AFRL-RY-WP-TR-2012-0227

THE PRANDTL PLUS SCALING PARAMETERS AND VELOCITY PROFILE SIMILARITY

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OCTOBER 2012
Interim Report

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AIR FORCE RESEARCH LABORATORY
SENSORS DIRECTORATE
WRIGHT-PATTERSON AIR FORCE BASE, OH 45433-7320
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REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YY)* October 2012

2. REPORT TYPE Final

3. DATES COVERED *(From - To)* 01 October 2011 – 01 May 2012

4. TITLE AND SUBTITLE
THE PRANDTL PLUS SCALING PARAMETERS AND VELOCITY PROFILE SIMILARITY

5a. CONTRACT NUMBER FA8650-00-M-0000

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER 62204F

6. AUTHOR(S)
David Weyburne

5d. PROJECT NUMBER 4916

5e. TASK NUMBER HC

5f. WORK UNIT NUMBER 4916HC10/Y0AX

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Optoelectronic Technology Branch
Electromagnetics Technology Division
Air Force Research Laboratory, Sensors Directorate
Wright-Patterson Air Force Base, OH 45433-7320
Air Force Materiel Command, United States Air Force

8. PERFORMING ORGANIZATION REPORT NUMBER AFRL-RY-WP-TR-2012-0227

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Air Force Research Laboratory
Sensors Directorate
Wright-Patterson Air Force Base, OH 45433-7320
Air Force Materiel Command
United States Air Force

10. SPONSORING/MONITORING AGENCY ACRONYM(S) AFRL/RYHC

11. SPONSORING/MONITORING AGENCY REPORT NUMBER(S) AFRL-RY-WP-TR-2012-0227

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

13. SUPPLEMENTARY NOTES
This is a work of the U.S. Government and is not subject to copyright protection in the United States. PAO Case Number 88ABW-2012-4071, Clearance Date 19 July 2012. Report contains color.

14. ABSTRACT
Numerous experimental results have demonstrated similar behavior of the velocity profile in the near-wall region of the turbulent boundary layer using Prandtl's "Plus" scaling variables. However, the implications for similarity behavior of the near-wall turbulent boundary velocity profiles using Prandtl's Plus scaling variables have not been carefully explored. In the following report, we apply the momentum balance type approach to study velocity profile similarity using Prandtl's Plus scaling variables. It is shown that the Plus scaling variables do in fact satisfy the requirements for similarity based on this approach so long as the friction velocity is proportional to 1/(x-x0) where x is the distance along the wall in the flow direction and x0 is a constant. Experimental results are examined and found to confirm that certain datasets we tested do in fact have the friction velocity values behaving as 1/(x-x0). We show the Plus scaling variables satisfy all of the conditions for similarity using the momentum balance type approach for these datasets. However, the same velocity profile dataset plots also confirm that Prandtl's Plus scaling variables do not show whole profile similarity. Hence, we conclude that the scaling variables discovered by the momentum balance type approach as presently constituted are a necessary but not sufficient condition for velocity profile similarity..

15. SUBJECT TERMS
Fluid Boundary Layer, Prandtl Plus Scaling, Boundary Layer Flow, Velocity Profile Similarity, Momentum Balance Equation

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT:	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON (Monitor)
Unclassified	Unclassified	Unclassified	SAR	28	Jonahira Arnold
					19b. TELEPHONE NUMBER *(Include Area Code)* N/A

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ACKNOWLEDGEMENT

The author would like to thank the following groups and individuals for making their experimental datasets available for inclusion in this manuscript: Francis Clauser; H. Herring, and J. Norbury; Per Egil Skåre and Per-Åge Krogstad; Peter Bradshaw and D. Ferriss; Malcolm Jones, Ivan Marusic, and Anthony Perry; Randall Smith and Alexander Smits; and Karl Wieghardt and W. Tillmann.

