REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*
14/08/2013

2. REPORT TYPE
Final

3. DATES COVERED *(From - To)*
15/05/2010-14/05/2013

4. TITLE AND SUBTITLE
Computational Methods For Design, Estimation And Real-Time Control Of PDE Systems With Applications To Mobile Sensor Networks

5a. CONTRACT NUMBER

5b. GRANT NUMBER
FA9550-10-1-0201

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
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5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
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8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
AF Office of Scientific Research
875 N Randolph St. Room 3112
Arlington, VA, 22203

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release.

13. SUPPLEMENTARY NOTES

14. ABSTRACT
The primary objectives of this work are the construction of a rigorous mathematical framework and the corresponding computational science tools that can be used to address problems of parameter identification, real-time tracking and estimation for spatially dependent systems. This includes determining optimized sensor/actuator locations for complex hybrid spatial systems to enhance tracking, estimation, information and effectiveness while limiting energy consumption. Reduced-order modeling techniques are implemented as an efficient way to compute the functional gains. We illustrate how a small number of strategically placed sensor/actuator is sufficient to stabilize the flow while inappropriate placement of these sensors could destabilize the flow. Additionally we consider information delays present in the sensor/actuator network. The models are complex multi-scale systems of coupled partial and delay differential equations. We show that under suitable conditions, the coupled delay PDE systems are well posed and we use this corresponding abstract formulation to construct efficient numerical methods for control design.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**			**19b. TELEPHONE NUMBER** *(include area code)*

COMPUTATIONAL METHODS FOR DESIGN, ESTIMATION AND REAL-TIME CONTROL OF PDE SYSTEMS WITH APPLICATIONS TO MOBILE SENSOR NETWORKS

Finall Report: AFOSR Grant FA9550–10–10201
for the period
15 May 2010–14 May 2013

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Objectives

The primary objectives of this work are the construction of a rigorous mathematical framework and the corresponding computational science tools that can be used to address problems of parameter identification, real-time tracking and estimation for spatially dependent systems. This includes determining optimized sensor/actuator locations for complex hybrid spatial systems to enhance tracking, estimation, information and effectiveness while limiting energy consumption. Reduced-order modeling techniques are implemented as an efficient way to compute the functional gains. We illustrate how a small number of strategically placed sensor/actuator is sufficient to stabilize the flow while inappropriate placement of these sensors could destabilize the flow. Additionally we consider information delays present in the sensor/actuator network. The models are complex multi-scale systems of coupled partial and delay differential equations. We show that under suitable conditions, the coupled delay PDE systems are well posed and we use this corresponding abstract formulation to construct efficient numerical methods for control design.

Overview

Sensor Placement

Feedback control and optimal state estimator problems for PDE systems lead to operator equations where the feedback and observer gain operators often have integral representations. These kernel functions are called functional gains and often have localized support that provides information about the controller and estimator. If these functional gains exist and can be computed, one can use this information to place sensors in those regions in space that are most important to the controller and estimator.

The functional gain operators are computed by solving appropriate approximations to the infinite dimensional Riccati equations arising from infinite dimensional estimation and control problems. Due to the large scale of the approximate problem, special techniques are required to solve these equations. We propose and demonstrate a reduced-order modeling technique to compute the functional gains and analyze these gains to obtain insight into the physical behavior of the feedback control mechanism.

For the discussion consider a flow past a circular cylinder at Re=200 and we use suction on the cylinder surface to suppress vortex shedding. The incompressible Navier-Stokes equations are solved using parallel algorithms and snapshots of actuated and unactuated flow data are collected for building a reduced-order model. In this case, the reduced0-order model is computed using proper orthogonal decomposition (POD).

The velocity field is written as the sum of the mean flow ($\bar{\mathbf{v}}$) and the velocity fluctuations ($\mathbf{v}'$). The fluctuations are expanded in terms of the POD eigenfunctions (Φ_i) computed from the collected snapshots and the control function method:

$$\mathbf{v}(x,t) \approx \bar{\mathbf{v}}(x) + \sum_{i=1}^{M} z_i^r(t)\Phi_i(x) + \sum_{i=1}^{m} u_i(t)\Gamma_i(x). \tag{1}$$

The number of POD modes is denoted by M and m is the total number of control modes. Each function $\Gamma_i(x)$ is a suitable divergence-free control function that satisfy the inhomogeneous boundary condition due to fluidic actuators, and $\mathbf{u}(t)$ is the variable control input.

We project the Navier Stokes equations onto the POD basis functions (or modes) to develop a reduced-order model. The reduced-order model for $m = 1$ has the form

$$\dot{\mathbf{z}}^r = \mathbf{A}_1^r + \mathbf{A}_2^r\mathbf{z}^r + \mathbf{A}_3^r u + \mathbf{F}_1^r(\mathbf{z}^r)\mathbf{z}^r + \mathbf{F}_2^r(\mathbf{z}^r)u + \mathbf{F}_3^r(u)u + \mathbf{B}^r\dot{u}. \tag{2}$$

The system is then linearized about its mean component to reduce the model to a standard linear time invariant (LTI) control problem. If the control is a linear function of the states, then the closed loop system then takes the form

$$\begin{bmatrix} \dot{\mathbf{z}}^r \\ \dot{u} \end{bmatrix} = \underbrace{\left(\begin{bmatrix} \mathbf{A}_2^r & \mathbf{A}_3^r \\ \mathbf{0}^T & 0 \end{bmatrix} - \begin{bmatrix} \mathbf{B}^r \\ 1 \end{bmatrix} \mathbf{K} \right)}_{\mathbb{A}_\mathbf{c}} \begin{bmatrix} \mathbf{z}^r \\ u \end{bmatrix}. \tag{3}$$

As the purpose of the control law is to stabilize the system, it is desired to have the poles of the system in the left-half of the complex plane. In other words, the eigenvalues of $\mathbb{A}_c$ must have negative real parts.

