Double-Exponential *LR* Circuit

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Simple *LR* and *RC* circuits are familiar to generations of physics students as examples of single-exponential growth and decay in the relevant voltages, currents, and charges. An element of novelty can be introduced by connecting *two* (instead of one) *LR* coils in parallel with a battery. The resulting circuit can still be treated using little more than the basic tools (Kirchhoff's rules plus a trial exponential solution) employed in the standard *LR* analysis. But the solution is now a double exponential, as can be verified by constructing such a circuit.

Consider the circuit shown in Fig. 1, which is adapted from Ref. 1. The resistors include the internal resistances of the coils and battery.[2] (This is the reason for the addition of the resistor R_2, which was absent from the original circuit in Ref. 1.) Assume the currents in the circuit have reached their steady-state values with switch S open. The switch is then closed at $t = 0$. The problem is to find the subsequent currents in the circuit as a function of time.

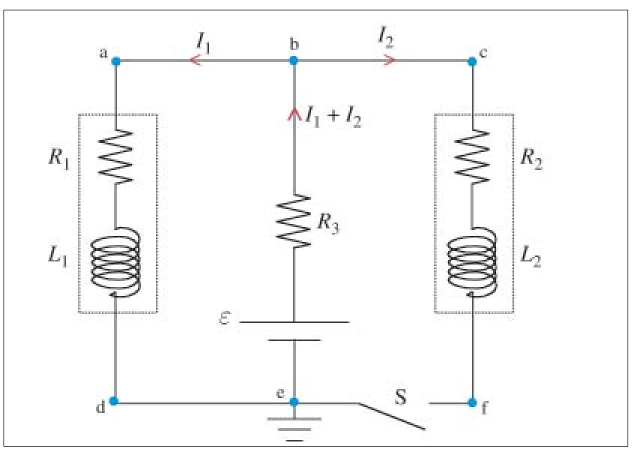

Fig. 1. A circuit consisting of two inductors L_1 and L_2, one battery ε, one switch S, and three resistors R_1 through R_3. The dotted rectangular boxes represent the coils used in the experimental measurements.

General Solution for the Currents After the Switch Is Closed

The inductors prevent the currents from suddenly changing and thus they instantaneously remain at the steady-state values they had at the instant before S was closed,

$$I_1(0) = \frac{\varepsilon}{R_1 + R_3} \quad \text{and} \quad I_2(0) = 0. \tag{1}$$

A long time after the switch is closed the currents attain new steady-state values, which can be derived using the parallel and series rules for resistors,

$$I_1(\infty) = \frac{\varepsilon R_2}{R_1R_2 + R_1R_3 + R_2R_3} \quad \text{and} \quad I_2(\infty) = \frac{\varepsilon R_1}{R_1R_2 + R_1R_3 + R_2R_3}. \tag{2}$$

Equations (1) and (2) are obtained by replacing the inductors with ideal wires, since there is no voltage across an inductor in steady state. Provided R_1 and R_3 are nonzero, then $I_1(0) > I_1(\infty)$ and $I_2(0) < I_2(\infty)$.

Now let us find the detailed functional forms of $I_1(t)$ and $I_2(t)$ from $t = 0$ to ∞. Kirchhoff's current junction rule is already built into Fig. 1, since the upward current in the middle branch is the sum of the

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downward currents in the outer two branches. We therefore only need to write down two of the three Kirchhoff's voltage loop rules for the circuit. For loop defcba in Fig. 1, one gets

$$L_2 \frac{dI_2}{dt} + R_2 I_2 = L_1 \frac{dI_1}{dt} + R_1 I_1, \tag{3}$$

while loop deba gives rise to

$$\varepsilon = R_3(I_1 + I_2) + R_1 I_1 + L_1 \frac{dI_1}{dt}. \tag{4}$$

Substitute $t = 0$ and Eq. (1) into Eq. (4) to find the initial condition

$$\frac{dI_1(0)}{dt} = 0. \tag{5}$$

Next solve Eq. (4) for I_2 and substitute that result into Eq. (3) to obtain

$$A\frac{d^2 I_1}{dt^2} + B\frac{dI_1}{dt} + I_1 = I_1(\infty)\ , \tag{6}$$

with $I_1(\infty)$ given by Eq. (2) and where

$$A = \frac{L_1 L_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

and

$$B = \frac{L_1(R_2 + R_3) + L_2(R_1 + R_3)}{R_1 R_2 + R_1 R_3 + R_2 R_3}. \tag{7}$$

