CONSISTENCY OF INVARIANT TESTS FOR THE MULTIVARIATE ANALYSIS OF VARIANCE

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THEODORE W. ANDERSON, PROJECT DIRECTOR

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Consistency of Invariant Tests for the Multivariate Analysis of Variance*

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1. INTRODUCTION

Consider the standard multivariate linear regression model (cf. Anderson (1984), Chapter 8):

$$Z = \beta D + \varepsilon , \tag{1.1}$$

where $Z : p \times N$ is the matrix of observations, $\beta : p \times q$ is the matrix of unknown regression coefficients, $D : q \times N$ is the design matrix, and $\varepsilon : p \times N$ is the matrix of unobservable random errors. Assume that $q \leq N$, D is of full rank q, and that

$$\varepsilon \sim N(0 , \Sigma \otimes I_N) , \tag{1.2}$$

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which indicates that the N columns of ε are mutually independent p-variate normal random vectors with zero mean and common unknown covariance matrix $\Sigma: p \times p$, assumed positive definite. We shall study the consistency of invariant tests of the general linear hypothesis

$$H_0: \beta D_0 = 0, \tag{1.3}$$

where $D_0: q \times r$ has full rank r $(1 \le r \le q)$. If β is partitioned as (β_1, β_2) with $\beta_1: q \times r$, an important special case of (1.3) is $H_0: \beta_1 = 0$.

The general multivariate analysis of variance (MANOVA) testing problem is that of testing (1.3) vs. (1.1). It is well known that this testing problem can be reduced by sufficiency and invariance to the following canonical form (cf. Anderson (1984), Section 8.3.3 or Lehmann (1986), Sections 8.1, 8.2): based on the independent observations

$$\begin{aligned} X(p \times r) &\sim \mathrm{N}(\xi, \Sigma \otimes I_r) \\ Y(p \times n) &\sim \mathrm{N}(0, \Sigma \otimes I_n), \end{aligned} \tag{1.4}$$

where $\xi : p \times r$ is a matrix of unknown means, test

$$H_0: \xi = 0 \quad \text{vs.} \quad H: \xi \neq 0 \qquad (\Sigma \ \text{unknown}). \tag{1.5}$$

We assume that n $(\equiv N - q) \ge p$, so that $\hat{\Sigma} \equiv \frac{1}{n} YY'$ is positive definite with probability one.

The canonical testing problem given by (1.4) and (1.5) is invariant under the group of nonsingular linear transformations

$$(X, Y) \to (BX\Psi_1, BY\Psi_2), \tag{1.6}$$

where $B : p \times p$ is nonsingular and $\Psi_1: r \times r$, $\Psi_2: n \times n$ are orthogonal. Under (1.6), the parameters of the model (1.4) are transformed according to

$$(\xi, \Sigma) \to (B\xi\Psi_1, B\,\Sigma B'). \tag{1.7}$$

The maximal invariant statistic and maximal invariant parameter may be represented as

$$
\begin{aligned}
c &\equiv c(X,Y) \equiv (c_1,\ldots,c_t) \\
\lambda &\equiv \lambda(\xi,\Sigma) \equiv (\lambda_1,\ldots,\lambda_t),
\end{aligned} \tag{1.8}
$$

respectively, where $t = p \wedge r$,

$$
\begin{aligned}
c_i &= ch_i[XX'(YY')^{-1}] \geq 0 \\
\lambda_i &= ch_i[\xi\xi'\Sigma^{-1}] \quad \geq 0,
\end{aligned} \tag{1.9}
$$

and where, for any real symmetric matrix S, $ch_1(S) \geq ch_2(S) \geq \cdots$ denote its (ordered) characteristic roots (necessarily real). It will be convenient also to use the equivalent representations

$$
\begin{aligned}
d &\equiv d(c) \equiv (d_1,\ldots,d_t) \\
\delta &\equiv \delta(\lambda) \equiv (\delta_1,\ldots,\delta_t)
\end{aligned} \tag{1.10}
$$

of the maximal invariant statistic and parameter, respectively, where

$$
\begin{aligned}
d_i &= \frac{c_i}{c_i+1} = ch_i[XX'(XX'+YY')^{-1}] \\
\delta_i &= \frac{\lambda_i}{\lambda_i+1} = ch_i[\xi\xi'(\xi\xi'+\Sigma)^{-1}].
\end{aligned} \tag{1.11}
$$

Note that

$$
\begin{aligned}
c,\lambda \in \mathbf{C}_t &\equiv \{x \equiv (x_1,\ldots,x_t) \mid \infty > x_1 \geq \cdots \geq x_t \geq 0\} \\
d,\delta \in \mathbf{D}_t &\equiv \{x \equiv (x_1,\ldots,x_t) \mid 1 > x_1 \geq \cdots \geq x_t \geq 0\}.
\end{aligned} \tag{1.12}
$$

(More precisely, $c \in \mathbf{C}_t$ and $d \in \mathbf{D}_t$ with probability one.)

