LOCAL PRINCIPAL COMPONENT PURSUIT FOR NONLINEAR DATASETS

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ABSTRACT

A robust version of Principal Component Analysis (PCA) can be constructed via a decomposition of a data matrix into low rank and sparse components, the former representing a low-dimensional linear model of the data, and the latter representing sparse deviations from the low-dimensional subspace. This decomposition has been shown to be highly effective, but the underlying model is not appropriate when the data are not modeled well by a single low-dimensional subspace. We construct a new decomposition corresponding to a more general underlying model consisting of a union of low-dimensional subspaces, and demonstrate the performance on a video background removal problem.

Index Terms— Compressive Sensing, Robust Principal Component Analysis, Low Rank, Sparse Representation, Group Sparse

1. INTRODUCTION

Matrix completion, which attempts to reconstruct a matrix with only a small fraction of its entries known [1], is a recent branch of the field of compressive sensing. (The assumption that the matrix has a low rank plays a role analogous to that of sparsity in compressive sensing.) An extension of this problem seeks to decompose a matrix D of high-dimensional data into a sum of two components, one having low rank, the other being sparse. This can be expressed as the optimization

$$\min_{L,S} \operatorname{rank}(L) + \lambda \|S\|_0 \text{ such that } L + S = D \;, \qquad (1)$$

where $\| \cdot \|_0$ counts the number of nonzero entries, and $\lambda > 0$ is a tuning parameter. We can regard L as a low-dimensional description of the data, while S consists of deviations from that model, which can be interesting in their own right.

We can compare (1) to Principal Component Analysis (PCA), which would compute the matrix L of desired rank that minimizes $\|D - L\|_2$, the entry-wise Euclidean norm of the residual. Because the second term of (1) penalizes only the number of deviations and not their size, the low-dimensional model L will not be perturbed by outliers among the entries of D, and hence will provide a more robust description of most of the dataset. This connection between sparse optimization and "robust PCA" was made by Candès *et al.* [2], who also provided a tractable, convex approximation, which they called Principal Component Pursuit, of the NP-hard problem (1)

$$\min_{L,S} \|\sigma(L)\|_1 + \lambda \|S\|_1 \text{ such that } L + S = D \;, \qquad (2)$$

where D is $m \times n$ and $\lambda = 1/\sqrt{\max\{m, n\}}$. The first term is the ℓ^1 norm of the vector $\sigma(L)$ of singular values of L, and is known as the *nuclear norm* of L. Applications considered thus far include automated background removal in video [3], text analysis [4], and image alignment [5].

This decomposition approach assumes that there is a single, low-dimensional model that describes most components of the elements of the dataset. In this work, we develop a more general method that is suitable, for instance, for data described by a *manifold* [6], except for a sparse set of possibly-large deviations. We will thus allow our low-dimensional description to vary across the dataset, while retaining the robustness given by having a second, sparse component. In the context of video background removal, this will allow us to handle the case of a moving camera, making the method suitable for a much larger class of surveillance problems.

2. LOCAL PRINCIPAL COMPONENT PURSUIT

The geometric intuition motivating our approach is that if the data lie within a nonlinear manifold, then every sample in the manifold may be represented (assuming adequate sampling density) as a sparse linear combination of neighboring samples spanning an approximation to the local tangent plane. This idea can be implemented as the problem

$$\min_{U,S} \alpha \|U\|_1 + \beta \|U\|_{2,1} + \|S\|_1 \text{ such that } DU + S = D \;, \quad (3)$$

in which the explicit notion of low rank, and its nuclear-norm proxy, is replaced by representability of a matrix as a sparse representation on itself. (The subspace segmentation algorithm of Liu *et al.* [7] also employs the concept of self-representability, but combines it with a nuclear-norm proxy for rank, as in (2).) The $2,1$-norm, defined as

This research was supported by the U.S. Department of Energy through the LANL/LDRD Program.

