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Statistics of Radial Ship Extent as Seen by a Seeker

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ABSTRACT

We present a geometrical study of a missile seeker illuminating a ship target. Modern seekers employ various methods to recognise their target amid background vessels and electronic countermeasures such as chaff and active decoys. We focus on one such possible method: the seeker's measurement of a ship's down-range extent. We aim to find a representative value for this extent given that the seeker can approach the ship, centre on, from any direction. This representative value can then be used in studies of electronic attack in the generation and presenting to the seeker of a realistic false ship target. Ensuring the false target has a statistically favourable range extent might make it appear more realistic to the seeker. On the other hand, a lack of any statistically dominant range extent implies that we needn't select any particular value to present to the seeker. Either way, this report's analysis gives guidance in what range extent to specify when constructing a false target.

The simple scenarios we consider turn out to present no statistically dominant range extents. Extrapolating from these, we conclude that it's probably not necessary to focus strongly on any one range extent when constructing a false target.

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Statistics of Radial Ship Extent as Seen by a Seeker

Executive Summary

This report studies the geometry and associated statistics of a missile seeker illuminating a ship target. The seeker must isolate its target from other nearby sea-going vessels that it has no interest in targeting. Modern seekers employ various methods to recognise their target amid background vessels and electronic countermeasures such as chaff and active decoys. In this report we focus on one of these possible methods: the seeker's measurement of a ship's down-range extent. A wide-beam radar on a seeker typically returns little or no information about a ship's cross-range extent, so down-range extent becomes the dominant extent parameter that it analyses.

The seeker measures the target's range extent in some way that we don't need to specify; we stipulate only that the seeker illuminates the target with a beam. This beam could be wide (from a radar), or narrow (from a laser). Our study aims to find a representative value for the measured range extent, given that the seeker can approach the ship, centre on, from any direction.

To implement a false-target electronic attack, we require a representative value of range extent in order to generate and present to the seeker a realistic ship image. If certain seeker-measured range extents turn out to be statistically favoured when all angles of the seeker's incoming direction have been examined, then creating our false target with one of these values might make it appear more realistic to the seeker. On the other hand, if no range extents are statistically dominant, we are not forced into selecting any particular value to apply to the range extent of the false target that we generate. Either way, the outcome of the following analysis gives guidance in what range extent to specify when constructing a false target.

We find that a wide seeker beam yields down-range extent values near to the ship's length, while a narrower beam yields extent values near to the ship's width. But no very strongly dominant range extents emerge from our statistical analysis. We can expect the seeker to be aware of these statistics when it makes assumptions about the nature of its target. Our conclusion is that it's probably unnecessary to focus on any one particular range extent when constructing a false target presentable to a seeker.

The results of this report allow a selection of more representative range extent values for ongoing work into false-target generation for countering incoming missiles.

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Don Koks completed a doctorate in mathematical physics at Adelaide University in 1996, with a thesis describing the use of quantum statistical methods to analyse decoherence, entropy and thermal radiance in both the early universe and black hole theory. He holds a Bachelor of Science from the University of Auckland in pure and applied mathematics and physics, and a Master of Science in physics from the same university with a thesis in applied accelerator physics (proton-induced X ray and γ ray emission for trace element analysis). He has worked on the accelerator mass spectrometry programme at the Australian National University in Canberra, and in commercial Internet development.

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1 Introduction

This report presents a study of the geometry and associated statistics of a missile seeker illuminating a ship target. The seeker must isolate the ship from a background that it has no interest in targeting, such as other nearby sea-going vessels and electronic countermeasures such as chaff and active decoys. In this report we focus on one of these possible methods: the seeker's measurement of a ship's down-range extent. A wide-beam radar on a seeker typically returns little or no information about a ship's cross-range extent, so down-range extent becomes the dominant extent parameter that it analyses.

The seeker measures the target's range extent in some way that we don't need to specify; we stipulate only that the seeker illuminates the target with a beam. This beam could be wide (from a radar), or narrow (from a laser). Our study aims to find a representative value for the measured range extent, given that the seeker can approach the ship, centre on, from any direction.

To implement a false-target electronic attack, we require a representative value of range extent in order to generate and present to the seeker a realistic image of the ship it seeks. If certain seeker-measured range extents turn out to be statistically favoured when all angles of the seeker's incoming direction have been examined, then creating our false target with one of these values might make it appear more realistic to the seeker. On the other hand, if no range extents are statistically dominant, we are not forced into selecting any particular value to apply to the range extent of the false target that we generate. Either way, the outcome of the following analysis gives guidance in what range extent to specify when constructing a false target.

As an example of relevant statistics, a seeker will presumably not reject a ship-sized target, but it might well reject targets that are too *short* in range extent, because these could be smaller vessels such as fishing boats. So a typical question to address is how likely it is that the range extent of a real ship will be measured to have its smallest possible value.

The geometrical analysis in this report is aimed at complementing other analyses that focus on the waveform aspects of false-target generation. Most work in this field seems to focus on the high-quality images produced by ISAR. For example, Pace et al. [1] describe a hardware implementation of an image synthesiser for countering ISAR. Despite their using a much higher granularity than our wide-beam analysis, along with Doppler processing, still only down-range information is considered. Benedek [2] proposes analysing the images of range-Doppler maps by correlating their successive frames; again, down-range information is being heavily relied upon. Kostis [3] describes the production of false-target images presentable to an ISAR, and focuses on creating as much realism as possible through a fine-grained analysis of how the ISAR analyses target characteristics.

2 Ship Models Used and Assumptions Made

We use two ship models and two beam models to construct four scenarios of a ship being illuminated by the radar beam of an incoming seeker. All ships and beams are two dimensional. First we consider a wide radar beam, the domain of current seekers, and analyse such a beam illuminating an elliptical ship and a rectangular ship. After this we consider a narrow beam, which could be the domain of future seekers, and again perform the analysis with this beam illuminating elliptical and rectangular ships.

The ship models have length L and width W, and without loss of generality we always assume $W \leqslant L$. (Note that the ship's width is often called its beam, but to avoid confusion with the seeker's radar beam, we don't use this term.) In all cases we describe the beam of the seeker as intersecting the centre of the ellipse or rectangle at some angle θ to the "length" line running fore to aft of the ship, measured clockwise from the forward direction. The seeker can approach from any bearing with equal likelihood, making all values of θ equally likely.

At some given moment the seeker measures the ship's range extent to be some length E. Given that the seeker can approach from any bearing, we wish to calculate the extent E and to examine its statistics as functions of L, W, and θ:

– its probability density $p(E)$,

– its mean $\langle E \rangle$, and

– the cumulative probability $C(E)$ of measuring any extent up to some given E.

The fore–aft and left–right symmetry of the two ship models ensures we need only consider incoming seeker angles in the first quadrant: $0 \leqslant \theta \leqslant 90°$. This is due to a combination of all quadrants being equally likely to contain the seeker, and the fact that the value of E for any value of θ is always equal to the E for the corresponding angle determined by symmetry in the quadrant $0 \to 90°$. In other words, we can replace any value of $\theta > 90°$ by a corresponding θ in the interval $\theta = 0 \to 90°$.

To construct $\langle E \rangle$ and $C(E)$ requires knowledge of $p(E)$. We will begin with the only probability density available, $p(\theta)$ the probability density for θ, and use it to construct $p(E)$. (It's conventionally understood that these are two different functions "p".) By definition of the probability density, the probability that a seeker will be inbound in the angle interval $\theta \to \theta + \mathrm{d}\theta$ is $p(\theta)\,|\mathrm{d}\theta|$. As discussed above, we will treat the first quadrant as being the only one that the seeker accesses, in which case its probability of being found there is one:

$$\int_0^{\pi/2} p(\theta)\,\mathrm{d}\theta = 1\,. \tag{2.1}$$

But $p(\theta)$ is independent of θ, so we conclude that $p(\theta) = 2/\pi$.

If the angle interval $\theta \to \theta + \mathrm{d}\theta$ were to map uniquely to a corresponding extent interval $E \to E + \mathrm{d}E$, the extent probability in the extent interval would equal the angle probability in the angle interval:

$$p(E)\,|\mathrm{d}E| = 2/\pi\,|\mathrm{d}\theta|\,, \tag{2.2}$$

since clearly the fraction of measurements $p(E)\,|\mathrm{d}E|$ falling into an extent bin must be the same as the fraction $2/\pi\,|\mathrm{d}\theta|$ falling into the corresponding angle bin—because the same measurements occupy both bins; the two bins are just different ways of labelling the same measurements. We'll find this to be the case for both ellipse models, but for the rectangle models there can be *two* angle bins that contribute to one extent bin. We'll encounter those in due course and will modify (2.2) to sum the probabilities in two angle bins instead of one.

In the next sections we determine the extent E and its statistics $p(E), \langle E \rangle$, and $C(E)$ for each of the four scenarios described on the preceding page.

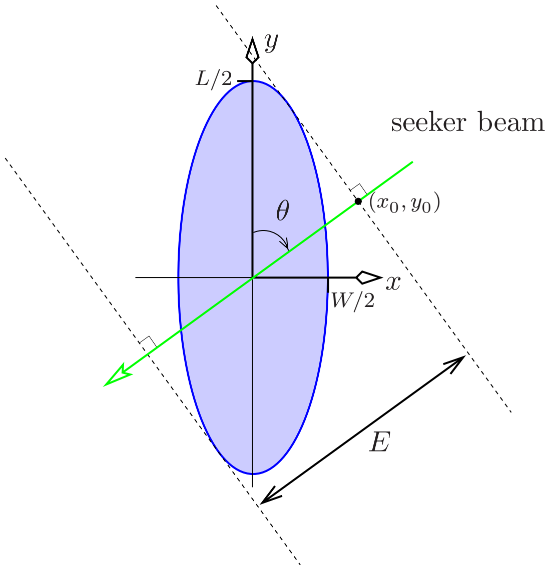


Figure 1: Definitions of terms used in this analysis

3 Scenario 1: Ellipse Illuminated by a Wide Beam

We model the ship as an ellipse in two dimensions, defining its extent E in Figure 1.

