

# ***A Mixed Boundary Value Problem That Arises in the Study of Adhesively Bonded Structures***

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# Report Documentation Page

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Given a physics-based problem, we often have choices in deciding

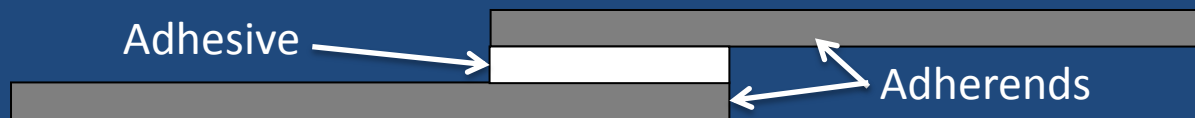
a) which Reduced-Order Model to use

b) which mathematical method to use to analyze the ROM

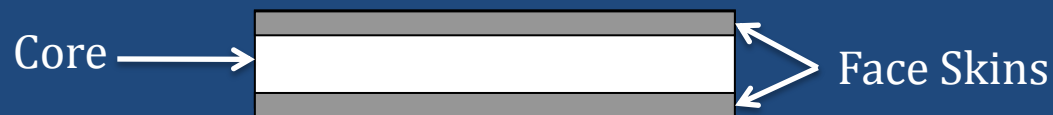
Examples come from Ocean Modeling, Electromagnetism, and Structural Mechanics

# Typical Problem Genesis:

- Adhesively Bonded Joints



- Aerospace Sandwich Structures



# Boundary Value Problem: Mixed form in terms of Stress, Infinitesimal Strain, and Displacement

$z$   
 $\eta$   
 $u(x,\eta) = u_t(x)$        $w(x,\eta) = w_t(x)$   
 $\sigma_{xx} = 0$        $\nabla \cdot \underline{\sigma} = 0$        $\frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial z}$        $\sigma_{xx} = 0$   
 $\sigma_{xz} = 0$        $u(x,0) = u_b(x)$        $w(x,0) = w_b(x)$        $L$        $x$        $\sigma_{xz} = 0$

# Boundary Value Problem: Expressed in terms of Displacement

$u(x, \eta) = u_i(x) \quad w(x, \eta) = w_i(x)$

$(\lambda + 2G) \frac{\partial u}{\partial x}(0, z) + \lambda \frac{\partial w}{\partial z}(0, z) = 0$

$\frac{\partial w}{\partial x}(0, z) + \frac{\partial u}{\partial z}(0, z) = 0$

$(\lambda + 2G) \frac{\partial^2 u}{\partial x^2} + G \frac{\partial^2 u}{\partial z^2} + (\lambda + G) \frac{\partial^2 w}{\partial x \partial z} = 0$

$(\lambda + 2G) \frac{\partial^2 w}{\partial z^2} + G \frac{\partial^2 w}{\partial x^2} + (\lambda + G) \frac{\partial^2 u}{\partial x \partial z} = 0$

$(\lambda + 2G) \frac{\partial u}{\partial x}(L, z) + \lambda \frac{\partial w}{\partial z}(L, z) = 0$

$\frac{\partial w}{\partial x}(L, z) + \frac{\partial u}{\partial z}(L, z) = 0$

$u(x, 0) = u_b(x) \quad w(x, 0) = w_b(x)$

$L \quad x$

# Boundary Value Problem: Expressed in terms of Displacement Potential

$$\frac{\partial \Psi}{\partial x}(x, \eta) = u_t(x) \quad \frac{\partial \Psi}{\partial z}(x, \eta) = w_t(x)$$

$$\frac{\partial \Psi}{\partial x}(x, 0) = u_b(x) \quad \frac{\partial \Psi}{\partial z}(x, 0) = w_b(x)$$

$$(\lambda + 2G) \frac{\partial^2 \Psi}{\partial x^2}(0, z) + \lambda \frac{\partial^2 \Psi}{\partial z^2}(0, z) = 0$$

$$\frac{\partial^2 \Psi}{\partial x \partial z}(0, z) = 0$$

$$(\lambda + 2G) \frac{\partial^2 \Psi}{\partial x^2}(L, z) + \lambda \frac{\partial^2 \Psi}{\partial z^2}(L, z) = 0$$

$$\frac{\partial^2 \Psi}{\partial x \partial z}(L, z) = 0$$

$$\nabla^4 \Psi = 0$$

$$\frac{\partial \Psi}{\partial x}(x, z) = u(x, z)$$

$$\frac{\partial \Psi}{\partial z}(x, z) = w(x, z)$$

# Boundary Value Problem: Expressed in terms of Airy's Stress Function and Solution Ansatz

$$\nabla^4 \Phi = 0; \{0 \leq x \leq L, 0 \leq z \leq \eta\}$$

$$\frac{\partial^2 \Phi}{\partial z^2}(0, z) = 0 \quad \frac{\partial^2 \Phi}{\partial x \partial z}(0, z) = 0$$