The feedback control gain, $\mathbf{K}$, represents an approximation to the feedback operator associated with the PDE control system. The Riesz Representation Theorem implies that there exists a divergent free vector field $\mathbf{h}(x)$ such that

$$u_c(t) = -\mathbf{K}_z \mathbf{z}^r(t) - K_u u(t) = -\int_\Omega \mathbf{h}(x) \cdot \sum_{i=1}^{M} z_i^r(t)\Phi_i(x) d\Omega - K_u u(t). \quad (4)$$

The efficiency of this approach was verified in the following way: For a specific controller where the eigenvalue is moved to $-0,05 \pm 1.16i$, 12 POD modes were used compute the functional gain, see Figure 1 . From this information, we simulate two

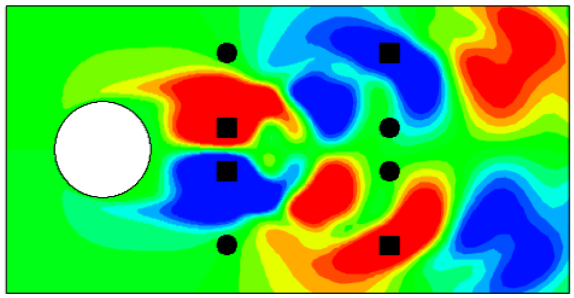


Figure 1: Functional gain approximated at four locations; *good* (square) and *bad* (circle) locations.

scenarios. In each of the scenarios we use four locations to approximate the feedback gain. These choices are motivated by the magnitude of the functional gain in the wake. Four locations are chosen where the functional gain is high (good choices) and four where the gain has small magnitude (bad choice). These locations are identified in Figure 1 with *good* choice and a *bad* choice scenarios represented by squares and circles, respectively. We approximate the feedback control input using quadrature as follows:

$$u_c(t) = \int_\Omega \mathbf{h}(\mathbf{x}) \cdot \sum_{i=1}^{12} z_i(t)\Phi_i(x) d\Omega + K_u u(t) \approx \sum_{j=1}^{4} \mathbf{h}_j \cdot \sum_{i=1}^{12} z_i(t)\Phi_i(x_j) dA_j + K_u u(t)$$

where $dA = dx \cdot dy$ and $dx = dy = 0.2$. The approximation of the feedback gains provided by the *good* data set stabilizes the system. Although the response is slow compared to the full-order feedback control, it is important to note that the gains were approximated only from four locations rather than using the complete domain. On the other hand, if the sensors are placed at *bad* locations, the feedback gain approximation is poor and fails to stabilize the closed-loop system. Rather, for long-time integration, it results in instability of the system. Thus, it is important to choose the correct locations of the sensors to measure or estimate the flow field.

By computing the functional gains and using the spatial information contained in these gains we could determine where to place sensors in the wake region to ensure stability and performance. The control is computed efficiently by using good quadrature points for the integral operator.

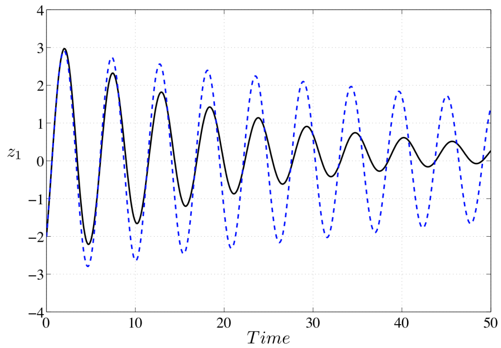


Figure 2: Closed-loop simulation of u with gains approximated at *good* locations.

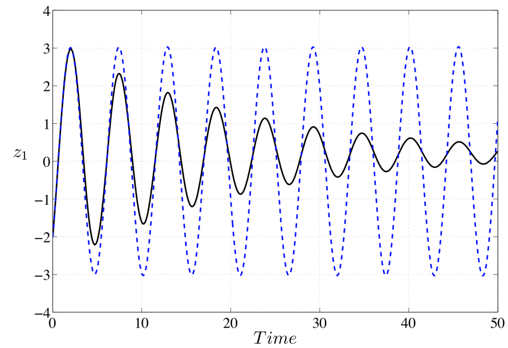


Figure 3: Closed-loop simulation of u with gains approximated at *bad* locations.

Observe that this analysis also provides valuable information concerning the construction of state estimators (filters). In particular, if one can not physically place a sensor in the desired location in the wake, then one would design a filter to estimate the flow **only** in that region. Thus, this approach can be used for designing reduced-order controllers without developing reduced-order models first.

Delayed actuator dynamics

This work investigates the inclusion of actuator dynamics with delays on a control problem. In many applications such as the control of energy efficient buildings where actuation is provided by a HVAC system, the inclusion of actuator dynamics results in a more realistic representation of the system. Including the actuator dynamics impacts the support of the functional gain $\mathbf{k}^{\theta}(\cdot)$ which in turn impacts sensor location problems. Parabolic partial differential equations with no actuator dynamics are typically formulated as an abstract control problem. Coupling the actuator dynamics defined by a delay equation with the abstract system at this stage results in a composite system that can be difficult to analyze. Computational schemes based on finite element methods often require high order "test functions".

We develop an abstract state space model that avoids many of these complexities by working directly with the PDE system. Standard finite element methods can be used to construct approximations of the composite control system and in many cases these approximations can be implemented with existing software packages.

