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## **SIMILARITY OF THE VELOCITY PROFILE**

**David Weyburne**

**Optoelectronics Technology Branch  
Aerospace Components & Subsystems Division**

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DAVID W. WEYBURN, Program Manager  
Optoelectronics Technology Branch  
Aerospace Components & Subsystems Division

*//Signature//*

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JONAHIRA R. ARNOLD, Branch Chief  
Optoelectronics Technology Branch  
Aerospace Components & Subsystems Division

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## Table of Contents

Section	Page
1. Introduction.....	1
2. Similarity of the Velocity Profile.....	2
2.1 The First Derivative Profile.....	2
2.2 Defect Profile .....	3
3. Discussion.....	5
4. Conclusion .....	8
5. References.....	9

# 1. INTRODUCTION

Similarity of the velocity profile is one of the basic concepts of fluid flow theory going back to the pioneering work of Reynolds [1] in the 1800's. Recently, Weyburne [2] introduced a new approach for studying similarity of the velocity profile for 2-D flow along a wall. The following provides a more concise derivation and fixes some minor omissions from that report.

Any discussion of similarity must start with the traditional definition of similarity of the velocity profile given formally by Schlichting [3]. For 2-D wall-bounded flows, the definition of similarity is that two velocity profile curves from different stations along the flow are similar if they differ only by scaling constants in  $y$  and  $u(x,y)$ , where  $y$  is the normal direction to the wall,  $x$  is the flow direction, and  $u(x,y)$  is the velocity parallel to the wall in the flow direction.

To put this in formal mathematical terms, we first take the length scaling variable as  $\delta_s$  and velocity scaling variable as  $u_s$ . These scaling variables can vary with the flow direction ( $x$ -direction) but not perpendicular to the wall ( $y$ -direction). Hence, according to Schlichting [3], the scaled velocity profile at a station  $x_1$  along the wall will be similar to the scaled profile at  $x_2$  if

$$\frac{u(x_1, y/\delta_s(x_1))}{u_s(x_1)} = \frac{u(x_2, y/\delta_s(x_2))}{u_s(x_2)} \quad \text{for all } y. \quad (1)$$

From a mathematical perspective, it is found that this definition of similarity leads to a number of properties of the scaled profiles that can be used to advantage. Consider the area under the scaled velocity profiles. Similarity necessarily requires that if the scaled profiles are plotted versus the scaled  $y$ -coordinate, then the area under the curves must be equal. In the following, we use this property to find a new way to define similarity of the velocity profile as well as discover some properties of similar velocity profiles of 2-D wall-bounded boundary layer flows.

The equal area concept will be defined in terms of some integral-based requirements. These results are derived directly from the definition of similar curves given by Eq. 1. We therefore do not need to know how the curves were generated. Hence, the flow governing equations do not have to be invoked. Furthermore, the new results apply to similar profile curves whether they are laminar, transitional, or turbulent velocity profiles as long as the velocity is taken as the Reynolds averaged velocity.

## 2. SIMILARITY OF THE VELOCITY PROFILE

### 2.1 The First Derivative Profile

We start by considering what can be learned from this equal area approach to similarity for 2-D flow along a wall. In the analysis below, the only assumptions that are necessary as to the functional form of the velocity profile is that the form is consistent with a physically realizable form, *i.e.* no discontinuities or singularities. The development presented herein is based on a simple concept; for similarity, the area under the scaled velocity profile curves plotted versus the scaled  $y$ -coordinate must be equal at each station along the wall.

Rather than starting directly with the consideration of the velocity profile, we will first consider the implications of similarity on the first derivative profile curves since this result is required later on. If similarity is present in a set of velocity profiles then it is self-evident that the properly scaled first derivative profile curves (derivative with respect to the scaled  $y$ -coordinate) must also be similar. It is also self-evident that the area under the scaled first derivative profiles plotted against the scaled  $y$ -coordinate must be equal for similarity.