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
MAY 2012	**N/A**	**-**

4. TITLE AND SUBTITLE	5a. CONTRACT NUMBER
Local Principal Component Pursuit For Nonlinear Datasets	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Los Alamos National Laboratory Los Alamos, NM 87545, USA**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release, distribution unlimited

13. SUPPLEMENTARY NOTES
See also ADA561051. AOARD-CSP-111007 International Conference on Acoustics, Speech and Signal Processing (37th) (ICASSP 2012) Held in Kyoto, Japan on March 25-30, 2012. U.S. Government or Federal Purpose Rights License.

14. ABSTRACT
A robust version of Principal Component Analysis (PCA) can be constructed via a decomposition of a data matrix into low rank and sparse components, the former representing a lowdimensional linear model of the data, and the latter representing sparse deviations from the low-dimensional subspace. This decomposition has been shown to be highly effective, but the underlying model is not appropriate when the data are not modeled well by a single low-dimensional subspace. We construct a new decomposition corresponding to a more general underlying model consisting of a union of low-dimensional subspaces, and demonstrate the performance on a video background removal problem.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			
unclassified	**unclassified**	**unclassified**	**SAR**	**4**	

$\|U\|_{2,1} = \sum_i \sqrt{\sum_j u_{ij}^2}$, encourages rows of U to be zero, but does not discourage nonzero values among the entries of a nonzero row [8]. This takes advantage of the *group-sparsity* structure that can arise when points of the dataset are near to each other. It also plays the vital role of penalizing away degenerate solutions in which U is approximately the identity matrix, which could arise if there were only a 1-norm penalty on U. The $\|U\|_1$ term is included, however, since we usually (an exception would be when the data lie within a single low-rank subspace) also wish to encourage zero values within each nonzero row of U.

To better handle noisy data, we replace (3) with a penalized form, and add a Total Variation (TV) penalty on the sparse deviations (for cases when we expect these deviations to form contiguous regions), giving the problem

$$\min_{U,S} \frac{1}{2}\|AU + S - D\|_2^2 + \alpha\|U\|_1 + \beta\|U\|_{2,1} + \gamma\|S\|_1 + \delta\|\nabla S\|_1 \,, \quad (4)$$

where the dictionary A is derived from the data D (*e.g.* by mean-subtraction and scaling), and ∇S is a vector-valued discretization of the 3-D gradient of S, interpreted as a data cube.

Eq. (4) can be solved efficiently using the Split Bregman method [9]. We introduce variables P, Q, and R, which are auxiliary versions of U, S, and ∇S, respectively. We add terms relaxing the equality constraints of each quantity and its auxiliary variable, and in order to enforce equality at convergence, we introduce Bregman variables B_p, B_q, and B_r [9]:

$$\begin{aligned}\min_{U,S,P,Q,R} \frac{1}{2}\|AU + S - D\|_2^2 + \alpha\|P\|_1 + \beta\|P\|_{2,1} \\ + \gamma\|Q\|_1 + \delta\|R\|_1 + \frac{\lambda}{2}\|P - U - B_p\|_2^2 \\ + \frac{\mu}{2}\|Q - S - B_q\|_2^2 + \frac{\nu}{2}\|R - \nabla S - B_r\|_2^2 \,. \quad (5)\end{aligned}$$

This allows the problem to be split into an alternating minimization of the following subproblems:

$$\min_U \frac{1}{2}\|AU - (D - S)\|_2^2 + \frac{\lambda}{2}\|U - (P - B_p)\|_2^2 \,, \quad (6)$$

$$\min_S \frac{1}{2}\|S - (D - AU)\|_2^2 + \frac{\mu}{2}\|S - (Q - B_q)\|_2^2 + \frac{\nu}{2}\|\nabla S - (R - B_r)\|_2^2 \,, \quad (7)$$

$$\min_P \frac{\lambda}{2}\|P - (U + B_p)\|_2^2 + \alpha\|P\|_1 + \beta\|P\|_{2,1} \,, \quad (8)$$

$$\min_Q \frac{\mu}{2}\|Q - (S + B_q)\|_2^2 + \gamma\|Q\|_1 \text{ , and} \quad (9)$$

$$\min_R \frac{\nu}{2}\|R - (\nabla S + B_r)\|_2^2 + \delta\|R\|_1 \,. \quad (10)$$