Consider the convection diffusion control system

$$\frac{\partial}{\partial t}\theta(t,\xi) = \mu\nabla^2\theta(t,\xi) + \kappa(\xi)\cdot\nabla\theta(t,\xi), \quad \xi \in \Omega \subset \mathbb{R}^d, \tag{5}$$

with Dirichlet boundary inputs $\theta(t,\xi)|_{\partial\Omega} = g(\xi)v_{\theta}(t)$, where $g(\cdot) \in L^2(\partial\Omega)$ is a fixed function defined on the boundary. In this setting $g(\cdot)$ can have compact support, in which case, the input to the PDE (5) is local. Following the development in [2], let $b(\cdot)$ be the weak solution of the boundary value problem

$$\mu\nabla^2 b(\xi) + \kappa(\xi)\cdot\nabla b(\xi) = 0, \quad b(\xi)|_{\partial\Omega} = g(\xi).$$

We are concerned in the case where the input $v_\theta(t)$ is given by the output of a finite dimensional delay differential equation describing the actuator dynamics. Assume that $v_\theta(t) = \mathbf{H}\mathbf{w}(t)$

where

$$\dot{\mathbf{w}}(t) = \mathbf{A}_0\mathbf{w}(t) + \mathbf{A}_1\mathbf{w}(t-r) + \mathbf{B}_a\mathbf{u}(t). \tag{6}$$

Here, $\mathbf{A}_0$ and $\mathbf{A}_1$ are $n\times n$ matrices, $\mathbf{B}_a$ is an $n\times m$ matrix and $r>0$ is a fixed time delay, and $\mathbf{u}(\cdot):[0,+\infty)\to\mathbb{R}^m$ is the command to the actuator. Assuming one can differentiate $v_\theta(\cdot)$ it follows that $\dot{v}_\theta(t) = \mathbf{H}\mathbf{A}_0\mathbf{w}(t) + \mathbf{H}\mathbf{A}_1\mathbf{w}(t-r) + \mathbf{H}\mathbf{B}_a\mathbf{u}(t)$. Let $\tilde{\theta}(t,\xi) = \theta(t,\xi) - b(\xi)v_\theta(t)$ then the composite system,

$$\frac{\partial}{\partial t}\tilde{\theta}(t,\xi) = \left[\mu\nabla^2 + \kappa(\xi)\cdot\nabla\right]\tilde{\theta}(t,\xi) + \mathbf{f}_0(x)\mathbf{w}(t) + \mathbf{f}_1(x)\mathbf{w}(t-r) + \mathbf{b}_\theta(x)\mathbf{u}(t), \tag{7}$$

$$\dot{\mathbf{w}}(t) = \mathbf{A}_0\mathbf{w}(t) + \mathbf{A}_1\mathbf{w}(t-r) + \mathbf{B}_a\mathbf{u}(t) \tag{8}$$

can be written as a distributed parameter delay system on the state space $X = L^2(\Omega)\times\mathbb{R}^n$

$$\dot{x}(t) = \mathcal{A}_0 x(t) + \mathcal{A}_1 x(t-r) + \mathcal{B}_0\mathbf{u}(t). \tag{9}$$

The following result follows. See [9] for the details on the proof and notation which are omitted due to lack of space.

Theorem 1 *The linear operator $\mathcal{A}_0$ generates a C_0-semigroup $T_0(t)$ on $X = L^2(\Omega)\times\mathbb{R}^n$. Moreover, if $\mathcal{A}_\theta$ and $\mathbf{A}_0$ are stable and $\sigma_p(\mathcal{A}_\theta)\cap\sigma_p(\mathbf{A}_0) = \emptyset$, then the composite semigroup is exponentially stable and the controlled system (9) is stabilizable.*

Let $z(t) = \begin{bmatrix} \tilde{\theta}(t,\cdot) & \mathbf{w}(t) & \mathbf{w}_t(\cdot) \end{bmatrix}^T = \begin{bmatrix} \tilde{\theta}(t,\cdot) & \Phi(t) \end{bmatrix}^T$ then one can also show, see [10] for the detail, the composite distributed parameter control system has the form

$$\dot{z}(t) = \mathcal{A}_c z(t) + \mathcal{B}_c\mathbf{u}(t) \in Z = L^2(\Omega)\times[\mathbb{R}^n\times L^2(-r,0;\mathbb{R}^n)], \tag{10}$$

with bounded (compact) input operator. The following result follows, see [10] for the detail.

Theorem 2 *The linear operator $\mathcal{A}_c$ generates a C_0-semigroup $T_c(t)$ on $Z = L^2(\Omega)\times\mathbb{R}^n\times L^2(-r,0;\mathbb{R}^n)$. Moreover, if $\mathcal{A}_\theta$ and $\mathcal{A}_d$ are stable and $\sigma_p(\mathcal{A}_\theta)\cap\sigma_p(\mathcal{A}_d) = \emptyset$, then the composite semigroup is exponentially stable and the controlled system (10) is stabilizable.*

Consider the Linear Quadratic Regulator (LQR) problem for the composite system (10) with cost

$$J_c = \int_0^{+\infty} \{\langle \mathcal{Q} z(t), z(t)\rangle_Z\} + \int_0^{+\infty} \{\langle \mathbf{R}_a \mathbf{u}(t), \mathbf{u}(t)\rangle_{\mathbb{R}^m}\} \, dt. \tag{11}$$

The details of this transformation are described in [9] and will not be presented here in order to save space. Implementing the Riesz representation theorem results in the feedback controller of the form $\mathbf{u}^{opt}(t) = -\int\limits_{\Omega} \mathbf{k}^\theta(\xi)\theta(t,\xi)d\xi - \mathbf{k}^a \mathbf{w}(t) - \int_{-1}^0 \mathbf{k}^d(s)\mathbf{w}(t+s)ds$, where $\mathbf{k}^a = \tilde{\mathbf{k}}^a - \int_0^1 \mathbf{k}^\theta(\xi) b(\xi) \mathbf{H} d\xi$. The functions $k_i^\theta(\cdot)$ and $k_{i,j}^d(\cdot)$ are called the functional gains and $\tilde{\mathbf{k}}^a$ and $\mathbf{k}^d(\xi)$ are $m \times n$ matrix functions. Consequently, once one computes $\mathbf{k}^\theta(\xi)$ and $\tilde{\mathbf{k}}^a$, computing the gain matrix $\mathbf{k}^a$ requires only a quadrature.