SUMMARY

Numerous experimental results have demonstrated similar behavior of the velocity profile in the near-wall region of the turbulent boundary layer using Prandtl's "Plus" scaling variables. However, the implications for similarity behavior of the near-wall turbulent boundary velocity profiles using Prandtl's Plus scaling variables have not been carefully explored. In the following report, we apply the momentum balance type approach to study velocity profile similarity using Prandtl's Plus scaling variables for the wall-bounded turbulent boundary layer. It is shown that the Plus scaling variables will in fact satisfy the requirements for similarity based on this approach so long as the friction velocity values in the flow direction are proportional to $1/(x-x_0)$ where x is the distance along the wall in the flow direction and x_0 is a constant. Experimental results are examined and found to confirm that certain datasets we tested do in fact have the friction velocity values behaving as $1/(x-x_0)$. For these datasets, we show the Plus scaling variables satisfy all of the conditions for similarity using the momentum balance type approach. However, the same velocity profile dataset plots also confirm that Prandtl's Plus scaling variables do not show whole profile similarity in general. Hence, we conclude that the scaling variables discovered by the momentum balance type approach as presently constituted are a necessary but not sufficient condition for velocity profile similarity.

1. INTRODUCTION

Beginning with the pioneering work of Reynolds [1], there has been a concerted effort to find coordinate scaling parameters that make the scaled velocity profiles and shear-stress profiles taken at different stations along the wall in the flow direction to appear to be similar. For turbulent boundary layers, the early search for similarity was mostly unsuccessful. This led to the practice of trying to find similarity in subregions of the whole profile. Perhaps the most successful case of partial similarity was developed by von Kármán [2] and Prandtl [3] and is known as the Logarithmic Law or "Log Law." The Log Law states that in a region adjacent to the wall of the turbulent boundary layer the average velocity in the streamwise *x*-direction of a wall-bounded turbulent flow is given by

$$\frac{u(x,y)}{u_\tau} \cong \frac{1}{\kappa}\ln\left(\frac{yu_\tau}{\nu}\right)+B \quad , \tag{1}$$

where y is the height perpendicular to the solid surface , ν is kinematic viscosity, κ and B are constants, and u_τ is the Prandtl velocity scaling parameter, the so-called friction velocity.

Experimental plots of the wall-bounded turbulent boundary layer velocity profiles taken at various stations along the wall using u_τ as the velocity scaling parameter and ν/u_τ as the length scaling parameter (the so-called "Plus" scaling factors) have consistently demonstrated similarity-like behavior in the near-wall region of the profile. There has been much debate as to whether this is due to the fact that the Log Law or, as some believe, a Power Law that is universally applicable for all wall-bounded turbulent flows. In any case, one area that has not been explored is the similarity implications using the friction velocity as the key scaling variable. For example, the question of whether the Prandtl's Plus scaling variables even satisfy the conditions for similarity has not been studied.

The standard approach for studying similarity is to use the dimensionless momentum balance equation. The search for similarity scaling behavior for the turbulent boundary layer using this approach began with the experimental and theoretical work of Clauser [4]. Using the friction velocity u_τ as the velocity scaling variable and the displacement thickness δ_1 as the length scaling variable, Clauser predicted that equilibrium (similar) boundary layers are only obtained for the nonzero pressure gradient case when the quantity

$$\beta_\tau = -\frac{\delta_1}{\rho u_\tau^2}\frac{dp_e}{dx} \tag{2}$$

is a constant. In this equation ρ is the density and p_e is the pressure at the boundary layer edge. Rotta [5] and Townsend [6] subsequently developed some additional theoretical considerations for turbulent boundary layer similarity using this approach.

More recently, Castillo and George [7] used this approach and found that similarity will exist only when the parameter

$$\Lambda \;=\; -\frac{\delta\, du_s/dx}{u_s\, d\delta/dx} \tag{3}$$

is a constant. In this equation δ is an as yet unspecified thickness scaling variable and $u_s(x)$ is an as yet unspecified scaling velocity. Following a similar scheme of keeping the thickness scaling variable and the velocity scaling variables as unspecified, Weyburne [8] added the dimensionless Reynolds stress transport equation to obtain additional restrictions with the result that it was shown that in the outer region of the turbulent boundary layer δ must be a linear function of the type $\delta = a_1(x - x_0)$ and u_s must be $u_s \propto u_e$ and a power law function of the type $u_e = a_2(x - x_0)^m$, such that a_1, a_2, m, and x_0 are constants.