The MANOVA problem (1.5) may be expressed in the following equivalent form: test

$$
H_0: \lambda = (0,\ldots,0) \quad \text{vs.} \quad H: \lambda \in \mathbf{C}_t,\ \lambda \neq (0,\ldots,0). \tag{1.13}
$$

We shall be concerned with the consistency of *invariant* tests for (1.5) $\equiv$ (1.13), i.e., tests that depend upon (X,Y) only through c (or, equivalently, through d) and whose power functions therefore depend upon (ξ,Σ) only through $\lambda \equiv (\lambda_1,\ldots,\lambda_t)$, the vector of *noncentrality parameters*. Since c_i estimates λ_i, a "good" invariant test should accept (reject) H_0 for small (large) values of $c_1,\ldots,c_t$ (equivalently, of

$d_1, \ldots, d_t$). In fact, Schwartz (1967b) has shown that every admissible invariant test for (1.5) ≡ (1.13) must have a *monotone acceptance region* A in terms of c or (equivalently) d. That is (in terms of d), if $d \equiv (d_1, \ldots, d_t) \in A \subseteq \mathbf{D}_t$ and $d' \equiv (d_1', \ldots, d_t') \in \mathbf{D}_t$ is such that $d' \leq d$ (i.e., $d_1' \leq d_1, \ldots, d_t' \leq d_t$), then $d' \in A$. Therefore, we shall restrict our attention to the class of *monotone invariant tests*, i.e., those with monotone acceptance regions.

Perlman and Olkin (1980) showed that every monotone invariant test is *unbiased* for testing H_0 vs. H. The criterion of unbiasedness, therefore, does not distinguish among admissible invariant tests. Likewise, neither does the classical notion of consistency, which we shall call *sample size consistency* (SSC). In this paper we introduce the notion of *parameter consistency* and show that it *does* distinguish among admissible invariant tests.

An invariant level α test with acceptance region A is said to be *parameter consistent* (PC) if, for fixed p, r, n, and α, its power

$$P_\lambda\{d \notin A\} \rightarrow 1$$

as one or more noncentrality parameters $\lambda_i \rightarrow \infty$. It will be seen that the well-known Bartlett-Nanda-Pillai trace test, which is both admissible and the locally most powerful invariant test for H_0 vs. H, fails to be PC unless α or n is sufficiently large, whereas the Roy maximum root test, the Lawley-Hotelling trace test, and the likelihood ratio test (≡ Wilks criterion) are both admissible and PC for every α and n (cf. Section 4).

It is important to notice that parameter consistency is defined in terms of the power of a *single* invariant level α acceptance region A at *sequences* of alternatives $\{\lambda\}$ with $\|\lambda\| \rightarrow \infty$, whereas sample size consistency is defined in terms of the limiting power of a *sequence* of invariant level α acceptance regions $\{A^{(n)}\}$ at a *fixed* alternative $\lambda^* \neq (0, \ldots, 0)$. Usually the sequence $\{A^{(n)}\}$ is defined in terms of a single invariant test statistic $f \equiv f(d)$ as follows:

$$A_f^{(n)} = A_f(c_\alpha) \equiv \{d \in \mathbf{D}_t \mid f(d) \leq c_\alpha\}, \qquad (1.14)$$

where $c_\alpha \equiv c_\alpha(p, r, n; f)$ satisfies

$$P_{\lambda=0}\{f(d) \le c_\alpha\} = 1 - \alpha. \tag{1.15}$$

If f is *monotone* on $\mathbf{D}_t$ (i.e., nondecreasing in each d_i) then $A_f(c_\alpha)$ is a monotone invariant acceptance region with power function given by $P_\lambda\{f(d) > c_\alpha\}$.

Necessary and sufficient conditions for the parameter consistency of monotone invariant tests are presented in Section 2, while sample size consistency is defined and characterized in Section 3. The relation between parameter consistency and sample size consistency of monotone invariant tests, in particular admissible invariant tests, is examined in Section 4. Few detailed proofs are given, as this paper is primarily expository. The proofs, together with extensions of the results to related multivariate testing problems, will appear in Anderson and Perlman (1988).

2. PARAMETER CONSISTENCY OF MONOTONE INVARIANT TESTS

In a general hypothesis-testing problem, a level α test of H_0 vs. H is said to be parameter consistent if, for fixed sample size, its power approaches one for sequences of alternatives in H whose Kullback-Leibler discrimination distance from H_0 becomes arbitrarily large. For the canonical MANOVA problem (1.5) $\equiv$ (1.13), this definition is equivalent to the following:

DEFINITION 2.1. For fixed p, r, n, and α, an invariant level α test for (1.5) $\equiv$ (1.13) is *parameter consistent* (PC) if its power at $\lambda \equiv (\lambda_1, \ldots, \lambda_t)$ approaches 1 as $\|\lambda\| \to \infty$, where $\|\lambda\| = \Sigma_1^t \lambda_i = \operatorname{tr} \xi\xi'\Sigma^{-1}$. For $i = 1, \ldots, t$, the test is *parameter consistent of degree i* (PC(i)) if its power at λ approaches 1 as $\lambda_i \to \infty$. □

Since $\lambda \ge \cdots \ge \lambda_t \ge 0$, obviously PC$(i) \Longrightarrow$ PC$(i{+}1)$, and PC $\Longleftrightarrow$ PC(1). It will be seen in Section 4 that parameter consistency is not equivalent to sample size consistency.

In order to study the power of an invariant test at the alternative $\lambda = (\lambda_1, \ldots, \lambda_t)$ we may assume that $(\xi, \Sigma) = (\mu, I_p)$, where $\mu : p \times r$ is any matrix such that $ch_i(\mu\mu') = \lambda_i$, $1 \le i \le t$, and where I_p is the $p \times p$ identity matrix. Under this assumption XX' and YY' have standard Wishart distributions, noncentral and central respectively, and are independent. Define

$$l_i \equiv l_i(X) \equiv ch_i(XX'), \quad 1 \le i \le t .$$

Our characterization in Theorem 2.3 of parameter consistency for monotone invariant tests is based upon the following technical result:

LEMMA 2.2.