Numerical Example

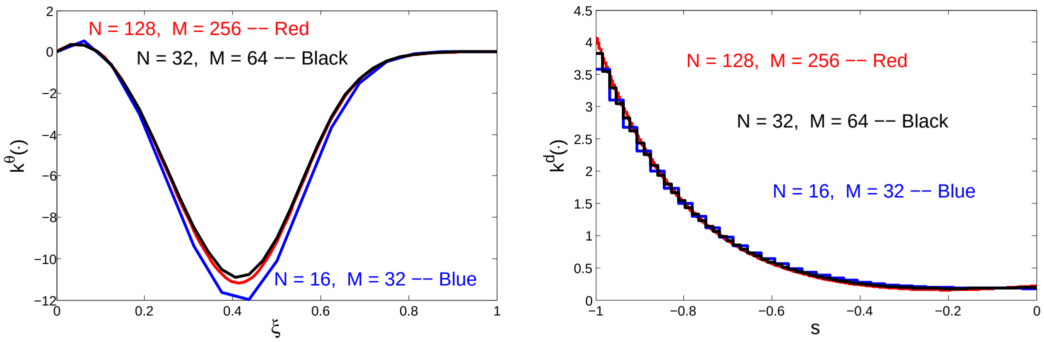
We consider the $1D$ control problem for the parabolic PDE

$$\frac{\partial}{\partial t}\theta(t,\xi) = \mu\theta_{\xi\xi}(t,\xi) + \kappa\theta_\xi(t,\xi), \quad \xi \in (0,1), \quad \text{and} \quad \theta(t,0) = v_\theta(t), \quad \theta(t,1) = 0. \tag{12}$$

Use the following parameters: $\mu = 1/120$, $\kappa = .1$, $R = 0.1$, $R_a = 0.1$, and the function $b(\cdot)$ is given by $b(x) = (e^{-kx} - e^{-k})/(1 - e^{-k})$. Set $Q = D^* \times D$ where $D\varphi(\cdot) = \int_0^1 d(x)\varphi(x)dx$ and $d(x) = \begin{cases} 5, & 0.4 < x < 0.6 \\ 0, & \text{elsewhere} \end{cases}$. The actuator is modeled by the simple scalar delay equation $\dot{w}(t) = -1.0w(t) + 0.95\dot{w}(t-r) + u(t)$ and $v_\theta(t) = \mathbf{H}w(t) = 5w(t)$. A standard piecewise linear finite element scheme (with upwinding) is used to approximate the convection-diffusion equation (see [6], [7] and [8]) and the "AVE" scheme is used to approximate the delay system (see [1] for details). Subdivide the interval $(0,1)$ into N equal subintervals and the interval $(-r,0)$ into M subintervals. Due to space limitations, we will not provide a rigorous proof of convergence here. However, the numerical results presented below illustrate this convergence.

Figure 4 illustrates the convergence of the functional gains $\mathbf{k}_{N,M}^\theta(\cdot)$ and $\mathbf{k}_{N,M}^d(\cdot)$. We note that the system (10) is stable since $max_{ol} = max(real(\mathcal{A}_c)) = -0.0255$. Also, the closed loop operator satisfies $max_{cl} = max(real([\mathcal{A}_c - \mathcal{K}\mathcal{B}_c])) = -0.1592$.

The inclusion of actuators offers a more "realistic" model of the dynamics of the coupled system and allows the integration of system components. Including the actuator dynamics impacts the support of the functional gain $\mathbf{k}^\theta(\cdot)$ which impacts sensor location problems (see [3], [4], [5]), and [12].) Similar results were observed for the delayed actuator case, see [9] and [10] for more detail.



(a) (b)

Figure 4: Plots of (a) ${k^{\theta}}_{N,M}(\cdot)$ and (b) ${k^{d}}_{N,M}(\cdot.)$

Infinite Dimensional Delay Differential Equations in Control and Sensitivity Analysis

Here we also consider a class of functional partial differential equations (FPDEs) which arise naturally in problems of control of systems governed by partial differential equations where delayed actuator dynamics are included and in the sensitivity analysis of such systems when one is concerned with sensitivities with respect to delays. Numerous examples motivating these models are presented in [11]. These examples include Boundary Control of PDE Systems with Actuator Dynamics, Boundary Control with Delayed Actuator Dynamics, Distributed Control with Neutral Delayed Actuator Dynamics, Boundary Control with Neutral Delayed Actuator Dynamics, and Sensitivity Systems. Well-posedness for a class of FPDE systems in product spaces is established and these formulations are used as a framework to develop efficient numerical approximations for control and simulation of the PDE problems. The theoretical results extend existing well-posedness results to problems where the standard range condition does not apply and we present a conjecture about a more general theorem.

We consider the control systems defined by a class of infinite dimensional neutral functional differential equations (NFDEs) of the form

$$\dot{z}(t) - \mathbb{D}\dot{z}(t-\tau) = \mathbb{A}_0 z(t) + \mathbb{A}_1 z(t-r) + \mathbb{B}u(t) \in \mathcal{Z} = \Theta \times \mathbb{R}^n \tag{13}$$

where Θ is a Hilbert space and $\mathbb{D}$, $\mathbb{A}_0$ and $\mathbb{A}_1$ have the specific structure

$$\mathbb{D}= \begin{bmatrix} 0 & D \\ 0 & \mathbf{E} \end{bmatrix}, \mathbb{A}_0 = \begin{bmatrix} \boldsymbol{A}_\theta & F_0 \\ 0 & \mathbf{A}_0 \end{bmatrix}, \mathbb{A}_1 = \begin{bmatrix} 0 & F_1 \\ 0 & \mathbf{A}_1 \end{bmatrix} \text{ and } \mathbb{B} = \begin{bmatrix} B_\theta \\ \mathbf{B} \end{bmatrix}, \tag{14}$$

respectively.

When the operator D is not zero, well-posedness for a generic system of the form (13) often requires assuming the "range condition"

$$Range(\mathbb{D}) \subseteq Domain(\mathbb{A}_0) \text{ and } \mathbb{A}_0\mathbb{D} \text{ is bounded.} \tag{15}$$

However, for certain applications to control and sensitivity analysis of PDE systems with delays, the range condition (15) fails. In particular, boundary control and sensitivity analysis of PDE systems where delayed actuator dynamics are included can lead to systems of the form (13) where (15) is not satisfied. We formulate the system (13) defined by (14) as an abstract system on a "reduced" product space and take advantage of the problem structure to obtain well-posedness without the range condition (15) and use this framework to develop efficient approximations. We assume the following hypothesis holds.

