# The Satellite Derotator

*Carl Mungan*

**W**hen introducing linear collision problems, physics texts contrast the elastic and inelastic cases. This distinction is useful because, for an interaction between two bodies in one dimension, equality of the kinetic energies and linear momenta enables one to solve for the final velocities from the initial velocities and masses. A number of important problems are approximately elastic, including collisions between aircarts having spring bumpers, "gravity boosting" of space probes around planets, billiard-ball and superball bounces, and atomic collisions in the absence of electronic or nuclear excitations. In contrast, with the exception of planetary orbits, there are no standard examples of elastic angular collisions. The figure skater pulling in her arms, spinning disks that stick together, a bike wheel manipulated on a turntable, the revolving ball on a string whose length can be varied, people walking or jumping onto a merry-go-round — all of

these are inelastic.

However, there is an elastic rotational demonstration apparatus dating back to at least the 1960s,[1] and more recently popularized by its inclusion in *The Video Encyclopedia of Physics Demonstrations* under the name of the "satellite derotator."[2] Although presented as an example of conservation of angular momentum, it is instructive to instead view it as a mechanism for transferring rotational kinetic energy from one object to another without loss. In particular, while intended solely as a means of halting the rotations of a satellite, one could imagine modifying the apparatus to controllably adjust its angular orientation or velocity, storing up mechanical energy for later use.

## The Video Demonstration

The actual device in the video demo[2] is as follows (see Fig. 1). A flat disk of radius *R* and mass *M* rotates about its center on low-friction bearings. Two small

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pucks, each of mass $m$, fit into C-shaped grooves on the periphery at opposite ends of a diameter. A pair of metal clips keep these pucks from flying off until a release button is pressed. The disk is set into rotation by unreeling a cord wound around the axle. Subsequently, the pucks are released from the disk but remain tethered to a separate set of axle bearings by long, light strings of length $l$. As the pucks spiral outward, the disk slows down until finally it is stationary and the pucks are rapidly rotating around the axle in a circle of radius $l$. At this point, all of the angular momentum of the disk has been transferred to the pucks. If the mass of the pucks is increased, even more angular momentum is transferred, so that the disk ends up counter-rotating.

One measurement suffices to explain this behavior in detail. Taking a plastic ruler to the video monitor, I find that the ratio of the string lengths to the disk's radius is $l/R \cong 2.7$. Let the system consist of the disk, strings, and pucks. Since we are neglecting friction and the system is rotationally balanced, there are no external torques on it. Therefore, angular momentum is conserved. Let the subscript "i" denote the initial values of the angular momentum, kinetic energy, moment of inertia, and angular speed before the pucks are released; subscript "f " the final values after the strings have reached their full radial extension; and the unsubscripted values an arbitrary instant of time, when the pucks are at some distance $r$ from the axle and have velocity $\vec{v}$. The radial and azimuthal components of this velocity are, respectively, $v_r$ and $v_\theta \equiv r\omega_{puck}$; at the same instant in time, the disk has angular speed $\omega_{disk}$. Conservation of angular momentum can be expressed as