In mathematical terms, the area under the scaled first derivative profile curve is expressed by

$$a(x) = \int_0^{h/\delta_s} d\left\{\frac{y}{\delta_s}\right\} \frac{d\{u(x,y/\delta_s)/u_s\}}{d\left\{\frac{y}{\delta_s}\right\}} \quad , \quad (2)$$

where  $a(x)$  is in general a non-zero numerical constant. For clarity,  $h(x)$ ,  $u_s(x)$ , and  $\delta_s(x)$  have been shortened to  $h$ ,  $u_s$ , and  $\delta_s$ . Using the 2-D boundary conditions  $u(x,0)=0$  and  $u(x,h)=u_e(x)$  where  $u_e(x)$  is the free-stream velocity above the boundary layer edge, and a simple variable switch ( $d\{y/\delta_s\} \Rightarrow (1/\delta_s)dy$ ), Eq. 2 can be shown to reduce to

$$\begin{aligned} a(x) &= \int_0^{h/\delta_s} d\left\{\frac{y}{\delta_s}\right\} \frac{d\{u(x,y/\delta_s)/u_s\}}{d\left\{\frac{y}{\delta_s}\right\}} \quad , \quad (3) \\ a(x) &= \int_0^h dy \frac{d\{u(x,y)/u_s\}}{dy} \\ a(x) &= \frac{1}{u_s} \int_0^h dy \frac{du(x,y)}{dy} \\ a(x) &= \frac{1}{u_s} [u(x,y)]_{y=h,0} \\ a(x) &= \frac{u_e}{u_s} \quad . \end{aligned}$$

Similarity at two stations along the flow located at  $x_1$  and  $x_2$  requires that  $a(x_1) = a(x_2) = \text{constant}$ . Note that similarity also requires that  $h(x_1)/\delta_s(x_1) = h(x_2)/\delta_s(x_2)$  but this is satisfied as long as  $h(x_1)$  or  $h(x_2)$  are chosen to both satisfy the boundary condition  $u(x, h) = u_e(x)$ . Therefore, for similarity of the velocity profiles for a 2-D boundary layer flow, the velocity scaling factor  $u_s(x)$  must be proportional to  $u_e(x)$ , the free-stream velocity above the boundary layer edge. Note that under the assumption of similarity, Eq. 3 also can be derived from Eq. 1 directly. The above derivation is offered as a simple introduction to the equal area approach.

## 2.2 Defect Profile

Starting with the formal definition of similarity given by Eq. 1 it is self-evident that for the profiles to be similar, the area under these scaled velocity profiles plotted versus the scaled  $y$ -coordinate must be equal. The area under the scaled profile curves, in integral form, is given by

$$b_0(x) = \int_0^{h/\delta_s} d\left\{\frac{y}{\delta_s}\right\} \frac{u_e - u(x, y/\delta_s)}{u_s} , \quad (4)$$

where  $b_0(x)$  is in general a nonzero numerical constant. Note that the integral is written using the scaled velocity difference rather than just the scaled velocity. It is simple to show that similarity of the defect profile  $u_e - u$  is equivalent to similarity of the velocity profile so long as Eq. 3 holds true (note the use of Eq. 3 was not explicit but subsumed in the original derivation given in Weyburne [2]). The use of the defect profile has two advantages. First, the integral value is not dependent on the value of  $h$  as long as  $h$  is chosen to satisfy the boundary condition  $u(x, h) = u_e(x)$ . This simplifies the application to experimental data. Secondly, using the same simple variable switch from above and simple algebra, Eq. 4 can be shown to reduce to

$$\begin{aligned} b_0(x) &= \int_0^{h/\delta_s} d\left\{\frac{y}{\delta_s}\right\} \frac{u_e - u(x, y/\delta_s)}{u_s} & (5) \\ b_0(x) &= \frac{1}{\delta_s} \int_0^h dy \frac{u_e - u(x, y)}{u_s} \\ b_0(x) &= \frac{u_e}{\delta_s u_s} \int_0^h dy \frac{u_e - u(x, y)}{u_e} \\ b_0(x) &= \frac{\delta_1 u_e}{\delta_s u_s} , \end{aligned}$$