Hypothesis H

- (H1) The linear operator $\boldsymbol{A}_\theta : D(\boldsymbol{A}_\theta) \subseteq \Theta \to \Theta$ generates a C_0-semigroup semigroup on Θ.
- (H2) The linear operators $D : \mathbb{R}^n \to \Theta$, $B_\theta : \mathbb{R}^m \to \Theta$ and $F_i : \mathbb{R}^n \to \Theta$, $i = 0, 1$, are bounded.
- (H3) The finite dimensional operators $\mathbf{E} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{A}_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, 1$, and $\mathbf{B} : \mathbb{R}^m \to \mathbb{R}^n$ are linear (hence bounded).

The state $z(\cdot) \in \mathcal{Z} = \Theta \times \mathbb{R}^n$ has the form $z(t) = \left[\begin{array}{cc} \theta(t,\cdot) & w(t) \end{array}\right]^T \in \mathcal{Z} = \Theta \times \mathbb{R}^n$ and because of the structure (14) only the finite dimensional state $w(\cdot)$ contains delays. With this in mind we define the (reduced) product space

$$\mathfrak{X}^R = \mathcal{Z} \times L^2((-r,0);\mathbb{R}^n) = \Theta \times \mathbb{R}^n \times L^2((-r,0);\mathbb{R}^n)$$

and note that $\mathfrak{X}^R$ may be viewed as a subspace of $\mathfrak{X} = \mathcal{Z} \times L^2((-r,0);\mathcal{Z})$ by the natural embedding. Consider the abstract control system on $\mathfrak{X}^R = \mathcal{Z} \times L^2((-r,0);\mathbb{R}^n) = \Theta \times \mathbb{R}^n \times L^2((-r,0);\mathbb{R}^n)$ defined by

$$\dot{\boldsymbol{x}}(t) = \mathcal{A}\boldsymbol{x}(t) + \mathcal{B}u(t) \in \mathfrak{X}^R, \tag{16}$$

where the domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \left[\begin{array}{c} \theta \\ \eta \\ \varphi(\cdot) \end{array}\right] \in D(A_\theta) \times \mathbb{R}^n \times H^1([-r,0];\mathbb{R}^n) : \varphi(0) - \mathbf{E}\varphi(-r) = \eta \right\}, \tag{17}$$

and for $\boldsymbol{x} = \left[\begin{array}{ccc} \theta & \eta & \varphi(\cdot) \end{array}\right]^T \in D(\mathcal{A})$

$$\mathcal{A}\boldsymbol{x} = \mathcal{A}\left[\begin{array}{c} \theta \\ \eta \\ \varphi(\cdot) \end{array}\right] = \left[\begin{array}{c} A_\theta\theta + F_0\varphi(0) + F_1\varphi(-r) \\ \mathbf{A}_0\varphi(0) + \mathbf{A}_1\varphi(-r) \\ \frac{d}{ds}\varphi(\cdot) \end{array}\right]. \tag{18}$$

The following result applies to the special case where $(H4)$ holds instead of the range condition on D.

Theorem 3 *Assume* $(H1)$ - $(H3)$ *hold. If either*
(HD) *the operator* D *satisfies the the reduced range condition*

$$Range(D) \subseteq Domain(\boldsymbol{A}_\theta) \text{ and } \boldsymbol{A}_\theta D \text{ is bounded or}$$

$(H4)$ *the bounded linear operators* F_0 *and* F_1 *are given by*

$$F_0 = F\mathbf{A}_0 \text{ and } F_1 = F\mathbf{A}_1 \tag{19}$$

then the operator $\mathcal{A}$ *defined by (17) - (18) generates a* C_0*-semigroup* $\mathfrak{T}(\mathfrak{t})$ *on* $\mathfrak{X}^R = \Theta \times \mathbb{R}^n \times L^2((-r,0);\mathbb{R}^n)$.

The detail and proof of this result is presented in [11].

Numrical example

We use the formulation (16) to construct approximating systems for simulation and control. For example, we apply this approach to the parabolic boundary control problem with actuator dynamics described by a (retarded) delayed differential equation. In one dimension this problem comes from the boundary control problem defined by the parabolic PDE

$$\frac{\partial}{\partial t}T(t,\xi) = \mu\frac{\partial^2}{\partial \xi^2}T(t,\xi) + \kappa\frac{\partial}{\partial \xi}T(t,\xi), \quad \xi \in (0,1), \tag{20}$$

with boundary control at the left end and Dirichlet condition at $\xi = 1$ so that

$$T(t,0) = v(t), \quad T(t,1) = 0. \tag{21}$$

Assume that the actuator dynamics are given by the finite dimensional retarded delay differential equation

$$\dot{w}(t) = \mathbf{A}_0 w(t) + \mathbf{A}_1 w(t-r) + \mathbf{B}u(t), \tag{22}$$

where $r > 0$. Also, for simplicity we assume

$$v(t) = \mathbf{H}w(t-r). \tag{23}$$

Consider an LQR problem defined by the boundary control problem (20) - (21) with cost function

$$J = \int_0^{+\infty} \left\{ \langle QT(t,\cdot), T(t,\cdot)\rangle_{L^2} + \langle Q_a \mathbf{w}(t), \mathbf{w}(t)\rangle \right\} dt,$$

where $Q_a = [\mathbf{H}]^T R\mathbf{H} \geq 0$.