$$L_i = (\tfrac{1}{2}M + 2m)R^2\omega_i = L = \tfrac{1}{2}MR^2\omega_{disk}$$

$$+ 2mr^2\omega_{puck} = L_f = 2ml^2\omega_f . \qquad (1)$$

Next we must consider the interaction between the disk and pucks. If the pucks were merely unclipped and then freely flew tangentially outward, there would be no mechanism to reduce the angular momentum of the disk. There must be a decelerating torque on the disk and a corresponding accelerating torque on the pucks. The nature of this coupling is not immediately obvious (in contrast to the x-ray view of Fig. 1) and is
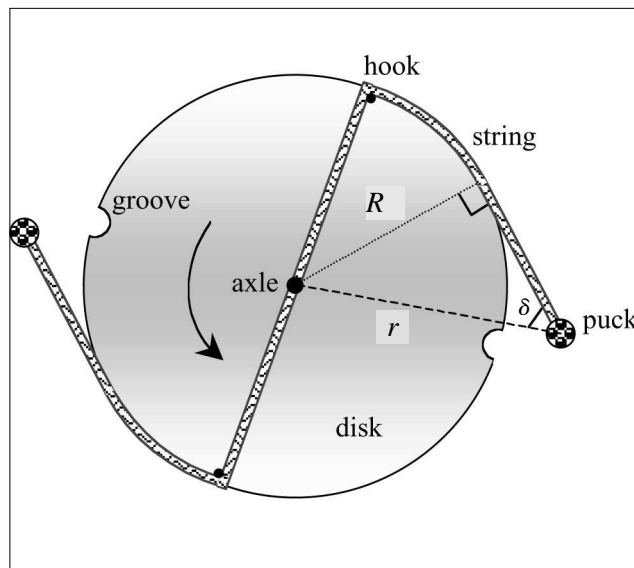


Fig. 1. Rotating disk after the pucks have been unclipped and have begun to symmetrically unwind from the circumference. In the video, the strings and hooks lie mostly behind the disk and so are not clearly visible.

not discussed in the video. But close inspection of the release and outward spiral of the pucks frame by frame reveals it. The strings are never slack. Prior to the release of the pucks, the strings proceed radially outward from the axle until they reach the periphery and then wrap around it to the C-grooves. There may be hooks at the points around which the strings bend as they turn from the radial into the azimuthal directions. The strings come free of the hooks as they straighten out. These hooks must be countersunk so that the strings do not snag on them in subsequent passes across the surface of the stationary disk. There is no mechanism for dissipating kinetic energy because the strings are always taut, have negligible elasticity, and are attached to a ring on the axle that does not stretch radially. We can therefore write
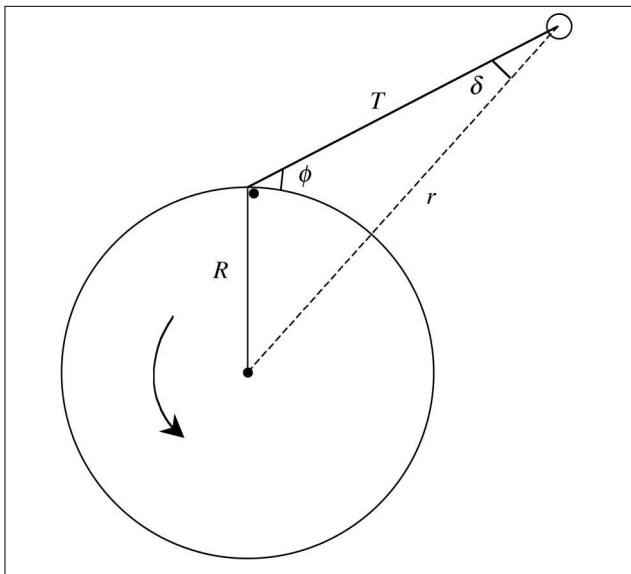
$$2K_i = (\tfrac{1}{2}M + 2m)R^2\omega_i^2 = 2K = \tfrac{1}{2}MR^2\omega_{disk}^2$$

$$+ 2m\, v_{puck}^2 = 2K_f = 2ml^2\omega_f{}^2 , \qquad (2)$$

where $v_{puck}^2 = v_r^2 + v_\theta^2$. By comparing the initial and final angular momenta and kinetic energies, we conclude that