$$\frac{\partial^2 \Phi}{\partial z^2}(L, z) = 0 \quad \frac{\partial^2 \Phi}{\partial x \partial z}(L, z) = 0$$

$$\left[ \frac{1}{E} \int \left( \frac{\partial^2 \Phi}{\partial x^2} - \nu \frac{\partial^2 \Phi}{\partial z^2} \right) dz \right]_{z=0} + C_1(x) = w_b(x)$$

$$\left[ \frac{1}{E} \int \left( \frac{\partial^2 \Phi}{\partial x^2} - \nu \frac{\partial^2 \Phi}{\partial z^2} \right) dz \right]_{z=\eta} + C_1(x) = w_t(x)$$

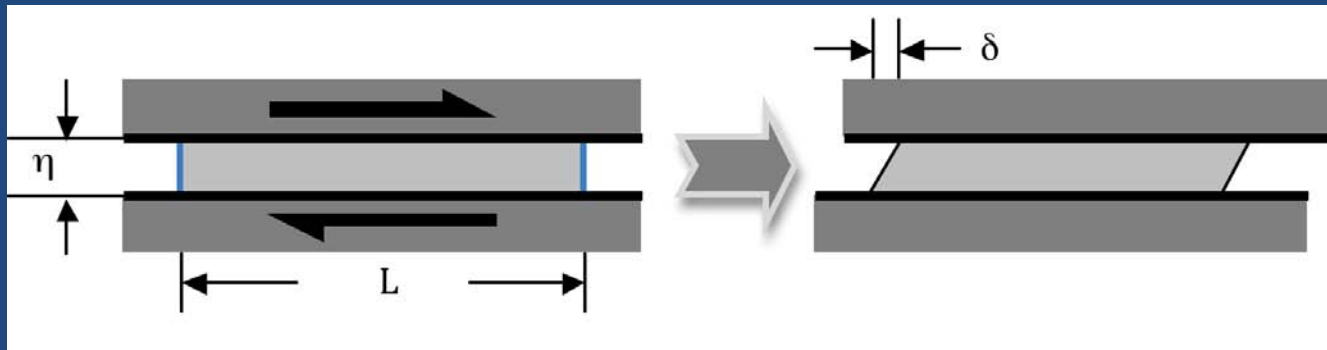
$$\left[ -\frac{1}{G} \int \left( \frac{\partial^2 \Phi}{\partial x \partial z} \right) dz - \frac{1}{E} \int \int \left( \frac{\partial^3 \Phi}{\partial x^3} - \nu \frac{\partial^3 \Phi}{\partial x \partial z^2} \right) dz dz \right]_{z=0} + C_2(x) = u_b(x)$$

$$\left[ -\frac{1}{G} \int \left( \frac{\partial^2 \Phi}{\partial x \partial z} \right) dz - \frac{1}{E} \int \int \left( \frac{\partial^3 \Phi}{\partial x^3} - \nu \frac{\partial^3 \Phi}{\partial x \partial z^2} \right) dz dz \right]_{z=\eta} + \eta \frac{dC_1}{dx}(x) + C_2(x) = u_t(x)$$

$$\Phi(x, z) = A_0(x) + zA_1(x) + \sum_{n=1}^{\infty} F_n(x) \sin\left(\frac{n\pi z}{\eta}\right)$$



