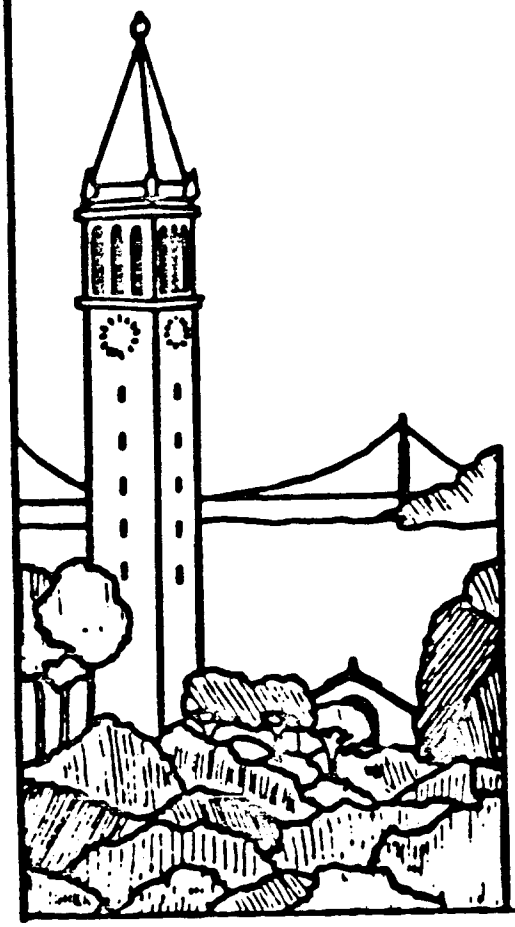
On Negations and Algebras in Fuzzy Set Theory

Francesc Esteva



Report No. UCB/CSD 87/330

March 19, 1986

Computer Science Division (EECS)
University of California
Berkeley, California 94720

Report Documentation Page

Form Approved
OMB No. 0704-0188

Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
19 MAR 1986		**00-00-1986 to 00-00-1986**

4. TITLE AND SUBTITLE	5a. CONTRACT NUMBER
On Negations and Algebras in Fuzzy Set Theory	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **University of California at Berkeley,Department of Electrical Engineering and Computer Sciences,Berkeley,CA,94720**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT
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15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**15**	

ON NEGATIONS AND ALGEBRAS IN FUZZY SET THEORY

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ABSTRACT

Dual automorphisms, involutions and intuitionistic negations on the lattice F(X) of Fuzzy Sets taking values on a complete distributive lattice L are characterized. All Symmetric, De Morgan and Intuitionistic Algebras on F(X) and their isomorphisms are described. Finally, negation functions and their conjugated classes are characterized.

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Introduction.

In Zadeh's definition of Fuzzy Sets [1] the operations are defined pointwise by max, min and 1-j, and Fuzzy Sets on an universe X with these operations form a De Morgan Algebra. In Goguen's generalization [2], that is, in Fuzzy Sets taking values on a lattice, intersection and union are defined pointwise by the lattice operation and negation is defined in several ways depending on the required properties. Usually, negation is defined as dual automorphism, or as involution, or as intuitionistic negation. Some of this negations on [0,1] and on the lattice of Fuzzy Sets are studied (see [4], [5], [6], [10] and [11]) and some of the Algebras of Fuzzy Sets defined by these operations and their isomorphisms are also studied (see [7], [8] and [9]).

In this repport a general overview on negations on Fuzzy Sets, and on Algebras of Fuzzy sets taking values on a distributive lattice and their isomorphisms is given.

The paper contains an introduction and three sections. In the first one a characterization of the different types of negations on the lattice of Fuzzy Sets is given. In the second one a isomorphic classes of Algebras of Fuzzy Sets are studied and in the third one the special case of Fuzzy Sets taking values on [0,1] is analised.

First at all, we must give some dfinitions, denotations and remarks which will be used from now on in this paper :

– The Center of a distributive lattice L is the set of complemented elements which form a complemented sublattice of L , that is, a Boolean Algebra.

– L = $(L,\wedge,\vee)$ denotes a complete and distributive lattice with universal bounds 0 and 1 such that Center (L) = {0,1}.

– P(X)= (P(X), $\bigcap$,$\bigcup$,C) denotes, as usual, the Boolean Algebra of classical subsets of a set X.

– F(X) = $(F(X), \bigcap, \bigcup)$ denotes the lattice of Fuzzy Sets on an universe X taking values on a distributive lattice L with the operations defined by $(A \bigcap B)(x) = A(x) \wedge B(x)$ and $(A \bigcup B)(x) = A(x) \vee B(x)$

* Visitor Scholar at the Computer Science Division at the University of California at Berkeley. Research partially supported by the Comissió Interdepartamental de Recerca i Innovaciò Tecnològica (C.I.R.I.T.) of the Generalitat de Catalunya and by N S F Grants IST-8320416 and DARPA Contract N 00039-83-C-0243

– σ_x , $\bar{\sigma}_x$, δ_α , σ_x^α and $\bar{\sigma}_x^\alpha$ denotes the Fuzzy Sets of F(X) defined by

$\sigma_x(y) = 0$ if $x \neq y$ and $\sigma_x(x) = 1$ (singleton or atom of P(X))

$\bar{\sigma}_x = C(\sigma_x)$ (antiatom of P(X))

$\delta_\alpha(x) = \alpha$ for every $x \varepsilon X$ (constant)

$\sigma_x^\alpha = \sigma_x \wedge \delta_\alpha$ and $\bar{\sigma}_x^\alpha = \bar{\sigma}_x \vee \delta_\alpha$

– H_s denotes the automorphism of F(X) or P(X) defined by $[H_s(A)](x) = A(s^{-1}(x))$

– n denotes a negation of any type. Specification will be given if necessary.

