



Global Resolution of Convex Programs with Complementarity Constraints

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Final Report**

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Final Report

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Project Title: Further Development in the Global Resolution of Convex Programs with Complementarity Constraints

Principal Investigators: John E. Mitchell (Rensselaer Polytechnic Institute) and Jong-Shi Pang (University of Illinois at Urbana-Champaign till August 15, 2013; substituted administratively by Angelia Nedich from August 16, 2013 to December 31, 2013)

Project Participants: Yu-Ching Lee (University of Illinois at Urbana-Champaign); Bin Yu and Lijie Bai (both at Rensselaer Polytechnic Institute). These students have successfully defended their doctoral dissertations whose topics are all concerned with various aspects of the project supported by this grant.

Project Summary: This collaborative project aims at the study of the global resolution of convex programs with complementarity constraints (CPCCs), which form a large subclass of the class of mathematical programs with complementarity constraints (MPCCs). Despite the large literature on the local properties of an MPCC, there is a lack of systematic investigation on the computation of a globally optimal solution of these constrained optimization problems, or in the case where such a solution does not exist, on the generation of a certificate demonstrating the solution non-existence. While this is by no means an easy task, the pervasiveness of the CPCC in applications provides an important motivation for the undertaking of this project. Extending our previous study on the subclass of linear programs with linear complementarity constraints (LPCCs), which is essentially a linear program with additional complementarity constraints on certain pairs of variables, we have investigated the class of convex quadratic programs with complementarity constraints (QPCCs) and begun to explore the global resolution of the class of ℓ_0 optimization problems. Our work has provided a solid foundation for the design of efficient algorithms for obtaining a solution with provable quality of the class of highly challenging complementarity constrained optimization problems.

Publications: Six papers have been published; all are available from <http://www.rpi.edu/~mitchj>:

- L. BAI, J.E. MITCHELL, AND J.S. PANG. On convex quadratic programs with linear complementarity constraints. *Computational Optimization and Applications* 54 (2013) 517–544.

The paper shows that the global resolution of a general convex quadratic program with complementarity constraints (QPCC), possibly infeasible or unbounded, can be accomplished in finite time. The method constructs a minmax mixed integer formulation by introducing finitely many binary variables, one for each complementarity constraint. Based on the

primal-dual relationship of a pair of convex quadratic programs and on a logical Benders scheme, an extreme ray/point generation procedure is developed, which relies on valid satisfiability constraints for the integer program. To improve this scheme, we propose a two-stage approach wherein the first stage solves the mixed integer quadratic program with pre-set upper bounds on the complementarity variables, and the second stage solves the program outside this bounded region by the Benders scheme. We report computational results with our method. We also investigate the addition of a penalty term $y^T D w$ to the objective function, where y and w are the complementary variables and D is a nonnegative diagonal matrix. The matrix D can be chosen effectively by solving a semidefinite program, ensuring that the objective function remains convex. The addition of the penalty term can often reduce the overall runtime by at least 50%. We report preliminary computational testing on a QP relaxation method which can be used to obtain better lower bounds from infeasible points; this method could be incorporated into a branching scheme. By combining the penalty method and the QP relaxation method, more than 90% of the gap can be closed for some QPCC problems.

- L. BAI, J.E. MITCHELL, AND J.S. PANG. Using quadratic convex reformulation to tighten the convex relaxation of a quadratic program with complementarity constraints. *Optimization Letters* 8 (2014) 811–822.

Abstract: Quadratic Convex Reformulation (QCR) is a technique that has been proposed for binary and mixed integer quadratic programs. In this paper, we extend the QCR method to convex quadratic programs with linear complementarity constraints (QPCCs). Due to the complementarity relationship between the nonnegative variables y and w , a term $y^T D w$ can be added to the QPCC objective function, where D is a nonnegative diagonal matrix chosen to maintain the convexity of the objective function and the global resolution of the QPCC. Following the QCR method, the products of linear equality constraints can also be used to perturb the QPCC objective function, with the goal that the new QP relaxation provides a tighter lower bound. By solving a semidefinite program, an equivalent QPCC can be obtained whose QP relaxation is as tight as possible. In addition, we extend the QCR to a general quadratically constrained quadratic program (QCQP), of which the QPCC is a special example. Computational tests on QPCCs are presented.

- J. HU, J.E. MITCHELL, AND J.S. PANG. An LPCC approach to nonconvex quadratic programs. *Mathematical Programming, Series A* 133 (2012) 243–277.

Abstract: Filling a gap in nonconvex quadratic programming, this paper shows that the global resolution of a feasible quadratic program (QP), which is not known a priori to be bounded or unbounded below, can be accomplished in finite time by solving two linear programs with linear complementarity constraints, i.e., LPCCs. Specifically, this task can be divided into two LPCCs: the first confirms whether the QP is bounded below on the feasible set and, if not, computes a feasible ray on which the QP is unbounded; the second LPCC computes a globally optimal solution if it exists, by identifying a stationary point that yields the best quadratic objective value. In turn, the global resolution of these LPCCs can

be accomplished by a parameter-free, mixed integer-programming based, finitely terminating algorithm developed recently by the authors, which can be enhanced in this context by a new kind of valid cut derived from the second-order conditions of the QP and by exploiting the special structure of the LPCCs. Throughout, our treatment makes no boundedness assumption of the QP; this is a significant departure from much of the existing literature which consistently employs the boundedness of the feasible set as a blanket assumption. The general theory is illustrated by 3 classes of indefinite problems: QPs with simple upper and lower bounds (existence of optimal solutions is guaranteed); same QPs with an additional inequality constraint (extending the case of simple bound constraints); and nonnegatively constrained copositive QPs (no guarantee of the existence of an optimal solution). We also present numerical results to support the special cuts obtained due to the QP connection.