where the  $\delta_1$  is the displacement thickness given by

$$\delta_1(x) = \int_0^h dy \{1 - u(x, y)/u_e\} , \quad (6)$$

and where  $h$  satisfies the boundary condition  $u(x, h) = u_e(x)$ . Eq. 5 is an exact equation that applies whether the profiles are similar or not. Similarity at two stations along the flow located at  $x_1$  and  $x_2$  requires that  $b_0(x_1) = b_0(x_2) = \text{constant}$ . Note that it is also necessary that  $h(x_1)/\delta_s(x_1) = h(x_2)/\delta_s(x_2)$  and, as above, either  $h(x_1)$  or  $h(x_2)$  can be freely chosen as long as they both satisfy the boundary condition  $u(x, h) = u_e(x)$ . The importance of Eq. 5 in regards to similar profiles is that it means that the thickness scaling factor and the velocity scaling factor cannot be independent for 2-D wall-bounded similarity flows.

Combing the result given by Eq. 5 with the result given by Eq. 3, then it is evident that

$$c(x) = \frac{\delta_1(x)}{\delta_s(x)}, \quad (7)$$

where  $c(x)$  is in general a non-zero numerical constant. Similarity at two stations along the flow located at  $x_1$  and  $x_2$  requires that  $c(x_1) = c(x_2)$ . Eq. 7 is important in that it states that if similarity exists, then the displacement thickness must be a length scale that results in similarity.

Having equal  $b_0(x)$  values at different stations along the flow is a necessary but not a sufficient condition for similarity of a set of profile curves. If the scaled velocity profiles are similar, then it is self-evident that the scaled velocity profile curves multiplied by the scaled  $y$ -coordinate raised to the  $n$ th power must also be similar. In area integral terms, the area under the scaled velocity profiles multiplied by the scaled  $y$ -coordinate raised to the  $n$ th power is equivalent to

$$b_n(x) = \int_0^{h/\delta_s} d\left\{\frac{y}{\delta_s}\right\} \left(\frac{y}{\delta_s}\right)^n \frac{u_e - u(x, y/\delta_s)}{u_s}, \quad (8)$$

where  $b_n$  are, in general, non-zero numerical constants. It is self-evident that a sufficient condition for similarity at two stations along the flow located at  $x_1$  and  $x_2$  is that

$$b_n(x_1) = b_n(x_2) \quad \text{for } n=0,1,2,3, \dots, \infty, \quad (9)$$

so long as  $h$  is chosen appropriately as discussed above.



### 3. DISCUSSION

An important point about the above derivations is that although the results are not presented formally, the results above are mathematically rigorous and can be easily substantiated in the form of mathematical proofs.

The approach used to generate these new results is different than the traditional method. In the traditional method, one starts with the flow governing equations, then one makes certain assumptions to reduce the governing equations to ordinary differential equations and tries to find appropriate analytical expressions for the solutions. Hence the traditional method is intimately bound to the physics of the flow through the flow governing equations. There may be a tendency to dismiss the new approach described above since the flow governing equations are not invoked. However, note that one defines similarity of a set of velocity profile curves using Eq. 1 [3]. This equation is asking a simple question; are the left and right sides equivalent or are they not. There are no flow governing equations invoked in posing this question or answering the question. If one experimentally measures the velocity profile at two stations along the flow, it is not necessary to invoke the flow governing equations to test if the scaled profile curves are similar. That is, the similarity solutions to the Navier-Stokes equation are not similar because they are derived from the Navier-Stokes equation; they are similar if one substitutes the velocity solutions from the Navier-Stokes equation into Eq. 1 and then finds that the scaled velocity profile curves are equivalent at different stations along the flow.