What is relevant here is that in none of these previous momentum balance type approaches have explicitly tested using u_τ as the velocity scaling parameter and ν / u_τ as the length scaling parameter to find out whether they satisfy the conditions for similarity of the velocity profiles of a turbulent boundary layer. In what follows, we take on this task. Using the momentum balance equation type analysis, we first confirm theoretically that the unknown thickness scaling variable δ and the unknown scaling velocity $u_s(x)$ must behave in a manner consistent with the sink flow/wedge flow solution. It is shown that the Plus scaling variables will in fact satisfy this requirement for similarity based on the momentum balance equation approach so long as the friction velocity values in the flow direction are proportional to $1/(x - x_0)$ where x is the distance along the wall in the flow direction and x_0 is a constant. Experimental results are examined and found to confirm that certain datasets we tested do in fact satisfy all the conditions of whole profile similarity based on the momentum balance type approach requirements. However, the same plots also confirm that Prandtl's Plus scaling variables based plots do not show velocity profile similarity over the whole boundary layer region. Hence, we conclude that the scaling variables discovered by the momentum balance type approach are a necessary but not sufficient condition for velocity profile similarity.

2. SIMILARITY EQUATIONS

The theoretical guidance for discovering similarity scaling laws for the turbulent boundary layer started with Clauser [4] who looked at the implications of similarity using the *x*-momentum balance equation. For a 2-D incompressible turbulent boundary layer that is steady state on the mean, the Reynolds-averaged stream-direction component (*x*-direction) of the momentum balance for flow along a plate is given by

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial}{\partial x}\left\{\overline{\tilde{u}\tilde{u}}\right\}+\frac{\partial}{\partial y}\left\{\overline{\tilde{u}\tilde{v}}\right\} \;=\; -\frac{1}{\rho}\frac{\partial P}{\partial x}+v\frac{\partial^2 u}{\partial y^2} \quad , \tag{4}$$

where the bar above a variable is the Reynolds average operator and the tilde operator designates the instantaneous velocity. What Clauser [4] did was to cast Eq. 4 into reduced units in a manner similar to what Blasius [9] and Falkner and Skan [10] did for laminar flow over a flat plate. What one finds in doing this reduction is that each term in Eq. 4 can be reduced to a product of a *x*-functional and a η-functional where η is the reduced *y*-variable given by $\eta \equiv y/\delta$. For similarity we must have that the η-functional terms do not change as one moves along the plate and that the various *x*-functional terms from Eq. 4 change in a proportional manner as one moves along the plate. Equivalently, we can divide through by one of the *x*-grouping terms and similarity would then require that the *x*-grouping ratios be constants. Clauser [4] assumed that for the turbulent boundary layer flow over a plate that the length scaled as δ_1 and the velocity scaled as u_τ which then led to Eq. 2 as one of the *x*-grouping ratio terms.

A generalized process of transforming *x*-momentum balance equation starts by using δ and u_s as the unknown similar scaling variables for the length and velocity. These unknowns must be functions of *x* but not *y*. Next we assume that the stress terms can be separated into the product of an *x*-dependent functional and an η-dependent functional as

$$\overline{\tilde{u}\tilde{v}} \;=\; uv(x)\,g_{12}(\eta) \quad \text{and} \quad \overline{\tilde{u}^2} \;=\; uu(x)\,g_{11}(\eta) \quad . \tag{5}$$

The task of casting *x*-momentum balance and the Reynolds stress transport equations into reduced units using these length, velocity, and stress scaling variables is detailed in Appendix 1.

The results of the transformation are that the *x*-momentum balance equation (Eq. 4) reduces to

$$-(1/\alpha)f''' - ff'' - \frac{\Lambda}{1-\Lambda}f'^2 - \tau_1\eta g_{11}' + \tau_2 g_{11} + \tau_3 g_{12}' \;=\; \phi \quad , \tag{6}$$

where *f* is a dimensionless function satisfying the Stream function (Eq. 1.2), where the prime represents differentiation with respect to η, and where

$$\alpha \;=\; \frac{\delta^2}{\nu}\frac{du_s}{dx}+\frac{\delta u_s}{\nu}\frac{d\delta}{dx} \quad , \quad \Lambda \;=\; -\frac{\delta}{u_s}\frac{du_s/dx}{d\delta/dx} \quad , \quad \phi \;=\; -\frac{u_e\,du_e/dx}{u_s\,du_s/dx}\frac{\Lambda}{1-\Lambda} \quad , \tag{7}$$

$$\tau_1 = \frac{uu(x)}{u_s^2(1-\Lambda)} \quad , \quad \tau_2 = \frac{duu(x)/dx}{u_s\,du_s/dx+\left(u_s^2/\delta\right)d\delta/dx} \quad ,\text{and} \quad \tau_3 = \frac{uv(x)}{\delta u_s\,du_s/dx+u_s^2\,d\delta/dx} \quad .$$

Similarity of the velocity profiles means that the parameters appearing in Eq. 6, such as $\alpha, \Lambda, \tau_1, \tau_2,$ *etc*. need to be constant as one moves along the wall in the flow direction.