It is interesting to remind that:

(a) The condition Center(L)= {0,1} imply Center(F(X)) = P(X)

(b) Any $A \varepsilon F(X)$ satisfy $A = \bigcup_{x \varepsilon X} \sigma_x^{A(x)}$

1. Negations on Fuzzy Sets

First at all, we remind the basic definitions and properties of negations on lattices (see [12]).

Definition 1. A decreasing mapping n: L → L is said to be :

A dual automorphism if n satisfy the De Morgan laws

An intuitionistic negation if $n^2 \geq Id$ and n(1)= 0

An involution if $n^2 = Id$

The main properties of these negations are the following:

A) Dual automorphism satisfy :

(i) For any S ⊂ L such that $\vee_{a \varepsilon S} a \varepsilon L$, $n(\vee_{a \varepsilon S} a) = \wedge_{a \varepsilon S}(n(a))$

(ii) For any S ⊂ L such that $\wedge_{a \varepsilon S} a \varepsilon L$, $n(\wedge_{a \varepsilon S} a) = \vee_{a \varepsilon S}(n(a))$

(iii) n(0) = 1 and n(1) = 0

(iv) n is one-to-one and onto

B) Involutions satisfy :

(i), (ii), (iii), (iv) and

(v) $n^2 = Id$

C) Intuitionistic negations satisfy :

(i), (iii) and

(vi) For any S ⊂ L such that $\wedge_{a \varepsilon S} a \varepsilon L$, $n(\wedge_{a \varepsilon S} a) \geq \vee_{a \varepsilon S} n(a)$

(vii) n(L) is a complete meet-subsemilattice containing X and ϕ.

(viii) $n|_{n(L)}$ is an involution $\bar{n}$ such that n(L) = $\bar{n}$ (n(L))

(ix) Let L' be a complete meet-subsemilattice of L that contain X and ϕ and let $\bar{n}$ be an involution on L' . Then, there exists an unique intuitionistic negation n on L such that $n|_{L'} = \bar{n}$ and n(L) = L'. This negation is defined by

$$n(a) = \bar{n}(\wedge \{b \varepsilon L' \mid b \geq a\}) \qquad (*)$$

D) A mapping n: L → L is an involution if, and only if, it is an intuitionistic negation and a dual automorphism.

E) Any dual automorphism n on P(X) is defined by $n = C \circ H_s$.

F) Any involution n on P(X) is defined by $n = C \circ H_s$ being s a permutation of X such that $s^2 = Id$

On the lattice F(X) of Fuzzy Sets, negations can be characterized by the following definitions and propositions :

Proposition 1. Any negation n on F(X) is univocally determined by the values of $n(\sigma_x^\alpha)$ for every $x \varepsilon X$ and every $\alpha \varepsilon L$.

Proof.

For any $A \varepsilon F(X)$, $A = \bigcup_{x \varepsilon X} \sigma_x^{A(x)}$ and so, $n(A) = \bigcap_{x \varepsilon X} n(\sigma_x^{A(x)})$.

Definition 2. A negation n on F(X) will be said to satisfy :

- The Extension Principle (E.P.) if $n|_{P(X)} = C$.
- The Generalized Extension Principle (G.E.P.) if $n|_{P(X)}$ is a dual automorphism of P(X).

Proposition 2. Any dual automorphism satisfy the Generalized Extension Principle.

Proof.

If $A \varepsilon P(X)$, then $\phi = n(X) = n(A \cup C(A)) = n(A) \cap n(C(A))$ and dualy , $X = n(A) \cup n(C(A))$. Then, for any $x \varepsilon X$, $\{n(A)(x), n(C(A))(x)\} = \{0, 1\}$. So n(A) and n(C(A)) belongs to P(X).

On the other hand if n is a dual automorphism, n^{-1} is also a dual automorphism. So, given any $A \varepsilon P(X)$, there exists $B = n^{-1}(A)$ such that $n(B) = A$ being B a classical subset of X.

Theorem 1. Any dual automorphism n on F(X) can be defined by

$$[n(A)](x) = n_x(A(s^{-1}(x))) \tag{1}$$

being s a permutation of X and $\{ n_x \mid x \varepsilon X \}$ a family of dual automorphisms of L.

Proof.

If $\{ n_x \mid x \varepsilon X \}$ is a family of dual automorphisms and if s is a permutation of X, then n defined by (1) is a dual automorphism as an easy computation show.