Subproblems (6) and (7) are simple ℓ^2 problems, and can be solved by standard techniques for solving linear systems (*e.g.*, conjugate gradient). The other three subproblems can be solved very cheaply using *shrinkage*. Subproblems (9) and (10) use standard shrinkage, also known as soft thresholding:

$$\text{shrink}(T, \zeta) = \text{sign}(T)\max\{0, |T| - \zeta\} \,, \quad (11)$$

where the operations are to be understood entrywise. Subproblem (8), which contains both the 1- and 2, 1-norm, uses a generalized shrinkage, defined row-wise by

$$\text{shrink}_{2,1}(T, \zeta, \eta)^i = \frac{\text{shrink}(T^i, \zeta)}{1 + \eta/\,\text{shrink}(\|\,\text{shrink}(T^i, \zeta)\|_2, \eta)} \,, \quad (12)$$

with the convention that $1/(1 + \eta/0) = 0$. The algorithm consists of iteratively solving the main variables and updating the Bregman variables as follows:

$$\begin{aligned}U^{(k+1)} &= (A^TA + \lambda I)^{-1}\big(A^T(D - S^{(k)}) + \lambda(P^{(k)} - B_p^{(k)})\big) \,, \\ S^{(k+1)} &= \big((1+\mu)I + \nu\nabla^T\nabla\big)^{-1}\big((D - AU^{(k+1)}) \\ &\quad + \mu(Q^{(k)} - B_q^{(k)}) + \nu\nabla^T(R^{(k)} - B_r^{(k)})\big) \,, \\ P^{(k+1)} &= \text{shrink}_{2,1}(U^{(k+1)} + B_p^{(k)}, \alpha/\lambda, \beta/\lambda) \,, \\ Q^{(k+1)} &= \text{shrink}(S^{(k+1)} + B_q^{(k)}, \gamma/\mu) \,, \\ R^{(k+1)} &= \text{shrink}(\nabla S^{(k+1)} + B_r^{(k)}, \delta/\nu) \,, \\ B_p^{(k+1)} &= B_p^{(k)} + U^{(k+1)} - P^{(k+1)} \,, \\ B_q^{(k+1)} &= B_q^{(k)} + S^{(k+1)} - Q^{(k+1)} \text{ , and} \\ B_r^{(k+1)} &= B_r^{(k)} + \nabla S^{(k+1)} - R^{(k+1)} \,.\end{aligned}$$

We initialize all of these variables with zero vectors, but convergence is not expected to be dependent on this choice.

3. ADAPTIVE, OUTLIER-REMOVED DICTIONARY

In (4), an appropriate sparse U can be viewed as generating a locally low-dimensional approximation AU of $D - S$. When the dictionary is simply the data (*i.e.*, $A = D$), the sparse deviations (or outliers) S are also the deviations of the dictionary A, so constructing the locally low-dimensional approximation as $(A - S)U$, implying an adaptive dictionary $A - S$, should allow U to be even sparser. This gives the modified problem

$$\min_{U,S} \frac{1}{2}\|(A - S)U + S - D\|_2^2 + \alpha\|U\|_1 + \beta\|U\|_{2,1} + \gamma\|S\|_1 + \delta\|\nabla S\|_1 \,. \quad (13)$$