3.1 The Extent E

To calculate the extent E as a function of θ, begin by noting that the slope of the seeker's line of sight of the ship at angle θ relative to the fore–aft axis (the y axis in Figure 1) is $\cot\theta$; hence the slope of the rightmost dotted perpendicular line in that figure is $-\tan\theta$. This dotted line therefore has equation $y = y_0 - \tan\theta\,(x - x_0)$. The requirement that this line intersect the ellipse at a single point determines (x_0, y_0), and the extent E is then given by $(E/2)^2 = x_0^2 + y_0^2$. The ellipse has equation $\left(\frac{x}{W/2}\right)^2 + \left(\frac{y}{L/2}\right)^2 = 1$, which becomes $y = L\sqrt{1/4 - x^2/W^2}$ in the first quadrant. So we require that the two equations, of the dotted line through (x_0, y_0) and of the ellipse, have *one* solution (x, y):

$$\begin{aligned} y &= y_0 - \tan\theta\,(x - x_0) && \text{(dotted line)}\,, \\ y &= L\sqrt{\frac{1}{4} - \frac{x^2}{W^2}} && \text{(ellipse)}\,. \end{aligned} \tag{3.1}$$

Equating the right-hand sides of (3.1) gives (with $t \equiv \tan\theta = x_0/y_0$ for shorthand)

$$y_0 - t(x - y_0 t) = L\sqrt{\frac{1}{4} - \frac{x^2}{W^2}}\,. \tag{3.2}$$

Setting $k \equiv 1 + t^2 = \sec^2\theta$ and squaring both sides of (3.2) gives a quadratic in x:

$$\left(t^2 + \frac{L^2}{W^2}\right)x^2 - 2y_0ktx + y_0^2k^2 - \frac{L^2}{4} = 0\,, \tag{3.3}$$

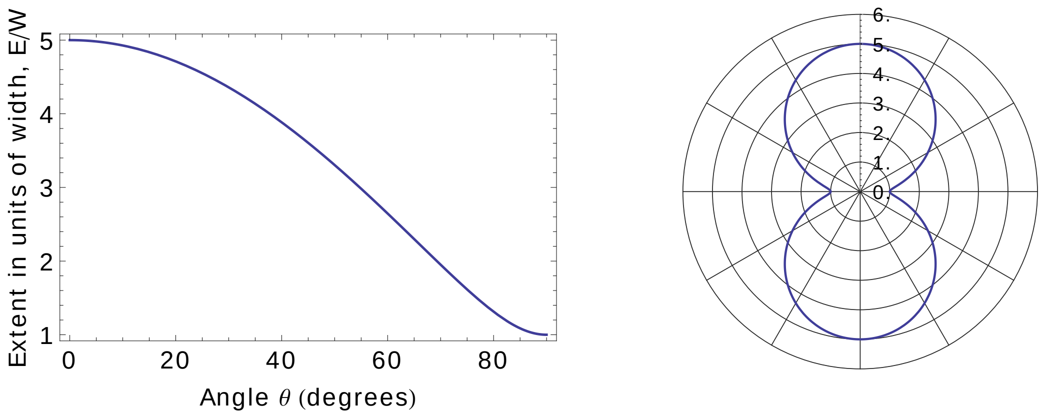


Figure 2: ***Left****: The extent in units of width for a length-to-width ratio $L/W = 5$, where it's sufficient to plot only angles in the first quadrant.* ***Right****: A polar plot of the same extent in all four quadrants. The vertical axis here runs along the length of the ship.*

which will have a unique solution (corresponding to the dotted line intersecting the ellipse at a single point) if its discriminant "$b^2 - 4ac$" equals zero; that is, if

$$4y_0^2k^2t^2 - 4\left(t^2 + \frac{L^2}{W^2}\right)\left(y_0^2k^2 - \frac{L^2}{4}\right) = 0\,. \tag{3.4}$$

Equation (3.4) simplifies to

$$y_0^2 = \frac{W^2}{4k^2}\left(t^2 + \frac{L^2}{W^2}\right). \tag{3.5}$$

But the extent is given by $(E/2)^2 = x_0^2 + y_0^2$, or

$$E^2 = 4(x_0^2 + y_0^2) = 4(t^2y_0^2 + y_0^2) = 4ky_0^2\,. \tag{3.6}$$

Now substitute the expression for y_0^2 from (3.5) into the last term of (3.6), giving

$$E^2 = \frac{W^2}{k}\left(t^2 + \frac{L^2}{W^2}\right) = \frac{W^2}{\sec^2\theta}\left(\tan^2\theta + \frac{L^2}{W^2}\right), \tag{3.7}$$

which simplifies to

$$\boxed{E = \sqrt{W^2\sin^2\theta + L^2\cos^2\theta}\,.} \tag{3.8}$$

The extent in units of width, E/W, is plotted as a function of θ in Figure 2 for a length-to-width ratio $L/W = 5$.

3.2 Probability Density $p(E)$

Equation (2.2) requires that we relate $\mathrm{d}\theta$ to $\mathrm{d}E$. Differentiating (3.8) with respect to θ leads to

$$\frac{\mathrm{d}\theta}{\mathrm{d}E} = \frac{E}{(W^2 - L^2)\sin\theta\cos\theta}\,. \tag{3.9}$$

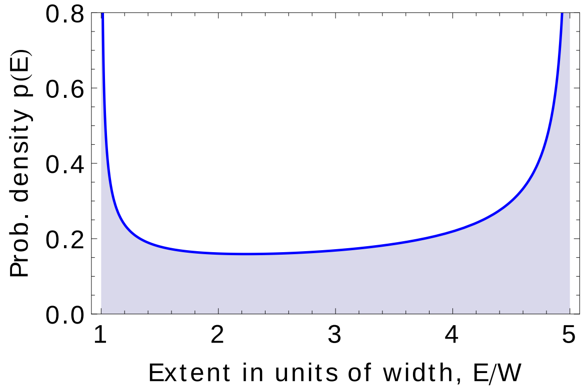


Figure 3: *Probability density $p(E)$ from* (3.13) *for a length-to-width ratio* $L/W = 5$

This expression is always negative in the interval $\theta = 0 \to \pi/2$, meaning E decreases while θ increases, which is expected and evident in Figure 2. This one-to-one mapping of angle to extent implies that every angle bin $\theta \to \theta + \mathrm{d}\theta$ corresponds uniquely to one extent bin $E \to E + \mathrm{d}E$, as in (2.2). We conclude that (2.2) can be applied as

$$\begin{aligned} p(E) &= \frac{2}{\pi}\left|\frac{\mathrm{d}\theta}{\mathrm{d}E}\right| = \frac{-2}{\pi}\frac{\mathrm{d}\theta}{\mathrm{d}E} \\ &= \frac{2E}{\pi(L^2 - W^2)\sin\theta\cos\theta}\,. \end{aligned} \tag{3.10}$$

To express $p(E)$ wholly in terms of E we need to write $\sin\theta$ and $\cos\theta$ in terms of E. Begin by writing (3.8) as

$$E^2 = W^2\sin^2\theta + L^2(1 - \sin^2\theta)\,, \tag{3.11}$$

which leads to

$$\sin\theta = \sqrt{\frac{L^2 - E^2}{L^2 - W^2}}\,, \quad \cos\theta = \sqrt{\frac{E^2 - W^2}{L^2 - W^2}} \tag{3.12}$$

in the first quadrant. These convert (3.10) to an expression wholly in terms of E:

$$\boxed{p(E) = \frac{2E}{\pi\sqrt{(L^2 - E^2)(E^2 - W^2)}}\,.} \tag{3.13}$$

The probability density $p(E)$ for a length-to-width ratio 5 is plotted in Figure 3. Although (3.13) shows that $p(E)$ goes to infinity when the extent equals either the ship's width or its length, the divergence at these extremes is not strong enough to prevent $p(E)$ from integrating to 1 between these two values, as is required of a probability density. The high density near the extreme values of W and L is due to the change in extent being of a smaller order as the angle θ sweeps through an infinitesimal range near these values, as compared with the change in extent for other angle values. It can be likened to the comparatively long time that a piston spends at the top and bottom of its travel.

This bi-modality of $p(E)$ is its key feature, and it implies that $p(E)$ cannot simply be rendered down to a single representative value. (Contrast this with a bell-shaped probability density, which *can* be characterised by its mean, together with a spread about

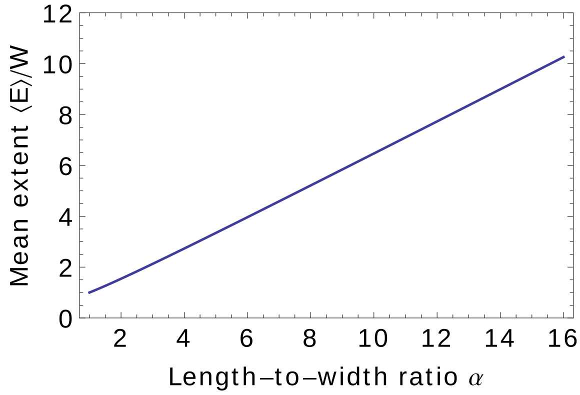


Figure 4: *The mean extent $\langle E\rangle$ in units of width W for a range of length-to-width ratios α, found by evaluating* (3.15) *numerically*

that mean.) So while we might choose to represent $p(E)$ by the resulting mean extent, $\langle E\rangle$, we must remember that this value is representative only in a limited context.

3.3 Mean Extent $\langle E\rangle$

Refer to (3.13) to write

$$\langle E\rangle = \int_W^L E\, p(E)\, \mathrm{d}E = \frac{2}{\pi}\int_W^L \frac{E^2\,\mathrm{d}E}{\sqrt{(L^2-E^2)(E^2-W^2)}}\,. \tag{3.14}$$

Write this integral dimensionlessly by setting $x \equiv E/W$ and $\alpha \equiv L/W$, giving

$$\boxed{\frac{\langle E\rangle}{W} = \frac{2}{\pi}\int_1^\alpha \frac{x^2\,\mathrm{d}x}{\sqrt{(\alpha^2-x^2)(x^2-1)}}\,.} \tag{3.15}$$

The result of the integration is an elliptic integral of the second kind:

$$\frac{\langle E\rangle}{W} = \frac{2\alpha}{\pi}\, E\left(\frac{\pi}{2}, \frac{\sqrt{\alpha^2-1}}{\alpha}\right), \tag{3.16}$$

where the "E" on the right-hand side of (3.16) is the elliptic integral function, not the extent[4]. In fact, the curve of $\langle E\rangle/W$ versus α is more or less straight for all reasonable values of α, as shown in Figure 4. This analysis is found to agree well with a Monte Carlo simulation in which $\langle E\rangle$ results from averaging values of E that are calculated from (3.8) for several thousand values of θ chosen uniform randomly.

Best-fitting the curve in Figure 4 with a line over a range of values of α gives

$$\boxed{\langle E\rangle/W \simeq 0.6\alpha + 0.4\,.} \tag{3.17}$$

This differs little from an approximation $\langle E\rangle/W \simeq 0.5\alpha + 0.5$, meaning that $\langle E\rangle$ is approximately just the average of the ship's length and width.