Reciprocally, let n be a dual automorphism of F(X). Then :

- n define a permutation s of X because by proposition 2, $n|_{P(X)}$ is a dual automorphism of P(X). So, $n|_{P(X)} = C \circ H_s$.
- For any $x \varepsilon X$ we define $n_x : L \to L$ by $n_x(\alpha) = [n(\delta_\alpha)](x)$
- For every x, it is easy to proof that n_x is a dual automorphism.
- n satisfy (1) being the family $\{ n_x \mid x \varepsilon X \}$ and the permutation s those defined above because,

$$n(\sigma_x^\alpha) = n(\sigma_x \wedge \delta_\alpha) = \bar{\sigma}_{s(x)} \vee n(\delta_\alpha) = \bar{\sigma}_{s(x)}^{n_{s(x)}(\alpha)}$$

and

$$n_y(\sigma_x^\alpha(s^{-1}(y))) = \begin{cases} n_y(0)=1, & \text{if } y \neq s(x) \\ n_y(\alpha), & \text{if } y = s(x) \end{cases}$$

So, the theorem is proved.

Corollary 1. Any involution n on F(X) can be defined by

$$[n(A)]\,(x) \;=\; n_x\,(A(s(x))) \tag{2}$$

being s an involutive permutation of X and being $\{\, n_x \mid x\varepsilon X \,\}$ a family of dual automorphisms of L such that $n_{s(x)} = n_x^{-1}$.

Proof.

The proof is an easy consecuence of theorem 1 taking into acount that in this case $n^2 = \mathrm{Id}$ which imply $s^2 = \mathrm{Id}$, that is, $s = s^{-1}$.

Theorem 2. Any intuitionistic negation n on F(X) satisfying the G.E.P. can be defined by :

$$[n(A)](x) \;=\; n_x\,(A(s(x))) \tag{3}$$

being s an involutive permutation of X and $\{\, n_x \mid x\varepsilon X \,\}$ a family of order-preserving mappings from L to itself such that $n_x \circ n_{s(x)} \geq Id$, $n_{s(x)} \circ n_x \geq Id$ and $n_x\,(1) = 0$, for every $x\varepsilon X$.

Lemma 1. Let $\{\, n_x \mid x\varepsilon X \,\}$ be a family of order-reversing mapping from L to itself such that $n_x\,(1) = 0$ and let s be a permutation of X. The mapping n from F(X) to itself defined by (3), is an intuitionistic negation if, and only if, s is involutive and $n_x \circ n_{s(x)} \geq Id$ and $n_{s(x)} \circ n_x \geq Id$ for every $x\varepsilon X$.

Proof of the lemma.

Applying n to the elements of type σ_x^α , we have,

$$[n(\sigma_x^\alpha)](y) \;=\; n_y\,(\sigma_x^\alpha\,(s^{-1}\,(y))) \;=\; \begin{cases} n_y\,(0) = 1 & \text{if } y \neq s(x) \\ n_y\,(\alpha) & \text{if } y = s(x) \end{cases}$$

So, $n(\sigma_x^\alpha) = \bar{\sigma}_{s(x)}^{\,n_{s(x)}(\alpha)}$ and

$$[n^2\,(\sigma_x^\alpha)](y) \;=\; n_y\,(\bar{\sigma}_{s(x)}^{\,n_{s(x)}(\alpha)}(s^{-1}\,(y))) \;=\; \begin{cases} n_y(1) = 0 & \text{if } y \neq s^2\,(x) \\ n_y\,(n_{s(x)}\,(\alpha)) & \text{if } y = s^2(x) \end{cases}$$

So, $n^2\,(\sigma_x^\alpha) = \sigma_{s^2(x)}^{\,n_{s^2(x)}\,(n_{s(x)}\,(\alpha))}$

If n is an intuitionistic negation, then $n^2(\sigma_x^\alpha) \geq \sigma_x^\alpha$ and this imply $s^2 = Id$ and $n_{s^2(x)} \circ n_{s(x)} \geq Id$

The reciproc can be proved by a simple computation.

Proof of the theorem.

(a) The negation n define a permutation s of X such that $s^2 = \mathrm{Id}$ because, if n satisfy the G.E.P., $n \mid_{P(X)} = C \circ H_s$.

(b) For every $x\varepsilon X$, we define $n_x : \mathrm{L} \to \mathrm{L}$ by

$$n_x\,(\alpha) \;=\; [n(\sigma_{s(x)}^\alpha)](x)$$

which imply

$$n(\sigma_x^\alpha) \;=\; \bar{\sigma}_{s(x)}^{\,n_{s(x)}(\alpha)}$$

Then,

$$[n(\sigma_y^a)](x) = (\overline{\sigma}_{s(y)}^{n_{s(y)}(a)})(x) = \begin{cases} 1 & \text{if } x \neq s(y) \\ n_x(\alpha) & \text{if } x = s(y) \end{cases}$$

and, $(n(\sigma_y^a(x)) = n_x(\sigma_y^a(s(x)))$

(c) So, n is defined by (2) being the family $\{ n_x \mid x \varepsilon X \}$ and the permutation s, those defined in (a) and (b).