This problem can be minimized as before, the only changes being to the subproblems for U and S:

$$\begin{aligned}\min_U \frac{1}{2}\|(A - S)U - (D - S)\|_2^2 + \frac{\lambda}{2}\|U - (P - B_p)\|_2^2 \,, \\ \min_S \frac{1}{2}\|S(I - U) - (D - AU)\|_2^2 + \frac{\mu}{2}\|V - (Q - B_q)\|_2^2 \\ + \frac{\nu}{2}\|\nabla S - (R - B_r)\|_2^2 \,,\end{aligned}$$

with solutions given by the linear systems

$$\begin{aligned}&((A-S)^T(A-S)+\lambda I)U \\ &\qquad = (A-S)^T(D-S)+\lambda(P-B_p)\ , \\ &S(I-U)(I-U)^T+(\mu I+\nu\nabla^T\nabla)S \\ &= (D-AU)(I-U)^T+\mu(Q-B_q)+\nu\nabla^T(R-B_r)\ .\end{aligned}$$

This adaptive-dictionary approach is still possible when $A \neq D$, depending on how A is derived from D, but the resulting equations for U and S are a little more complicated.

4. RESULTS

We test our algorithm on the video background removal problem addressed by Wright *et al.* [3], using a 288-frame traffic video sequence from the Lankershim Boulevard Dataset [10, camera 4, 8:45–9:00 AM]. (This problem provides a convenient comparison between these two general data decomposition techniques, but while the performance of our method is subjectively quite good, we do not claim that it is competitive when compared with application-specific algorithms for this problem.) We use the modified-dictionary form (13) of the algorithm since it gives better results. A and D were both constructed from the data by subtracting the mean from each column and scaling so that the maximum value was 1.

The first test sequence is a reduced-resolution (240×320 pixel frames) version of the data, with each frame of the video being a column of D, giving a 76800×288 matrix. Because the traffic camera is stationary, this dataset is well-modeled by a single low-dimensional subspace. Our algorithm gives a decomposition (see Fig. 1) that is visually almost indistinguishable from the result (omitted here due to space constraints) obtained by solving (2), using the algorithm of [11].

Our second test sequence simulates a panning camera by taking a moving 240×320 pixel cropping window within the original sequence. This window moves slowly to the left, and then back to the original position, at a rate of 1/4 pixel/frame. In this case the background is poorly approximated by any single low-dimensional subspace, but since the background motion is slow with respect to the foreground motion, a locally low-dimensional model provides a much better approximation. A comparison of a single frame of the sparse components of different methods applied to this data is provided in Fig. 2. The "local" sparse components computed using our algorithm clearly have far less residual background than the "global" sparse component resulting from (2).

5. CONCLUSION

We have proposed a new decomposition, together with a Split Bregman type algorithm, for high-dimensional data, generalizing the Robust PCA of Candès *et al.* [2] to certain nonlinear data. The ability of this generalization to model data that does not conform to the globally low-dimensional restriction has been demonstrated on the video background removal problem. Future work will include development of automatic parameter selection methods, and application of the decomposition to additional problems in which the relaxed constraints on the data can be expected to provide an advantage.