The similarity scaling result given by Eq. 5 is important in that it states that if similarity exists, then the similarity length scale factor and the similarity velocity scale factor cannot be independent. The form of this interdependence is new and different from than that found previously. This interdependence is not like pressure gradient parameter  $\Lambda$  found in Castillo and George [4] for example. The pressure gradient parameter  $\Lambda$  provides a required relationship for similarity between the boundary layer thickness, the velocity at the boundary layer edge, and the gradient of these two parameters in the flow direction. The new interdependence factor given by Eq. 5, on the other hand, provides a necessary similarity relationship on the proposed scaling factors in terms of two experimentally accessible variables. However, the biggest difference between the two parameters is that whereas Eq. 5 is invoked directly from the definition of similarity (Eq. 1), the Castillo and George parameter is based on looking at the x-momentum balance equation in the outer region of the turbulent boundary layer. Note that if one looks at the inner region of the turbulent boundary layer, a different, much more restrictive parameter is obtained. Castillo and George, nor anyone else, has explained why it is okay to ignore the inner region restrictions when dealing with velocity profile similarity using their approach. In contrast, there are no restrictions in the approach developed herein. It applies to wall-bounded turbulent boundary layers, laminar flows, and transition flows. If similarity exists in any of these flows then the results described above must hold true.

One remarkable finding is that if we take  $\delta_s$  in Eq. 5 as the boundary layer thickness  $\delta$ , then  $u_s(x)$  (with  $b_0 = \text{constant}$ ) is the empirically derived velocity scale developed by Zagarola and Smits [5] for turbulent boundary layer flow. Zagarola and Smits and others have shown that the velocity scaling factor given by Eq. 5 with  $\delta_s$  as the boundary layer thickness can collapse certain experimental turbulent profiles to a single curve (*i.e.*, they behave similarly). The above

results now give a solid theoretical foundation to the empirical results. Note that although this equation explains the empirical approach of Zagarola and Smits, it is important to point out that the fact that the similarity length scale factor and the similarity velocity scale factor must follow Eq. 5 (with  $b_0(x) = \text{constant}$ ) is not something appreciated in the fluid flow community. Indeed, if one checks the recent literature dealing with similarity of wall-bounded turbulent flows [4-6], it is apparent that no one has made the connection that similarity requires that the length and velocity similarity variables must be coupled through Eq. 5.

By far the most significant new finding is that for the thickness scaling variable, Eq. 7 indicates that if similarity exists in a set of velocity profiles, then the displacement thickness  $\delta_1$  must be a similarity length scale parameter. This new result applies to all 2-D wall bounded boundary layer flows. This is the first time that a particular parameter has been identified as a similarity length scale and most significantly, it is the first time that the displacement thickness has been identified as a similarity length scale parameter. The importance of this result cannot be overstated. Consider that the 2-D wall-bounded similar flows encompass the Blasius [7] and the Falkner-Skan [8] similarity solutions. In the more than 100 years since Blasius published his work no one has ever made the connection that the displacement thickness  $\delta_1$  is a similarity length scale for the Blasius flow solution. The same is true for the Falkner-Skan similarity solution. Up until now, the length scale for the Blasius solution has been taken as the square root of the kinematic viscosity times the distance along the wall divided by the velocity at the boundary edge (infinity). The connection to the displacement thickness has never been made theoretically but was posited as a possibility by Weyburne [9] based on examining certain experimental data sets.

Another finding from above is that Eq. 3 indicates that the appropriate velocity scaling constant for 2-D wall bounded flow similarity is  $u_e$ . We note that Castillo and George [4] came to a similar conclusion using a momentum balance type approach to similarity for flows with a pressure gradient. In contrast, the results above apply to all 2-D wall bounded flows displaying similarity regardless of whether a pressure gradient exists or not.

It must be emphasized that the results obtained above do not exclude other parameters from being similarity scaling variables. Consider for example the Prandtl Plus scaling variables. Jones, *et al.*, [6] used a flow governing scaling approach to show that for the case of zero-pressure-gradient turbulent boundary layer in the limit of infinite Reynolds number, the flow friction velocity  $u_\tau$  leads to a valid similarity solution. For the non-infinite limit case, we can postulate that  $u_\tau$  is a possible similarity variable. That would mean that, based on Eq. 3,

$$\frac{u_e(x)}{u_\tau(x)} = \text{constant} . \quad (10)$$

This is of course the well-known Rotta [10] condition stating that for similarity, the inner and outer regions velocity scaling variables must evolve at the same rate as one moves along the wall in the flow direction. What is interesting here, is that we can now also establish a length scale ratio that must hold if the Prandtl scaling variables are to result in similarity, that is

$$\frac{\delta_1(x)u_\tau(x)}{\nu} = \text{constant} . \quad (11)$$

This means that if similarity is present in set of velocity profiles and if the Prandtl scaling variables are to be similarity scaling variables, then Eqs. 10 and 11 must apply.