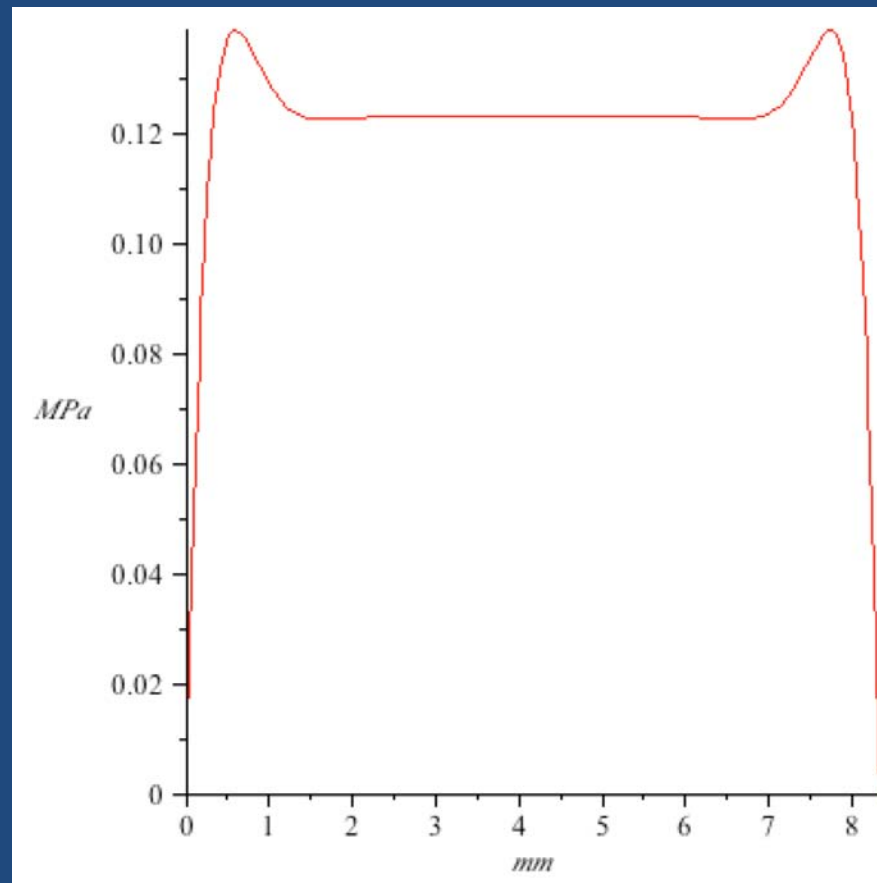
# Example Loading Case: Displacements, Geometry, and Properties



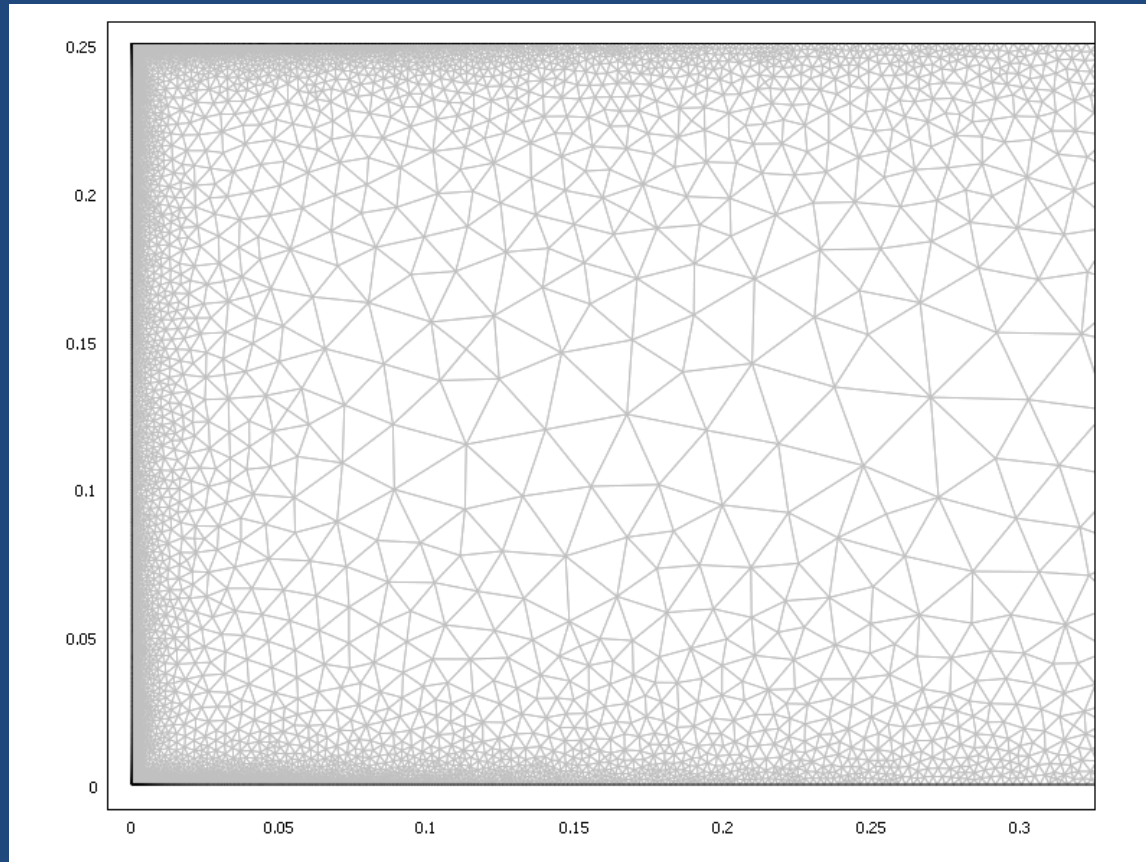
$E$	$\nu$	$L$	$\eta$	$\delta$
344.6 MPa	0.3	8.333 mm	0.25 mm	0.00025 mm