(i) $l_i \xrightarrow{p} \infty$ iff $\lambda_i \to \infty$.

(ii) $c_i \xrightarrow{p} \infty$ iff $\lambda_i \to \infty$.

(iii) $d_i \xrightarrow{p} 1$ iff $\lambda_i \to \infty$.

PROOF. The result (i) follows from appropriate stochastic bounds for l_i in terms of λ_i, (ii) follows from (i) by conditioning on Y, while (ii) and (iii) are equivalent by (1.11). See Anderson and Perlman (1988) for details. □

It is most convenient to state our results for acceptance regions A defined in terms of the statistic d. For any subset $A \subseteq \mathbf{D}_t$ (recall (1.12)), we denote the closure of A in $\overline{\mathbf{D}}_t$ by $\overline{A}$, where

$$\overline{\mathbf{D}}_t = \{x \mid 1 \ge x_1 \ge \cdots \ge x_t \ge 0\} . \tag{2.1}$$

If A is monotone then $\overline{A}$ is also monotone and $\overline{A} \setminus A$ has Lebesgue measure 0. Since the distribution of d is absolutely continuous with respect to Lebesgue measure for every $\lambda \in H_0 \cup H$, the power (and hence the consistency) of the invariant test with acceptance region A is the same as that of the test with acceptance region $\overline{A}$.

Let $e_0, e_1, \ldots, e_t$ denote the vertices of $\overline{\mathbf{D}}_t$, i.e.,

$$e_i = (\underbrace{1, \ldots, 1}_{i}, \underbrace{0, \ldots, 0}_{t-i}), \tag{2.2}$$

and, for $0 \le \eta \le 1$, define $e_i(\eta) \in \overline{\mathbf{D}}_t$ by

$$e_i(\eta) = e_i + \eta(e_t - e_i) = (\underbrace{1, \ldots, 1}_{i}, \underbrace{\eta, \ldots, \eta}_{t-i}). \tag{2.3}$$

If A is a monotone subset of $\mathbf{D}_t$, then $e_i(\eta) \in \bar{A}$ implies $e_{i-1}(\eta) \in \bar{A}$.

THEOREM 2.3. Fix p, r, n, and α, let $A \subseteq \mathbf{D}_t$ be a monotone level α acceptance region for the testing problem (1.5) $\equiv$ (1.13), and fix $i \in \{1, \ldots, t\}$. A necessary and sufficient condition that the invariant test with acceptance region A be PC(i) is that $e_i(\eta) \notin \bar{A}$ for all $\eta > 0$. A sufficient condition is that $e_i \notin \bar{A}$. □

If the upper boundary of A is not too irregular, Theorem 2.3 essentially states that the test with acceptance region A is PC(i) if either $e_i \notin \bar{A}$ (cf. Fig. 2.1a) or e_i lies in the upper boundary of A (Fig. 2.1b), whereas it fails to be PC(i) if $e_i \in A$ but $e_i \notin$ (upper boundary of A) (Fig. 2.1c). Thus, the test is PC(i) if e_i lies either in the *rejection* region or in its lower boundary. These three cases are illustrated in Figures 2.1a,b,c where $i = 1$ and $t = 2$.

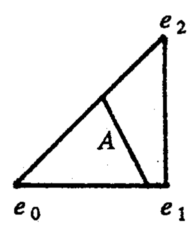

Fig. 2.1a:
test is PC(1)

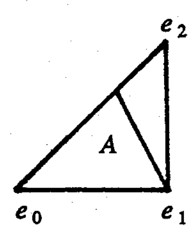

Fig. 2.1b:
test is PC(1)

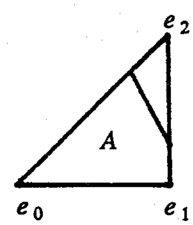

Fig. 2.1c:
test is *not* PC(1)

Note that if $\alpha > 0$, every monotone invariant test must be PC(t).

It is convenient to restate Theorem 2.3 for the case where $A = A_f(c_\alpha)$ as in (1.14) and f is a monotone invariant test statistic defined on $\mathbf{D}_t$. For any extended real-valued function f on $\mathbf{D}_t$ define $\bar{f}$ on $\bar{\mathbf{D}}_t$ as follows:

$$\bar{f}(x) = f(x-) \equiv \lim_{\varepsilon \downarrow 0} f((1-\varepsilon)x). \tag{2.4}$$

Then $\bar{f}$ is defined (possibly extended real-valued), monotone, and lower semicontinuous on $\overline{\mathbf{D}}_t$, hence $A_{\bar{f}}(c_\alpha) \equiv \overline{A_f(c_\alpha)}$ is a closed and monotone subset of $\overline{D}_t$. Furthermore, $\{x \in \overline{D}_t \mid \bar{f}(x) \neq f(x)\}$ has Lebesgue measure zero, hence so does $A_{\bar{f}}(c_\alpha) \setminus A_f(c_\alpha)$ for every c_α. Thus, the level α tests determined by f and $\bar{f}$ are equivalent and have identical power functions.

COROLLARY 2.4. Fix p, r, n, and α, let f be a monotone test statistic defined on $\mathbf{D}_t$, let $c_\alpha \equiv c_\alpha(p, r, n; f)$ satisfy (1.15), and fix $i \in \{1, \ldots, t\}$.

(i) A necessary and sufficient condition that the invariant level α test based on f be PC(i) is that $\bar{f}(e_i(\eta)) > c_\alpha$ for all $\eta > 0$. A sufficient condition is that $\bar{f}(e_i) > c_\alpha$.