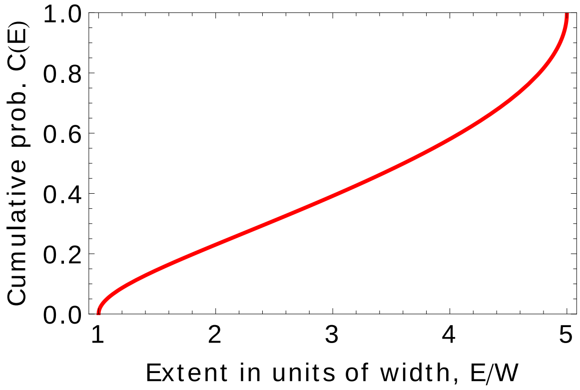


Figure 5: *$C(E)$ for a length-to-width ratio 5, found by evaluating* (3.19) *numerically with $\alpha = 5$*

3.4 Cumulative Probability of the Extent, $C(E)$

The cumulative probability of the extent is the probability that the extent will be measured to have some value between the minimum possible, the width, and a given value E:

$$C(E) \equiv \int_W^E p(E)\,\mathrm{d}E \overset{(3.13)}{=\!=} \frac{2}{\pi}\int_W^E \frac{E\,\mathrm{d}E}{\sqrt{(L^2-E^2)(E^2-W^2)}}\,. \tag{3.18}$$

As in the previous section, set $x \equiv E/W$ and $\alpha \equiv L/W$ to write

$$\boxed{C(E) = \frac{2}{\pi}\int_1^{E/W} \frac{x\,\mathrm{d}x}{\sqrt{(\alpha^2-x^2)(x^2-1)}}\,.} \tag{3.19}$$

$C(E)$ has a reasonably fixed form for length-to-width ratios of $\alpha = 5, 10$, and 15. The case of $\alpha = 5$ is shown in Figure 5.

As a concrete example we calculate $C(\langle E\rangle)$, the probability of measuring any extent less than the mean extent, for length-to-width ratios of $\alpha = 5, 10$, and 15.

Length-to-width ratio $\alpha = 5$: Equation (3.15) with $\alpha = 5$ gives $\langle E\rangle/W \simeq 3.34$. [Compare this to the simpler estimate (3.17) which gives a value of 3.4.] Now set E/W to 3.34 in (3.19) and evaluate that integral numerically, giving $C(\langle E\rangle) \simeq 0.451$.

As a sensitivity comparison, set E/W to the simpler estimate of 3.4 and evaluate (3.19) numerically to find $C(\langle E\rangle) \simeq 0.462$. Also, setting E/W equal to the mean of length and width, 3 [according to the comment just after (3.17)] and evaluating (3.19) numerically gives $C(\langle E\rangle) \simeq 0.392$.

Length-to-width ratio $\alpha = 10$: Equation (3.15) with $\alpha = 10$ gives $\langle E\rangle/W \simeq 6.47$. [Compare this to the simpler estimate (3.17) which gives a value of 6.4.] Now set E/W to 6.47 in (3.19) and evaluate that integral numerically, giving $C(\langle E\rangle) \simeq 0.444$.

Length-to-width ratio $\alpha = 15$: Equation (3.15) with $\alpha = 15$ gives $\langle E\rangle/W \simeq 9.63$. [Compare this to the simpler estimate (3.17) which gives a value of 9.4.] Now set E/W to 9.63 in (3.19) and evaluate that integral numerically, giving $C(\langle E\rangle) \simeq 0.442$.

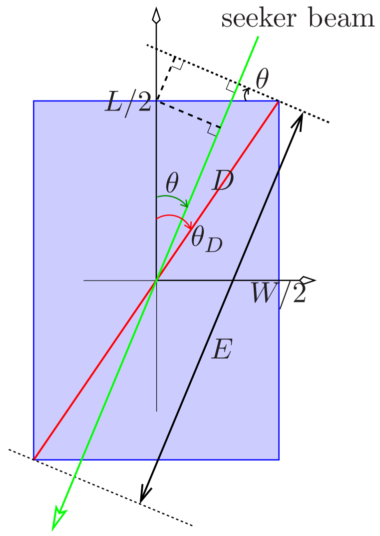


Figure 6: *Definitions of terms used in this analysis*

These three cumulative probabilities $C(\langle E\rangle) = 0.451, 0.444, 0.442$ are all fairly similar, and we conclude that for a wide range of length-to-width ratios, on average about 44% of the extent measurements will yield values between the minimum possible (the width) and the mean extent.

4 Scenario 2: Rectangle Illuminated by a Wide Beam

4.1 The Extent E

We now replace the ellipse of Scenario 1 with a rectangle of width W and length $L \geqslant W$, again illuminated by a wide beam, shown in Figure 6. For shorthand, define D as the diagonal length of the (full) rectangle:

$$D \equiv \sqrt{W^2 + L^2}\,. \tag{4.1}$$

Two key angles will be needed in our analysis: θ_D, the angle where the extent attains its maximum D, and θ_L, the nonzero angle less than 90° where the extent has value L (shown in Figure 7). Figure 6 shows that for $0 \leqslant \theta \leqslant \theta_D$ the semi-extent is given by $E/2 = W/2 \sin\theta + L/2 \cos\theta$. We can treat the case of $\theta_D < \theta \leqslant \pi/2$ using Figure 6 by swapping L and W and setting $\theta \to \pi/2 - \theta$, which yields the same expression for E. So we conclude that in the whole first quadrant,

$$\boxed{E = W \sin\theta + L \cos\theta\,.} \tag{4.2}$$

This is shown schematically in Figure 7 and quantitatively in Figure 8.

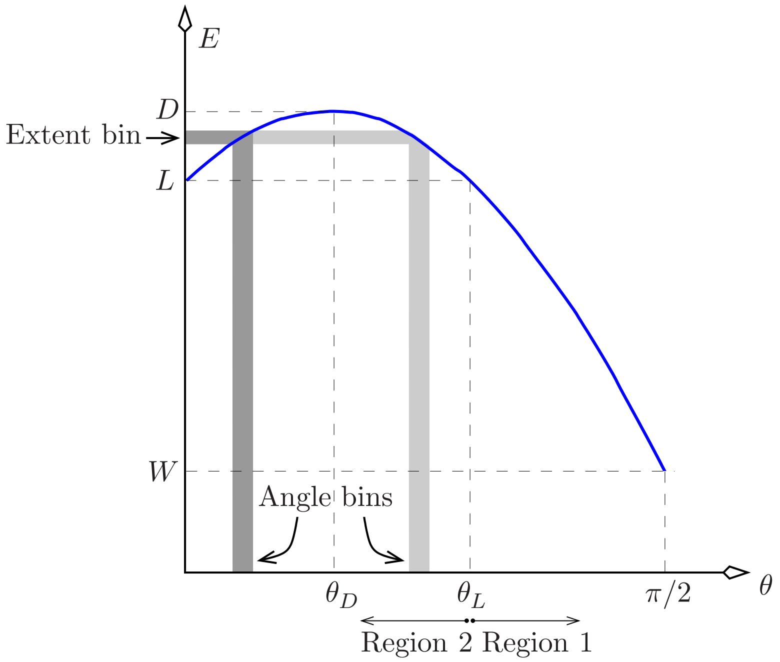


Figure 7: *A schematic of the extent E as a function of angle θ from* (4.2), *with angle contributions to the interval $E \to E + \mathrm{d}E$ shown in grey*

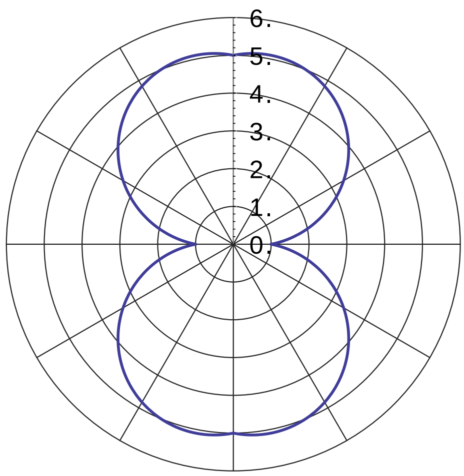


Figure 8: *A polar plot of the four-quadrant extent for $L/W = 5$ from* (4.2)*; the vertical axis here runs along the length of the ship*

4.2 Probability Density $p(E)$

The calculation of $p(E)$ is now more involved than in the ellipse model. For $\theta < \theta_L$ ($E > L$), there are *two* angle-bin contributions to $p(E)\,|\mathrm{d}E|$, because the same value of E can result from either of two angles. In that case we add the probabilities for the two angle contributions, which is simply the requirement that the number of "trials" $p(E)\,|\mathrm{d}E|$ in the extent bin be the sum of the numbers of trials in the two angle bins that contribute to that extent bin. The analysis becomes more complicated, but this is purely due to the extra bookkeeping required to keep track of the two angle contributions to the extent probability. The two angle intervals that contribute to a given extent interval are shown as grey regions in Figure 7.

The angle $\theta_D = \tan^{-1} \frac{W}{L}$ marks the greatest possible value of the extent, being along the diagonal of the rectangle. The angle θ_L marks a dividing line: to its left in Figure 7 is the region where there are two contributions to $p(E)$ (where the function $E(\theta)$ is not "one to one"), and to its right is the region where there is only contribution (where $E(\theta)$ *is* one to one). We must calculate $p(E)$ in each region separately.

Note that (4.2) gives

$$\begin{aligned} E = W \sin\theta + L\cos\theta &= D\left(\frac{W}{D}\sin\theta + \frac{L}{D}\cos\theta\right) \\ &= D(\sin\theta_D \sin\theta + \cos\theta_D\cos\theta) \\ &= D\cos(\theta - \theta_D)\,, \end{aligned} \tag{4.3}$$

which is a cosine shifted to the right by θ_D. Hence $E(\theta)$ is symmetrical about θ_D, so we can see from Figure 7 that $\theta_L = 2\theta_D$. Equation (4.2) also produces

$$|\mathrm{d}E/\mathrm{d}\theta| = \sqrt{D^2 - E^2} \tag{4.4}$$

over the whole range of θ, a result that will be required shortly.

