accurate; however, we would like the number of pieces and the degree of the polynomial functions comprising the MOP densities to be as small as possible to avoid overfitting and speed up computation of optimal inventory policies. Thus, we will consider several possible values for d and n for each PDF and select the approximation that maximizes the Bayesian information criterion (BIC) calculated as

$$BIC\left(\hat{f}_{X|L=l}(x), D\right) = L\left(D | \hat{f}_{X|L=l}(x)\right) - \frac{(m-1)logN_l}{2}. \tag{11}$$

The second term in the BIC expression is a penalty for adding parameters to the model. The approach we will take is to find the values of d and n that maximize the BIC score for the PDF of LTD given the most likely value $L=l$, then use those parameters to estimate each of the conditional PDFs for LTD given each possible lead time value. In practice, once we settle on good values for d and n, this step could be avoided. Alternatively, we have found that d=3 and n=3 seem to be adequate for many problems.

Notice from Figure 10 that L=7 is the lead time value that occurs most frequently in the empirical data. Thus, we will begin by constructing the PDF $\hat{f}_{X|L=7}$ for LTD given L=7 with an MOP function constructed from B-splines with the values

of d and n that maximize the BIC score. We tested all possible combinations of parameter values where d and n are in the set {2,3,4,5,6}.

The best value for the BIC score is –290.4 and is achieved for n=2 and d=3. The PDF is shown graphically in the left panel of Figure 11 overlaid on the $N(7{\cdot}40,7{\cdot}30)$ distribution. Recall that the MOP is not fit to the normal PDF, but rather a small sample of data generated from the normal PDF. The four B-splines used to construct the MOP function are shown in the right panel of Figure 11. The mixing coefficients determined via 37 iterations of equation (10) are α_1=0.04, α_2=0.07, α_3=0.83, and α_4=0.06.
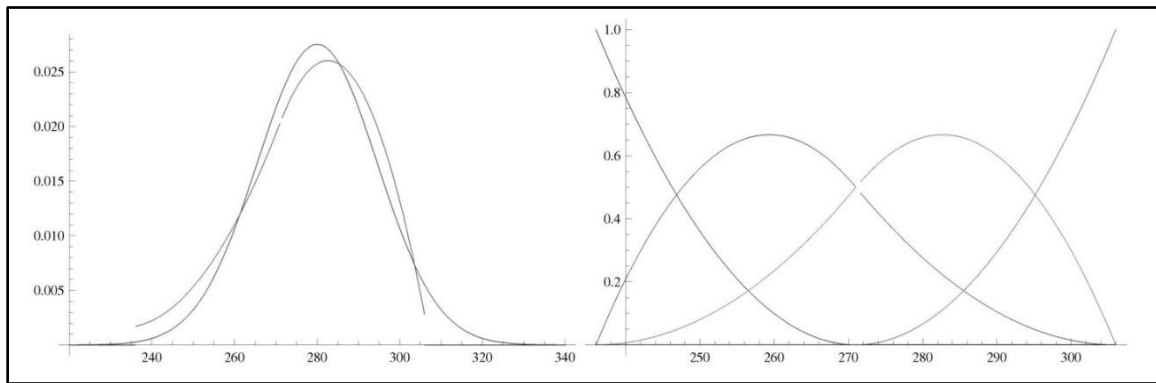

Figure 11. PDF for LTD Given a Lead Time of 7 days (left) and B-splines Used to Construct the MOP Function (right)

The MOP fitting process is repeated for l=12,14,15,16,25 to find each conditional PDF $\hat{f}_{X|L=l}$. The mixture distribution for LTD is then calculated (Cobb, 2013) as

$$\hat{f}_X(x) = \sum_{k=1}^{6} P\big(L = l^{(k)}\big) \cdot \hat{f}_{X|L=l^{(k)}}(x).$$

The superscript (k) has been added as an index for the number of possible LT values. Here, k=1 corresponds with $L = l^{(1)} = 7, k = 2$ corresponds with $L = l^{(2)} = 12$, and so on. The resulting distribution $\hat{f}_X$ is displayed in Figure 12 overlaid on the actual distribution for LTD. Again, $\hat{f}_X$ was created from the sample data without knowledge of the underlying LTD distribution. The function $\hat{f}_X$ is relatively compact---it contains 15 pieces and is a 2nd degree polynomial. Since the class of MOP functions is closed under addition and multiplication, the resulting mixture distribution is also an MOP function.
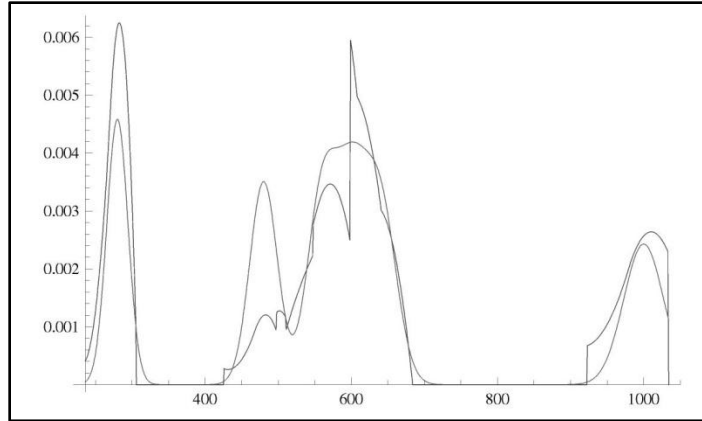
Figure 12. MOP Mixture Distribution for LTD Overlaid on the Actual LTD Distribution.