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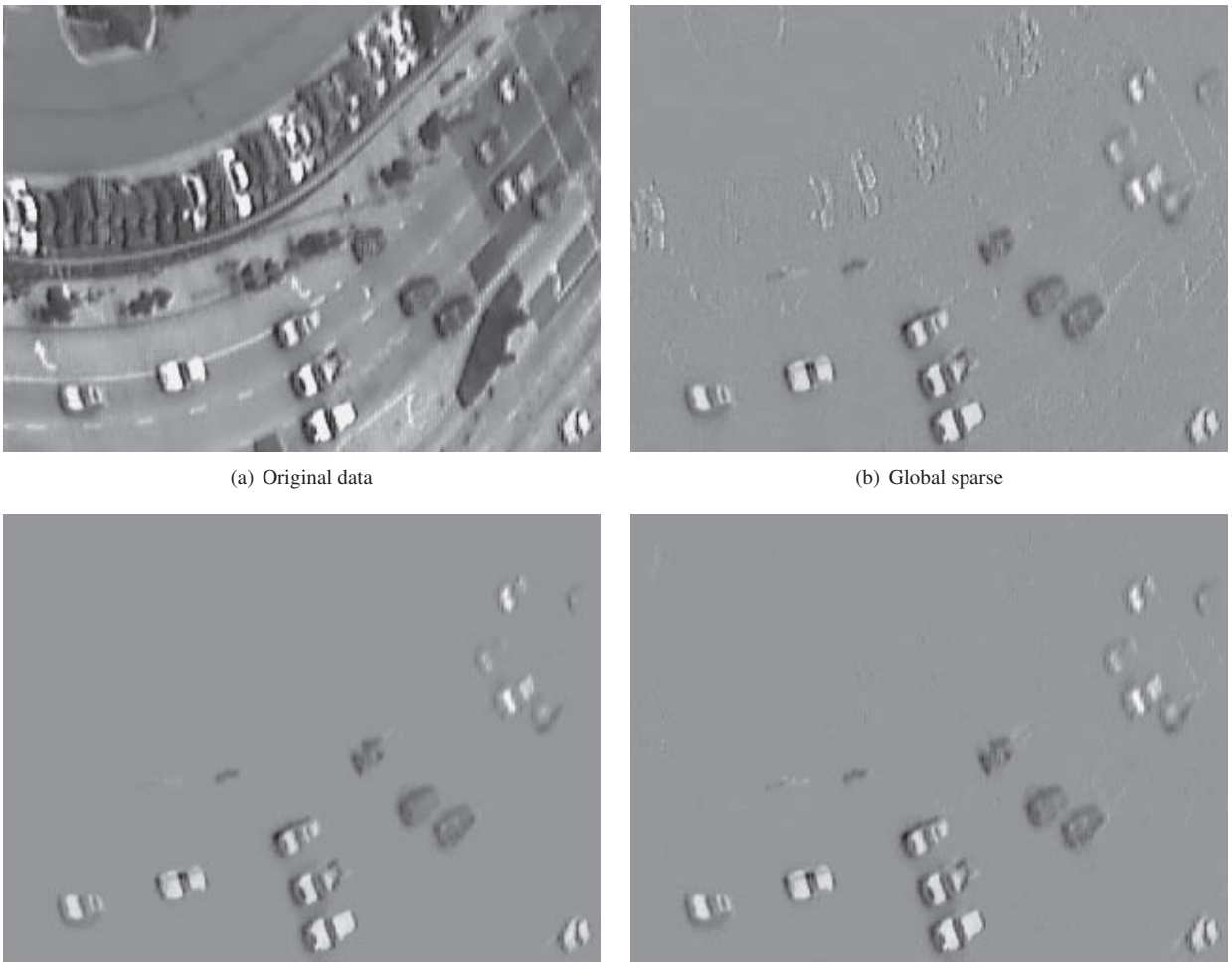
(a) Original data (b) Local low rank (c) Local sparse

Fig. 1. Results for frame 166 from the stationary test video sequence. The decomposition was computed using our algorithm with parameters $\alpha = 1.0 \times 10^{-5}, \beta = 1.0 \times 10^{-2}, \gamma = 3.0 \times 10^{-5}$, and $\delta = 1.0 \times 10^{-4}$.



(a) Original data (b) Global sparse

(c) Local sparse (d) No-TV local sparse

Fig. 2. Results for frame 166 from the slowly-panning test video sequence. The global sparse component (b) is obtained using decomposition (2) (as in [3]), and the local sparse component (c) is generated by our algorithm with parameters $\alpha = 4.0 \times 10^{-3}, \beta = 8.0 \times 10^{-2}, \gamma = 5.0 \times 10^{-4}$, and $\delta = 3.0 \times 10^{-4}$. Component (d) is generated in the same was as (c), except that $\delta = 0$, so that there is no TV regularization. This example demonstrates that the performance advantage of our algorithm is primarily due to the local-linear model, and not the TV regularization.