This second-order linear differential equation must have two independent solutions. Substituting a trial solution of the form $I_1(t) - I_1(\infty) = C \exp(-kt)$, where C is a current amplitude and k is a rate constant leads to

$$Ak^2 - Bk + 1 = 0. \tag{8}$$

This quadratic equation does indeed have two (real and positive[3]) decay rate constants:

$$k_{\text{fast}} = \frac{B + \sqrt{B^2 - 4A}}{2A}$$

and

$$k_{\text{slow}} = \frac{B - \sqrt{B^2 - 4A}}{2A}. \tag{9}$$

Therefore, the general solution of Eq. (6) is

$$I_1(t) - I_1(\infty) = C_{\text{fast}} \exp(-k_{\text{fast}} t) + C_{\text{slow}} \exp(-k_{\text{slow}} t). \tag{10}$$

The values of the two current amplitudes are found by fitting to the initial conditions of Eqs. (1) and (5) to obtain

$$C_{\text{fast}} = -\frac{k_{\text{slow}}}{k_{\text{fast}} - k_{\text{slow}}}[I_1(0) - I_1(\infty)]$$

and

$$C_{\text{slow}} = +\frac{k_{\text{fast}}}{k_{\text{fast}} - k_{\text{slow}}}[I_1(0) - I_1(\infty)]. \tag{11}$$

Note that $C_{\text{slow}} > -C_{\text{fast}} > 0$ (for any nonzero values of the circuit elements).

To find $I_2(t)$, the analysis starting at Eq. (4) can be repeated by instead considering loop febc. By symmetry, Eqs. (4) and (6)–(9) remain the same as before, provided one interchanges subscripts "1" and "2." However the initial condition analogous to Eq. (5) is

$$\frac{dI_2(0)}{dt} = \frac{\varepsilon R_1 / L_2}{R_1 + R_3}, \tag{5a}$$

and Eq. (10) becomes

$$I_2(t) - I_2(\infty) = C'_{\text{fast}} \exp(-k_{\text{fast}} t) + C'_{\text{slow}} \exp(-k_{\text{slow}} t), \tag{10a}$$

with the same rate constants as for I_1. The current amplitudes C'_{fast} and C'_{slow} can again be found by fitting to the initial conditions of Eqs. (1) and (5a), but this time both turn out to be negative.

The key result is that the two currents are *double* rather than *single* exponentials. This is an uncommon but interesting function.[4] In the case of I_1 the more rapidly decaying exponential has a small, negative amplitude while the slower exponential has a large, positive amplitude. This enables I_1 to surprisingly start out with a positive value and zero slope,[5] yet still manage to decrease monotonically with time, as graphed in blue in Fig. 2. In contrast, I_2 starts out with a zero value and positive slope and increases monotonically with time, which is not particularly surprising and hence is not plotted.

For comparison, Fig. 2 also graphs $I_1(t)$ for the case of $L_1 = 0$ in red. Then the current is a single exponential with a decay rate constant of $1/B$, as one can see from Eq. (6) with $A = 0$.