- J. HU, J.E. MITCHELL, J.S. PANG, AND B. YU. On linear programs with linear complementarity constraints. *Journal of Global Optimization* 53 (2012) 29–51.

Abstract: The paper is a manifestation of the fundamental importance of the linear program with linear complementarity constraints (LPCC) in disjunctive and hierarchical programming as well as in some novel paradigms of mathematical programming. In addition to providing a unified framework for bilevel and inverse linear optimization, nonconvex piecewise linear programming, indefinite quadratic programs, quantile minimization, and zero-norm minimization, the LPCC provides a gateway to a mathematical program with equilibrium constraints, which itself is an important class of constrained optimization problems that has broad applications. We describe several approaches for the global resolution of the LPCC, including a logical Benders approach that can be applied to problems that may be infeasible or unbounded.

- J.E. MITCHELL, J.S. PANG, AND B. YU. Convex quadratic relaxations of nonconvex quadratically constrained quadratic programs. *Optimization Methods and Software* 29 (2014) 120–136.

Abstract: Nonconvex quadratic constraints can be linearized to obtain relaxations in a well-understood manner. We propose to tighten the relaxation by using second order cone constraints, resulting in a convex quadratic relaxation. Our quadratic approximation to the bilinear term is compared to the linear McCormick bounds. The second order cone constraints are based on linear combinations of pairs of variables. With good bounds on these linear combinations, the resulting constraints strengthen the McCormick bounds. Computational results are given, which indicate that the convex quadratic relaxation can dramatically improve the solution times for some problems.

- J.E. MITCHELL, J.S. PANG, AND B. YU. Obtaining tighter relaxations of mathematical programs with complementarity constraints. In T. Terlaky and F. Curtis, editors. *Modeling and Optimization: Theory and Applications*. Springer Proceedings in Mathematics and Statistics (New York 2012) Volume 21, pp. 1–23.

Abstract: The class of mathematical programs with complementarity constraints (MPCCs)

constitutes a powerful modeling paradigm. In an effort to find a global optimum, it is often useful to examine the relaxation obtained by omitting the complementarity constraints. We discuss various methods to tighten the relaxation by exploiting complementarity, with the aim of constructing better approximations to the convex hull of the set of feasible solutions to the MPCC, and hence better lower bounds on the optimal value of the MPCC. Better lower bounds can be useful in branching schemes to find a globally optimal solution. Different types of linear constraints are constructed, including cuts based on bounds on the variables and various types of disjunctive cuts. Novel convex quadratic constraints are introduced, with a derivation that is particularly useful when the number of design variables is not too large. A lifting process is specialized to MPCCs. Semidefinite programming constraints are also discussed. All these constraints are typically applicable to any convex program with complementarity constraints. Computational results for linear programs with complementarity constraints (LPCCs) are included, comparing the benefit of the various constraints on the value of the relaxation, and showing that the constraints can dramatically speed up the solution of the LPCC.

Two additional papers have been submitted for publication:

- L. BAI, J.E. MITCHELL, AND J.S. PANG. On QPCCs, QCQPs and completely positive programs. *Mathematical Programming, Series A*, submitted January 2014.

In this paper, we show that any quadratically constrained quadratic program is equivalent to a convex optimization problem. The result requires no assumptions on the boundedness of the feasible region or on the convexity of the quadratic constraints. The result also holds if a nonnegativity assumption on the variables is replaced by a more general convex conic constraint; for example, the result holds if the variables satisfy a second order cone constraint or if the variables satisfy a semidefiniteness constraint. The proof exploits the relationship between this class of problems and quadratic programs with complementarity constraints.

The results show that many important practical problems can be represented as convex optimization problems through the use of a single lifting. This includes rank-constrained semidefinite programs. The latter class includes problems such as factor analysis problems for finding a low-rank covariance matrix, sensor array processing, minimizing the rank of a Hankel matrix in model identification problems in system theory and signal processing, problems in systems and control, and combinatorial optimization problems. The convex optimization problem is not easy to solve, because it is defined over a convex cone that is hard to work with. Nonetheless, the exact reformulation offers the possibility of designing new and effective algorithms for solving a broad class of optimization problems.

- M. FENG, J.E. MITCHELL, J.S. PANG, X. SHEN, AND A. WAECHTER. Complementarity formulations of ℓ_0 -norm optimization problems. *Mathematical Programming, Series A*, submitted September 2013.

In this paper, we developed a nonlinear programming formulation of the problem of minimizing the number of nonzero components in the solution of a system of linear equalities

and inequalities, also known as minimizing the ℓ_0 -norm. This is the problem of compressed sensing that is currently of great interest. It also arises in the search for sparse solutions to support vector machine problems in classification. Our method is slower than a linear programming approximation to the ℓ_0 -norm problem, but it has the advantage that it often gives a sparser solution than the L1-norm approximation. This seems to be especially true when the system involves inequality constraints, as is the case for sparse support vector machines.

The following doctoral thesis supervised by the PI was successfully defended in December 2012:

- YUCHING LEE. *Global Solution to Parametric Complementarity Constrained Programs and Applications in Optimal Parameter Selection*. Ph.D. thesis, Department of Industrial and Enterprise Systems Engineering, University of Illinois at Urbana-Champaign (December 2012).

Further papers are in preparation, including documentation of methods for determining global optima for cross-validation problems, and improved methods for finding global optima of problems with complementarity constraints. Multiple talks have been given at conferences and universities by students and collaborator Mitchell.