(ii) If $\eta = 0$ is a point of increase of $\bar{f}(e_i(\eta))$ (i.e., $\bar{f}(e_i(\eta)) > \bar{f}(e_i)$ for all $\eta > 0$) then $\bar{f}(e_i) \geq c_\alpha$ is a sufficient condition that the level α test based on f be PC(i). If $\bar{f}(e_i(\eta))$ is continuous at $\eta = 0$ then $\bar{f}(e_i) \geq c_\alpha$ is a necessary condition. Thus, if $\eta = 0$ is both a point of increase and a continuity point of $\bar{f}(e_i(\eta))$, then $\bar{f}(e_i) \geq c_\alpha$ is both necessary and sufficient for the level α test to be PC(i). □

If $f(d)$ depends only on $d_{i+1}, \ldots, d_k$ for some $0 \leq i < k \leq t$, it follows from Corollary 2.4(i) that for $0 < \alpha < 1$, the level α test based on f is PC(k) but not PC(i). It may or may not be PC(j) for $i < j < k$: for example, the level α test based on $f(d) \equiv \prod_{i+1}^{k} d_j$ is PC(k) but not PC$(k-1)$, while those based on $\prod_{i+1}^{k} d_j(1-d_j)^{-1}$ and $\sum_{i+1}^{k} d_j(1-d_j)^{-1}$ are PC$(i+1)$ but not PC(i). Furthermore, parameter consistency may depend on the value of α: the level α test based on $f(d) \equiv \sum_{i+1}^{k} d_j$ is PC(j) but not PC$(j-1)$ for α satisfying $j-i-1 < c_\alpha \leq j-i$, $j = i+1, \ldots, k$. Further examples are considered in Section 4.

3. SAMPLE SIZE CONSISTENCY OF MONOTONE INVARIANT TESTS

For fixed p, r, and α, we shall study the consistency of the sequence of invariant level α tests determined by the acceptance regions $A_f^{(n)}(n \geq p)$ based on a monotone test statistic $f = f(d)$ defined on $\mathbf{D}_t$ (see (1.14)).

DEFINITION 3.1. For the testing problem (1.5) $\equiv$ (1.13), the sequence of invariant level α tests based on f is *sample size consistent* (SSC) at the fixed alternative $\lambda^* \neq (0, \ldots, 0)$ if the power

$$P_{\lambda^{(n)}}\{f(d) > c_\alpha(p, r, n; f)\} \to 1$$

as $n \to \infty$ for every sequence of alternatives $\{\lambda^{(n)}\}$ such that $\lambda^{(n)} = (n + o(n))\lambda^*$, i.e., $n^{-1}\lambda^{(n)} \to \lambda^*$. If this holds for every $0 < \alpha < 1$, we say that f is *sample size consistent* at λ^* (SSC *at* λ^*). The test statistic f is called *sample size consistent* (SSC) if it is SSC at every $\lambda^* \neq (0, \ldots, 0)$. □

REMARK 3.2. The condition $\lambda^{(n)} = (n + o(n))\lambda^*$ stems from the fact that X in (1.4) is typically of the form $\sqrt{n}\,(1 + o(1))\bar{X}$ with $\bar{X}$ a sample mean vector. In the simplest case, for example, $X_1, \ldots, X_N$ are independent univariate observations from $N(\mu, \sigma^2)$ and we wish to test $\mu = 0$ vs. $\mu \neq 0$. Here the two-sided t-test rejects $\mu = 0$ for large values of $T^2 \equiv (\sqrt{N}\,\bar{X})^2 / s^2$, where $s^2 = \Sigma(X_i - \bar{X})^2 / N-1$. In this case, $p = q = r = 1$, $t = 1$, $n = N-1$, and, when $\mu \neq 0$, T^2 has the noncentral F distribution with 1 and n degrees of freedom and noncentrality parameter $\lambda = N\mu^2/\sigma^2$. This is of the form $\lambda = (n + o(n))\lambda^*$ with $\lambda^* = \mu^2/\sigma^2$. □

Theorem 3.5, our main result on the sample size consistency of a monotone invariant test statistic f, is based on the following two elementary lemmas, which summarize the limiting behavior of d and $c_\alpha \equiv c_\alpha(p, r, n; f)$ as $n \to \infty$ with p, r, and α fixed. Their proofs follow directly from the definitions (1.4) and (1.11) of d_i, X, and Y.

LEMMA 3.3. Fix $p, r, \lambda^* \in \mathbf{C}_t$, and $i \in \{1, \ldots, t\}$.

(i) If $\lambda = 0$, then $d_i = O_p(1/n)$ as $n \to \infty$.