Special Case of Identical Coils

If $L_1 = L_2 \equiv L$ and $R_1 = R_2 \equiv R$, then the form of the solution can be simplified. Equation (9) becomes

$$k_{\text{fast}} = \frac{1+2r}{\tau} \quad \text{and} \quad k_{\text{slow}} = \frac{1}{\tau}, \tag{12}$$

where $r \equiv R_3/R$ and $\tau \equiv L/R$. Equation (11) and the analogous equation for the current amplitudes of I_2 reduce to

$$C_{\text{fast}} = C'_{\text{fast}} = -\frac{i}{(1+r)(1+2r)}$$
$$\text{and} \tag{13}$$
$$C_{\text{slow}} = -C'_{\text{slow}} = \frac{i}{1+r},$$

where $i \equiv \varepsilon/2R$. Finally we can combine Eqs. (10) and (10a) to find

$$I_1 + I_2 = \frac{2i}{(1+r)(1+2r)}\left[2+2r+e^{-k_{\text{fast}}t}\right]$$
$$\text{and} \tag{14}$$
$$I_1 - I_2 = \frac{2i}{1+r}e^{-k_{\text{slow}}t}.$$

The fast rate constant thereby describes the exponential equilibration of the battery current $I_1 + I_2$. This is consistent with the fact that we can write this decay rate in terms of the equivalent circuit resistance and inductance. Specifically the two inductors in parallel give $L_{\text{eq}} = L/2$. Similarly the coil resistors are in parallel and that pair is in series with R_3, so that $R_{\text{eq}} = R/2 + R_3$. Now $k_{\text{fast}} = R_{\text{eq}}/L_{\text{eq}}$. On the other hand, the slow constant describes the equilibration of either coil alone, i.e., $k_{\text{slow}} = R/L$.

Furthermore if $R_3 = 0$ then $r = 0$, implying in turn[6] that $k_{\text{slow}} = k_{\text{fast}} = 1/\tau$ and $C_{\text{slow}} = -C_{\text{fast}} = i$. The currents through the coils would then reduce to the usual independent single-exponential solutions as each is connected in turn to the battery. (In fact, however, the first coil was assumed to have been connected long before $t = 0$, so only I_2 is here found to be time dependent.) The role of R_3 in the circuit is therefore to couple I_1 and I_2 together. This explains why R_3 is not chosen to be small (compared to R_1 and R_2) in the following demonstration circuit.

Thinking of $r \equiv R_3/R$ as the coupling strength between the two coils suggests that we can interpret $I_1 - I_2$ as the symmetric mode and $I_1 + I_2$ as the antisymmetric mode (keeping in mind the opposite directions for positive flows of I_1 and I_2 in Fig. 1) in a

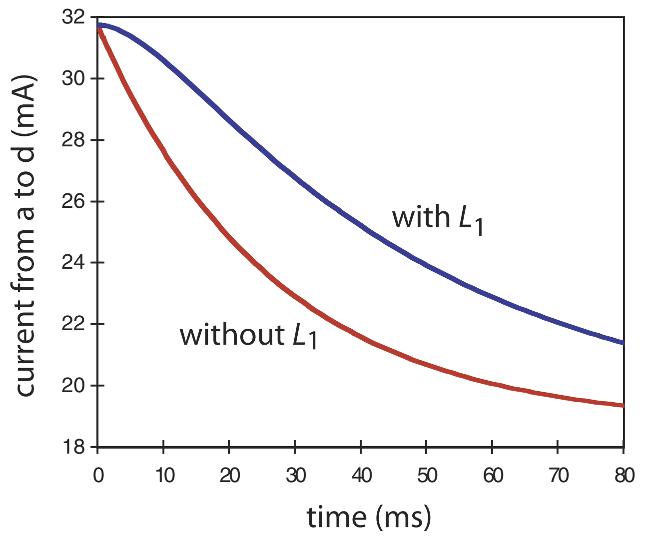

Fig. 2. Theoretical plots of $I_1(t)$ for the circuit of Fig. 1 using the component values specified in the caption of Fig. 3. The blue curve (with both inductors present) is a double exponential with zero slope at the instant after switch S is closed, in striking contrast to the usual single-exponential decay (red curve) obtained when L_1 (but not R_1) is shorted out.