(d) Lemma 1 prove that s and the family $\{ n_x \mid x \varepsilon X \}$ satisfy the required properties.

Corollary 2. A negation n of F(X) satisfy the E.P. if, and only if, there exists a family $\{ n_x \mid x \varepsilon X \}$ of negations of F(X) such that $[n(A)](x) = n_x(A(x))$ (4)

Proof.

It is easy to prove that n satisfy the E.P. if, and only if, s= Id. So (4) is obtained from (1), (2) and (3) taking into account this condition.

2. Algebras of Fuzzy Sets: isomorphic classes.

First at all, we recall the characterization of the automorfisms of the lattice F(X) (see [7] and [8])

Definiton 3. An automorphism h of F(X) is said to be pointwise functionally expressable (p.f.e.) by the family $\{ h_x \mid x \varepsilon X \}$ of automorphisms of L if it is defined by

$$[h(A)](x) = h_x(A(x)).$$

Proposition 3. Any automorphism H of F(X) can be defined by $H = h \circ H_t$, that is,

$$[H(A)](x) = h_x(A(t^{-1}(x))) \tag{4}$$

being t a permutation of X and $\{ h_x \mid x \varepsilon X \}$ a family of automorphisms of L which define h.

Proof.

If h is a p.f.e. automorphism defined by the family $\{ h_x \mid x \varepsilon X \}$ of automorphisms of L and if t is a permutation of X, the mapping H defined by (4) is an automorphism of F(X) as an easy computation show.

If H is an automorphism of F(X), then:

(a) H defines a permutation t of X because $H|_{P(X)}$ is an automorphism of P(X) and so, it is equal to H_t for some permutation t of X.

(b) For any $x \varepsilon X$, we define $h_x: L \to L$ by $h_x(\alpha) = [H(\sigma_{t^{-1}(x)}^a)](x)$. It is easy to proof that every h_x is an automorphism of L.

(c) $H = H_t \circ h$ being h the p.f.e. automorphism defined by $\{ h_x \mid x \varepsilon X \}$ because,

$$[H(\sigma_x^a)](y) = \begin{cases} 0 & \text{if } y \neq t^{-1}(x) \\ h_{t(x)}(\alpha) & \text{if } x = t(y) \end{cases}$$

and so

$$H(\sigma_x^a) = \sigma_{t(x)}^{h_{t(x)}(\alpha)} = (h \circ H_t)(\sigma_x^a)$$

Definition 4. Two permutations s and s' of X are said to have isomorphic orbital graphs if there exists a permutation t of X such that $t \circ s = s' \circ t$.

Definition 5. For any $x \varepsilon X$ and any permutation s of X, the s-orbit of x is the sequence x, s(x), $s^2(x)$, ... $s^n(x)$, ... This s-orbit is said to be of order n if $s^n(x) = x$ and $s^k(x) \neq x$ for any $k < n$.

Definition 6. Two mappings f and g from L to itself are said to be equivlents ($f \equiv g$) if there exists an automorphism h of L such that $f \circ h = h \circ g$.

The following definition and theorems give us the Algebras of Fuzzy Sets and characterize their isomorphisms.

Definition 7. An Algebra (F(X), $\bigcap, \bigcup$,n) is said to be :

- A Symmetric Algebra if n is a dual automorphism
- An Intuitionistic Algebra if n is an intuitionistic negation
- A De Morgan Algebra if n is an involution.

Let n and n' be negations on F(X) defined by the permutations s and s' and by families $\{ n_x \mid x \varepsilon X \}$ and $\{ n'_x \mid x \varepsilon X \}$ that satisfies the required conditions for each type of negations.

Theorem 3. Two Symmetric Algebras (F(X), $\bigcap, \bigcup$,n) and (F(X), $\bigcap, \bigcup$, n') are isomorphic if, and only if, the orbital graphs of s and s' are isomorphic and, for every $x \varepsilon X$ with s-orbit of finit order m, the following equivalence is satisfied,

$$n_{s^{m-1}(x)} \circ .... \circ n_x \equiv n'_{t \circ s^{m-1}(x)} \circ .... \circ n'_{t(x)}$$

Proof.

Any isomorphism H of these algebras satisfy:

(i) $H(A \bigcup B) = H(A) \bigcup H(B)$

(ii) $H(A \bigcap B) = H(A) \bigcap H(B)$

(iii) $H(n(A)) = n'(H(A))$

By proposition 3, conditions (i) and (ii) are satisfied if, and only if, $H = h \circ H_t$ being t any permutation of X and h a p.f.e. automorphism defined by any family $\{ h_x \mid x \varepsilon X \}$ of automorphisms of L.