Mathematically it is evident that a set of arbitrary curves are similar if the condition given by Eq. 9 exists for all  $n$ . However, for flow similarity of velocity profiles taken at various stations along the wall in the flow direction, the profile curves are not arbitrary and so it may only be necessary to insure that  $b_0(x_1) = b_0(x_2)$ . In any case, this approach to similarity has an advantage from an experimental standpoint since the equal area test method would allow for statistical testing for similarity by comparing  $b_n(x)$  values at various stations along the wall. From a practical standpoint, this method is superior to the use of Eq. 1 since to use Eq. 1 the experimentalist needs to insure the velocity at each  $y/\delta_s(x)$  value is equal at each measurement station. This is a very difficult task since, in general, the actual values of  $\delta_s(x)$  are not known *a priori*. As a result of the  $y/\delta_s(x)$  issue, the usual imprecise method the flow community presently uses to judge whether a set of velocity profiles are similar is to plot the profiles (with lines connecting the data points) and use the subjective “chi-by-eye” method to judge the success or failure. If the plotted curves fall on top of one another, they are considered similar. In contrast, the equal area test method would allow for statistical testing by performing simple numerical integrals (Eq. 8) of the profile data while insuring Eqs. 3 and 7 are constant at the stations along the flow.

## 4. CONCLUSION

A new approach for studying similarity of the velocity profile was outlined. It starts from the equation used to define similarity of the velocity profile. This method was used to discover fundamentally new results for the similarity of the 2-D wall bounded boundary layer velocity profiles. It was shown that if similarity exists, then the similarity velocity and length scaling constants cannot be independent. Furthermore, it was shown that if similarity exists, the displacement thickness  $\delta_1$  must be a length scaling variable and the velocity at the boundary layer edge  $u_e$  must be a velocity scaling variable.

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## 5. REFERENCES

- [1] Reynolds, O., 1883, "An experimental investigation of the circumstances which determine whether the motion of water in parallel channels shall be direct or sinuous, and of the law of resistance in parallel channels," *Philosophical Transactions of the Royal Society of London*, **174**, pp. 935-982.
- [2] Weyburne, D., 2008, "The Mathematics of Flow Similarity of the Velocity Boundary Layer," Technical Report AFRL-RY-HS-TR-2010-0014, <http://www.DTIC.mil>.
- [3] Schlichting, H., 1979, *Boundary Layer Theory*, 7th edn., McGraw-Hill, New York, USA, pp. 152, ISBN 0-07-055334-3.
- [4] Castillo, L., and George, W., 2001, "Similarity Analysis for Turbulent Boundary Layer with Pressure Gradient: Outer Flow," *AIAA J.*, **39**, pp. 41-47.
- [5] Zagarola, M., and Smits, A., 1998, "Mean-Flow Scaling of Turbulent Pipe Flow," *J. Fluid Mech.*, **373**, pp. 33-79.
- [6] Jones, M., Nickels, T., and Marusic, I., 2008, "On the asymptotic similarity of the zero-pressure-gradient turbulent boundary layer," *J. Fluid Mech.*, **616**, 195-203(2008).
- [7] Blasius, H., 1908, "Grenzschichten in Flüssigkeiten mit kleiner Reibung," *Zeitschrift für Mathematik und Physik*, **56**, pp. 1-37.
- [8] Falkner, V., and Skan, S., 1931, "Some Approximate solutions of the boundary layer solutions," *Philosophical Magazine*, **12**, pp. 865-896.
- [9] Weyburne, D., 2009, "Similarity of the Outer Region of the Turbulent Boundary Layer," Technical Report AFRL-RY-HS-TR-2010-0013, <http://www.DTIC.mil>.
- [10] Rotta, J., 1962, "Turbulent boundary layers in incompressible flow," *Prog. Aero. Sci.*, **2**, pp. 1-219.