$$\omega_i = \omega_f \quad \text{and} \quad I_i = (\tfrac{1}{2}M + 2m)R^2 = I_f = 2ml^2. \quad (3)$$
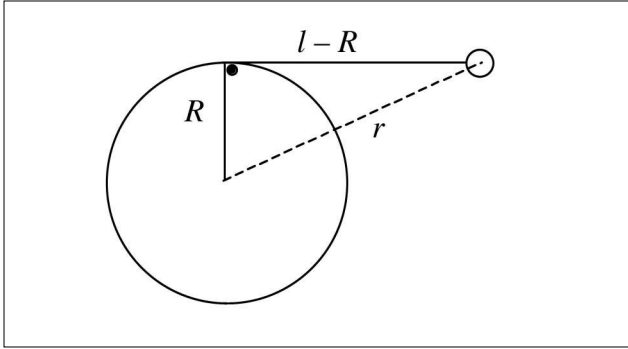
**Fig. 2. Sketch of the instant at which either string has finished unwrapping from the disk's periphery. At later times, the string will no longer intersect the disk tangentially.**



**Fig. 3. Geometry of a puck spiraling outward to its maximum radial extension. During this phase of the motion, the angle $\phi$ is increasing from 0° to 90°, while the angle $\delta$ is decreasing to 0°.**

Rearranging the second equality leads to

$$\frac{4m}{M} = \frac{1}{(l/R)^2 - 1}. \tag{4}$$

Hence, $l/R$ dictates how massive the pucks need to be relative to the mass of the disk to just stop it. In the calculations below, I will assume each puck has exactly 4% of the mass of the disk, as this gives the measured value of $l/R$.

## Transferring Rotational Energy

As long as the strings bend around the hooks, they do positive work on the pucks and an equal amount of negative work on the disk, $dW_{puck} = -dW_{disk}$. It is convenient to divide both sides by the time interval $dt$ and balance powers, computed by taking the dot product of the tension (with magnitude $T$) and the velocities (of the rim in the case of the disk, which is equivalent to the product of the torque and angular speed). The unfurling of the string is divided into two ranges of motion. First, the string unwraps from the rim, ending up as two straight segments making a 90° bend at the hook, as in Fig. 2. Then this bend angle opens up to 180°.

Consider some arbitrary instant while the string is unreeling from the rim, as sketched in Fig. 1. From the geometry, it follows that

$$-\frac{dW_{disk}}{dt} = TR\omega_{disk} = \frac{dW_{puck}}{dt}$$

$$= T(v_\theta \sin \delta - v_r \cos \delta), \tag{5}$$

where

$$\sin \delta = R/r \Rightarrow \cos \delta = \sqrt{1 - (R/r)^2}. \tag{6}$$

We begin the second phase of the motion when the string is fully unwrapped, as in Fig. 2. Application of Pythagoras' theorem implies that

$$r_{unwrap} = \sqrt{R^2 + (l - R)^2}. \tag{7}$$

Beyond this point in time, the geometry of the string is as sketched in Fig. 3. Now Eqs. (5) and (6) become

$$-\frac{dW_{disk}}{dt} = TR\omega_{disk} \cos \phi = \frac{dW_{puck}}{dt}$$

$$= T(v_\theta \sin \delta - v_r \cos \delta), \tag{8}$$

where

$$\frac{\sin \delta}{R} = \frac{\sin(90° + \phi)}{r} \Rightarrow \cos \phi = \frac{r}{R}\sin \delta \tag{9a}$$

and

$$R^2 = r^2 + (l - R)^2 - 2r(l - R)\cos \delta, \tag{9b}$$

which can be solved for $\delta$.

## Numerical Solution

The simultaneous solution of Eqs. (1), (2), and (8) using the quadratic formula is

$$\frac{\omega_{\text{disk}}}{\omega_i} =$$

$$\frac{1+k}{1+ky^2}\left[1- \sqrt{1+ \frac{1+ky^2}{1+k}\left(\frac{y^2-1}{y^2-y^2\cos^2\delta+1/k}-1\right)}\right]$$

$$\frac{\omega_{\text{puck}}}{\omega_i} = \frac{1+k-\omega_{\text{disk}}/\omega_i}{ky^2} \; ; \tag{10}$$