We note that the *x*-direction momentum balance equation is but one of a handful of equations that govern the flow behavior. In an earlier report, Weyburne [6] pointed out that another important flow governing equation appropriate to wall-bounded turbulent flow similarity was the Reynolds stress transport equation given by

$$u\frac{\partial\overline{\tilde{u}\tilde{v}}}{\partial x}+v\frac{\partial\overline{\tilde{u}\tilde{v}}}{\partial y}-2\overline{\tilde{u}\tilde{v}}\frac{\partial u}{\partial y}-\nu\frac{\partial^2\overline{\tilde{u}\tilde{v}}}{\partial y^2}+\frac{\partial}{\partial y}\left(\overline{\tilde{u}\tilde{v}\tilde{v}+\frac{\tilde{P}\tilde{u}}{\rho}}\right)+2\nu\overline{\frac{\partial\tilde{u}}{\partial y}\frac{\partial\tilde{v}}{\partial x}}-\overline{\frac{\tilde{P}}{\rho}\left(\frac{\partial\tilde{u}}{\partial y}+\frac{\partial\tilde{v}}{\partial x}\right)}=0 \quad . \tag{8}$$

It is generally known that turbulent velocity profile similarity will only occur if the Reynolds shear stress profiles also show similarity. Hence it makes sense to consider the similarity implications of Eq. 8 in addition to the *x*-direction momentum balance equation**.** If one applies the same reduction technique used above on the momentum balance equation, one finds that the Reynolds shear stress equation becomes

$$-\kappa f g_{12}'-\varepsilon\eta f' g_{12}'+\tau_4 f' g_{12}-\varepsilon f g_{12}'+2g_{12} f''+\chi g_{12}''+\{3\,additional\ terms\}=0 \quad , \tag{9}$$

where the three additional terms are not written out expressly because they cannot be written in terms of $\overline{\tilde{u}\tilde{v}}$ (see Appendix 1) and where

$$\chi = \frac{\nu}{\delta u_s} \quad , \quad \varepsilon = \frac{d\delta}{dx} \quad , \text{ and } \quad \kappa = \frac{\delta}{u_s}\frac{du_s}{dx} \quad . \tag{10}$$

2.1 Near-Wall Flow Similarity

The requirements for similarity using the momentum balance-type approach consists of ensuring the parameters identified in Eqs. 7 and 10 are constant as one moves along the wall in the flow direction. By inspection one solution is to have

$$\delta = a_1(x-x_0) \quad , \quad u_s(x) = \frac{a_2}{x-x_0} \quad , \tag{11}$$

where a_1, a_2, and x_0 are constants. This is the so-called sink flow (or wedge flow) solution.

For flows with a pressure gradient in the flow direction, the momentum balance requires that ϕ be a constant. Taking Λ as a constant, the general solution to have ϕ (Eq. 7) equal to a nonzero constant is given by $u_s=\sqrt{a+b\,u_e^2}$ where *a* and *b* are constants. With only a slight loss of generality (taking *a*=0), we see that for similarity, this reduces to $u_s \propto u_e$ as first pointed out in this context by Castillo and George [4].

2.2 Outer Region Flow Similarity

In the outer region of a turbulent boundary layer on a wall we will assume that the viscous forces are negligible. This means that the viscous terms in Eqs. 4 and 8 are negligible (eliminating α and χ). Therefore, the functional form that u_s and δ may take is now governed by the fact that Λ, ϕ, ε, and κ from Eqs. 7 and 10 must be constants. This means that the functional form that δ may take is a linear function of the type $\delta = a_1(x-x_0)$ such that a_1 and x_0 are constants. This reduces κ to

(12)

$$\frac{a_1(x-x_0)}{u_s}\frac{du_s}{dx} = \kappa \quad ,$$

which has the general solution of

$$u_s(x) = a_2(x-x_0)^{\kappa} \tag{13}$$

where a_2 is a constant.