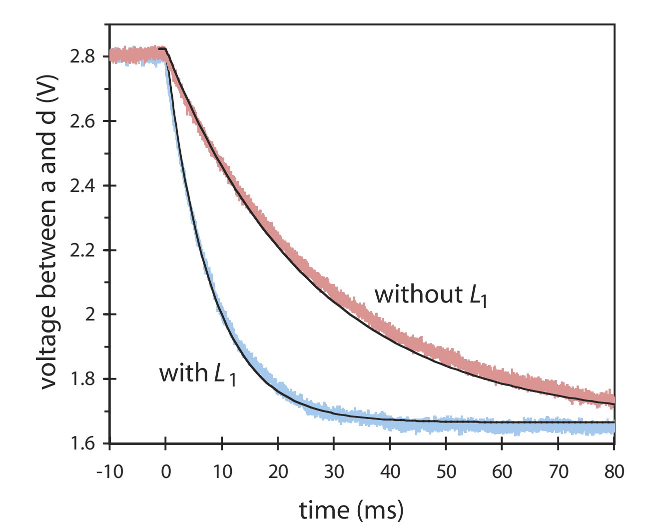

Fig. 3. Measured voltage as a function of time (blue curve) for the circuit in Fig. 1 with $L_1 = L_2 = 4.0$ H, $R_1 = R_2 = 89\ \Omega$, $R_3 = 200\ \Omega$, and $\varepsilon = 9.3$ V. For the red curve labeled "without L_1," coil 1 was removed from the circuit and replaced by a variable resistor adjusted to 89 Ω. The smooth black curves are theoretical plots of Eq. (15) using these component values.

normal-mode analysis. Just as is the case for two identical oscillators coupled together,[7] the rate constant for the slow (low-frequency) mode $I_1 - I_2$ does not involve the coupling but only the "natural" rate constant $1/\tau$, while that for the fast (high-frequency) mode $I_1 + I_2$ is increased by twice the dimensionless coupling constant r.

Experimental Confirmation

These theoretical results were verified by actually constructing the circuit. Two 12-cm long, 8-cm inner diameter, 4000-turn solenoids were used and their bores were filled with stacks of iron rods to increase their inductance. (It then proved important to set up the circuit on a nonmetallic table to avoid stray flux linkages.) The inductances, L_1 and L_2, and internal resistances, R_1 and R_2, of these two coils were measured using a multimeter. The emf ε was set at about 9 V using a dc power supply, and a variable resistor box was utilized to adjust R_3 to a convenient value for measurements. (The exact values of ε under load and of R_3 were also measured using a multimeter.) A telegraph switch was used for S and care was taken to avoid "bounces" during its closing.

It is less intrusive to measure the voltage across coil 1 rather than the current through it by simply connecting an oscilloscope across it and pre-triggering off the switch closure. (This explains why point e is grounded in Fig. 1.) This voltage between points a and d is related to the current through the coil by

$$V_{\text{a-d}} = R_1 I_1 + L_1 \frac{dI_1}{dt}. \tag{15}$$

In general $V_{\text{a-d}}$ is double exponential, because I_1 has that form. But unlike the current, the voltage has a nonzero initial slope $L_1 d^2 I_1(0)/dt^2$. However $V_{\text{a-d}}$ reduces to a single exponential (for nonzero circuit parameters in Fig. 1) if and only if $L_1/R_1 = L_2/R_2$. In particular for the case of identical coils, Eq. (15) becomes

$$V_{\text{a-d}} = \frac{\varepsilon}{(1+r)(1+2r)}\left[1 + r + re^{-k_{\text{fast}}t}\right], \tag{16}$$

which has the expected limiting values for $t = 0$ and ∞.

The experimental curves plotted in color in Fig. 3 are in good agreement with this prediction (plotted in black) with no adjustable parameters. (The small discrepancies can be explained by a few percent error in the component values, well within their measurement tolerances.) If $L_1 = 0$, the last term in Eq. (15) is absent, which explains why the red curves in Figs. 2 and 3 have the same shapes, in striking contrast to the blue curves when this inductor is present. Also, when L_1 is in place, note that dI_1/dt is zero both the instant after and long after switch S is closed, thus explaining why the red and blue curves in Fig. 3 share the same starting and ending voltages.

