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14. ABSTRACT **This paper applies the Maximum Entropy Principle (MEP) to convert point estimates to probability distributions to be used as priors for Bayesian reliability analysis of missile data, and illustrate this approach by applying the priors to a Bayesian reliability model of a missile system.**

15. SUBJECT TERMS **priors, Bayesian, missile reliability, maximum entropy**

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Bayesian Missile System Reliability from Point Estimates

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Introduction

Bayesian estimation of missile system reliability requires a prior distribution to be specified for a parameter of interest, such as the system failure rate or the system reliability for example. However, many current parameter estimates available to the missile community are in the form of point estimates or design values without uncertainty assessments such as the MIL-HDBK-217 reliability predictions (MIL-HDBK-217FN2, February 1995). This lack of error bounds makes it problematic to utilize the point estimates as priors for Bayesian reliability assessment. In this paper, we propose using the Maximum Entropy Principle (MEP) to convert point estimates or design values, as obtained *pro forma* from external sources, to prior probability distributions convenient for Bayesian analysis. This approach utilizes the physical concept of entropy to search for the appropriate distribution that maximizes this entropy, which characterizes the most uncertain prior distribution with the available point estimates as constraints (e.g. Bayesian Inference and Maximum Entropy Methods in Science and Engineering, July 2007). We illustrate the utility of the above priors by applying them to a Bayesian reliability model of a missile system.

Maximum Entropy Density

The entropy[2] (aka relative entropy, Kullback-Leibler divergence), of a continuous

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probability density function $\rho(\theta)$ is defined by the physicist E.T. Jaynes as (Berger, 2010)

$$\mathcal{E}(\rho) = -E^\rho \left(\ln \frac{\rho(\theta)}{\rho_0(\theta)} \right) = -\int \rho(\theta) \ln \frac{\rho(\theta)}{\rho_0(\theta)} \, d\theta \ ,$$

where $\rho_0(\theta)$ is the natural invariant uninformative density in the Bayesian sense for the class of distribution under consideration. Missile pass/fail data are assumed to belong to the class of Bernoulli distributions $Binomial(1, \theta)$ (where $Binomial(n, \theta)$ denotes the binomial distribution with sample size n and probability θ) with θ as the reliability parameter. For this class of distribution, the natural parameterization invariant uninformative prior ρ_0 is given by the Jeffreys' prior

$$\rho_0(\theta) = \frac{1}{\pi\sqrt{\theta(1-\theta)}} \ .$$

Without any additional constraints, we note that ρ_0 is the pdf that maximizes the entropy. Suppose in addition, we have some information about the moments of ρ (such as the mean, variance, quantiles, etc.), it is desired to look for a pdf that maximizes the entropy subject to these constraints. In other words, we are looking for the most uncertain pdf that is consistent with the given prior information. Using variational calculus, this density can be written as (Berger, 2010)

$$\tilde{\rho}(\theta) = \frac{\rho_0(\theta) \, e^{\sum_i g_i(\theta)\omega_i}}{\int \rho_0(\theta) \, e^{\sum_i g_i(\theta)\omega_i} \, d\theta} \ , \tag{1}$$

where the parameters ω_i are determined by the known information

$$E^{\tilde{\rho}}(g_i(\theta)) = \mu_i, \quad i = 1 \dots n,$$

and $g_i(\theta)$ are predetermined functions of the parameter θ. To illustrate, let us suppose that only the mean is known (i.e., mean reliability μ). Then we have a single constraint $g_1(\theta) = \theta$, and

$$E^{\tilde{\rho}}(\theta) = \mu \ .$$

Claim: The maximum entropy probability density with mean μ is given by

$$\tilde{\rho}(\theta) = \frac{e^{\left(-\frac{\omega}{2}+\omega\theta\right)}}{I_0\left(\frac{\omega}{2}\right)\pi\sqrt{\theta(1-\theta)}} \, ,$$

where $I_n(x)$ is the modified Bessel function of the first kind of order n.