For flows with a pressure gradient in the flow direction, the momentum balance requires that ϕ be a constant. The general solution to have ϕ (Eq. 7) equal to a nonzero constant is given by $u_s = \sqrt{a + b\,u_e^2}$ where *a* and *b* are constants. With only a slight loss of generality (taking *a*=0), we see that for similarity, this reduces to $u_s \propto u_e$ as first pointed out in this context by Castillo and George [4].

3. PRANDTL PLUS SCALING SIMILARITY

The exact requirements for momentum-balance type whole-profile similarity can now be applied to the Prandtl Plus friction velocity-based scaling parameters. Consider first the near-wall region. Substituting the length scale $\delta \propto 1/u_\tau$ and the velocity scale $u_s \propto u_\tau$ into Eqs. 7 and 10, then it is possible to show that similarity in the near-wall region requires that the friction velocity must be a function of the type

$$u_\tau(x) = \frac{a}{x-x_0} \tag{14}$$

where *a* and x_0 are constants. That is, if the friction velocity behaves as Eq. 14, then the Prandtl Plus scaling parameters insure all of the parameters in Eqs. 7 and 10 are equal to a constant value. Next, consider the Outer region. In the Outer region, the parameters given in Eqs. 7 and 10 (neglecting α and χ, and setting $\kappa = -1$) will also be constant value so long as the friction velocity follows Eq. 14. Hence, according to the momentum balance type approach, all wall-bounded turbulent similarity requirements are satisfied using the Prandtl Plus scaling variables so long as the friction velocity behaves as Eq. 14 and ϕ is a constant.

Having ϕ constant will have an additional constraint on the similarity requirements for the Prandtl Plus scaling parameters. We note this parameter subsumes the Euler equation given as

$$-\frac{1}{\rho}\frac{\partial p_e}{\partial x} = u_e\frac{du_e}{dx} \tag{15}$$

which therefore only applies to flows with a pressure gradient. For these cases, as noted above, the momentum balance type similarity requires $u_s \propto u_e$. Since we are testing whether u_τ is also a similarity parameter, then we must have

$$\frac{u_e}{u_\tau} = \text{constant}, \tag{16}$$

the so-called Rotta constraint.

4. EXPERIMENTAL VERIFICATION

We are now in a position to answer the question as to whether the Prandtl Plus friction velocity-based scaling parameters satisfy the momentum-balance type whole-profile similarity requirements. A number of datasets from the literature were selected based on satisfying Eq. 16. In Figs. 1a-7a we plot some sets of velocity profiles in which the plot axes are put in reduced units using the Prandtl Plus friction velocity-based scaling parameters. In these figures the wall is located at $y^+ = 0$. In all cases there is a region near the wall for which all of the profiles overlap, *i.e*. they display near-wall similarity behavior. The question we want to answer is whether these same datasets also satisfy the condition for whole-profile similarity, that is does the friction velocity values fit to Eq. 14 and is Eq. 16 satisfied. First, consider the friction velocity fits. In Figs. 1b-7b we plot the experimental friction velocity as points (+) versus the station location for each corresponding dataset. The red line in each plot is the least squares fit to the points using Eq. 14. In general, the fits are good indicating that Prandtl Plus scaling factors do in fact satisfy the condition for constant parameter ratios (Eqs. 7 and 10).

For the flows considered in Figs. 1-5, which do have a pressure gradient present in the flow direction, we must also have Eq. 16 hold if u_τ is to be a similarity parameter. This condition is checked in Figs. 1c-5c which plots this velocity ratio along with average velocity ratio value as the red line. As is evident from the plots, this constraint does in fact hold for these datasets (recall that in fact these datasets were selected based on this criterion). It is necessary to note that datasets presented in Figs. 6 and 7 are for flat plate flows for which dp/dx=0, and therefore the ϕ parameter constraint is not applicable.

The experimental results presented above indicate that all of these datasets satisfy the conditions for similarity using the momentum balance type approach when using u_τ as the velocity scaling parameter and $1/u_\tau$ as the length scaling parameter. However, it also apparent from Figs. 1a-7a that these datasets do not in general display velocity profile similarity over the whole profile.