$$\frac{v_r}{r\omega_i} = \left(\frac{\omega_{\text{puck}}}{\omega_i} - \frac{\omega_{\text{disk}}}{\omega_i}\right)\sqrt{\cos^{-2}\delta-1} \; ,$$

where $y \equiv r/R$, $k \equiv 4m/M$, and $\cos\delta$ is given by Eq. (9b). This solution also holds for the first phase of the motion if Eq. (6) is instead used for $\cos\delta$; Eq. (7) determines the value of $r$ at which one switches from one expression for $\cos\delta$ to the other. Equation (10) is written in a form that can be evaluated in a spreadsheet. The first two quantities are plotted together with $v_{\text{puck}}/r\omega_i \equiv v_{\text{puck}}/v_i$ in Fig. 4.

Finally, we can compute the timescales and angular displacements of the system. The radial velocity of the pucks is $v_r \equiv dr/dt$. This can be expressed in terms of a finite-difference approximation that can be iterated in a spreadsheet,

$$t(r) \cong t(r-\Delta r) + \Delta r/v_r. \tag{11}$$

It is convenient to calculate this in units of the initial period, $T \equiv 2\pi/\omega_i$. We also have

$$\omega_{\text{puck}} \equiv d\theta_{\text{puck}}/dt \Rightarrow$$

$$\theta_{\text{puck}}(r) \cong \theta_{\text{puck}}(r-\Delta r) + \omega_{\text{puck}}\,\Delta r/v_r \tag{12}$$

and similarly for $\theta_{\text{disk}}$. Note that these angular displacements are independent of the initial angular speed of the system, i.e., the pucks must rotate through the same total angle in order to stop the disk regardless of how hard you initially tug the pull cord. These are graphed in Fig. 5, beginning when the pucks are released and ending when they have reached their maximum radial extension. A step size of $\Delta r = 0.01R$ was used.
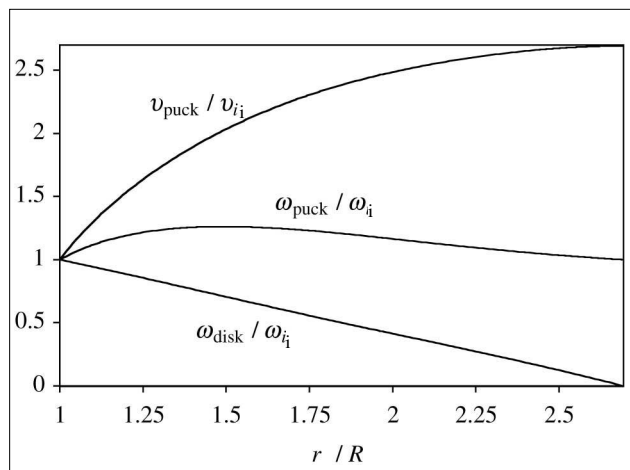


**Fig. 4. Graph of select angular and linear speeds beginning when the pucks are released and ending when the strings are fully extended and the disk has stopped. Notice that $\omega_{\text{puck}}$ ends up at its initial value, while $v_{\text{puck}}$ monotonically increases until it finally equals $l/R$.**
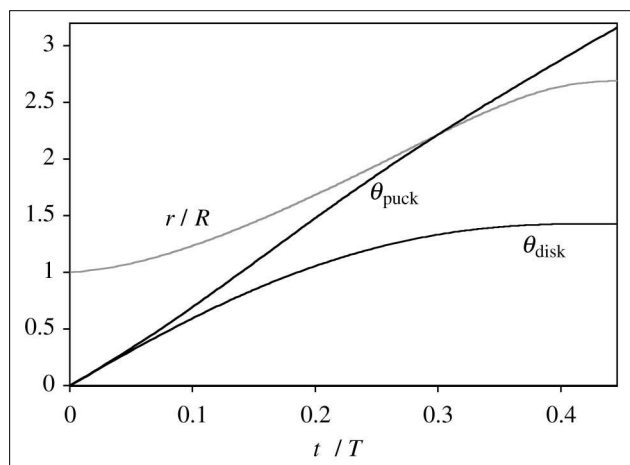