To prove this, we only need to show that

$$\int_0^1 \rho_0(\theta)\, e^{\omega\theta}\, d\theta = e^{\frac{\omega}{2}} I_0\left(\frac{\omega}{2}\right).$$

From (Abramowitz & Stegun, 1972, 13.2.1), the following fact holds

$$\int_0^1 \rho_0(\theta)\, e^{\omega\theta}\, d\theta = M\left(\frac{1}{2},1,\omega\right),$$

where *M(a,b,z)* is the Kummer's function (Abramowitz & Stegun, 1972, 13.1.2). Finally, from (Abramowitz & Stegun, 1972, 13.6.3)

$$M\left(\frac{1}{2},1,\omega\right) = e^{\frac{\omega}{2}} I_0\left(\frac{\omega}{2}\right).$$

QED.

As mentioned before in (1), the parameter ω is determined by solving the equation

$$\int_0^1 \theta\tilde{\rho}(\theta)\, d\theta = \mu.$$

In fact, this is the same as solving the equation

$$\left(1 + \frac{I_1\left(\frac{\omega}{2}\right)}{I_0\left(\frac{\omega}{2}\right)}\right) = 2\mu, \tag{2}$$

which can be done using numerical techniques such as Newton's method. Above formula follows immediately from the fact that $I_0{}'(x) = I_1(x)$ (Abramowitz & Stegun, 1972, 9.6.27) and

$$\int_0^1 \theta \rho_0(\theta)\, e^{\omega\theta}\, d\theta = \frac{d}{d\omega} \int_0^1 \rho_0(\theta)\, e^{\omega\theta}\, d\theta = \frac{d}{d\omega}\left(e^{\frac{\omega}{2}} I_0\left(\frac{\omega}{2}\right)\right)$$

$$= \frac{e^{\frac{\omega}{2}}}{2}\left(I_0\left(\frac{\omega}{2}\right) + I_1\left(\frac{\omega}{2}\right)\right).$$

Example. Suppose a missile point reliability estimate has the value .92. Then equation (2) gives $\omega = 6.96$.

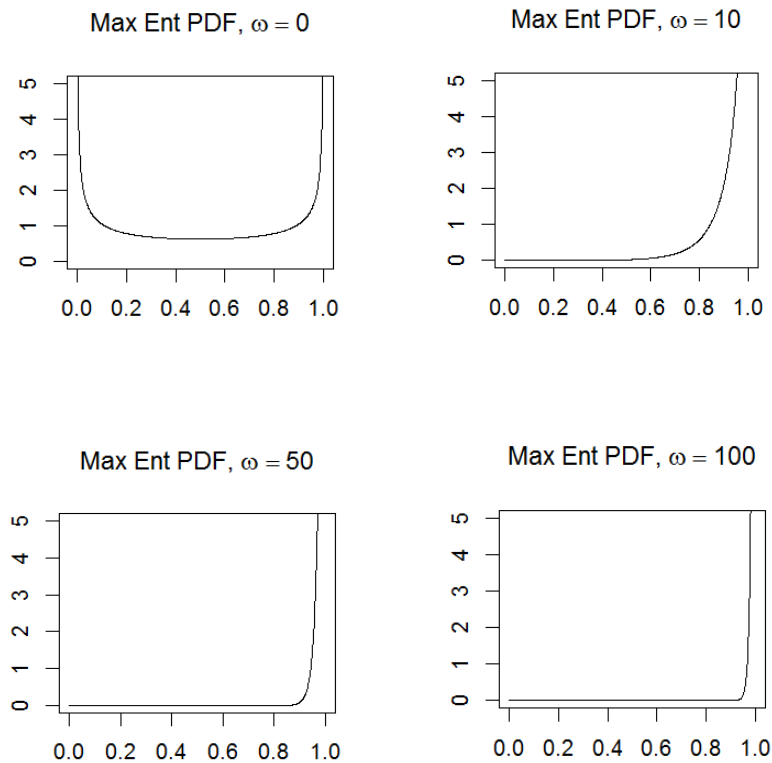Some representative max entropy pdf's are plotted below.

Figure 1. Examples of maximum entropy priors with different values of ω.