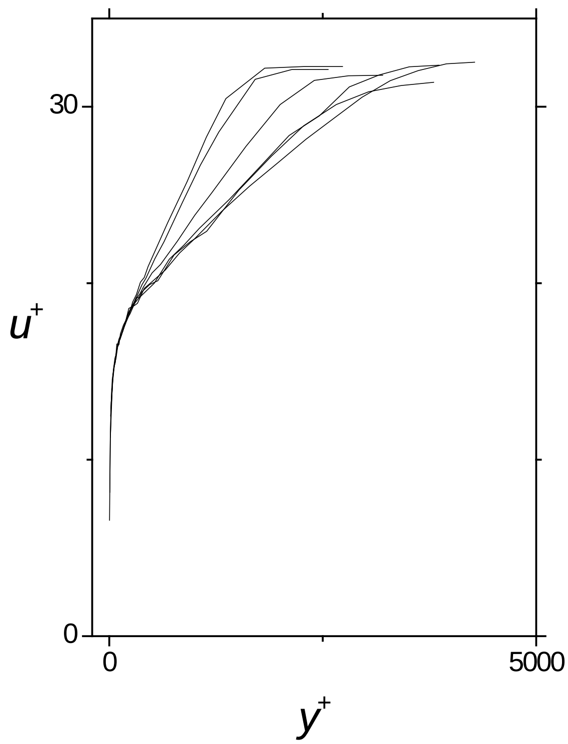

Figure 1a: The solid lines are eight Clauser [4] profiles plotted in Plus units.

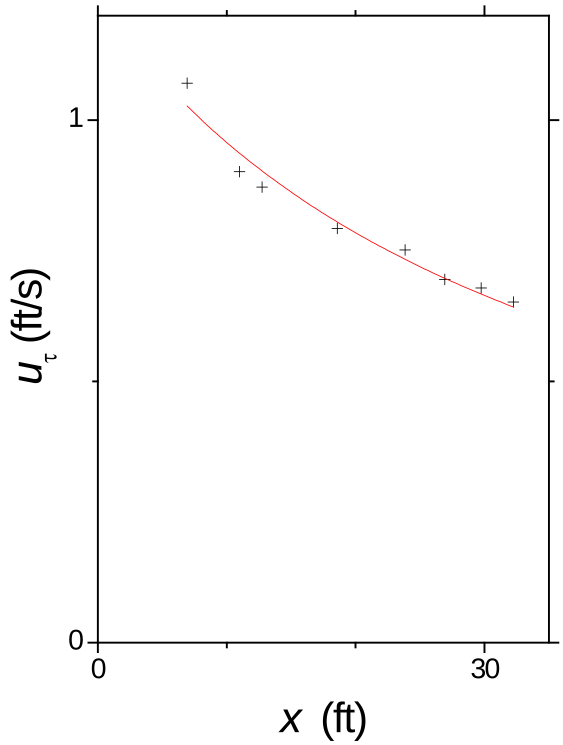

Figure 1b: Clauser [4] friction velocity (+) along with the $a/(x-x_0)$ fitted line.

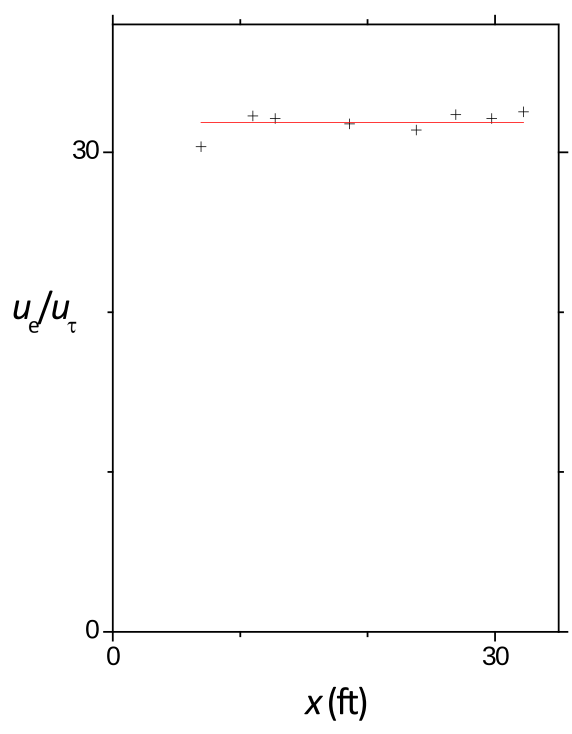

Figure 1c: Clauser [4] velocity ratio (+) along with the average value as the red line.