4.2.1 The Region $W \leqslant E \leqslant L$

Figure 7 shows that this extent interval corresponds to the angle interval $\theta_L \leqslant \theta \leqslant \pi/2$. There is a single contribution to $p(E)\,|\mathrm{d}E|$ here of $2/\pi\,|\mathrm{d}\theta|$. It follows from (4.4) that

$$p(E)\,|\mathrm{d}E| = \frac{2}{\pi}\,|\mathrm{d}\theta| = \frac{2}{\pi}\frac{|\mathrm{d}E|}{\sqrt{D^2 - E^2}}\,, \tag{4.5}$$

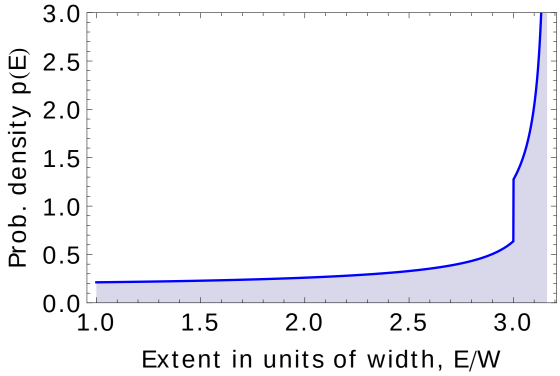
and therefore

$$p(E) = \frac{2}{\pi\sqrt{D^2 - E^2}}\,. \tag{4.6}$$

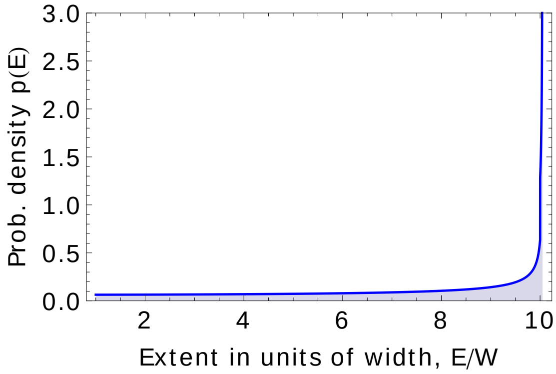
4.2.2 The Region $L < E \leqslant D$

Figure 7 shows that this extent interval corresponds to the angle interval $0 \leqslant \theta < \theta_L$. The probability $p(E)\,|\mathrm{d}E|$ now comes from two angle contributions (two angle bins) each with probability $p(\theta)\,|\mathrm{d}\theta| = 2/\pi\,|\mathrm{d}\theta|$. In both bins (4.4) holds, so that each gives the contribution on the right-hand side of (4.5); hence these bins add to give

$$p(E)\,|\mathrm{d}E| = \frac{4}{\pi}\frac{|\mathrm{d}E|}{\sqrt{D^2 - E^2}}\,, \tag{4.7}$$



Length-to-width ratio 3



Length-to-width ratio 10

Figure 9: *Probability densities for two length-to-width ratios*

so that

$$p(E) = \frac{4}{\pi\sqrt{D^2 - E^2}}\,. \tag{4.8}$$

The probability density for all values of θ is then

$$\boxed{p(E) = \begin{cases} \frac{2}{\pi\sqrt{D^2-E^2}}\,, & W \leqslant E \leqslant L \\ \frac{4}{\pi\sqrt{D^2-E^2}}\,, & L < E \leqslant D \end{cases}} \tag{4.9}$$

This is plotted for two length-to-width ratios in Figure 9.

4.3 Mean Extent $\langle E\rangle$, and Spread of the Extent σ_E^2

The mean extent is

$$\begin{aligned}\langle E\rangle &= \int_W^D E\,p(E)\,\mathrm{d}E \\ &= \frac{2}{\pi}\int_W^L \frac{E\,\mathrm{d}E}{\sqrt{D^2 - E^2}} + \frac{4}{\pi}\int_L^D \frac{E\,\mathrm{d}E}{\sqrt{D^2-E^2}} \\ &= \frac{2}{\pi}\left[-\sqrt{D^2-E^2}\right]_W^L + \frac{4}{\pi}\left[-\sqrt{D^2-E^2}\right]_L^D \end{aligned} \tag{4.10}$$

which simplifies to

$$\boxed{\langle E\rangle = 2/\pi\ (W + L)\,.} \tag{4.11}$$

This is slightly more than the mean of W and L, and is plotted in Figure 10.

As an exercise in calculating the standard deviation of E, consider that the variance

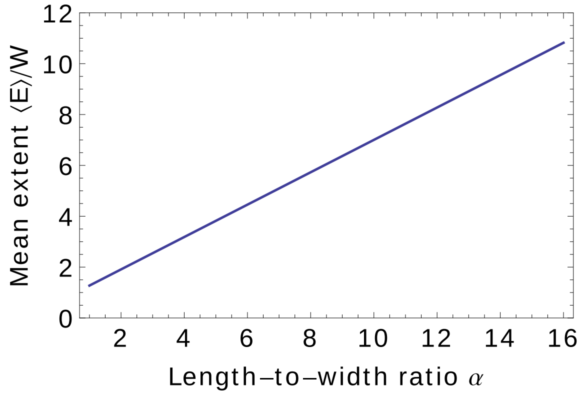


Figure 10: *The mean extent $\langle E\rangle$ in units of width W for a range of length-to-width ratios α, found from* (4.11)

is $\sigma_E^2 = \langle E^2\rangle - \langle E\rangle^2$. We require:

$$\begin{aligned}
\langle E^2\rangle = \int_W^D E^2\, p(E)\,\mathrm{d}E &= \frac{2}{\pi}\int_W^L \frac{E^2\,\mathrm{d}E}{\sqrt{D^2-E^2}} + \frac{4}{\pi}\int_L^D \frac{E^2\,\mathrm{d}E}{\sqrt{D^2-E^2}} \\
&= \frac{2}{\pi}\left[-\frac{E}{2}\sqrt{D^2-E^2} + \frac{D^2}{2}\sin^{-1}\frac{E}{D}\right]_{E=W}^{L} + \frac{4}{\pi}\left[-\frac{E}{2}\sqrt{D^2-E^2} + \frac{D^2}{2}\sin^{-1}\frac{E}{D}\right]_{E=L}^{D} \\
&= \frac{D^2}{2} + \frac{2WL}{\pi}\,. \qquad (4.12)
\end{aligned}$$

Now we can write

$$\begin{aligned}
\sigma_E^2 = \langle E^2\rangle - \langle E\rangle^2 &= \frac{D^2}{2} + \frac{2WL}{\pi} - \frac{4}{\pi^2}(W+L)^2 \\
&= \left(\frac{1}{2} - \frac{4}{\pi^2}\right)\left[W^2 + L^2 + 2WL\frac{\pi-4}{\pi^2/2-4}\right] \\
&\simeq 0.1\,(L-W)^2\,, \qquad (4.13)
\end{aligned}$$

and the standard deviation of the extent is $\sigma_E \simeq 0.3\,(L-W)$, remembering that $L \geqslant W$. The extent is not peaked close to its mean; for example, if $W = 1$ and $L = 10$, then $\langle E\rangle \simeq 7$ and $\sigma_E \simeq 3$—and of course there is no peak at all around 7 in the right-hand plot in Figure 9.

4.4 Cumulative Probability of the Extent, $C(E)$

The cumulative probability of measuring any extent up to some value E is

$$C(E) = \int_W^E p(E)\,\mathrm{d}E \overset{(4.9)}{=\!=} \begin{cases} \frac{2}{\pi}\int_W^E \frac{\mathrm{d}E}{\sqrt{D^2-E^2}}\,, & W \leqslant E \leqslant L \\ \frac{2}{\pi}\int_W^L \frac{\mathrm{d}E}{\sqrt{D^2-E^2}} + \frac{4}{\pi}\int_L^E \frac{\mathrm{d}E}{\sqrt{D^2-E^2}}\,, & L < E \leqslant D \end{cases}$$

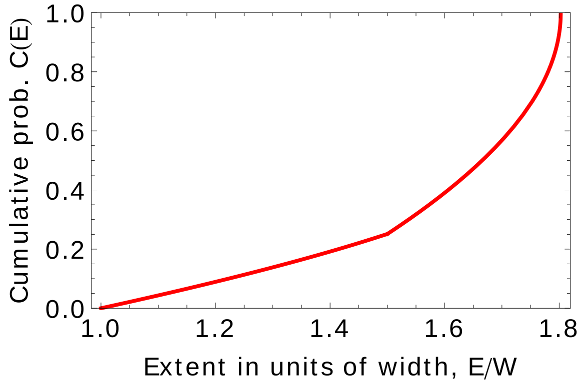
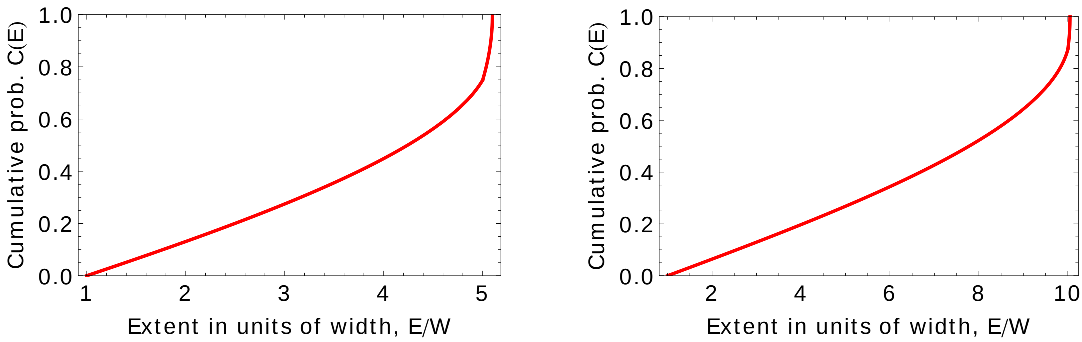


Figure 11: *$C(E)$ for a length-to-width ratio 1.5, from (4.15) with $L = 1.5W$*



Length-to-width ratio 5 Length-to-width ratio 10

Figure 12: *Cumulative probabilities for realistic length-to-width ratios, from (4.15). The rightmost part of the curve becomes less significant for higher ratios*

$$= \begin{cases} \frac{2}{\pi}\left[\sin^{-1}\frac{E}{D}\right]_W^E , & W \leqslant E \leqslant L \\ \frac{2}{\pi}\left[\sin^{-1}\frac{E}{D}\right]_W^L + \frac{4}{\pi}\left[\sin^{-1}\frac{E}{D}\right]_L^E , & L < E \leqslant D \end{cases} \tag{4.14}$$

leading to

$$\boxed{C(E) = \begin{cases} \frac{2}{\pi}\left(\sin^{-1}\frac{E}{D} - \theta_D\right) , & W \leqslant E \leqslant L \\ \frac{4}{\pi}\sin^{-1}\frac{E}{D} - 1 , & L < E \leqslant D\,. \end{cases}} \tag{4.15}$$

where $\theta_D = \tan^{-1}\frac{W}{L}$ as before. The cumulative probability is plotted in Figure 11 for a length-to-width ratio 1.5. More realistic examples are given in Figure 12.