So, look for an isomorphism H is equivalent to look for a permutation t and a family $\{ h_x \mid x \varepsilon X \}$ of automorphisms of L such that

$$(h \circ H_t) \circ n = n' \circ (h \circ H_t) \tag{3.1}$$

By proposition 1, it is enough to compute the equality (3.1) for the elements of type σ_x^α for every $x \varepsilon X$ and for every $\alpha \varepsilon L$.

$$(h \circ H_t)(n(\sigma_x^\alpha)) = (h \circ H_t)(\overline{\sigma}_{s(x)}^{n_{s(x)}(\alpha)})$$

$$(h \circ H_t)(n(\sigma_x^\alpha)) = \overline{\sigma}_{(t \circ s)(x)}^{h_{(t \circ s)(x)}(n_{s(x)}(\alpha))} \tag{3.2}$$

$$n'((h \circ H_t)(\sigma_x^\alpha)) = n'(\sigma_{t(x)}^{h_{t(x)}(\alpha)}) = \overline{\sigma}_{(s' \circ t)(x)}^{n_{(s' \circ t)(x)}(h_{t(x)}(\alpha))} \tag{3.3}$$

Then, H is an isomorphism if, and only if, (3.2) equal to (3.3) which is equivalent to,

$$t \circ s = s' \circ t \tag{3.4}$$

and

$$h_{(t\circ s)(x)} \circ n_{s(x)} = n'_{(s'\circ t)(x)} \circ h_{t(x)} \qquad \text{for every } x\varepsilon X \tag{3.5}$$

It follows from (3.4) that if H exists, then the orbital graphs of s and s' are isomorphic.

On the other hand it is necessary to study condition (3.5) for every type of elements $x\varepsilon X$ respect to the permutation s :

(a) If **x** is invariant by s, that is, if s(**x**)= **x**.

In this case condition (3.5) is satisfied if, and only if, $n_x \equiv n'_{t(x)}$

(b) If **x** has an s-orbit of order m

In this case condition (3.5) is equivalent to the following sistem of equations (taking into account condition (3.4) and s^m (**x**)= **x**)

$$\begin{aligned} h_{(t\circ s)(x)} \circ n_{s(x)} &= n'_{(t\circ s)(x)} \circ h_{t(x)} \\ h_{(t\circ s^2)(x)} \circ n_{s^2(x)} &= n'_{(t\circ s^2)(x)} \circ h_{(t\circ s)(x)} \\ \cdots\cdots\cdots\cdots &= \cdots\cdots\cdots\cdots \\ h_{t(x)} \circ n_x &= n'_{t(x)} \circ h_{(t\circ s^{m-1})(x)} \end{aligned} \tag{3.6}$$

which is equivalent to

$$h_{(t\circ s^{m-1})(x)} \circ n_{s^{m-1}(x)} \circ \cdots \circ n_{s(x)} \circ n_x = n'_{(t\circ s^{m-1})(x)} \circ \cdots \circ n'_{t(x)} \circ h_{(t\circ s^{m-1})(x)} \tag{3.7}$$

So, there exist functions $h_{(t\circ s^k)(x)}$ for k = 1,2,..., m-1 verifying (3.6) if, and only if,

$$n_{s^{m-1}(x)} \circ \cdots \circ n_{s(x)} \circ n_x \equiv n'_{(t\circ s^{m-1})(x)} \circ \cdots \circ n'_{t(x)} \tag{3.8}$$

(c) **x** has an s-orbit of infinit order

In this case, condition (3.5) is always satisfied. Given any automorphism f = $h_{t(x)}$ all the automorphisms of the family $\{ h_{(t\circ s^k)(x)} \mid k \varepsilon N \}$ satisfying condition (3.5) can be calculated by recurrence.

So, the theorem is proved.

Given an involutive permutation s of a set X, we will denote by I and S the subsets of X defined by $I = \{ x\varepsilon X \mid s(x) = x \}$ and $S = \{ x\varepsilon X \mid s(x) \neq x \}$. The subsets S, I forms a partition of X because s is involutive.

Lemma 2. The orbital graphs of two involutive permutations s and s' of a set X are isomorphic if, and only if, the sets I, S have the same cardinal then the sets I' and S' respectively.

Proof.

If s and s' satisfy the conditions of the lemma any permutation t of X such that t(I)= I' and t(S)= S' satisfy $s' \circ t = t \circ s$.

Corollary 2. Two De Morgan Algebras $(F(X), \bigcap, \bigcup, n)$ and $(F(X), \bigcap, \bigcup, n')$ are isomorphic if, and only if,

(i) Cardinal(I)= Cardinal(I') and Cardinal(S)= Cardinal(S'),

(ii) For every **x** ε I , $n_x \equiv n'_{t(x)}$ being t a permutation of X such that t(I) = I' and t(S) = S'.

Proof.

The proof is an easy consecuence of theorem 3 and lemma 2 taking into account that for every element $x \in S$, $n_x \circ n_{s(x)} = Id$.

Theorem 4. Two intuitionistic Algebras $(F(X), \bigcap, \bigcup, n)$ and $(F(X), \bigcap, \bigcup, n')$ are isomorphic if, and only if,

(i) Cardinal(I) = Cardinal(I') and Cardinal(S) = Cardinal (S')

(ii) For every $x \in I$, $n_x \equiv n'_{t(x)}$

(iii)' For every $x \in S$, there exist two automorphisms f,g of L such that

$$f \circ n_x = n'_{t(x)} \circ g \quad \text{and} \quad g \circ n_{s(x)} = n'_{(t \circ s)(x)} \circ f$$

Proof.