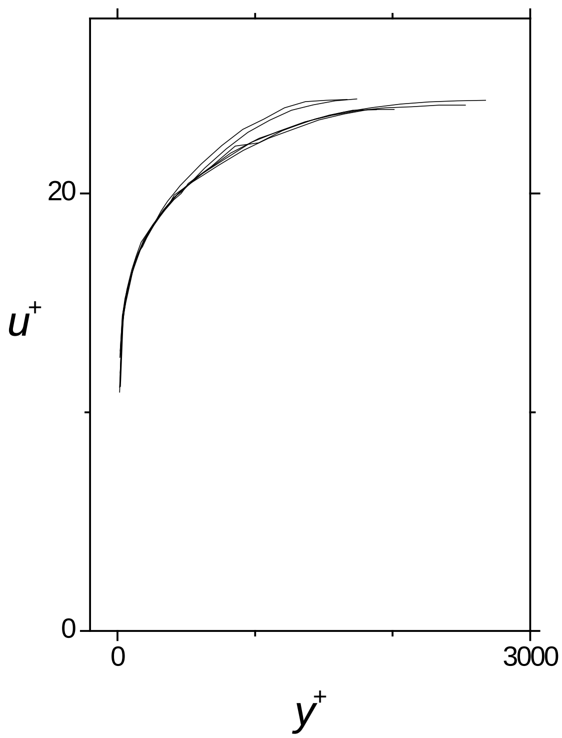

Figure 2a: The solid lines are six Herring and Norbury [11] profiles plotted in Plus units.

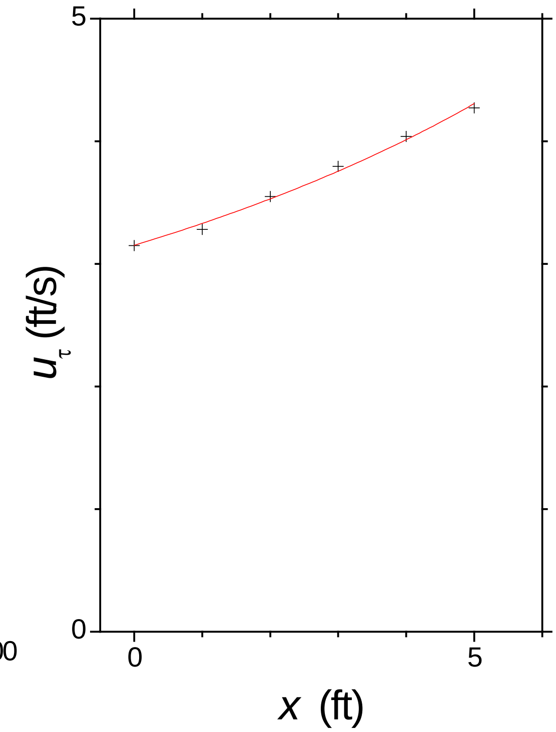

Figure 2b: Herring and Norbury [11] friction velocity (+) and the $a/(x-x_0)$ fitted line.

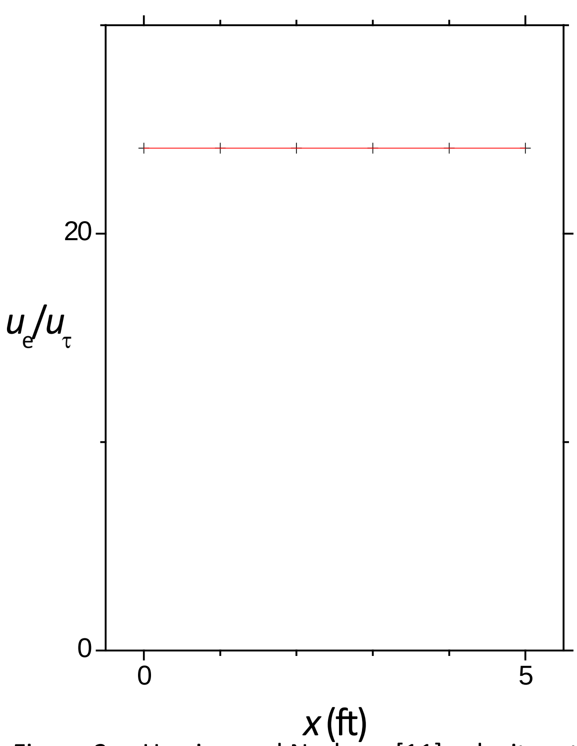

Figure 2c: Herring and Norbury [11] velocity ratio (+) along with the average value as the red line.