As we did for the elliptical ship, we again calculate $C(\langle E\rangle)$, the cumulative probability of measuring an extent up to the mean $\langle E\rangle$. Applying (4.15) requires us to know whether $\langle E\rangle$ is greater than or less than L. Use (4.11) to write, after some short algebra,

$$\langle E\rangle < L \iff L > \frac{W}{\pi/2 - 1} \simeq 1.75\,W. \tag{4.16}$$

(So $\langle E\rangle$ will be less than L for all practical length-to-width ratios, which is reasonable.) The required cumulative probability becomes

$$C(\langle E\rangle) = \begin{cases} \frac{2}{\pi}\left(\sin^{-1}\frac{\langle E\rangle}{D} - \theta_D\right), & L > \frac{W}{\pi/2-1} \simeq 1.75\,W \\ \frac{4}{\pi}\sin^{-1}\frac{\langle E\rangle}{D} - 1\,, & L \leqslant \frac{W}{\pi/2-1} \simeq 1.75\,W\,. \end{cases} \tag{4.17}$$

Equation (4.17) requires $\langle E\rangle/D$, which is given by (4.11) in terms of $\alpha = L/W \geqslant 1$ as

$$\frac{\langle E\rangle}{D} = \frac{\frac{2}{\pi}(\alpha+1)}{\sqrt{\alpha^2+1}}\,. \tag{4.18}$$

We can use (4.17) and (4.18) to calculate $C(\langle E\rangle)$ for representative values of α:

$$\begin{aligned} \alpha = 5 &\implies C(\langle E\rangle) \simeq 0.41, \\ \alpha = 10 &\implies C(\langle E\rangle) \simeq 0.43, \\ \alpha = 15 &\implies C(\langle E\rangle) \simeq 0.43. \end{aligned} \tag{4.19}$$

These values don't differ markedly from those calculated for an elliptical ship on page 7: there the corresponding values were $C(\langle E\rangle) \simeq 0.45,\ 0.44,\ 0.44$.

5 Scenario 3: Ellipse Illuminated by a Narrow Beam

In this section and the next, we again use elliptical and rectangular ship models in two dimensions, but with the seeker now illuminating them using a narrow beam. The elliptical-ship scenario is shown in Figure 13.

5.1 The Extent E

Again it's sufficient to deal with incoming missile angles in the first quadrant. The extent is given by $E = 2\sqrt{x_0^2 + y_0^2}$, where the point $(x_0, y_0) = (y_0 \tan\theta, y_0)$ lies on the ellipse. This point must satisfy the equation for the ellipse,

$$\left(\frac{x}{W/2}\right)^2 + \left(\frac{y}{L/2}\right)^2 = 1\,, \tag{5.1}$$

in which case simple substitution leads to

$$y_0 = \frac{LW/2}{\sqrt{L^2\tan^2\theta + W^2}}\,, \qquad x_0 = y_0 \tan\theta\,. \tag{5.2}$$

Substituting these into $E = 2\sqrt{x_0^2 + y_0^2}$ gives the extent:

$$\boxed{E = \frac{LW}{\sqrt{L^2\sin^2\theta + W^2\cos^2\theta}}\,.} \tag{5.3}$$

This is plotted for the case of $L/W = 5$ in Figure 14.

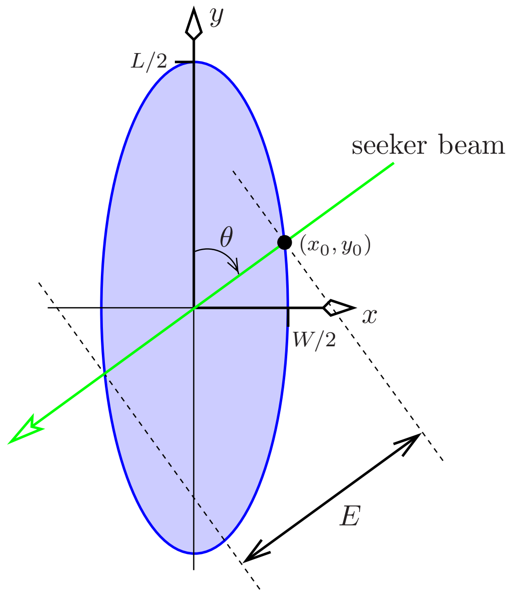
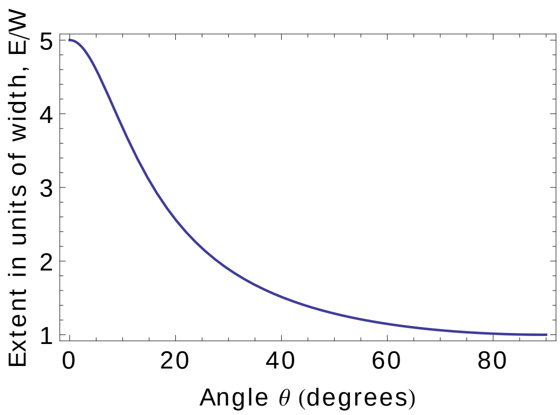


Figure 13: *Definitions of terms used in this analysis*



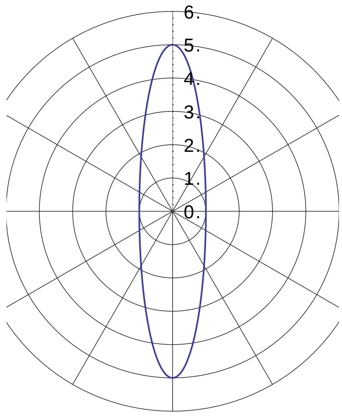


Figure 14: Left*: The extent in units of width for a length-to-width ratio $L/W = 5$, where it's sufficient to plot only angles in the first quadrant.* ***Right****: A polar plot of the same extent in all four quadrants. Not surprisingly this is just the outline of the ship, albeit twice as long and wide as the actual ship, due to the way the polar plot presents the data.*

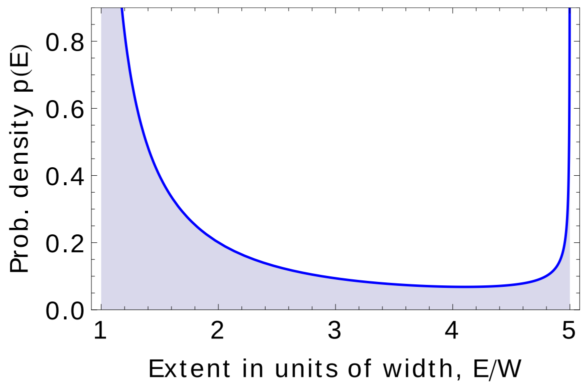


Figure 15: *Probability density $p(E)$ from (5.6) for a length-to-width ratio $L/W = 5$*

5.2 Probability Density $p(E)$

Because E is always decreasing as θ increases, only one angle bin contributes to any range bin. Hence, using

$$\frac{\mathrm{d}E}{\mathrm{d}\theta} = \frac{-E}{LW}\sqrt{(L^2 - E^2)(E^2 - W^2)}\,, \tag{5.4}$$

we can write

$$p(E)\,|\mathrm{d}E| = p(\theta)\,|\mathrm{d}\theta| = \frac{2}{\pi}\frac{|\mathrm{d}E|}{\frac{E}{LW}\sqrt{(L^2 - E^2)(E^2 - W^2)}}\,, \tag{5.5}$$

from which it follows that

$$\boxed{p(E) = \frac{2LW}{\pi E\sqrt{(L^2 - E^2)(E^2 - W^2)}}\,.} \tag{5.6}$$

This certainly integrates numerically to one for various random choices of L and W. It's plotted in Figure 15 for the case of $L/W = 5$.

5.3 Mean Extent $\langle E\rangle$

The mean of E is

$$\langle E\rangle = \int_W^L E\,p(E)\,\mathrm{d}E \overset{(5.6)}{=\!=} \frac{2LW}{\pi}\int_W^L \frac{\mathrm{d}E}{\sqrt{(L^2 - E^2)(E^2 - W^2)}}\,. \tag{5.7}$$

As before, set $x \equiv E/W$ and $\alpha \equiv L/W$ to write this integral dimensionlessly:

$$\boxed{\frac{\langle E\rangle}{W} = \frac{2\alpha}{\pi}\int_1^\alpha \frac{\mathrm{d}x}{\sqrt{(\alpha^2 - x^2)(x^2 - 1)}}\,.} \tag{5.8}$$

The mean extent is plotted in Figure 16 for practical values of α. As can be seen, it ranges typically from about 1.5 times the width for shorter vessels to about 2.7 times the width for the longest ones. Why is it comparatively small and insensitive to the ship's length? This is due to the use of a narrow beam by the seeker. Such a beam is only capable of sampling

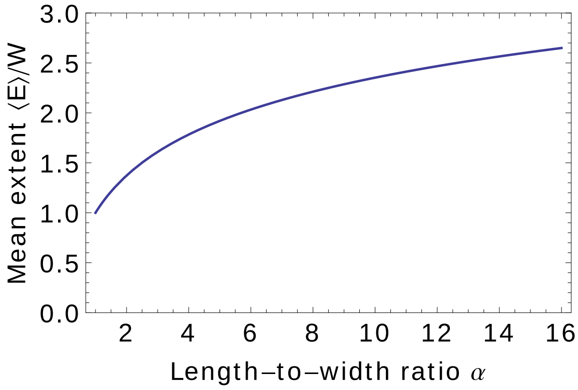


Figure 16: *The mean extent $\langle E\rangle$ in units of width W for a range of length-to-width ratios α, found by evaluating* (5.8) *numerically*

(or illuminating) approximately the full length of the ship for a small range of incoming seeker angles, which makes these larger extent values comparatively rare, in contrast to the wide-beam case. This is also shown by the probability density in Figure 15, which gives the larger extents a lower probability density. There is certainly a density peak at an extent equal to the ship's length, but this peak is too narrow to significantly modify the extent's mean.