For the numerical computations we use the parameters $\mu = 1/240$, $\kappa = .01$, $R = .1$ and set $Q = D^*D$ where $D : L^2(0,1) \to \mathbb{R}^1$ is defined by

$$D\varphi(\cdot) = \int_0^1 d(x)\varphi(x)dx$$

and $d(\cdot) \in L_2(0,1)$ is given by

$$d(x) = \begin{cases} 5, & 0.4 < x < 0.6 \\ 0, & \text{elsewhere} \end{cases}.$$

Also, observe that since $k = \frac{\mu}{\kappa} = 24 > 0$, the function $b(\cdot)$ is given by $b(x) = (e^{-kx} - e^{-k})/(1 - e^{-k})$. The actuator is modeled by the simple scalar delay equation

$$\dot{w}(t) = -1.25w(t) + 1.0\dot{w}(t-r) + u(t).$$

The actuator dynamics are assumed to be given by (22) and

$$v_\theta(t) = 0.5w(t-1)$$

so that $\mathbf{H} = 0.5$. The weight on the input $v(\cdot)$ is set to $R = 0.1$ and the actuator control weight is set to $R_a = 0.1$. A standard piecewise linear finite element scheme (with upwinding) is used to approximate the convection-diffusion equation.

Again, it is important to note that only $w(\cdot)$ is delayed which implies that the feedback control law does not require the past history of $\theta(t,\cdot)$ so that

$$\mathbf{u}^{opt}(t) = -\int_\Omega k^\theta(\xi)\theta(t,\xi)dx - k^a w(t) - \int_{-1}^0 k^d(s)w(t+s)ds$$

and the goal is to compute the functions $k^\theta(\cdot)$ and $k^d(\cdot)$ and a constant gain k^a. A second implication of the fact that only $w(\cdot)$ is delayed is that this allows us to use a reduced finite volume method (the so called "reduced AVE" scheme) for approximating the composite delay system.

We subdivided the interval $(0,1)$ into N equal subintervals for $N = 16, 32, 64, 128, 256, 512, 1024$ and the interval $(-r, 0)$ into M equal subintervals for $M = 32, 64, 128, 256, 512, 1024, 2048$. A standard finite element approximation of the convection diffusion equation on $(0,1)$ is then coupled to a piecewise constant approximation of the past history terms on $(-r, 0)$. Note that we used $M = 2N$ since the finite element scheme has a convergence rate twice the AVE scheme. We solve the algebraic Riccati equation using Matlab and compute the functional gains and gain parameters that define the optimal LQR feedback controller.

N	M	k^a
16	32	0.9897
32	64	0.9812
64	128	1.0240
128	256	1.0726
256	512	1.0994
512	1024	1.1115
1024	2048	1.1118

Table 1: Values of $k^a_{N,M}$

Figure 5 illustrates the convergence of the functional gains ${k^\theta}_{N,M}(\cdot)$ and in Figure 6 we see the same for the functional gains ${k^d}_{N,M}(\cdot)$. Note that the functional gains ${k^\theta}_{N,M}(\cdot)$ for the convection diffusion equation converge at $N = 64$ so we only show plots of ${k^\theta}_{N,M}(\cdot)$ for $N = 16, 32, 64$ and $M = 32, 64, 128$. As shown in Figure 6, convergence of the delay functional gains is much slower. Finally, Table 1 contains values of k^a as N and M increase.

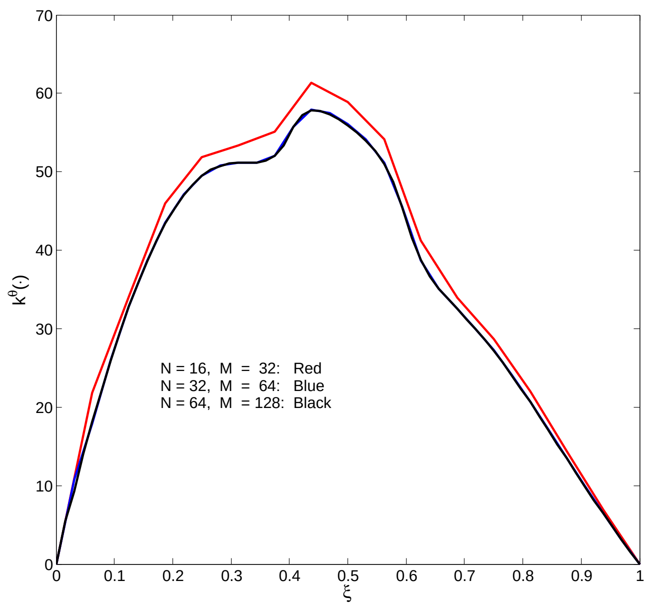


Figure 5: Plots of ${k^\theta}_{N,M}(\cdot)$

The reduced AVE scheme for the delayed terms was selected for two reasons. First, this scheme meets all the conditions necessary to establish the type of convergence required for gain convergence. The second reason is that the reduced scheme greatly

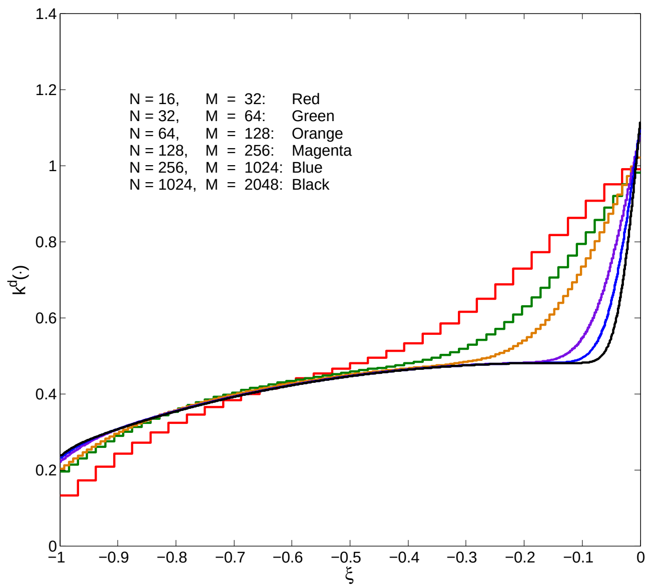


Figure 6: Plots of ${k^d}_{N,M}(\cdot)$

"reduces" the overall size of the approximation scheme. For example, if we had used the full AVE scheme where $N = 1,024$ and $M = 2,048$, then the size of the approximating system would have been $N(M+1) = 2,098,176$. However, the reduced system has is $N + M = 3,072$ so there is a huge reduction in model size. Finally, to obtain reasonable convergence for this specific example, we needed to use $M = 2,048$ approximations for the delay system.