A mapping H from $(F(X), \bigcap, \bigcup, n)$ to $(F(X), \bigcap, \bigcup, n')$ is an isomorphism of this algebras if conditions (3.4) and (3.5) are satisfied. So the theorem is an easy consecuence of this conditions and lemma 2.

3. Negations on [0,1] : conjugated classes.

Let $n:[0,1] \to [0,1]$ be a negation on the unit interval.

Proposition 4. A function $n:[0,1] \to [0,1]$ is a dual automorphism if, and only if, n is a continuous, one-to one and onto function.

Graphically, n is any stricly decreasing curve joining the points (0,1) and (1,0).

Proposition 5. A function $n:[0,1] \to [0,1]$ is an involution if, and only if, n is a continuous, one-to-one and onto function such that it is symmetric respec to $y = x$.

Proof.

If n is involutive, then $n(n(x)) = x$ for any $x \in X$. Then $(x, n(x))$ is a point of the graph of n if, and only if, $(n(x),x)$ is a point of the graph of n. So n, is symmetric respec to the line $y = x$.

Graphically, an involution can be defined by any strictly decreasing curve joining (0,1) and any point (a,a) of the line $y = x$. The graph of n is the line above defined and its symmetric respect to $y = x$.

Definition 8. A non-increasing function $n: [0,1] \to [0,1]$ is said to be quasi-symmetric respec to $y = x$ if :

(i) For any x of [0,1] such that n is continuous and strictly decreasing $n(n(x)) = x$.

(ii) If n discontinuous at a, then n is a constant function on $(n(a^+), n(a^-))$ with value a.

(iii) If (a,b) is a maximal open interval such that n is constant with value k, Then n is discontinuous at k and $n(k^+) = a$ and $n(k^-) = b$.

Proposition 6. A function $n:[0,1] \to [0,1]$ is an intuitionistic negation if, and only if, n is decreasing, left-continuous, quasi-symmetric respect to $y = x$ and $n(0) = 1$, $n(1) = 0$.

Lemma 3. If n is an intuitionistic negation, then n is left-continuous.

Proof.

Let $\{x_i\}$ a non-decreasing sequence such that $\lim_{i \to \infty} x_i = \vee_{i \in N} x_i = x$. Then $\{n(x_i)\}$ is a non-increasing sequence auch that $n(x) = n(\vee_{i \in N} x_i) = \wedge_{i \in N} n(x_i)$. So $\lim_{i \to \infty} n(x_i) = n(x)$ and this imply that n is a left continuous function.

Lemma 4. Let n be an intuitionistic negation. Then :

(a) If $x \in n([0,1])$, then $n(n(x)) = x$.

(b) If n is discontinuous at $a \in [0,1]$, then n is a constant function on $(n(a^+),n(a^-))$ with value a.

(c) If (a,b) is a maximal open interval in which n is a constant function, then n is discontinuous at n(b) and $n(n(b)^+) = a$, $n(n(b)^-) = b$.

Proof.

(a) This statement is an easy consecuence of property (viii) of negations.

(b) If n is discontinuosus at a , $I = (n(a^+),n(a^-))$ do not belong to n([0,1]). Then by property (ix) of negations, n is a constant function on this interval.

By lemma 3, n is left continuous at $n(a^-)$. Then $n(x) = n(n(a^-) = n^2(a)$ for any $x \in I$.

We are going to prove that $n^2(a) = a$.

- By definition $n^2(a) \geq a$.

- If $n^2(a) > a$, then $n(n^2(a)) \leq n(a^+) < n(a^-) = n(a)$, that is, $n^2(n(a)) < n(a)$ in contradiction with the definition of n.

(c) By the left continuity of n, for any x of (a,b), $n(x) = n(b)$.

Because (a,b) is maximal, if $n^2(b) > b$,then $n(n^2(b)) < b$ that is, $n^2(n(b)) < n(b)$ in contradiction with the definition of n. So, $n^2(b) = b$.

- If $x > n(b)$, then $n(x) \geq a$ because if $n(x) < a$, then $n^2(x) \geq n(a) = n(b) < x$ which is a contradiction. So, $n(n(b)^+) \leq a$.

- By the left continuity $n(n(b)^-) = n^2(b) = b$.

Then n is discontinuous at n(b) and $n(n(b)^+) \leq a$ and $n(n(b)^-) = b$. However, by (b) n is a constant function on the interval $(n(n(b)^+),(n(n(b)^-))$ containing (a,b).

So , $n(n(b)^+) = a$ because (a,b) is maximal.

So, the theorem is proved.

It is interesting to remark that graphically any intuitionistic negation function can be defined by any non-increasing function joining (0,1) and a point (a,a) of the line $y = x$. The graph of n is formed by the curve defined above and its quasi-symmetric respect to $y = x$.

Remark.

- Any dual automorphism or involution n on [0,1] has an unique level of symmetry or fixed point a, that is, there exist a unique $a \in [0,1]$ such that $n(a) = a$ which is denoted by a_n .

- If n is an intuitionistic negation function, two different cases are possible:

A) There exists a fixed point by n which is denoted by a_n .