(ii) If $\lambda = (n + o(n))\lambda^*$, then $d \xrightarrow{p} \delta^*$ as $n \to \infty$, where

$$\begin{aligned} \delta^* &\equiv \delta(\lambda^*) \equiv (\delta_1^*, \ldots, \delta_t^*) \in \mathbf{D}_t\,, \\ \delta_i^* &= \lambda_i^* / (\lambda_i^* + 1) \qquad \square. \end{aligned} \tag{3.1}$$

For $0 \le \eta < 1$, define

$$f_0(\eta) = f(e_0(\eta)) \equiv f(\eta, \ldots, \eta). \tag{3.2}$$

To avoid technicalities, we shall assume that

$$f_0 \quad \text{is continuous and strictly increasing,} \tag{3.3}$$

hence the inverse function f_0^{-1} is well-defined, continuous, and strictly increasing. Since

$$f_0(d_t) \le f(d) \le f_0(d_1) \tag{3.4}$$

for $d \equiv (d_1, \ldots, d_t) \in \mathbf{D}_t$, it follows from (1.15) that

$$P_{\lambda=0}\{d_t > f_0^{-1}(c_\alpha)\} \le \alpha \le P_{\lambda=0}\{d_1 > f_0^{-1}(c_\alpha)\}. \tag{3.5}$$

Because $P_{\lambda=0}\{0 < d_t < d_1 < 1\} = 1$, this implies that $0 < f_0^{-1}(c_\alpha) < 1$ for $0 < \alpha < 1$, which, together with Lemma 3.3(i), yields the following result:

LEMMA 3.4. Fix p, r and α $(0 < \alpha < 1)$. Then

$$0 < \liminf_{n \to \infty} n f_0^{-1}(c_\alpha) \le \limsup_{n \to \infty} n f_0^{-1}(c_\alpha) < \infty. \qquad \square$$

For $\delta \equiv (\delta_1, \ldots, \delta_t) \in \mathbf{D}_t$, define

$$\tilde{f}(\delta) = \lim_{\eta \downarrow 0} \frac{1}{\eta} f_0^{-1}[f(\delta \vee e_0(\eta))] \tag{3.6}$$

provided the limit exists (possibly infinite), where $x \vee y = (x_1 \vee y_1, \ldots, x_t \vee y_t)$. Then by the monotonicity of f,

$$\tilde{f} \quad \text{is monotone on} \quad \mathbf{D}_t, \tag{3.7}$$

$$1 = \tilde{f}(0) \le \tilde{f}(\delta) \le \infty. \tag{3.8}$$

By (3.7),

$$\tilde{f}(\delta-) \le \tilde{f}(\delta) \le \tilde{f}(\delta+), \tag{3.9}$$

where $\tilde{f}(\delta-)$ is defined as in (2.4) and where

$$\bar{f}(\delta+) = \lim_{\varepsilon \downarrow 0} \bar{f}((1+\varepsilon)\delta) .$$

By (3.7), $\bar{f}(\delta\pm)$ exists provided that $\bar{f}((1\pm\varepsilon)\delta)$ exists for sufficiently small $\varepsilon > 0$. We say that $\bar{f}$ is *radially continuous* at δ if

$$\bar{f}(\delta-) = \bar{f}(\delta) = \bar{f}(\delta+) . \tag{3.10}$$

Radial continuity is a weaker requirement than continuity: for example, if $\bar{f}$ depends on $\delta \equiv (\delta_1, \ldots, \delta_t)$ only through

$$\begin{aligned} \operatorname{rank}(\delta) &\equiv \text{number of nonzero } \delta_i \\ &\equiv \max\{i \mid \delta_i > 0\} , \end{aligned} \tag{3.11}$$

then $\bar{f}$ is radially continuous on $\mathbf{D}_t$ but not necessarily continuous.

The following characterization of the sample size consistency of f in terms of $\bar{f}$ may be proved by applying Lemma 3.3(ii) and Lemma 3.4. It also follows from a slightly stronger result in Anderson and Perlman (1988).

THEOREM 3.5. Fix p, r, and $\lambda^* \in \mathbf{C}_t$, and set $\delta^* = \delta(\lambda^*)$ as in (3.1). Let f be a monotone invariant test statistic defined on $\mathbf{D}_t$ and satisfying (3.3).

(i) If $\bar{f}(\delta^*-) = \infty$, then f is SSC at λ^*.

(ii) If $\bar{f}(\delta^*+) < \infty$, then f is not SSC at λ^*.

(iii) Suppose, in addition, that $\bar{f}$ exists and is radially continuous at δ^*. Then f is SSC at λ^* iff $\bar{f}(\delta^*) = \infty$. □

It is important to note that when Theorem 3.5 is applicable, the sample size consistency of the sequence of level α tests based on f does not depend on the value of α $(0 < \alpha < 1)$.

REMARK 3.6. In addition to the hypotheses of Theorem 3.5, suppose that $f_0(0) = 0$ and that

$$f_0(\eta) \sim a\eta^b \quad \text{as} \quad \eta \downarrow 0 \tag{3.12}$$

for some $a, b > 0$. Then it may be shown that for any $\delta \in \mathbf{D}_t$,

$$\bar{\bar{f}}(\delta) \equiv \lim_{\eta \downarrow 0} \frac{f(\delta \vee e_0(\eta))}{f_0(\eta)} = [\bar{f}(\delta)]^b . \tag{3.13}$$

That is, $\bar{\bar{f}}(\delta)$ exists iff $\bar{f}(\delta)$ exists, in which case (3.13) holds. Then all conclusions of Theorem 3.5 remain valid with $\bar{f}$ replaced by $\bar{\bar{f}}$, which is usually easier to calculate. The condition (3.12) is satisfied, for example, if $f(d_1, \ldots, d_t)$ is monotone on $\mathbf{D}_t$, and admits a power series expansion about $(0, \ldots, 0)$. □

LEMMA 3.7. Suppose that f is monotone on $\mathbf{D}_t$ and satisfies (3.3). Then for $\delta \in \mathbf{D}_t$,

$$f(\delta) > f_0(0) \implies \bar{f}(\delta) = \infty . \tag{3.14}$$

PROOF. It follows from (3.3) and the monotonicity of f that for every $M > 0$ and sufficiently small $\eta > 0$,

$$\begin{aligned} f(\delta) > f(0) &\implies f(\delta) > f_0(\eta M) \\ &\implies \frac{1}{\eta} f_0^{-1}[f(\delta \vee e_0(\eta))] \geq \frac{1}{\eta} f_0^{-1}[f(\delta)] \geq M . \end{aligned} \tag{3.15}$$