$u_b(x)$	$u_t(x)$	$w_b(x)$	$w_t(x)$
0 mm	0.00025 mm	0 mm	0 mm

# Spectral-Collocation Analysis Results: Mid-plane Shear Stress $\sigma_{xz}(x, \eta/2)$ and Interfacial Shear Stresses $\sigma_{xz}(x, 0)$



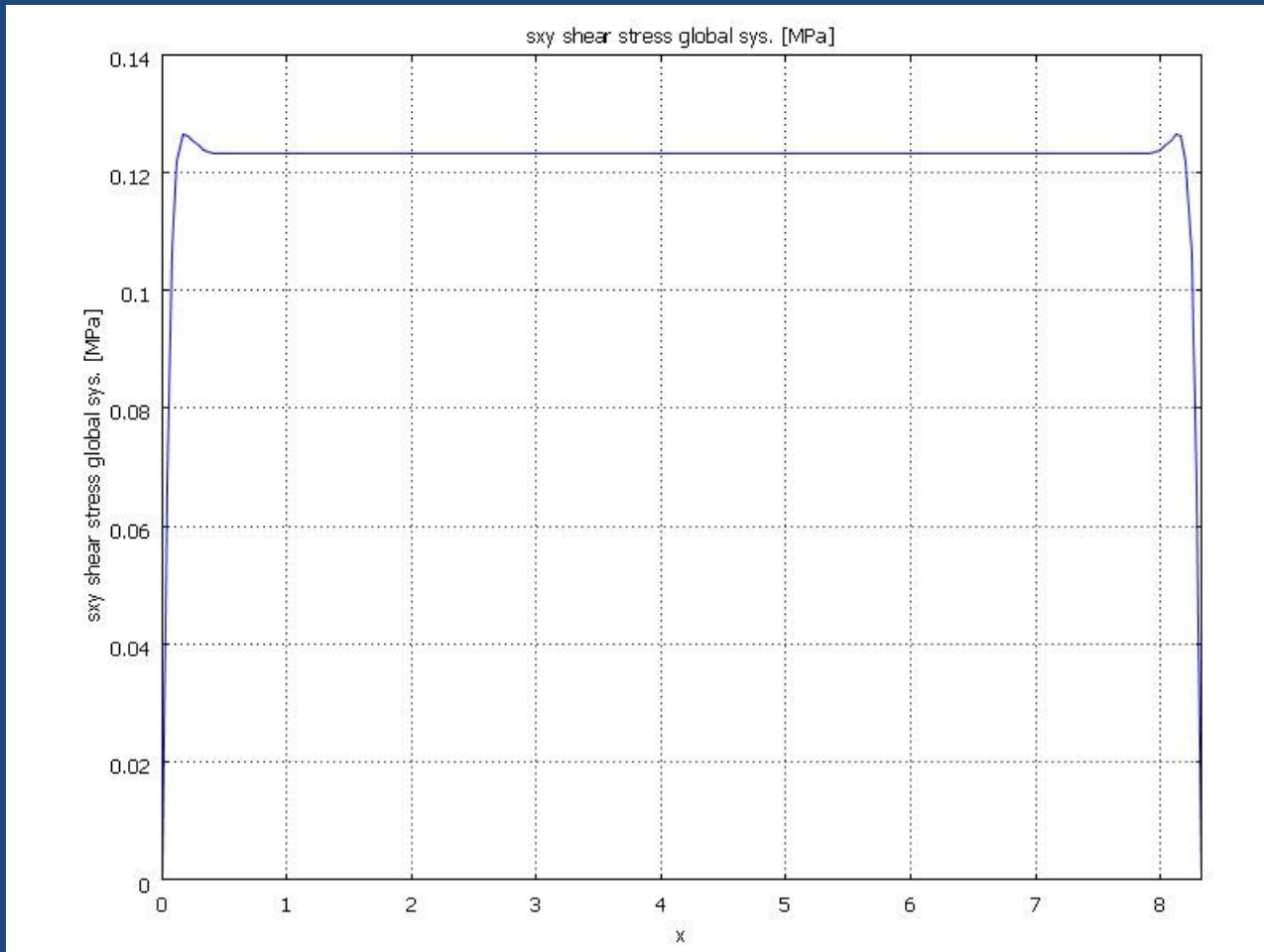
# COMSOL Structural Mechanics Mesh: Shown in vicinity of stress free surface