5.4 Cumulative Probability of the Extent, $C(E)$

The cumulative probability of measuring any extent up to some value E is

$$C(E) = \int_W^E p(E)\,\mathrm{d}E \overset{(5.6)}{=\!=} \int_W^E \frac{2LW\,\mathrm{d}E}{\pi E\sqrt{(L^2-E^2)(E^2-W^2)}}\,, \tag{5.9}$$

and when again we write $x \equiv E/W$ and $\alpha \equiv L/W$, this becomes

$$\boxed{C(E) = \frac{2\alpha}{\pi}\int_1^x \frac{\mathrm{d}x}{x\sqrt{(\alpha^2-x^2)(x^2-1)}}\,.} \tag{5.10}$$

Figure 17 plots $C(E)$ for $\alpha = 5$. Mean extents and cumulative probabilities for representative values of α are

$$\begin{aligned}
\alpha = 5 \quad &\Longrightarrow \langle E\rangle \simeq 1.92 \;\text{ and }\; C(\langle E\rangle) \simeq 0.67\,,\\
\alpha = 10 \quad &\Longrightarrow \langle E\rangle \simeq 2.35 \;\text{ and }\; C(\langle E\rangle) \simeq 0.73\,,\\
\alpha = 15 \quad &\Longrightarrow \langle E\rangle \simeq 2.61 \;\text{ and }\; C(\langle E\rangle) \simeq 0.75\,.
\end{aligned} \tag{5.11}$$

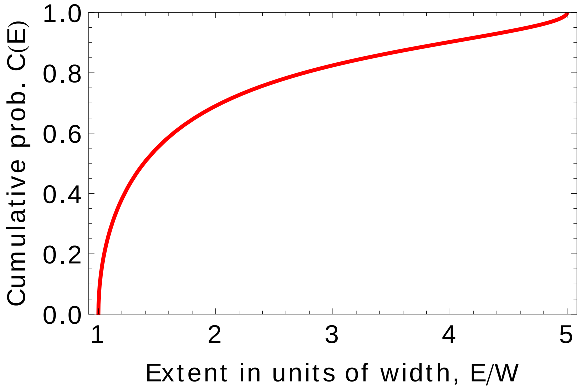


Figure 17: *$C(E)$ for a length-to-width ratio 5, found by evaluating (5.10) numerically with $\alpha = 5$*

6 Scenario 4: Rectangle Illuminated by a Narrow Beam

In this section we model the ship as a rectangle illuminated with a narrow beam. The scenario is shown in Figure 18 with the plot of the extent shown in Figure 19, which we'll derive shortly.

6.1 The Extent E

Just as in the previous rectangle model of Section 4, θ_D marks the greatest possible value of the extent, and points along the diagonal of the rectangle:

$$\theta_D = \tan^{-1} \frac{W}{L} . \tag{6.1}$$

Unlike the expressions for the extent E for the previous two ship/illumination models in (3.8) and (4.2), the current extent must be specified separately for angles less than θ_D ("D-region 1") and greater than θ_D ("D-region 2")

The extent of the narrow beam across the rectangle is easily found by inspecting Figure 18:

$$\boxed{E = \begin{cases} L/\cos\theta\,, & 0 \leqslant \theta \leqslant \theta_D \qquad (D\text{-region } 1) \\ W/\sin\theta\,, & \theta_D \leqslant \theta \leqslant \pi/2 \quad (D\text{-region } 2)\,. \end{cases}} \tag{6.2}$$

This is plotted in Figure 20; not surprisingly, the polar plot simply gives a picture of the rectangle! It's one corner of this rectangle that is being shown in Figure 19; of course, the cartesian nature of Figure 19 warps the corner, as compared to the right-angled version in the polar plot of Figure 20.

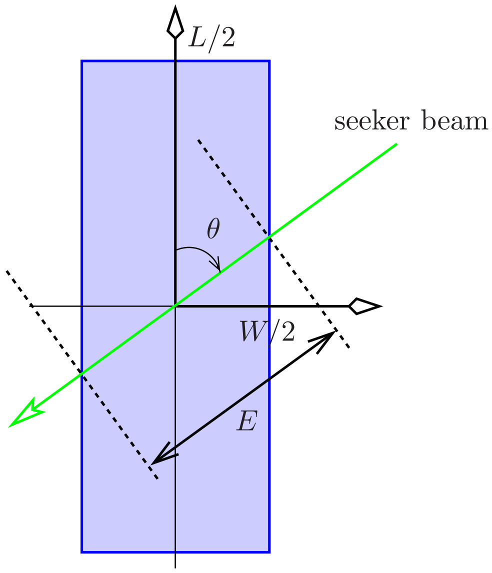


Figure 18: *Definitions of terms used in this analysis*

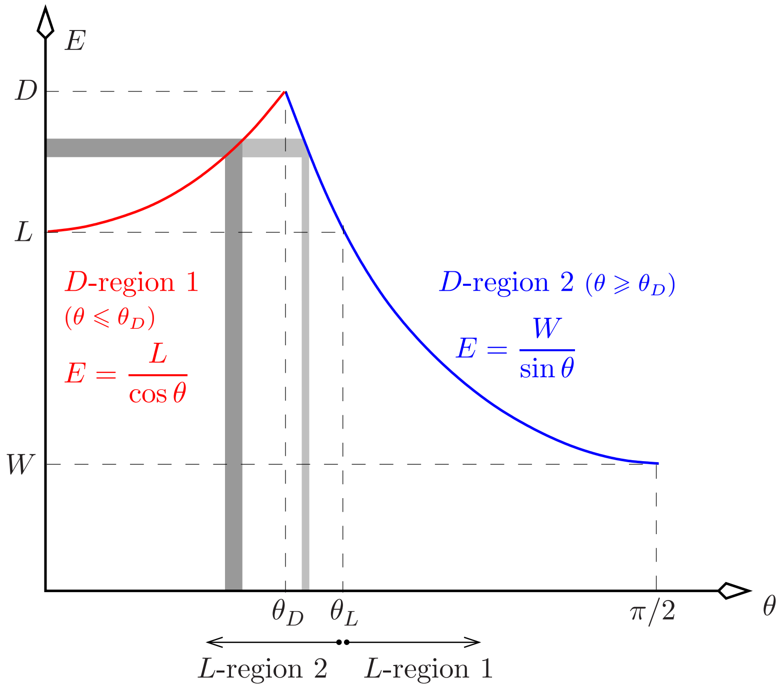


Figure 19: *Schematic of the extent E as a function of angle θ from* (6.2)*, with angle contributions to the interval $E \to E + \mathrm{d}E$ shown in grey for L-region 2*

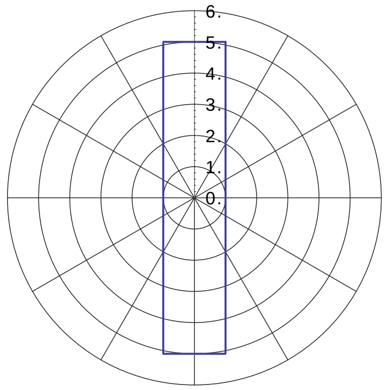


Figure 20: *A polar plot of the four-quadrant extent* (6.2) *for* $L/W = 5$*; the vertical axis here runs along the length of the ship. As in Figure 14, this plot shows the outline of the ship.*

6.2 Probability Density $p(E)$

Here we will need

$$\left|\frac{\mathrm{d}E}{\mathrm{d}\theta}\right| = \begin{cases} E\tan\theta = E\sqrt{E^2 - L^2}/L\,, & D\text{-region } 1 \\ E\cot\theta = E\sqrt{E^2 - W^2}/W\,, & D\text{-region } 2\,. \end{cases} \tag{6.3}$$

Also as in the previous rectangle model, an angle θ_L marks a dividing line: to its *right* in Figure 19 is "L-region 1" where there is only one angle contribution to $p(E)$, and to its *left* is "L-region 2" where there are two angle contributions to $p(E)$. Note that the two L-regions *don't* coincide with the two D-regions.

$$\theta_L = \sin^{-1}\frac{W}{L}; \quad \text{compare (6.1): } \theta_D = \tan^{-1}\frac{W}{L}\,. \tag{6.4}$$

We must construct $p(E)$ separately for each L-region.

6.2.1 L-region 1: $W \leqslant E \leqslant L$

Figure 19 shows that in this region there is a single contribution of $2/\pi\,|\mathrm{d}\theta|$ to $p(E)\,|\mathrm{d}E|$, coming from D-region 2 in (6.3):

$$p(E)\,|\mathrm{d}E| = \frac{2}{\pi}\,|\mathrm{d}\theta|_{D\text{-region } 2} = \frac{2}{\pi}\frac{W\,|\mathrm{d}E|}{E\sqrt{E^2 - W^2}}\,. \tag{6.5}$$

6.2.2 L-region 2: $L < E \leqslant D$

Figure 19 shows that the probability $p(E)\,|\mathrm{d}E|$ comes from two contributions of $2/\pi\,|\mathrm{d}\theta|$, one each from D-regions 1 and 2. The contribution from D-region 1 in (6.3) is

$$\frac{2}{\pi}\,|\mathrm{d}\theta|_{D\text{-region } 1} = \frac{2}{\pi}\frac{L\,|\mathrm{d}E|}{E\sqrt{E^2 - L^2}}\,, \tag{6.6}$$

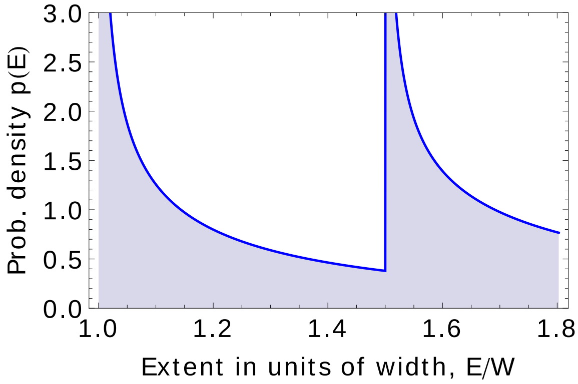


Figure 21: *Probability density* $p(E)$ *of ship extent from* (6.9) *for a ship of length-to-width ratio 1.5, clipped at the top to display well*

and the contribution from D-region 2 in (6.3) is

$$\frac{2}{\pi}\,|\mathrm{d}\theta|_{D\text{-region }2} = \frac{2}{\pi}\frac{W\,|\mathrm{d}E|}{E\sqrt{E^2-W^2}}\,. \tag{6.7}$$

Adding these two contributions (6.6) and (6.7) gives

$$p(E)\,|\mathrm{d}E| = \frac{2}{\pi}\left[\frac{L}{E\sqrt{E^2-L^2}} + \frac{W}{E\sqrt{E^2-W^2}}\right]|\mathrm{d}E|\,. \tag{6.8}$$

Equations (6.5) and (6.8) give the complete probability density as

$$\boxed{p(E) = \begin{cases} \frac{2}{\pi}\frac{W}{E\sqrt{E^2-W^2}}\,, & W \leqslant E \leqslant L \\ \frac{2}{\pi}\left[\frac{L}{E\sqrt{E^2-L^2}} + \frac{W}{E\sqrt{E^2-W^2}}\right], & L < E \leqslant D \end{cases}} \tag{6.9}$$

A plot of $p(E)$ versus E is shown in Figure 21 for a ship of length 1.5 times its width. (This unrealistic ratio is chosen to better highlight the details of the function.) The probability density goes to infinity when the extent equals either the ship's width or its length. The high density near these two values is caused by the change in extent being of a smaller order as the angle θ sweeps through an infinitesimal range near each of these two values, as compared with what θ sweeps through near other extent values. That is, the extent change is negligible compared to the angle change for angles in the vicinity of the ship's width and length, which has the effect that the probability in the *extent* bins around the width and length includes many more angles and so must be correspondingly large. Note that because $p(E)$ is bi-modal, the comment at the end of Section 3.2 applies to its mean value, which we calculate shortly.