Now let $\eta \downarrow 0$ and $M \uparrow \infty$. □

If $\lambda^* = \lambda(\xi^*, \Sigma^*)$ and $\delta^* = \delta(\lambda^*)$, then from (1.8)-(1.11) and (3.1),

$$\operatorname{rank}(\lambda^*) = \operatorname{rank}(\delta^*) = \operatorname{rank}(\xi^*) . \tag{3.16}$$

COROLLARY 3.8. Suppose that f is monotone on $\mathbf{D}_t$ and satisfies (3.3). Fix $\lambda^* \in \mathbf{C}_t$ and set $\delta^* = \delta(\lambda^*)$.

(i) If $f(\delta^* -) > f_0(0)$, then f is SSC at λ^*.

(ii) If $\operatorname{rank}(\delta^*) = t$, then f is SSC at λ^*.

PROOF. (i) For sufficiently small $\varepsilon > 0$,

$$f(\delta^*-) > f_0(0) \implies f((1-\varepsilon)\delta^*) > f_0(0)$$
$$\implies \bar{f}((1-\varepsilon)\delta^*) = \infty$$

by Lemma 3.7, hence $\bar{f}(\delta^*-) = \infty$. By Theorem 3.5(i), f is SSC at λ^*.

(ii) For sufficiently small $\varepsilon > 0$,

$$\operatorname{rank}(\delta^*) = t \iff \operatorname{rank}((1-\varepsilon)\delta^*) = t$$
$$\iff (1-\varepsilon)\delta_t^* > 0$$
$$\implies f((1-\varepsilon)\delta^*) \geq f_0((1-\varepsilon)\delta_t^*) > f_0(0)$$

by (3.4) and (3.3). As in the proof of part (i) it follows that $\bar{f}(\delta^*-) = \infty$, hence that f is SSC at λ^*. □

REMARK 3.9. The converse of Corollary 3.8(i) is not necessarily true; in fact, it is not necessarily true that if $f(\delta^*) = f_0(0)$ then f fails to be SSC at λ^*. For example, if $f(d) = \prod_1^t d_i$ then $f(\delta^*) = f_0(0) = 0$ whenever $\operatorname{rank}(\delta^*) < t$; however, $\bar{f}(\delta^*) = \bar{\bar{f}}(\delta^*) = \infty$ for every $\delta^* \neq (0, \ldots, 0)$, hence f is SSC at every $\lambda^* \neq (0, \ldots, 0)$ by Theorem 3.5(i) or Remark 3.6. □

The following definition is suggested by Corollary 3.8(ii):

DEFINITION 3.10. For $i = 1, \ldots, t$, the invariant test statistic f is said to be *sample size consistent of degree* i (SSC(i)) for the testing problem (1.5) ≡ (1.13) if it is SSC at every λ^* such that $\operatorname{rank}(\lambda^*) \geq i$. □

Clearly, SSC$(i) \implies$ SSC$(i+1)$ and SSC $\iff$ SSC(1). If f is monotone on $\mathbf{D}_t$, satisfies (3.3), and is SSC at every λ^* such that $\operatorname{rank}(\delta^*) = i$, then by Theorem 3.5(ii), $\bar{f}(\delta^*+) = \infty$ for every $\delta^* \in \mathbf{D}_t$ of rank i. By the monotonicity of $\bar{f}$ this implies that $\bar{f}(\delta^*-) = \infty$ for every δ^* of rank i, hence that $\bar{f}(\delta^*-) = \infty$ for every δ^* such that $\operatorname{rank}(\delta^*) \geq i$, and therefore, by Theorem 3.5(i), that f is SSC(i).

The following result is similar to Corollary 3.8. Note, however, that in part (i) only the weaker condition $f(\delta) > f_0(0)$ need be assumed, rather than $f(\delta-) > f_0(0)$.

COROLLARY 3.11. Suppose that f is monotone on $\mathbf{D}_t$ and satisfies (3.3).

(i) If $f(\delta) > f_0(0)$ for every δ such that rank$(\delta) = i$, then f is SSC(i).

(ii) If $f(d) = g(d_1, \ldots, d_i)$ for some g, then f is SSC(i).

(iii) If $f(d) = g(d_{i+1}, \ldots, d_t)$ for some g, then f is *not* SSC(i). In fact, f is not SSC at *any* λ^* such that rank$(\lambda^*) \le i$.

(iv) f is SSC(t).

PROOF. (i) By Lemma 3.7, $\bar{f}(\delta) = \infty$ for every δ of rank i, hence $\bar{f}(\delta-) = \infty$ for every δ such that rank$(\delta) \ge i$, so the result follows from Theorem 3.5(i).

(ii) If rank$(\delta) = i$, then $\delta_i > 0$ and $f(\delta) = g(\delta_1, \ldots, \delta_i) \ge g(\delta_i, \ldots, \delta_i) = f_0(\delta_i) > f_0(0)$ by (3.3), so the result follows from part (i).

(iii) If rank$(\delta) \le i$, then $\delta_{i+1} = \cdots = \delta_t = 0$ and $f(\delta \vee e_0(\eta)) = g(\eta, \ldots, \eta) = f_0(\eta)$, hence $\bar{f}(\delta) = 1$ by (3.6). Thus $\bar{f}(\delta^*+) = 1$ whenever rank$(\delta^*) \le i$, so f cannot be SSC at the corresponding λ^*.