# COMSOL Structural Mechanics

## 2D Plane-Stress Analysis Results:

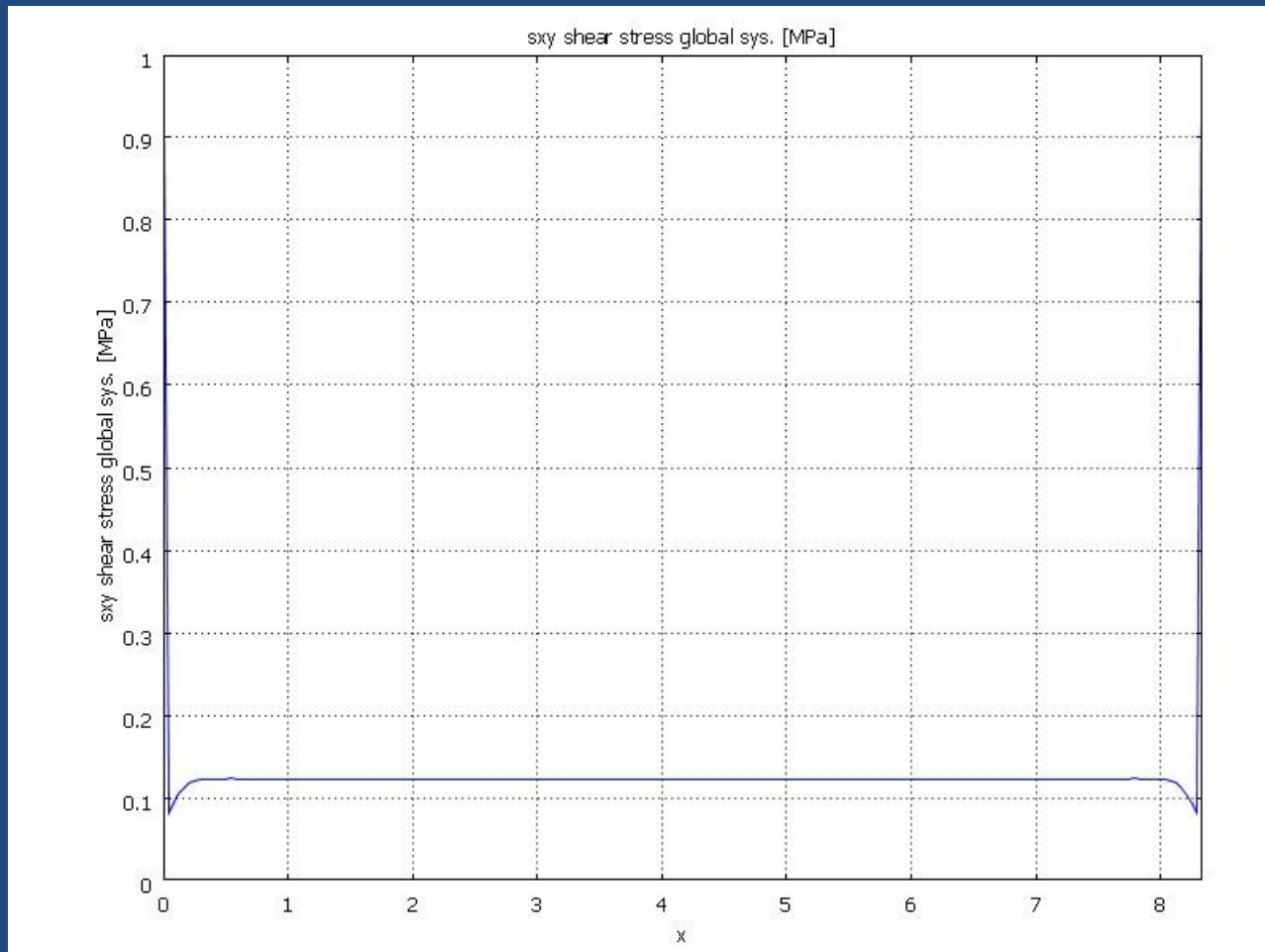
### Mid-plane Shear Stress $\sigma_{xz}(x, \eta/2)$