The MOP distribution for LTD is utilized to find a closed-form expression for expected shortage per cycle as

$$\hat{S}_R(R) = \int_R^{X_{max}} (x - R) \cdot \hat{f}_X(x) \ dx \ .$$

The value X_{max} is the largest value for which $\hat{f}_X$ is defined as non-zero. In the previous example, $\hat{S}_R$ is a 9-piece, 5th degree polynomial. Again, both (x-R) and $\hat{f}_X$ are MOP functions, so the resulting expression for $\hat{S}_R$ is an MOP function because this class of functions is closed under multiplication and integration. Since $\hat{S}_R$ has a closed-form, the cost function TC_b can be derived and used to find optimal inventory policies. The function TC_b is a 9-piece function with polynomial terms in Q and R and some terms that have a polynomial numerator and a Q term in the denominator.

An outline of the process for using empirical data on DPUT and LT to construct the LTD distribution $\hat{f}_X$ and buyer's cost function is as follows:

1. Collect datasets of observations for DPUT and LT. Each of the possible observations for LT form the set Ω_L of values $l^{(1)}$, …, $l^{(K)}$

2. Create datasets for LTD given each possible LT value by summing $l^{(K)}$ consecutive values as many times as possible from the DPUT dataset.

3. For the most likely LT value in the discrete empirical distribution, use the corresponding LTD dataset to calculate the MOP distribution $\hat{f}_{X|L=l^{(k)}}$ that maximizes the BIC score in (11) by testing different values of d and n and using equation (10) iteratively until convergence.

4. Using the values of d and n that maximize BIC in the previous step for $L= l^{(K)}$ create MOP distributions $\hat{f}_X$ for each additional value in Ω_L.

5. Calculate the LTD distribution as

$$\hat{f}_X(x) = \sum_{k=1}^{K} P\left(L = l^{(k)}\right) \cdot \hat{f}_{X|L=l^{(k)}}(x). \tag{12}$$

6. Calculate the expected shortage per cycle function $\hat{S}_R$ and insert this in the buyer's cost function to create TC_b.

Optimal Policies and Results

This example again assumes K_b=50, K_s=150, h_b=5, h_s=12.5, and π=6. We assume 250 working days per year so Y = 250·39.66 ≈ 10000. The results obtained for optimal inventory policies using the MOP approximation developed from empirical data using the B-spline estimation method are again compared to the solution developed by Chaharsooghi and Heydari (2011) (the CH solution) that assumes a normal distribution for LTD. The total costs are obtained by simulating the expected total costs on 10,000 simulation trials using the actual normal distributions for daily demand and the discrete distribution for lead time. Clearly, the cost savings are dependent on the parameters.

Table 4.   Optimal Solutions and Total Costs for the Supply Chain

Decentralized	Q	R	N	TC_b	% Dec.	TC
CH (Normal)	558	999	1	4531	--	683
MOP (Mixture Dist.)	455	1019	1	4455	2.0%	649
Coordinated	Q	R	N	TC^c	% Dec	TC
CH (Normal)	1012	926	1	6872	--	678
MOP (Mixture Dist.)	909	1004	1	6628	3.7%	648

Conclusions

This paper serves as an introduction to using a mixture distribution approach to modeling the probability density function for lead time demand in problems where a continuous review inventory system is implemented. First, construction of the lead time distribution was illustrated. This distribution was then utilized to determine optimal order policies in cases where a buyer makes its decisions alone, and then when members of a two-level supply chain coordinate their actions.

Several approaches to modeling the LTD distribution were illustrated. We first considered using the normal CDF for each possible lead time value and weighting the results with discrete probabilities. This method allows calculation of service level probabilities, but does not provide a closed-form approximation to the LTD distribution that can be used for determining optimal inventory policies. Next, a mixture of truncated exponentials approximation was utilized to model the LTD distribution. The distributions for each possible lead time value were estimated, then the results were weighted to create a mixture distribution for LTD. Finally, mixture of

polynomial approximations were considered. The use of this model was demonstrated as an approximation to both standard probability density functions and as a fit to actual historical data. Use of either approximation technique allows determination of optimal inventory policies and provides significant cost savings as compared to a solution where demand over the entire lead time period is assumed to be normal.

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