Missile System Reliability Model

Missile system test data are usually provided as $(x_1, t_1), (x_2, t_2), \ldots, (x_m, t_m)$, where each x_i is Bernoulli random variables with values 0 (fail) and 1 (success) and t_i is the time that the data are collected. Our goal is to estimate the reliability θ using this data set. Recall that reliability can be dependent on other factors (e.g., time) and the method used to specify such dependence is to construct a reliability regression model (Hamada, Wilson, Reese, & Martz, 2010). For simplicity's sake, we will only consider time as the dependent variable and construct an intercept-exponential model for reliability,

$$R(t) = \theta(t) = e^{a_0 + a_1 t},$$

where a_0 and a_1 are the unknown parameters to be estimated (the notations R and θ are interchangeable). For this to be a valid reliability model, there is an implicit assumption that $a_0 + a_1 t \leq 0$ at all times, so that $R(t) \leq 1$. This condition is satisfied whenever a_0 (the intercept, which provides an initial reliability estimate e^{a_0} at $t = 0$) and a_1 (the rate of failure) are both less than or equal to 0. The computed likelihood function becomes

$$L(\text{data}| \ a_0, a_1) = \prod_{i=1}^{m} R(t_i)^{x_i} \big(1 - R(t_i)\big)^{1-x_i}.$$

Note that missile pass/fail data are usually left and right censored data, since a failed test usually indicate the failure occurred sometime in the past, while a successful test means the system is still good for some time after the test. Therefore, this is the appropriate form of the likelihood equation given censored data (National Institute of Standards, 2013). We now describe some approaches on how the max entropy distributions can be used as priors for the parameters a_0 and a_1. We identify two main methods for Bayesian reliability estimation, the independent endpoints method and the independent decrement methods (e.g. Lawrence & Van der Wiel, 2006).

Independent Endpoints Method

In many cases dealing with missile reliability, we are provided with the point estimate of reliability at certain time points only. Suppose we know the system reliabilities $\overline{R_1}$ and $\overline{R_2}$ at two different times T_1 and T_2 respectively. In this case, we can use the above results to specify a max entropy distribution as prior with reliability R_1 at time T_1 having mean $\overline{R_1}$, and another max entropy prior with reliability R_2 at time T_2 having mean $\overline{R_2}$. The independent endpoints method allows one to obtain the priors for a_0 and a_1 under the assumption that the joint density is the product of the two priors, i.e.

$$\pi(R_1, R_2) = C\, \tilde{\rho}(R_1)\tilde{\rho}(R_2)$$
$$= Ce^{\omega_1 R_1}R_1^{-1/2}(1-R_1)^{-1/2}e^{\omega_2 R_2}R_2^{-1/2}(1-R_2)^{-1/2},$$

where C is some normalizing constant. Since $R_1 = e^{a_0+a_1 T_1}$ and $R_2 = e^{a_0+a_1 T_2}$, by a standard change of variables, we derived the prior joint density of (a_0, a_1) as

$$\pi(a_0, a_1) = C\, \frac{e^{\left(\omega_1 e^{a_0+a_1 T_1} + \omega_2 e^{a_0+a_1 T_2}\right)}e^{\left(a_0 + \frac{a_1(T_2+T_1)}{2}\right)}|T_1 - T_2|}{\sqrt{1-e^{a_0+a_1 T_1}}\sqrt{1-e^{a_0+a_1 T_2}}}, \quad (3)$$

and the posterior density is

$$\rho(a_0, a_1|\text{data}) \propto \pi(a_0, a_1)L(\text{data}|\, a_0, a_1).$$

Bayesian mean estimates and confidence limit (credible intervals) calculations for the reliability model can now be computed from the above posterior distribution, i.e.

$$\overline{R(t)} = E\big(R(t)\big) = \int_{-\infty}^{0} \int_{-\infty}^{0} e^{a_0+a_1 t}\, \rho(a_0, a_1|\text{data})\, da_0 da_1.$$