More realistic probability densities are plotted in Figure 22, using length-to-width ratios of 5 and 10 respectively.

Note that the entire density still integrates to 1 as required: equation (6.24) ahead—the cumulative probability $C(E)$—returns a value of 1 when evaluated at the largest value of extent, $E = D$.

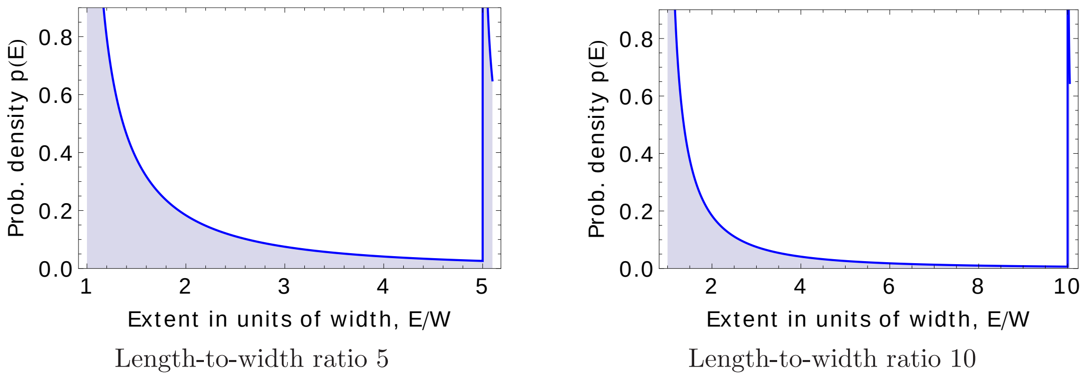


Figure 22: *Probability densities from* (6.9) *for more realistic length-to-width ratios, showing that the rightmost peak in the density has less support at higher ratios*

6.3 Mean Extent $\langle E \rangle$

The mean of E is

$$\langle E \rangle = \int_W^D E\, p(E)\, \mathrm{d}E\,. \tag{6.10}$$

Use (6.9) to write this as

$$\langle E \rangle = \frac{2}{\pi} \int_W^L \frac{W \mathrm{d}E}{\sqrt{E^2 - W^2}} + \frac{2}{\pi} \int_L^D \left[\frac{L\,\mathrm{d}E}{\sqrt{E^2 - L^2}} + \frac{W\,\mathrm{d}E}{\sqrt{E^2 - W^2}} \right]. \tag{6.11}$$

The first and third terms in this sum have the same integrand and so can be combined:

$$\langle E \rangle = \frac{2W}{\pi} \int_W^D \frac{\mathrm{d}E}{\sqrt{E^2 - W^2}} + \frac{2L}{\pi} \int_L^D \frac{\mathrm{d}E}{\sqrt{E^2 - L^2}}\,. \tag{6.12}$$

These integrals are inverse hyperbolic cosines (ch^{-1}): the general identity for $a > 0$ is

$$\int \frac{\mathrm{d}x}{\sqrt{x^2 - a^2}} = \mathrm{ch}^{-1} \frac{x}{a} = \ln \left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right). \tag{6.13}$$

With this identity, (6.12) becomes

$$\langle E \rangle = \frac{2}{\pi} \left(W\, \mathrm{ch}^{-1} \frac{D}{W} + L\, \mathrm{ch}^{-1} \frac{D}{L} \right). \tag{6.14}$$

The mean extent $\langle E \rangle$ can be expressed using logarithms by way of (6.13). Writing the length-to-width ratio as $\alpha \equiv L/W$, (6.14) becomes

$$\boxed{\frac{\langle E \rangle}{W} = \frac{2}{\pi} \left[\ln \left(\sqrt{1 + \alpha^2} + \alpha \right) + \alpha \ln \left(\sqrt{1 + 1/\alpha^2} + 1/\alpha \right) \right].} \tag{6.15}$$

Figure 23 plots $\langle E \rangle / W$ versus α from (6.15), and shows that values are typically around 2. That these values are reasonably insensitive to the ship's length is due to the seeker's narrow beam, as discussed in Section 5.3.

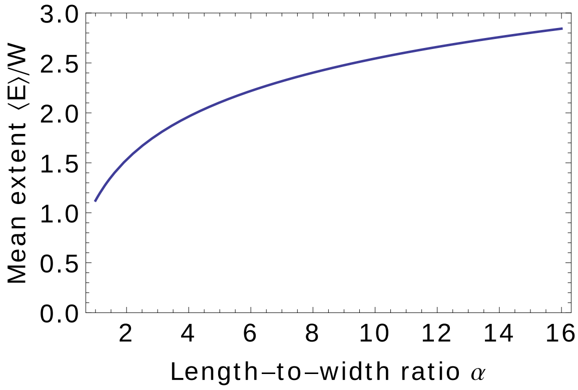


Figure 23: *The mean extent $\langle E\rangle$ in units of width W for a range of length-to-width ratios α, from* (6.15)

For $\alpha \gtrsim 1$, the term in brackets in (6.15) is approximately $\ln(2\alpha) + \alpha \ln(1 + 1/\alpha)$. Now applying the identity $\ln(1 + x) \simeq x$ for $|x| \ll 1$ produces

$$\frac{\langle E\rangle}{W} = \frac{2}{\pi}\,(1 + \ln 2 + \ln\alpha) \approx (1.1 + 0.6\ln\alpha). \tag{6.16}$$

So, for length-to-width ratios between 5 and 10, $\langle E\rangle$ ranges from $2.1\,W$ to $2.5\,W$, as indeed is shown in Figure 23. But measured values of E are not especially peaked about $\langle E\rangle$ because the probability density $p(E)$ is bi-modal.

6.4 Cumulative Probability of the Extent, $C(E)$

The cumulative probability of measuring any extent up to some value E is

$$C(E) \equiv \int_W^E p(E)\,\mathrm{d}E\,. \tag{6.17}$$

In principle we could use (6.9) to evaluate this integral in different extent regimes. In practice it's perhaps easier to relate $p(E)\,\mathrm{d}E$ to $2/\pi\,|\mathrm{d}\theta|$ as we did when calculating $p(E)$, which amounts to a change of variables in the integration in (6.17).

$C(E)$ for $W \leqslant E \leqslant L$ (L-region 1)

Integrating from W to L means that E is increasing (i.e. $\mathrm{d}E > 0$ so $|\mathrm{d}E| = \mathrm{d}E$), which corresponds to θ *decreasing*; that is, $\mathrm{d}\theta < 0$. Hence $|\mathrm{d}\theta| = -\mathrm{d}\theta$, and

$$p(E)\,\mathrm{d}E = p(E)\,|\mathrm{d}E| = 2/\pi\,|\mathrm{d}\theta| = -2/\pi\,\mathrm{d}\theta\,. \tag{6.18}$$

With the aid of (6.2), the integral (6.17) becomes

$$C(E) = \int_{E=W}^{E} \frac{-2}{\pi}\,\mathrm{d}\theta = \int_{\pi/2}^{\sin^{-1}(W/E)} \frac{-2}{\pi}\,\mathrm{d}\theta = \frac{-2}{\pi}\left(\sin^{-1}\frac{W}{E} - \frac{\pi}{2}\right) = \frac{2}{\pi}\cos^{-1}\frac{W}{E}\,, \tag{6.19}$$

using the identity $\sin^{-1} x + \cos^{-1} x = \pi/2$ for all relevant x.

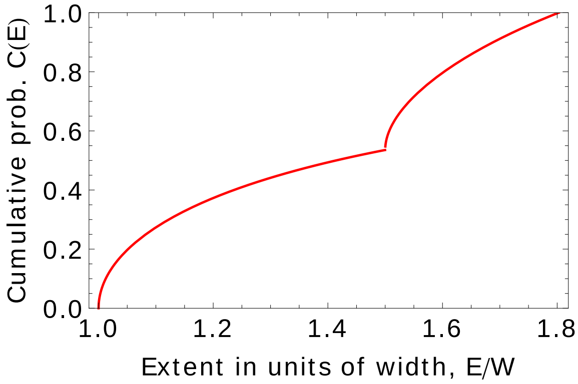


Figure 24: *Cumulative probability density $C(E)$ for length-to-width 1.5, from* (6.24)

$C(E)$ for $L < E \leqslant D$ (L-region 2)

The integration (6.17) now requires both D-region 1 and D-region 2. Again the integration is over increasing E, so $\mathrm{d}E > 0$, meaning $|\mathrm{d}E| = \mathrm{d}E$:

$$p(E)\,\mathrm{d}E = p(E)\,|\mathrm{d}E| = 2/\pi\,|\mathrm{d}\theta|_{D\text{-region 1}} + 2/\pi\,|\mathrm{d}\theta|_{D\text{-region 2}}\,. \tag{6.20}$$

In D-region 1, increasing E means increasing θ (i.e. $\mathrm{d}\theta > 0$ so $|\mathrm{d}\theta| = \mathrm{d}\theta$). In D-region 2, increasing E means decreasing θ (i.e. $\mathrm{d}\theta < 0$ so $|\mathrm{d}\theta| = -\mathrm{d}\theta$). Thus (6.20) becomes

$$p(E)\,\mathrm{d}E = 2/\pi\,\mathrm{d}\theta_{D\text{-region 1}} - 2/\pi\,\mathrm{d}\theta_{D\text{-region 2}}\,. \tag{6.21}$$

The cumulative probability (6.17) is now written

$$C(E) = C(L) + \int_L^E p(E)\,\mathrm{d}E = C(L) + \int_{E=L}^E \frac{2}{\pi}\,\mathrm{d}\theta_{D\text{-region 1}} - \int_{E=L}^E \frac{2}{\pi}\,\mathrm{d}\theta_{D\text{-region 2}}\,, \tag{6.22}$$