(iv) This is immediate from Corollary 3.8(ii). □

By Remark 3.9, however, it is quite possible that f is SSC(i) even though $f(\delta) = f_0(0)$ for every δ of rank i. To illustrate this more fully, consider the four test statistics $f(d)$ appearing in the final paragraph of Section 2. Each is monotone on $\mathbf{D}_t$, satisfies (3.3), and, by Corollary 3.11(ii) and (iii), is SSC(k) but not SSC(i). In fact, however, Theorem 3.5 or Remark 3.6 implies that each is SSC$(i+1)$ but not SSC(i), even though the first two statistics ($\prod_{i+1}^{k} d_j$ and $\prod_{i+1}^{k} d_j(1-d_j)^{-1}$) satisfy $f(\delta) = f_0(0)$ for every δ of rank $< k$.

4. COMPARISON OF PARAMETER CONSISTENCY AND SAMPLE SIZE CONSISTENCY

When comparing these two properties for a monotone invariant test statistic f defined on $\mathbf{D}_t$, it is important first to examine their differences. Throughout this discussion the dimensions p and r remain fixed, while we write $d = d(n)$, $c_\alpha = c_\alpha(n)$, and $A_f(c_\alpha) = A_f(c_\alpha(n))$ to stress the dependence of these quantities on n (the number of degrees of freedom for estimating Σ) — cf. (1.4), (1.11), (1.14), (1.15).

First (recall Definition 2.1), parameter consistency is defined for the *single* level α acceptance region $A_f(c_\alpha(n))$ with n and α fixed: we say that $A_f(c_\alpha(n))$ is PC(i) (or simply "f is PC(i) for n, α") if

$$\lim_{\lambda_i \to \infty} P_\lambda\{d(n) \notin A_f(c_\alpha(n))\} = 1 . \tag{4.1}$$

It was seen at the end of Section 2 that the PC(i) property may depend non-trivially on the value of α. By Corollary 2.4, this property is determined not only by the values of $\bar{f}(e_i(\eta))$ but also by that of $c_\alpha(n) \equiv c_\alpha(p, r, n; f)$, therefore by the *global* behavior of f on $\mathbf{D}_t$.

On the other hand (recall Definitions 3.1 and 3.10), sample size consistency is defined for the *sequence* of acceptance regions $\{A_f(c_\alpha(n)) \mid n \ge p\}$ for a fixed α: this sequence is said to be SSC(i) if

$$\lim_{n \to \infty} P_{\lambda^{(n)}}\{d(n) \notin A_f(c_\alpha(n))\} = 1 \tag{4.2}$$

for every sequence $\{\lambda^{(n)}\}$ of the form $\lambda^{(n)} = (n + o(n))\lambda^*$ with $\operatorname{rank}(\lambda^*) \ge i$. Whenever Theorem 3.5 applies, this property does not depend on the value of α $(0 < \alpha < 1)$, so we then simply say that f is SSC(i). Again by Theorem 3.5, this property is determined by the values of $\hat{f}(\delta)$ for every $\delta \in \mathbf{D}_t$ of rank i, hence only by the *local* behavior of f in a neighborhood of the set $\{x \in \mathbf{D}_t \mid x_t = 0\}$. (This is because f is *always* SSC(t) (cf. Corollary 3.11(iv)), while for $1 \le i \le t-1$, if $\operatorname{rank}(\delta) = i$ then the value of $\hat{f}(\delta)$ is determined by the values of f in a neighborhood of $\{x \in \mathbf{D}_t \mid x_t = 0\}$.

In view of these differences, it is not surprising that the properties PC(i) and SSC(i) are not equivalent, even for a monotone test statistic f. In general, neither property implies the other, as demonstrated by the following examples. Define (cf. (1.11))

$$f_1(d) = d_1 \equiv ch_{\max}[XX'(XX'+YY')^{-1}]$$

$$f_2(d) = \sum_{j=1}^{t} d_j(1-d_j)^{-1} \equiv \operatorname{tr}[XX'(YY')^{-1}]$$

$$f_3(d) = \prod_{j=1}^{t} (1-d_j)^{-1} \equiv \det(XX'+YY')/\det(YY')$$

$$f_4(d) = \prod_{j=1}^{t} d_j(1-d_j)^{-1} \equiv \det(XX')/\det(YY')$$

$$f_5(d) = \sum_{j=1}^{t} d_j \equiv \operatorname{tr}[XX'(XX'+YY')^{-1}]$$

$$f_6(d) = \prod_{j=1}^{t} d_j \equiv \det(XX')/\det(XX'+YY')$$

$$f_7(d) = d_t \prod_{j=1}^{t-1} (1-d_j)^{-1}$$

$$f_8(d) = d_t \prod_{j=1}^{t-1} (1+d_j)$$

$$f_9(d) = d_t \equiv ch_{\min}[XX'(XX'+YY')^{-1}] .$$

The statistics f_1, f_2, f_3, and f_5 are well-known (cf. Anderson (1984), Chapter 8): f_1 is the Roy maximum root statistic, f_2 is the Lawley-Hotelling trace statistic, f_3 is the likelihood ratio (LR) statistic ($\equiv$ Wilks statistic), and f_5 is the Bartlett-Nanda-Pillai trace statistic. Each of the statistics $f_1 - f_9$ is monotone on $\mathbf{D}_t$ and satisfies (3.3), (3.10), and (3.12) (more precisely, $f_3 - 1$ satisfies (3.12)). The parameter consistency and sample size consistency of the invariant level α tests based on $f_1 - f_9$ are readily determined from the results of Sections 2 and 3:

f_1, f_2, f_3, f_4 are PC(1) for all n, α	*and*	f_1, f_2, f_3, f_4 are SSC(1)
f_5 is PC(i), not PC($i-1$), for $i-1 < c_\alpha(n) \le i,\ i = 1, \ldots, t$	*but*	f_5 is SSC(1)
f_6 is PC(t), not PC($t-1$), for all n, α	*but*	f_6 is SSC(1)
f_7 is PC(1) for all n, α	*but*	f_7 is SSC(t), not SSC($t-1$)
f_8, f_9 are PC(t), not PC($t-1$), for all n, α	*and*	f_8, f_9 are SSC(t), not SSC($t-1$).

Although "f is PC(i) for all n, α" and "f is SSC(i)" are thus seen to be inequivalent, their defining properties (4.1) and (4.2) have an important common feature: since $\lambda_i^{(n)} = (n + o(n))\lambda_i^*$, we see that $\text{rank}(\lambda_i^*) \ge i$ iff $\lambda_i^{(n)} \to \infty$ as $n \to \infty$. This suggests the following definition:

DEFINITION 4.1. For fixed α, the sequence $\{A_f(c_\alpha(n)) \mid n \ge p\}$ of level α acceptance regions based on f is said to be *eventually parameter consistent of degree i* (*eventually* PC(i)) if there exists $n_0(\alpha)$ such that $A_f(c_\alpha(n))$ is PC(i) for every $n \ge n_0(\alpha)$. If $\{A_f(c_\alpha(n)) \mid n \ge p\}$ is eventually PC(i) for every $0 < \alpha < 1$, then the test statistic f is *eventually* PC(i). □

It is now natural to ask whether, for a monotone test statistic f, the properties "f is eventually PC(i)" and "f is SSC(i)" are equivalent. Again this is not true in general, as shown by the behavior of f_6 and f_7 above, but it is more nearly true: although f_5 is not PC(1) for some values of n, α, it is both eventually PC(1) and SSC(1). It is interesting to note that this is indeed *true for all admissible test statistics f*. More precisely, we introduce the following definition:

DEFINITION 4.2. For fixed α, the sequence $\{A_f(c_\alpha(n)) \mid n \ge p\}$ is said to be *eventually admissible* for the original testing problem (1.5) if there exists $n_1(\alpha)$ such that $A_f(c_\alpha(n))$ is an admissible acceptance region for (1.5) whenever $n \ge n_1(\alpha)$. If $\{A_f(c_\alpha(n)) \mid n \ge p\}$ is eventually admissible for every $0 < \alpha < 1$, then the test statistic f is *eventually admissible* for the testing problem (1.5). □

The proof of the following theorem is based on Schwartz's (1967b) necessary condition for the admissibility of an invariant acceptance region for problem (1.5)—see Anderson and Perlman (1988).

THEOREM 4.3. Let f be a monotone invariant test statistic. If f is eventually admissible for the testing problem (1.5), then f is SSC(1) and eventually PC(1). □

Schwartz's (1967b) sufficient condition for admissibility implies that the level α tests based on f_1, f_2, f_3, f_5 are admissible for every n and α, hence *a fortiori* f_1, f_2, f_3, f_5 are eventually admissible. It has already been noted that f_1, f_2, f_3, f_5 are SSC(1) and eventually PC(1), in agreement with Theorem 4.3. Both f_4 and f_7 are eventually *in*admissible (although admissible for sufficiently small n or α) and both are PC(1), but f_4 is SSC(1) whereas f_7 is not SSC(1). The level α tests based on f_6, f_8, and f_9 are *in*admissible for every n and α and none of f_6, f_8, f_9 is eventually PC(1), but f_6 is SSC(1) whereas f_8, f_9 are not SSC(1).

Thus, neither the requirement that f be eventually PC(1) nor the requirement that f be SSC(1) distinguishes among invariant tests. For *fixed* n and α, however, the requirement that f be PC(1) for n and α is *not* satisfied by every admissible test statistic. In particular, *the Bartlett-Nanda-Pillai statistic* f_5 *fails to be parameter consistent* unless its critical value satisfies $c_\alpha(p,r,n;f_5) \le 1$, i.e., unless n or α is sufficiently large. (Values of $c_\alpha(p,r,n;f_5)$ are tabulated in Anderson and Perlman (1988).) Despite the facts that the Bartlett-Nanda-Pillai test is admissible (Schwartz (1967b)), proper Bayes (Kiefer and Schwartz (1965)), locally most powerful invariant and locally minimax (Schwartz (1967a)), and robust (Olson (1974)), its failure to be parameter consistent is a potentially serious drawback. Injudicious or routine use of such a test (for example, in a statistical computer package) could result in failure to detect a sizable departure from the null hypothesis H_0.

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Two notions of consistency of invariant tests for the MANOVA testing problems are examined and compared: sample size consistency (the classical notion) and parameter consistency, which requires that for fixed sample size, the power of the test approaches one for any sequence of alternatives whose distance from the null hypothesis approaches infinity. The Roy, Lawley-Hotelling, and likelihood ratio (≡ Wilks) tests are consistent in both senses, whereas the Bartlett-Nanda-Pillai trace test, although sample size consistent, is not parameter consistent unless the significance level α or the error degrees of freedom n is sufficiently large.