We establish the well-posedness for a class of these systems in product spaces and use this formulation as a framework to develop efficient numerical approximations for control and simulation of the PDE control problems.

Project Summary

Personnel Supported During Duration of Grant

Jeff Borggaard	Professor, Virginia Tech, Blacksburg
John A. Burns	Professor, Virginia Tech, Blacksburg
Eugene Cliff	Professor, Virginia Tech, Blacksburg
Lizette Zietsman	Professor, Virginia Tech, Blacksburg
Imran Akhtar	Postdoc, Virginia Tech, Blacksburg
Saifon Chaturantabut	Postdoc, Virginia Tech, Blacksburg
Omer San	Postdoc, Virginia Tech, Blacksburg
Weiwei Hu	Graduate Student, Virginia Tech, Blacksburg
Boris Kraemer	Graduate Student, Virginia Tech, Blacksburg
Hans-Werner van Wyk	Graduate Student, Virginia Tech, Blacksburg

Publications

1. Optimal Sensor Design for Estimation and Optimization of PDE Systems, J. A. Burns, E. M. Cliff, C. N. Rautenberg, and L. Zietsman, *Proceedings of the 2010 American Control Conference*, pages 4127–4132, 2010.

2. Linear Feedback Control of a von Kármán Street by Cylinder Rotation, J. Borggaard, M. Stoyanov and L. Zietsman, *Proceedings of the 2010 American Control Conference*, pages 5674–5681, 2010.

3. On Commutation of Reduction and Control: Linear Feedback Control of a von Kármán Street, I. Akhtar, J. Borggaard, M. Stoyanov and L. Zietsman, *Proceedings of the 5th AIAA Flow Control Conference*, Paper Number 2010-4832, 2010.

4. Reduced-Order Modeling in Control and Optimization of High Performance and energy Efficient Building, J. A. Burns, *Proceedings of the International Conference on Power Generation Systems and Renewable Energy Technologies*, Islamabad, Pakistan, Nov 29–Dec 2, 2010, 1–5.

5. High Performance Computing for Energy Efficient Buildings, I. Akhtar, J. Borggaard, and J. A. Burns, *Proceedings of the International Conference on Frontiers of Information Technology*, Islamabad, Pakistan, Dec 21–23, 2010, 1–6.

6. On Control Strategies for Fluid Flows using Model Reduction, I. Akhtar, J. Borggaard, and J. A. Burns, *Proceedings of the International Bhurban Conference on Applied Sciences and Technology (IBCAST)*, Islamabad, Pakistan, Jan 10-13, 2011, 1–10.

7. Bochner Integrable Solutions to Riccati Partial Differential Equations and Sensor Placement, J. A. Burns, E. M. Cliff , and C. N. Rautenberg, *Proceedings*

of the 2011 American Control Conference*, San Francisco, CA, June 29-July 01, 2011, 2368–2373.

8. A State Space Error Estimate for POD-DEIM Nonlinear Model Reduction, S. Chaturantabut and D. C. Sorensen, *SIAM J. Numer. Anal.*, 50(1), pp. 46-63, 2012.

9. On Using LQG Performance Metrics for Sensor Placement, Jeff Borggaard, J. A. Burns and Lizette Zietsman, *Proceedings of the 2011 American Control Conference*, San Francisco, CA, June 29-July 01, 2011, 2381–2386.

10. Bochner Integrable Solutions to Riccati Partial Differential Equations and Sensor Placement, John A. Burns, Eugene. M. Cliff and Carlos N. Rautenberg, *Proceedings of the 2011 American Control Conference*, San Francisco, CA, June 29-July 01, 2011, 2368–2373.

11. On Control Strategies for Fluid Flows using Model Reduction, I. Akhtar, J. Borggaard and J. A. Burns, *Proceedings of the International Bhurban Conference on Applied Sciences and Technology (IBCAST)*, Islamabad, Pakistan, Islamabad, Pakistan, Jan 10-13, 2011, 1–10.

12. An Inverse Method for Bounded Error Parameter Identification, J. A. Burns and A. Childers, *J. Inverse Ill-Posed Problems.* 19 (2011), 549–572.

13. An Optimal Control Approach to Sensor / Actuator Placement for Optimal Control of High Performance Buildings, J. Borggaard, J. A. Burns, E. M. Cliff, and L. Zietsman, *Proceedings of the 2nd International High Performance Buildings Conference*, Purdue University, 34661-34667, 2012.

14. Control of the Boussinesq Equations and Implications for Sensor Location in Energy Efficient Buildings, John A. Burns and Weiwei Hu, *Proceedings of the 2012 American Control Conference*, Montreal, CA, June 27-June 29, 2012, 2232-2237.

15. On the Inclusion of Actuator Dynamics in Boundary Control of Distributed Parameter Systems, John A. Burns and Lizette Zietsman, *Proceedings 4th IFAC Workshop on Lagrangian and Hamiltonian Methods for Non Linear Control*, University of Bologna, Bertinoro, Italy, August, 2012, 126-130.

16. An Optimal Control Approach to Sensor / Actuator Placement for Optimal Control of High Performance Buildings, Jeff Borggaard, John A. Burns, E. M. Cliff and Lizette Zietsman, in *Proceedings of the 2nd International High Performance Buildings Conference*, Purdue University, July, 2012, 34661-34667.

17. Coupled CFD/Building Envelope Model for the Purdue Living Lab, Kim, D., Braun, J., Borggaard, J., Cliff, E. and Gugercin, S., in *Proceedings of the 2nd International High Performance Buildings Conference*, Purdue University, July 2012.

18. Control and Sensitivity Reduction for a Viscous Burgers Equation, E. Allen, J. A. Burns and D. S. Gilliam, in *Proceedings of the 51st IEEE Conference on Decision and Control*, Maui, HI, December 2012, 967-972.