Comparison with Previous Work

Art Hovey,[8] in analyzing the circuit in Fig. 1 when $R_2 = 0$, assumed that the currents are described by *single* exponentials with time constant T,

$$I_1(t) = \frac{\varepsilon}{R_1 + R_3} e^{-t/T}$$

and

$$I_2(t) = \frac{\varepsilon}{R_3}[1 - e^{-t/T}], \tag{17}$$

where the prefactors are chosen to agree with Eqs. (1) and (2). Equation (17) satisfies Eq. (3) provided that

$$T = L_1 R_1^{-1} + L_2(R_1^{-1} + R_3^{-1}). \tag{18}$$

But Eq. (17) does *not* satisfy Eq. (4). The original goal of the problem in Ref. 1 was to find the total charge Q that flows through R_1 between $t = 0$ and ∞. Simply using Eqs. (17) and (18), one obtains

$$Q = \int_0^\infty I_1(t)\,dt = \frac{\varepsilon}{R_1}\left[\frac{L_1}{R_1 + R_3} + \frac{L_2}{R_3}\right]. \tag{19}$$

However, $k_{\text{fast}} \neq k_{\text{slow}}$ in Eq. (9) even for[3] $R_2 = 0$. Thus the currents are actually *double* exponentials. Nevertheless when one integrates Eq. (10) with $R_2 = 0$, one gets the same solution (19) as did Hovey![9] This is a good illustration of the fact that one can get the right final answer to a problem even when intermediate steps are wrong. Note in particular that Eq. (17) erroneously predicts that $I_1(t)$ has a negative slope at $t = 0$, in contrast to Eq. (5). It is not possible for a single exponential to satisfy Eq. (5).

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References

1. Boris Korsunsky, "Physics challenges for teachers and students: Double closure," *Phys. Teach*. **42**, 312 (May 2004).
2. One can dispense with R_3 and trivially get a double exponential if one moves the switch into the middle branch and measures the current through the battery as a function of time. In this case, one simply obtains the sum of the standard single-exponential currents through each coil.
3. It can be shown that B^2 exceeds $4A$ for *any* positive values of L_1, L_2, R_1, R_2, and R_3. This remains true if either R_1 or R_2 is zero.
4. An analogous example of double exponentials is population lifetimes of species in fluorescence and radioactive decay chains, as in L. Moral and A.F. Pacheco, "Algebraic approach to the radioactive decay equations," *Am. J. Phys*. **71**, 684–686 (July 2003). In the present circuit, one might imagine current cascading down from the pump ε to the two coils and exponentially seeking a new steady state whenever equilibrium is disturbed (by flipping the switch).
5. Another example of a double exponential with zero initial slope is the position versus time of an overdamped, undriven harmonic oscillator released from rest with a positive initial displacement.
6. The current amplitudes in Eq. (11) have the form 0/0 for this case, but nevertheless have well-defined values. The point is that if $R_3 = 0$, then Eq. (4) and the analogous equation for loop febc give *decoupled* equations for I_1 and I_2 with single-exponential solutions.
7. See for example the discussion of two equal masses on two equal springs joined by a different middle spring in Sec. 11.3 of J.R. Taylor, *Classical Mechanics* (University Science Books, Sausalito CA, 2005).
8. Art Hovey, "Solutions to physics challenges for teachers and students: Double closure," *Phys. Teach*. 42, S-1 (May 2004 online, as revised 12 July 2004).
9. The reason for this coincidence of the final answers for Q becomes clear if one expresses I_1 in terms of dI_1/dt and dI_2/dt from Eq. (3) when $R_2 = 0$, and substitutes that result into the integral in Eq. (19). One then discovers that Q only depends on the values of the currents at $t = 0$ and ∞, and not on their functional forms at intermediate times.

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Carl Mungan *enjoys puzzling over the* Physics Challenges *published in each issue of* TPT*, as he always grows in his appreciation of the rich diversity of introductory physics.*

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