B) There exist an interval (a,b) called interval of symmetry, such that n is a constant function in (a,b] with value a and $n(a) = b$.

Proposition 6. Two algebras ([0,1],max,min,n) and ([0,1],max,min,n') are isomorphic if, and only if, there exists a continuous strictly increasing function $f:[0,1] \rightarrow [0,1]$ such that $fon = n'of$.

Proof.

f is an isomorphism of these algebras if:

(1) f(max(a,b)) = max(f(a),f(b))

(2) f(min(a,b)) = min(f(a),f(b))

(3) f(n(a)) = n' (f(a))

- (1) and (2) imply that f is no-decreasing. Then, f must be a continuous strictly increasing function because it is an isomorphism .So, f is an isomorphism if, and only if, satisfy the conditions of the proposition.

In other words, the proposition says that two algebras are isomorphic if n and n' are equivalent (see definition 6), that is, n and n' are in the same conjugated class respect to the grup of continuous strictly incresing functions from [0,1] to itself, which is denoted by C[0,1].

The following propositions characterize this conjugate classes.

Definition 9. Given a function f:[0,1] → [0.1], a point x of [0,1] is said to be:

- negative respect to f if $f(x) > x$,

- positive respect to f if $f(x) < x$,

- invariant respect to f if f(x) = x.

Given a negation function n the set of negative, positive and invariant points respect to n are denoted by N_n , P_n and I_n respectively, and the set of negative, positive and invariant points respect to n^2 are denoted by M_n^+ , M_n^- and S_n respectively.

Theorem 5. Two dual automorphism n and n' are equivalents if, and only if, there exists a function f of C[0,1] such that $f(M_n^+) = M_{n'}^+$, $f(M_n^-) = M_{n'}^-$ and $f(S_n) = S_{n'}$.

Lemma 5. If n' = $fonof^{-1}$, then $f(M_n^+) = M_{n'}^+$, $f(M_n^-) = M_{n'}^-$ and $f(S_n) = S_{n'}$.

Proof.

If $x \varepsilon M_n^+$ then $f(x) \varepsilon M_{n'}^+$ because $(n')^2 = fon^2of^{-1}$ and so $(n')^2(f(x)) = f(n^2(x)) < f(x)$.

The proof of the other cases is doing in a similar way.

Lemma 6. If n is a dual automorphism, then:

(1) S_n is a closed set and M_n^+ and M_n^- are open sets in the usual topology of [0,1],

(2) For any $x \varepsilon M_n^+$ (M_n^-) , there exists a maximal open interval (a,b) such that $(a,b) \subset M_n^+$ (M_n^-) and $a,b \varepsilon S_n$.

(3) If $x \varepsilon M_n^+$, then $n(x) \varepsilon M_n^-$ and reciprocally.

(4) If $x \varepsilon S_n$, then $n(x) \varepsilon S_n$.

Proff.

Conditons (1) and (2) are consequence that n^2 is a continuous function.

Conditions (3) and (4) are consequence that n is a strictly decreasing function.

Lemma 7. If there exists h: $[0,a_{n'}] \to [0,a_n]$ such that $h^{-1} \circ n^2 \circ h = (n')^2$, then there exists f ε C[0,1] such that $n' = f^{-1} \circ n \circ f$.

Proof.

We define f by the formula,

$$f(x) = \begin{cases} h(x) & \text{if } x \in [0,a_{n'}] \\ (n \circ h \circ (n')^{-1})(x) & \textit{otherwise} \end{cases}$$

An easy computation show that the theorem is true.

Lemma 8. If (a,b) is a maximal open interval of M_n^+ (M_n^-) and (c,d) is a maximal open interval of M_n^+ (M_n^-), there exists a function h: (a,b) → (c,d) such that $(n')^2 = h^{-1} \circ n^2 \circ h$ on the interval (a,b).

Proof.

If x ε (c,d), then:

$x, n^2(x), \ldots, (n^2)^m(x), \ldots \rightarrow c$

$x, (n^2)^{-1}(x), \ldots, (n^2)^{-m}, \ldots \rightarrow d$

So, $(c,d) = \bigcup_{m \in Z} (n^2)^m [x, n^2(x))$

Similarly, if y ε (a,b) ⊂ M_n^+, then $(a,b) = \bigcup_{m \in Z} ((n')^2)^m [y, (n')^2(y))$

Let g a strictly increasing and bijective function from [y, $(n')^2$ (y)) to [x, n^2 (x)).

We define h: (a,b) → (c,d) by

$$h(z) = \begin{cases} g(z) & \text{if } z \in [y,(n')^2(y)) \\ (n^2)^m(g(z')) & \text{if } z' \in [y,(n')^2(y)) \quad \text{and} \quad z = ((n')^2)^m(z') \end{cases}$$

An easy computation show that h satisfy the required condition.

Proof of the theorem.

Lemma 5 prove that the condition is necessary.

If there exists f satisfying the conditions of the theorem, then { I| I interval maximal of M_n^+ *or* M_n^- *or* S_n } form a partition of [0,1].