While these numerical integrations can be complicated, these calculations are now handled routinely using the Markov Chain Monte Carlo (MCMC) procedure, available for example on popular statistical packages such as SAS. The following figure illustrates the calculated system reliability with the above concepts.
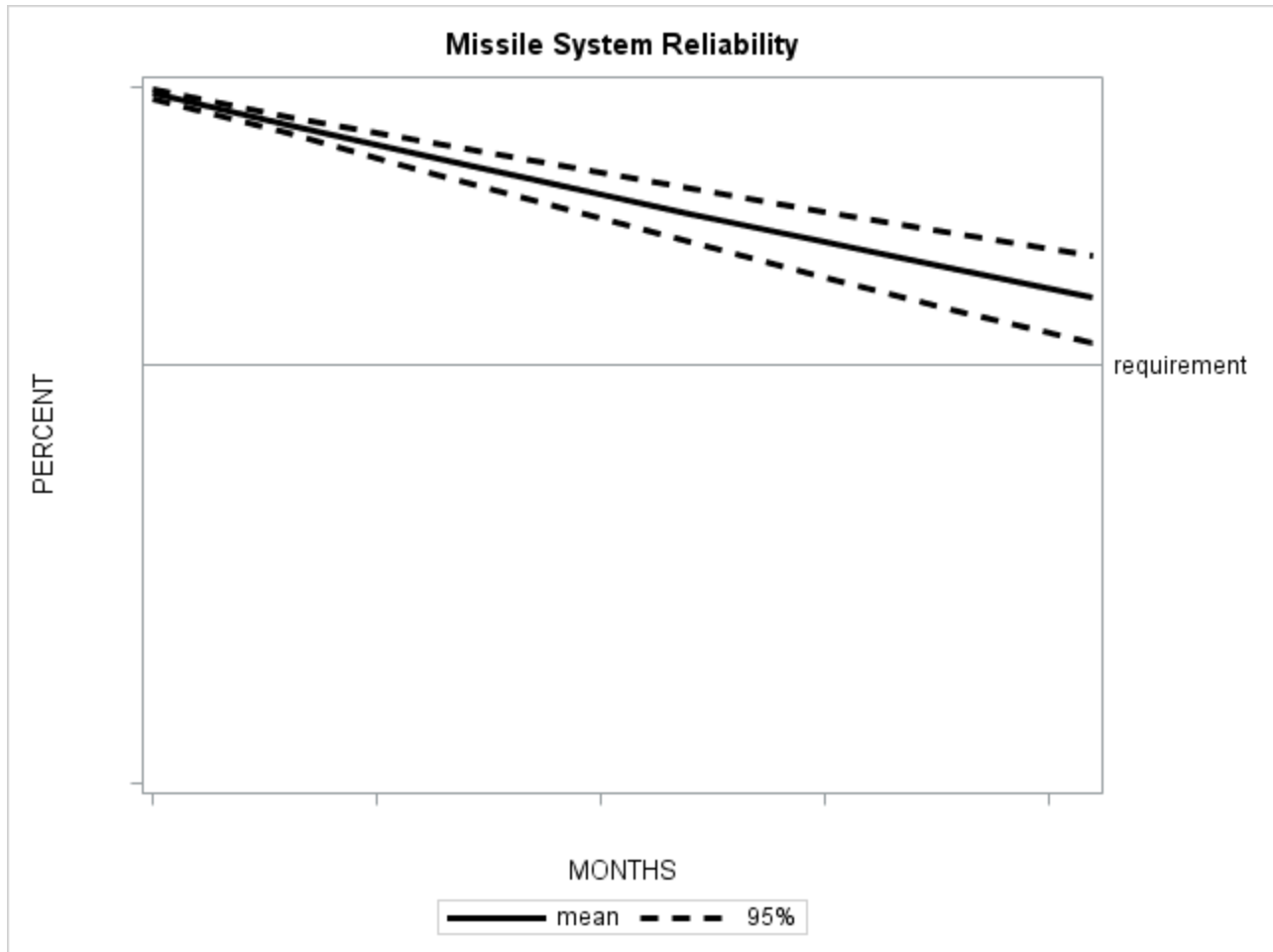
Missile System Reliability

Figure 2. Missile system reliability from the independent endpoints method. Shown are the mean reliability curve (solid) and the 95% two-sided credible limits (dash).