# COMSOL Structural Mechanics

## 2D Plane-Stress Analysis Results:

### Interfacial Shear Stress $\sigma_{xz}(x,0)$



# Back to Displacement BVP

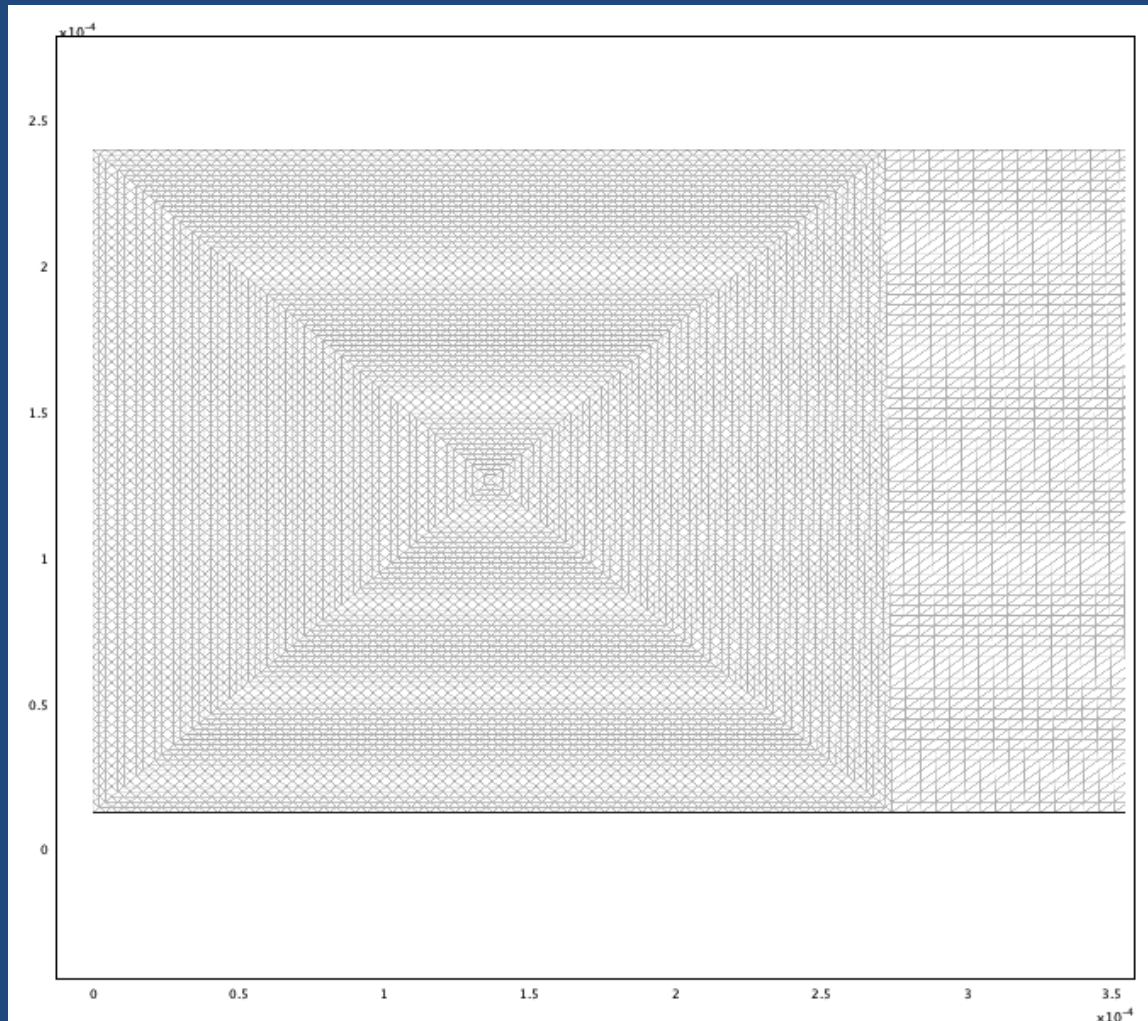
$u(x,\eta) = u_i(x) \quad w(x,\eta) = w_i(x)$   
 $(\lambda + 2G) \frac{\partial u}{\partial x}(0,z) + \lambda \frac{\partial w}{\partial z}(0,z) = 0$   
 $\frac{\partial w}{\partial x}(0,z) + \frac{\partial u}{\partial z}(0,z) = 0$   
 $(\lambda + 2G) \frac{\partial^2 u}{\partial x^2} + G \frac{\partial^2 u}{\partial z^2} + (\lambda + G) \frac{\partial^2 w}{\partial x \partial z} = 0$   
 $(\lambda + 2G) \frac{\partial^2 w}{\partial z^2} + G \frac{\partial^2 w}{\partial x^2} + (\lambda + G) \frac{\partial^2 u}{\partial x \partial z} = 0$   
 $(\lambda + 2G) \frac{\partial u}{\partial x}(L,z) + \lambda \frac{\partial w}{\partial z}(L,z) = 0$   
 $\frac{\partial w}{\partial x}(L,z) + \frac{\partial u}{\partial z}(L,z) = 0$   
 $u(x,0) = u_b(x) \quad w(x,0) = w_b(x)$