**Fig. 5. Numerical integration of the time and angles (in radians) from the release of the pucks to the stopping of the disk.**

The angle in radians that each puck is initially wrapped around the circumference of the disk is $\theta_{\text{wrap}} = l/R - 1$. Hence, the total angle that each puck must move through to reach its full extension is the sum of this plus the total angle that the disk rotates through,

$$\theta_{\text{wrap}} + \theta_{\text{disk,f}} = \theta_{\text{puck,f}}. \tag{13}$$

Substituting the values from Fig. 5 and Eq. (4), we find that this equality is verified — both sides equal $\pi$ radians, indicating that our step size for $\Delta r$ was

suitably small. This half revolution agrees with what one observes on the video. It further agrees with the abscissa in Fig. 5: It takes just under half the initial period of rotation to bring the disk to rest after releasing the pucks. This is not surprising. Figure 4 shows that the rotational speed of the pucks is roughly constant with an average speed slightly larger than the initial angular speed of the system. Noting from this same graph that the angular deceleration of the disk is approximately constant, we conclude that

$$\theta_{\text{disk,f}} \cong \frac{1}{2}\omega_{\text{i}}t_{\text{f}} \cong \frac{1}{2}\theta_{\text{puck,f}}. \tag{14}$$

Substituting these two angles into Eq. (13) implies that

$$\frac{t_{\text{f}}}{T} \cong 2\frac{\theta_{\text{wrap}}R}{2\pi R}. \tag{15}$$

This says that the fraction of the initial period required to stop the disk is roughly equal to the fraction of the circumference initially wrapped by the two strings. Hence, if we wish to stop the disk more rapidly (for a given initial angular speed), we should decrease the length of the strings (i.e., move the hooks closer to the C-grooves). In turn, Eq. (4) then implies we would have to increase the mass of the pucks. This agrees with the second half of the video, where the mass of the pucks is increased (but the string lengths remain unchanged) and the disk is consequently brought to rest before the strings reach their full extension.

## With Your Students

This video demonstration can be analyzed in class or homework at three different levels. As a springboard for discussion, it illustrates simultaneous conservation of angular momentum and kinetic energy. Ideally, rotational energy can be stored up without loss in the pucks for later recovery. By writing $K = \frac{1}{2}L\omega$ and requiring both $K$ and $L = I\omega$ to be the same ini-

tially as finally, Eq. (3) is immediately obtained. A smaller or larger value of $m$ would then necessarily slow down the disk or change its direction of rotation, respectively. Applications other than to satellites can be brainstormed.

As a homework problem, students could be asked to write down expressions for the initial and final values of $K$ and $L$ without assuming the disk stops. The general simultaneous solution for the final angular velocities of the disk and pucks is then straightforward yet inelegant. But there are some simpler special cases. For example, the disk ends up counter-rotating with an angular velocity equal and opposite to its initial velocity when $M/2m = l/R - 1$. As a second example, if $4m << M$ then one can neglect the contribution of the pucks to the initial values of $K$ and $L$. It follows, for instance, that $M/m \cong 12(l/R)^2$ to halve the initial angular speed of the disk.

Finally, as a spreadsheet exercise, the graphical solutions in this paper could be reproduced. For this purpose, I would recommend providing students with a copy of this article. There are a number of other interesting graphs that could then be prepared, such as of Eq. (4) or of the radial velocity of the pucks. Other values of $m/M$ and of $l/R$ could also be explored.

## References

1. *Physics Demonstration Experiments*, edited by H.F. Meiners (Ronald Press, New York, 1970), Vol. 1, pp. 318–320 and 596–599. Complete construction details are included.
2. *The Video Encyclopedia of Physics Demonstrations* (The Educational Group, Los Angeles, 1992), Demo 07-09.