Independent Decrements Method

Another method that has seen some application is the independent decrement method. Instead of having reliability priors specified at two different times as above, we have the following scenario, a point estimate $\overline{R_1}$ of the reliability is known at an initial time T_1, and a point estimate of reliability decrement $\overline{\Delta}(T_2, T_1)$ is also known at a later time T_2, so that $\overline{R(T_2)} = \overline{R(T_1)}\overline{\Delta}$. As in the independent endpoints method, we specify a max entropy distribution as prior with R_1 at time

T_1 having mean $\overline{R_1}$ and specify another max entropy prior for the reliability decrement (which we assume to be expressible in the same model form) with $\Delta = \Delta(T_2, T_1)$ having mean $\overline{\Delta}$. Under the assumption that the decrement is independent of the initial reliability, the joint density is given by

$$\boldsymbol{\pi}(R_1, \Delta) = C\tilde{\rho}(R_1)\tilde{\rho}(\Delta) = Ce^{\omega_1 R_1}R_1^{-1/2}(1 - R_1)^{-1/2}e^{\omega_2 \Delta}\Delta^{-1/2}(1 - \Delta)^{-1/2}.$$

As before, a standard change of variables with $R_1 = e^{a_0 + a_1 T_1}$ and $\Delta = e^{a_1(T_2 - T_1)}$ gives the joint density of (a_0, a_1) as

$$\boldsymbol{\pi}(a_0, a_1) = C\frac{e^{(\omega_1 e^{a_0 + a_1 T_1} + \omega_2 e^{a_1(T_2 - T_1)})}e^{\left(\frac{a_0}{2} + \frac{a_1(T_2 - T_1)}{2}\right)}|T_1 - T_2|}{\sqrt{1 - e^{a_0 + a_1 T_1}}\sqrt{1 - e^{a_1(T_2 - T_1)}}}, \quad (4)$$

and the posterior density is

$$\rho(a_0, a_1|\text{data}) \propto \boldsymbol{\pi}(a_0, a_1)L(\text{data}|a_0, a_1).$$

As before, Bayesian mean estimates can be computed from the above posterior distribution, i.e.

$$\overline{R(t)} = E\big(R(t)\big) = \int\limits_{-\infty}^{0}\int\limits_{-\infty}^{0} e^{a_0 + a_1 t} \, \rho(a_0, a_1|\text{data}) \, da_0 da_1.$$

Failure rate as point estimate

A special application of the above methods worth mentioning is the use of MIL-HDBK-217 point estimates for reliability predictions. These estimates are not reliability values in itself, but rather point estimates of failure rates. We now demonstrate how to apply the above methods to obtain the maximum entropy priors for this case.

Suppose we are given a reliability point estimate $\overline{R_0}$ at $t = 0$, and the failure rate λ as a point estimate similar to what is provided in MIL-HDBK-217 data. We can write

$$R(t) = e^{a_0 + a_1 t} = R_0 e^{a_1 t}.$$

Applying the independent endpoints method to this case, we deduce that the maximum entropy prior is given by

$$\boldsymbol{\pi}(R_0, a_1) = C(t)\, \tilde{\rho}(R_0)\tilde{\rho}(R_0 e^{a_1 t}) t R_0 e^{a_1 t}$$

$$= C(t) e^{\omega_0 R_0} R_0^{-\frac{1}{2}} (1 - R_0)^{-\frac{1}{2}} e^{\omega_1 R_0 e^{a_1 t}} (R_0 e^{a_1 t})^{-\frac{1}{2}} (1 - R_0 e^{a_1 t})^{-\frac{1}{2}} t R_0 e^{a_1 t}$$

$$= C(t) \frac{e^{(\omega_0 R_0 + \omega_1 R_0 e^{a_1 t})} t}{\sqrt{1 - R_0}\sqrt{e^{-a_1 t} - R_0}},$$

for some normalizing factor $C(t)$ that depends on t. ω_0, ω_1 are determined by solving the equations

