Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **2006**	2. REPORT TYPE	3. DATES COVERED **00-00-2006 to 00-00-2006**
4. TITLE AND SUBTITLE **Introducing the exponential function**		5a. CONTRACT NUMBER
		5b. GRANT NUMBER
		5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)		5d. PROJECT NUMBER
		5e. TASK NUMBER
		5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **US Naval Academy,Physics Department,Annapolis,MD,21402-5002**		8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)		10. SPONSOR/MONITOR'S ACRONYM(S)
		11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**2**	

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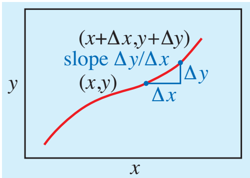
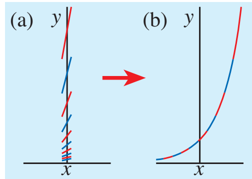
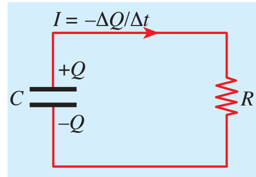
Introducing the exponential function







From the left, **Figure 1.** *Geometrical interpretation of the slope at an arbitrary point on a curve.* **Figure 2.** *(a) Short line segments whose slopes are equal to their* y*-intercepts. (b) Continuous curve constructed by displacing the segments with slopes that are greater than one to the right and those with slopes less than one to the left. The* x*- and* y*-axes are plotted to the same scale, so that unit slope corresponds to a 45° angle of inclination.* **Figure 3.** *An initially charged capacitor* C *discharging through a resistor* R. *The negative sign in the expression for* I *reflects the fact that the capacitor's charge* Q(t) *decreases with time. Equation (2) can be obtained from Kirchhoff's loop rule by traversing the circuit in the clockwise direction.*

The exponential function appears in a number of places in introductory physics, such as damped oscillations, radioactive populations, *LR* and *RC* circuits, isothermal and adiabatic processes for an ideal gas, and atmospheric pressure as a function of altitude. The purpose of this article is to introduce the exponential function, without using calculus, for the benefit of students in an algebra-based course. The key point is that the exponential is that special function $y(x)$ whose slope at any point x is equal to the value y of the function at that same point.

An instructor can proceed as follows. Begin by graphing an arbitrary monotonically increasing function $y(x)$, such as that shown in figure 1. Choose any convenient point on the curve and label it with the coordinates (x, y). Then label a neighbouring point as $(x + \Delta x,\ y + \Delta y)$. Draw a small right triangle connecting the two points, with run labelled Δx, rise of Δy, and hypotenuse having a slope of $\Delta y/\Delta x$ equal to the tangent of its angle above the horizontal. This slope is therefore small where the curve is nearly flat, and large where the curve rises sharply. Now suppose we wish the slope to be equal to the value of the function at each point,

$$\frac{\Delta y}{\Delta x} = y. \qquad (1)$$

Draw a fresh set of axes. Then sketch, at intervals along the y-axis, short line segments that each have a slope of $\Delta y/\Delta x$ that is equal to the value of y at that point along the axis, as in figure 2(a). Let us divide these segments into two groups: an upper set with slopes greater than one (i.e. inclined at angles relative to the horizontal exceeding 45°), and a lower set consisting of the line segments with slopes that are less than unity. Finally, displace the upper segments to the right and the lower segments to the left in such a fashion that all the segments smoothly join to form one continuous monotonic curve, as in figure 2(b). The result is a graph of the exponential function, $y(x) = \mathrm{e}^x$.

As an example to illustrate the application of this result in physics, consider the discharging *RC* circuit sketched in figure 3. The capacitor has charges $\pm Q$ on its plates, whose magnitude decreases with time t as a result of a current $I = -\Delta Q/\Delta t$ carrying increments of charge $\Delta Q < 0$ off the positive plate through the resistor and onto the negative plate in each time interval Δt. The potential rise Q/C across the capacitor must equal the potential drop IR across the resistor,

$$\frac{Q}{C} = IR = -R\frac{\Delta Q}{\Delta t}. \qquad (2)$$

Multiplying both sides by the capacitance C and dividing by the initial charge Q_0 of the capacitor, equation (2) assumes the dimensionless form

$$\frac{Q}{Q_0} = -\frac{RC}{\Delta t}\frac{\Delta Q}{Q_0} = \frac{\Delta(Q/Q_0)}{\Delta(-t/\tau)} \qquad (3)$$

where $\tau = RC$ is the time constant characterizing the decaying charge and current. Defining $x \equiv -t/\tau$ and $y \equiv Q/Q_0$, equation (3) becomes equation (1). Therefore,

$$y = \mathrm{e}^x \quad \Rightarrow \quad Q = Q_0\,\mathrm{e}^{-t/\tau} \qquad (4)$$

and if this result is substituted into the first equality in equation (2), we obtain $I(t) = I_0\,\mathrm{e}^{-t/\tau}$, where $I_0 = Q_0/\tau$ is the initial current in the circuit.

This example demonstrates how to get the standard expressions for the time dependences of the charge and current in an *RC* circuit without using calculus. A similar algebraic approach can be used to derive the exponential dependences of the other phenomena that were mentioned in the first sentence of this article.

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