$$\nabla \cdot \Gamma = \mathbf{F}$$

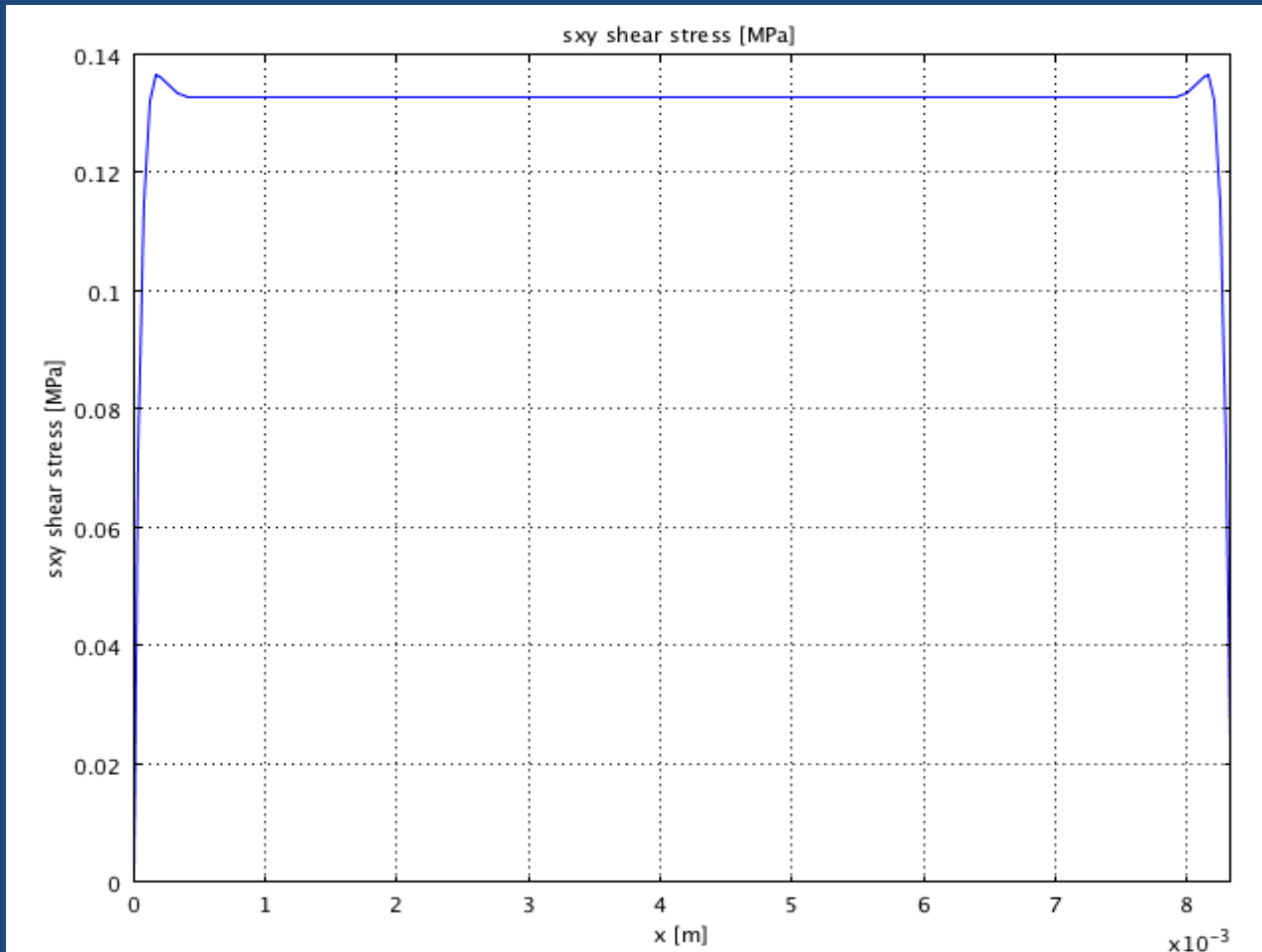
Use COMSOL's General PDE Solver

Gamma is a 2 x 2 tensor

# COMSOL PDE (General Form) Solver Mesh: Shown in vicinity of stress free surface

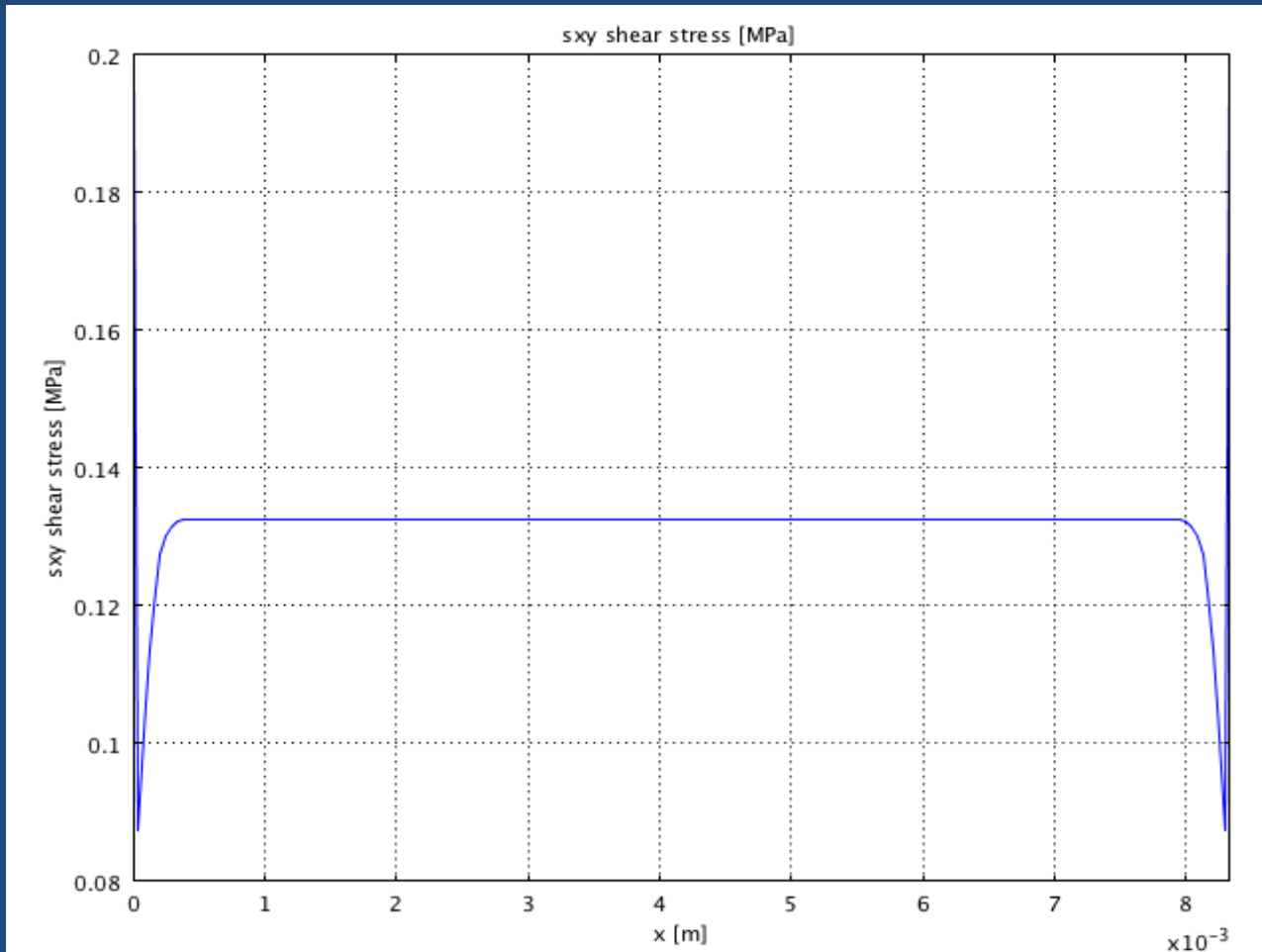


# COMSOL PDE Solver Analysis Results: Mid-plane Shear Stress $\sigma_{xz}(x, \eta/2)$





# COMSOL PDE Solver Analysis Results: Interfacial Shear Stress $\sigma_{xz}(x,0)$



# Some Conclusions/Observations

The Spectral Collocation method describes a shear stress that does not have a singularity at the corners; this seems to be the expected result from a mechanical point of view.

The Structural Mechanics result seems to suffer from artificially large singularities at the corners; did we implement it poorly?

The General PDE solver seems to be a natural way to pose this problem in COMSOL. It gives good result, although it seems to still suffer from some level of Numerical singularity at the corners.