$$\left(1 + \frac{I_1\left(\frac{\omega_0}{2}\right)}{I_0\left(\frac{\omega_0}{2}\right)}\right) = 2\overline{R_0}, \quad \left(1 + \frac{I_1\left(\frac{\omega_1}{2}\right)}{I_0\left(\frac{\omega_1}{2}\right)}\right) = 2\overline{R_0}e^{-\lambda t},$$

with the understanding that the mean $E(e^{a_0 + a_1 t}) \triangleq e^{E(a_0) + E(a_1)t} \triangleq \overline{R_0} e^{-\lambda t}$ is valid (as first-order Taylor approximation). Note that $\omega_1 = \omega_1(t)$ is also a function of t (for fixed λ). Equivalently, the change of variables $R_0 = e^{a_0}$ gives the max entropy prior as

$$\boldsymbol{\pi}(a_0, a_1) = \boldsymbol{\pi}(a_0, a_1, t) = C(t) \frac{e^{(\omega_0 e^{a_0})} e^{a_0}}{\sqrt{e^{a_0}(1 - e^{a_0})}} \frac{e^{(\omega_1 e^{a_0 + a_1 t})} t e^{a_0 + a_1 t}}{\sqrt{e^{a_0 + a_1 t}(1 - e^{a_0 + a_1 t})}}$$

$$= C(t) \frac{e^{(\omega_0 e^{a_0} + \omega_1 e^{a_0 + a_1 t})} t e^{a_0}}{\sqrt{1 - e^{a_0}}\sqrt{e^{-a_1 t} - e^{a_0}}}.$$

To see this, we just set $T_1 = 0$, $T_2 = t$, $\omega_1 = \omega_0$, $\omega_2 = \omega_1$ in equation (3).

The posterior density is given by

$$\rho(a_0, a_1 | \text{data}, t) \propto \boldsymbol{\pi}(a_0, a_1, t) L(\text{data} \,|a_0, a_1),$$

and the corresponding Bayesian mean reliability computed from the above posterior distribution is

$$\overline{R(t)} = E(R(t)) = \int\limits_{-\infty}^{0} \int\limits_{-\infty}^{0} e^{a_0 + a_1 t} \, \rho(a_0, a_1, t | \text{data}) \, da_0 da_1.$$

Now given the same prior information $\overline{R_0}$ at $t = 0$, and the failure rate λ, let us assume that the initial reliability R_0 is independent of the failure rate a_1. We can then apply the independent decrement method and write

$$R(t) = e^{a_0 + a_1 t} = R_0 e^{a_1 t} = R_0 \Delta(t),$$

and deduce that the maximum entropy prior is

$$\boldsymbol{\pi}(R_0, a_1) = C(t)\tilde{\rho}(R_0)\boldsymbol{\pi}(a_1) = C(t)e^{\omega_0 R_0} R_0^{-\frac{1}{2}}(1 - R_0)^{-\frac{1}{2}} \frac{e^{(\omega_1 e^{a_1 t})} t e^{a_1 t}}{\sqrt{e^{a_1 t}(1 - e^{a_1 t})}},$$

for some normalizing factor $C(t)$ that depends on t. ω_0, ω_1 are determined by solving the equations

$$\left(1 + \frac{I_1\left(\frac{\omega_0}{2}\right)}{I_0\left(\frac{\omega_0}{2}\right)}\right) = 2\overline{R_0} \,, \left(1 + \frac{I_1\left(\frac{\omega_1}{2}\right)}{I_0\left(\frac{\omega_1}{2}\right)}\right) = 2e^{-\lambda t} \,,$$

where we make the usual Taylor approximation $E(e^{a_1 t}) \triangleq e^{E(a_1)t} = e^{-\lambda t}$. Equivalently, the change of variables $R_0 = e^{a_0}$ gives the max entropy prior as

$$\boldsymbol{\pi}(a_0, a_1) = C(t) \frac{e^{(\omega_0 e^{a_0})} e^{a_0}}{\sqrt{e^{a_0}(1 - e^{a_0})}} \frac{e^{(\omega_1 e^{a_1 t})} t e^{a_1 t}}{\sqrt{e^{a_1 t}(1 - e^{a_1 t})}}.$$

To see this, we just set $T_1 = 0$, $T_2 = t$, $\omega_1 = \omega_0$, $\omega_2 = \omega_1$ in equation (4).