19. An example of thermal regulation of a two dimensional non-isothermal incompressible flow, E. Aulisa, J. A. Burns and D. S. Gilliam, in *Proceedings of the 51st IEEE Conference on Decision and Control*, Maui, HI, December 2012, 1578- 1583.

20. Optimization-Based Estimation of Random Distributed Parameters in Elliptic Partial Differential Equations, Borggaard, J., and van Wyk, H.-W., in *Proceedings of the 51st IEEE Conference on Decision and Control*, Paper TuB07.5, pages 2926–2933, December.

21. Approximating Parabolic Boundary Control Problems with Delayed Actuator Dynamics,John A. Burns, Terry L. Herdman and Lizette Zietsman, in *Proceedings of the 2013 American Control Conference*, Paper MoC14.4., June 2013.

22. Using Fréchet Sensitivity Analysis to Interrogate Distributed Parameters in Random Systems, Borggaard, J., Leite Nunes, V. and van Wyk, H.-W., in *Proceedings of the 2013 American Control Conference*, Paper Number MoA14.4, June 2013.

23. Using Dominant Modes for Optimal Feedback Control of Aerodynamic Forces, Akhtar, I., Naqvi, M., Borggaard, J. and Burns, J.,*Journal of Aerospace Engineering*, 2013, (available in early view).

24. Sensitivity and Uncertainty Quantification of Random Distributed Parameter Systems, Borggaard, J., Leite Nunes, V., and van Wyk, H.-W., *Mathematics in Engineering, Science and Aerospace*, Vol. 4, No. 2, pages 117–129, 2013

25. Infinite Dimensional Delay Differential Equations in Control and Sensitivity Analysis, John A. Burns, Terry L. Herdman and Lizette Zietsman, accepted Nonlinear Studies/MESA.

26. Numerical Approximations of the Dynamical System Generated by Burgers' Equation with Neumann Boundary Conditions, Edward J. Allen, John A. Burns and David S. Gilliam.

Professional Talks and Presentations

John Burns

1. IEEE American Control Conference, Baltimore, Maryland, June, 2010.
2. Workshop on Computational Science for Building Energy Efficiency, Arlington, VA, July, 2010.
3. SIAM National Meeting, Pittsburgh, Pennsylvania, July, 2010.
4. AFOSR Conference on Computational Science, Arlington, VA, July, 2010.
5. AFOSR Conference on Control, Arlington, VA, August, 2010.
6. Texas Tech Workshop on Control, Lubbock, Texas, September, 2010.
7. Worcester Polytechnic Institute, Worcester, MA, October, 2010.
8. DOE Fall Creek Falls workshop on Applications of High Performance Computing to Energy Efficiency, Memphis, TN, October, 2010.
9. ISE Informs Lecture, Virginia Tech, November, 2010.
10. Symposium on Analysis & Control of Infinite-Dimensional Systems, Max Planck Institute, Magdeburg, Germany, November, 2010.
11. International Research Forum: What Can the Academic Community Learn from the Global Crisis? Models, Methods and Transfer, Hong Kong Polytechnic University, December, 2010.
12. Auburn University, Auburn, AL, March, 2011.
13. Workshop on Future Directions in Applied Mathematics, NC State University, Raleigh, NC, March, 2011.
14. Missouri University of Science and Technology, Rolla, MO, April, 2011.
15. University of Colorado, Boulder, CO, April, 2011.
16. National Summit on Advancing Clean Energy Technologies - Entrepreneurship and Innovation through High Performance Computing , Washington, DC , May 2011.
17. National Summit on Advancing Clean Energy Technologies - Entrepreneurship and Innovation through High Performance Computing, Washington, DC , May 2011.
18. SIAM Conference on Dynamical Systems, Snowbird, UT, May, 2011.
19. Workshop on Building Modeling and Control, Philadelphia, PA, June, 2011.
20. American Control Conference, San Francisco, CA, June, 2011.
21. 7th Workshop on Control of Distributed Parameter Systems, Wuppertal, Germany, July, 2011.
22. Third Istanbul Conference on Mathematical Methods and Modeling in Life Sciences and Biomedicine, Sile, Turkey, August, 2011.

23. University of Waterloo, Waterloo, Canada, April, 2012.
24. University of Illinois, Champaign, IL, October, 2012.
25. Fifth International Conference on High Performance Computing, Hanoi, Vietnam, March, 2012.
26. 2012 American Control Conference, Montreal, Canada, June, 2012.
27. The 2nd International High Performance Buildings Conference, Purdue University, July, 2012.
28. AFOSR Conference on Control, Arlington, VA, August, 2012.
29. 4th IFAC Workshop on Lagrangian and Hamiltonian Methods for Non Linear Control, Bertinoro, Italy, August 2012.
30. IEEE 43rd Conference on Decision and Control, Maui, HI, December, 2012.
31. SIAM Control and Its Applications, San Diego, CA, July 2013.

Lizette Zietsman

1. DSPDES'10: Emerging Topics in Dynamical Systems and Partial Differential Equations, Barcelona, Spain, June, 2010.
2. SIAM Annual Meeting, Pittsburgh, PA, July, 2010.
3. AFOSR Dynamics and Control Program Review, Washington, DC, August, 2010.
4. AFIT, Department of Mathematics and Statistics, WPAFB, Ohio, April 2011.
5. ICNPAA 2012 Congress, Vienna Technical University, Vienna, Austria, July, 2012.
6. SIAM Control and Its Applications, San Diego, CA, July 2013.

Honors and Rewards Received

John Burns was awarded the 2010 W.T. and Idalia Reid Prize in Mathematics by the Society of Industrial and Applied Mathematics, July 2010.

AFRL Point of Contact

Presentation at AFIT, Department of Mathematics and Statistics, WPAFB, Ohio, April 2011.

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[12] J. A. Burns and W. Hu, Control of the Boussinesq Equations and Implications for Sensor Location in Energy Efficient Buildings, *Proceedings of the 2012 American Control Conference*, Montreal, CA, (2012), 2232–2237.