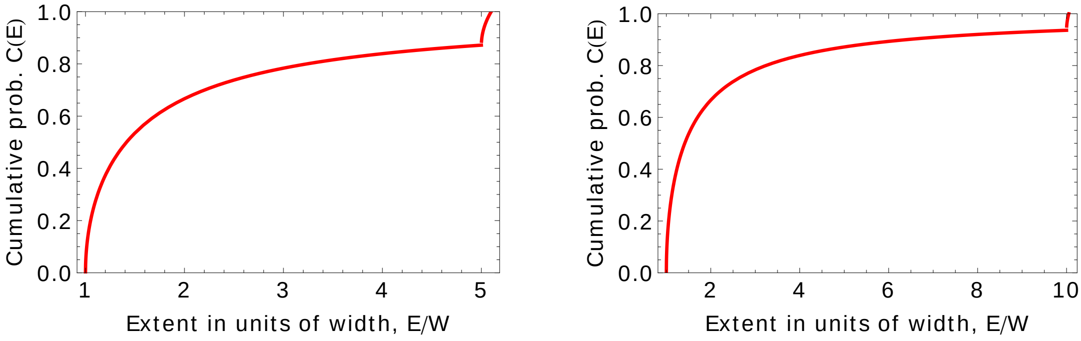
which becomes, with the aid of (6.2),

$$\begin{aligned} C(E) &= C(L) + \int_0^{\cos^{-1}(L/E)} \frac{2}{\pi}\,\mathrm{d}\theta - \int_{\sin^{-1}(W/L)}^{\sin^{-1}(W/E)} \frac{2}{\pi}\,\mathrm{d}\theta \\ &= \frac{2}{\pi}\left(\cos^{-1}\frac{W}{E} + \cos^{-1}\frac{L}{E}\right). \end{aligned} \tag{6.23}$$

The complete cumulative probability is then

$$\boxed{C(E) = \begin{cases} \frac{2}{\pi}\cos^{-1}\frac{W}{E} & W \leqslant E \leqslant L \\ \frac{2}{\pi}\left(\cos^{-1}\frac{W}{E} + \cos^{-1}\frac{L}{E}\right) & L < E \leqslant D\,. \end{cases}} \tag{6.24}$$

This is plotted in Figure 24 for a ship of length 1.5 times its width. (Again, this unrealistic ratio is chosen to better highlight the details of the function.) More realistic cumulative probabilities are plotted in Figure 25, using length-to-width ratios of 5 and 10 respectively.



Length-to-width ratio 5 Length-to-width ratio 10

Figure 25: *Cumulative probabilities from* (6.24) *for realistic length-to-width ratios. The rightmost part of the curve becomes less significant for higher ratios*

As a concrete example, we ask for $C(\langle E\rangle)$, the probability of measuring any extent less than the mean, for the case of $\alpha = L/W = 5$. Here, either of (6.15) and (6.16) gives $\langle E\rangle \simeq 2.10\,W$. Now use (6.24) to compute $C(2.10\,W)$, referring to the case $W \leqslant E \leqslant L$:

$$C(2.10\,W) \overset{(6.24)}{=\!=} \frac{2}{\pi}\cos^{-1}\frac{1}{2.10} \simeq 0.68\,. \tag{6.25}$$

That is, on average 68% of the extent measurements will yield values between W and $2.10\,W$.

Similarly for $\alpha = 10$ and 15 we have (with the above $\alpha = 5$ result included for completeness)

$$\begin{aligned} \alpha = 5 \quad &\Longrightarrow \langle E\rangle \simeq 2.10 \;\text{ and }\; C(\langle E\rangle) \simeq 0.68\,,\\ \alpha = 10 \quad &\Longrightarrow \langle E\rangle \simeq 2.54 \;\text{ and }\; C(\langle E\rangle) \simeq 0.74\,,\\ \alpha = 15 \quad &\Longrightarrow \langle E\rangle \simeq 2.80 \;\text{ and }\; C(\langle E\rangle) \simeq 0.77\,. \end{aligned} \tag{6.26}$$

These don't differ markedly from the corresponding values for an ellipse with narrow beam in (5.11).

7 Concluding Comments

The expressions given in this report form a base for more extensive analyses of how a seeker might interact with a ship based solely on the range extent measured by the seeker.

Any analysis must use a tractable model of the ship and a choice of beam type for the seeker. It's instructive to compare the statistics for the elliptical and rectangular models for each beam type; in doing so, we find the two sets of statistics to be similar as discussed in the next paragraphs.

For the wide beam, we see similar values for extent as a function of incoming seeker angle (Figures 2 and 8), and mean extent versus length-to-width ratio (Figures 4 and 10). The cumulative probabilities in Figures 5 and 12 (left) don't differ greatly.

For the narrow-beam models, the extent as a function of incoming seeker angle is just the outline of the ship: Figures 14 and 20. The other narrow-beam results cannot be compared with the wide-beam results, but within the two narrow-beam models we find that the two probability densities versus extent are similar (Figures 15 and 22 left), as are the mean extents versus length-to-width ratio (Figures 16 and 23), and the cumulative probabilities (Figures 17 and 25 left).

With regard to computational complexity for a ship model, the ellipse is generally more straightforward to analyse than the rectangle because the ellipse does not demand contributions from two angle bins to one extent bin, unlike the rectangle; this is a very big advantage of the ellipse model. However, the integrations required for the ellipse are generally not expressible in terms of elementary functions, unlike those for the rectangle. But these ellipse integrals are easily evaluated numerically so, on balance, the ellipse model appears to be the better choice.

The wide beam is obviously of most relevance to current seekers, but this could well change for future seekers. Mean ship extents for the wide beam are shown in Figures 4 and 10; referring to (3.17) and (4.11) with the comments immediately following them, we see that the mean extent will tend to have a value near the ship's *length*, especially when the length is much larger than the width. In contrast, a narrow beam produces a mean extent closer to the ship's *width.* But unfortunately, the situation is not quite as simple as this. The four probability densities, ellipse and rectangle wide beam, and ellipse and rectangle narrow beam are repeated in Figure 26 for convenience, and show that $p(E)$ is mostly bi-modal. This implies that in any one measurement, we do *not* expect to measure the mean extent, even though the mean extent is traditionally called the extent's "expected value". Such a term only applies when $p(E)$ is uni-modal. Even so, Figure 26 indicates that a rough rule of thumb is still

– wide beams tend to yield extents of approximately the ship's length (because the probability densities have more support at the right-hand end of the graphs)

– narrow beams tend to yield extents of approximately the ship's width (because the probability densities have more support at the left-hand end of the graphs)

In the absence of any other information except beam type and expected ship type, the seeker might well estimate the ship's size by assuming that a value "in the neighbourhood of" the mean extent has been returned by an extent measurement. For the wide beam, refer to Figures 4 and 10; for example if the seeker is searching for a ship with a length-to-width ratio of 10, then it could divide its extent measurement by 6 to give an estimate of the ship's width. (And of course, the length would then follow: $L = 10W$.) Similarly for a narrow beam: Figures 16 and 23 show that the seeker can divide its extent measurement by about 2.4 to give an estimate of the ship's width. But this idea assumes that the mean extent has been measured, which is only a first estimate of a seeker's strategy.

The seekers in this report approach the ship's centre directly. Having a seeker approach off to one side would not give very different results for a wide beam, but certainly would yield new values for a narrow beam. The question of what extents a narrow-beam seeker would measure if it explored different look angles on approach would extend the work in this report—but does become very model dependent.

In the final analysis the seeker sees no very strongly dominant range extents, so that it's probably not necessary for a jammer to focus heavily on constructing any one particular range extent when assembling a false target presentable to a seeker. Of course, a caveat is that measuring a ship's radial extent is just one strategy that a seeker might employ.

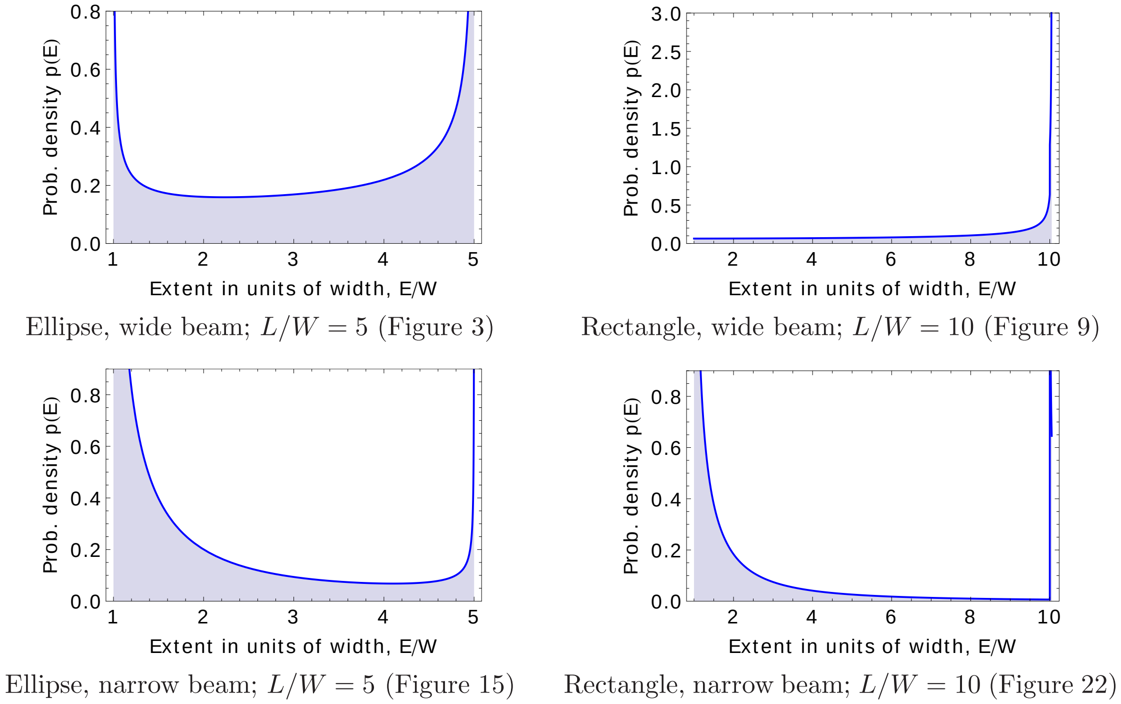


Figure 26: *All probability densities repeated here for convenience*

The results of this report allow a selection of more representative range extent values for ongoing work into false-target generation for countering incoming missiles. The main result here is that they indicate we have no strong obligation to manufacture a waveform that mimics the full length of a ship. This allows more economical use to be made of available false-target generation resources.

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19. ABSTRACT

We present a geometrical study of a missile seeker illuminating a ship target. Modern seekers employ various methods to recognise their target amid background vessels and electronic countermeasures such as chaff and active decoys. We focus on one such possible method: the seeker's measurement of a ship's down-range extent. We aim to find a representative value for this extent given that the seeker can approach the ship, centre on, from any direction. This representative value can then be used in studies of electronic attack in the generation and presenting to the seeker of a realistic false ship target. Ensuring the false target has a statistically favourable range extent might make it appear more realistic to the seeker. On the other hand, a lack of any statistically dominant range extent implies that we needn't select any particular value to present to the seeker. Either way, this report's analysis gives guidance in what range extent to specify when constructing a false target.

The simple scenarios we consider turn out to present no statistically dominant range extents. Extrapolating from these, we conclude that it's probably not necessary to focus strongly on any one range extent when constructing a false target.