Notice that $\boldsymbol{\pi}(a_0, a_1) = \boldsymbol{\pi}(a_0)\boldsymbol{\pi}(a_1)$, with

$$\boldsymbol{\pi}(a_0) = C_0 \frac{e^{(\omega_0 e^{a_0})} e^{a_0}}{\sqrt{e^{a_0}(1 - e^{a_0})}} \,, \boldsymbol{\pi}(a_1) = C_1(t) \frac{e^{(\omega_1 e^{a_1 t})} t e^{a_1 t}}{\sqrt{e^{a_1 t}(1 - e^{a_1 t})}},$$

for some normalizing factors C_0 and $C_1(t)$. As noted previously, $\omega_1 = \omega_1(t)$ is a function of t (for fixed λ). Let us write $\boldsymbol{\pi}(a_1) = \boldsymbol{\pi}(a_1, t)$. The posterior density satisfies

$$\rho(a_0, a_1, t|\text{data}) = \rho(a_0| \text{ data}, t = 0)\rho(a_1|\text{data}, t > 0)$$

$$\propto \boldsymbol{\pi}(a_0)L(\text{data}, t = 0| a_0) \boldsymbol{\pi}(a_1, t)L(\text{data}, t > 0| a_1).$$

In this case, the corresponding Bayesian mean estimates has a simple form given by

$$\overline{R(t)} = E\big(R(t)\big) = \int\limits_{-\infty}^{0} \int\limits_{-\infty}^{0} e^{a_0 + a_1 t} \; \rho(a_0, a_1, t|\text{data}) \; da_0 da_1$$

$$= \int_{-\infty}^{0} e^{a_0}\rho(a_0| \text{ data}, t = 0) \; da_0 \int_{-\infty}^{0} e^{a_1 t}\rho(a_1| \text{ data}, t > 0)da_1$$

$$= E(R_0)E(e^{a_1 t}).$$

Comparing the two approaches, we observe that for the independent decrement method, $0 \le \overline{\Delta(t)} \le 1$, so that $\overline{R(t_2)} \le \overline{R_1(t_1)}$ whenever $t_2 \ge t_1$. This method therefore guarantees that the mean reliability $\overline{R(t)}$ is monotonically decreasing with time, which is not necessarily the case with the independent endpoints method. This makes the latter a more attractive choice for reliability estimation.

Perfect intercept case

We would like to mention that in some cases, it is useful to consider the special model

$$R(t) = e^{a_1 t},$$

where the intercept term a_0 is ignored (e.g. system is essentially assumed to be perfect at $t = 0$). With known failure rate λ as point estimate, the max entropy prior for a_1 is just

$$\boldsymbol{\pi}(a_1) = \boldsymbol{\pi}(a_1, t) = C(t)\frac{e^{(\omega_1 e^{a_1 t})}te^{a_1 t}}{\sqrt{e^{a_1 t}(1 - e^{a_1 t})}} \; ,$$

with ω_1 determined by

$$\left(1 + \frac{I_1\left(\frac{\omega_1}{2}\right)}{I_0\left(\frac{\omega_1}{2}\right)}\right) = 2e^{-\lambda t}.$$

The posterior density is given by

$$\rho(a_1, t|\text{data}) \propto \boldsymbol{\pi}(a_1, t)L(\text{data}|\, a_1)\,,$$

with mean reliability

$$\overline{R(t)} = E\big(R(t)\big) = \int_{-\infty}^{0} e^{a_1 t}\, \rho(a_1, t|\text{data})\, da_1.$$

Concluding Remarks

In the absence of additional information regarding point estimates of reliability numbers, we illustrate a natural method for obtaining a reasonable probability distribution by employing the principle of maximum entropy (MEP) to the point estimates. MEP-based distributions characterize the least informative distributions with the available point estimates as constraints. These distributions are then applied as priors to a Bayesian system reliability model to obtain the posterior reliability distribution, which can be used to assess missile reliability and answer other important reliability related questions such as missile availability, maintenance cycle, logistic policy, firing doctrine, etc.

Acknowledgement

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