We are going to construct a function h:[0, a_n] → [0, $a_{n'}$] such that $(n')^2 = h \circ n^2 \circ h^{-1}$ (*)

This function h is constructed for any pair of intervals I, f(I) in the following way:

- If [a,b] $\in S_n$, then [f(a),f(b)] $\in S_{n'}$. In this intervals $n^2 = Id[a,b]$ and $(n')^2 = Id[f(a),f(b)]$. So any strictly increasing function from [a,b] to [f(a),f(b)] satisfy (*) on the interval [a,b].

- If (a,b) $\subset (M_n^+ \cup M_n^-) \cap N_n$, then, by lemma 8 there exists a function h_a:(a,b) → (f(a),f(b)) such that (*) is satisfied on (a,b).

So, we can define $h \in C[0,1]$ by

$$h(x) = \begin{cases} x & \text{if } x \in S_n \\ h_a(x) & \text{if } x \in (a,b) \subset (M_n^+ \cup M_n^-) \cap N_n \end{cases}$$

This function obviously satisfy (*) and by lema 7 this imply that n and n' are equivalents.

Corollary 3 Two involutions n and n' on [0,1] are equivalent.

Proof.

For any involution n , $M_n^+ = M_n^- = \emptyset$ and $S_n = [0,1]$. So, the theorem say that any pair of involutions are equivalent.

Let n be an intuitionistic negation on [0,1]. We denote by D the set D = {a ε [0,1]| n is discontinuous at a }.

Theorem 6. Two intuitionistic negations n and n' are equivalents if, and only if, there exists a function f of C[0,1] such that f(D) = D' and f(n(a^+), n(a^-)) = $(n'(f(a)^+), n'(f(a)^-))$ for every a ε D.

If n' = $h \circ n \, h^{-1}$, then:

(i) If n is a constant function on an interval I with value p, then n' is a constant function on h(I) with value h(p)

(ii) If n is discontinuous at a, then n' is discontinuous at h(a) and $n'(h(a)^+) = h(n(a^+))$ and $n'(h(a)^-) = h(n(a^-))$.

(iii) If n has a fixed point a_n , then n' has a fixed point $h(a_n)$ and if n has an interval of symmetry (s,t) , then n' has an interval of symmetry (h(s), h(t)).

The proof is an easy computation.

Lemma 10. Let n and n' be intuitionistic negations such that :

(i) n([0,1]) = n' ([0,1])

(ii) for every x ∉ n([0,1]) , n(x) = n'(x) Then, n and n' are equivalents.

Proof.

The conditions of the lemma imply that n and n' have the same intervals in which the negations are constant functions (this intervals are the complement of n([0,1]) and n'([0,1]) respectively) with the same value and also imply that n and n' has the same points of discontinuity with the same jumps. So both negations have the same interval of symmetry or both negations has point of symmetry. We are going to study the different cases:

(1) If n and n' have the same interval of symmetry or the same fix point p.

In this case $N_n = N_{n'}$ and we define the function f by

$$f(x) = \begin{cases} x, & \text{if } x \in N_n \bigcup (P_n - n([0,1])), \\ p, & \text{if } x = p \\ n'(n(x)), & \text{if } x \in P_n \bigcap n([0,1]) \end{cases}$$

It is easy to prove that f ε C[0,1] and n' = $f \circ n \circ f^{-1}$.

(2) If n and n' have points of symmetry a_n and $a_{n'}$ such that $a_n \neq a_{n'}$.

In this case let a be the point a = $\wedge\{ b \in n([0,1]) \mid$ n is continuous and strictly decreasing on [b, a_n] }. Then n and n' will be decreasing bijections from [a, n(a)] to itself. So, we can define the function f ε C[0,1] by

$$f(x) = \begin{cases} n'(n(x)), & \text{if } x \in (P_n - (a_n, n(a))) \bigcap n([0,1]) \\ h(x), & \text{if } x \in (a, a_n) \\ n'(h(n(x))), & \text{if } x \in (a_n, n(a)) \\ x, & \textit{otherwise} \end{cases}$$

being h any increasing bijection from (a , a_n) to (a , $a_{n'}$).

An easy computation show that n' = $f \circ n \circ f^{-1}$.

Proof of the theorem.

The condition is necessary by lemma 9.

If n and n' are intuitionistic negations satisfying the conditions of the theorem we are going to prove that n' and $f \circ n \circ f^{-1}$ satisfy the conditions of the lema 10.

By lemma 9, n' and $f \circ n \circ f^{-1}$ have the same points of discontinuity with the same jumps. So n'([0,1]) = ($f \circ n \circ f^{-1}$) ([0,1]) . If $x \notin$ n'([0,1]) , then x belongs to an interval I in which n' and $f \circ n \circ f^{-1}$ are constant functions. This interval is (f(a),f(b)) being (a,b) an interval in which n is a constant function and n'(x) = n'(f(b)) , ($f \circ n \circ f^{-1}$)(x) = f(n(b)). Because n is discontinuous at b , $f \circ n \circ f^{-1}$ and n' are discontinuous at f(b) with the same values.

So, this two negations satisfy the conditions of lemma 10, which imply n equivalent to n' by transsitivity.

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