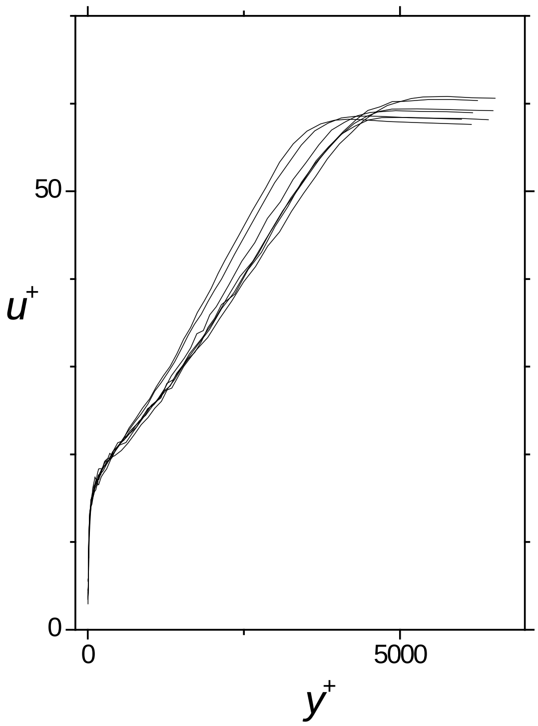

Figure 3a: The solid lines are seven Skåre and Krogstad [12] profiles plotted in Plus units.

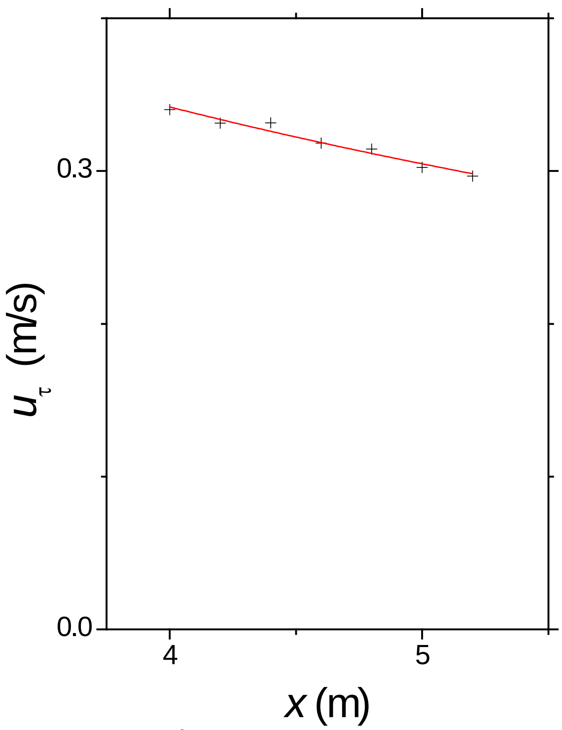

Figure 3b: Skåre and Krogstad [12] friction velocity (+) along with the $a/(x-x_0)$ fitted line.

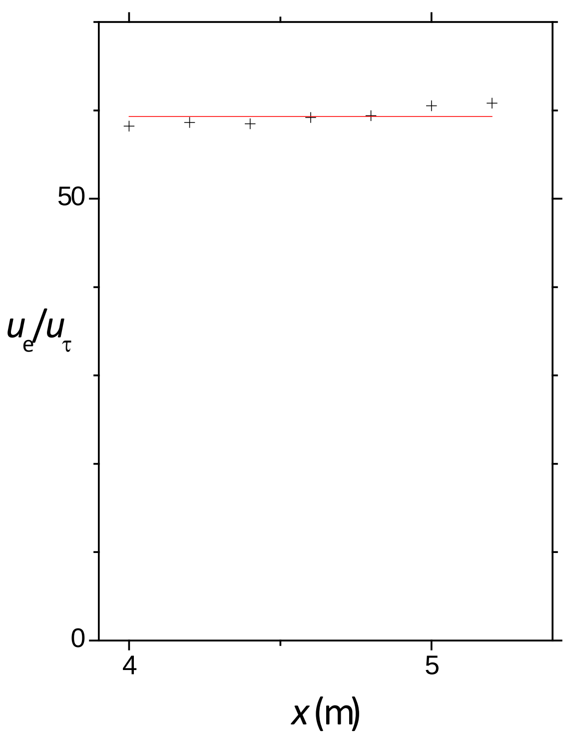

Figure 3c: Skåre and Krogstad [12] velocity ratio (+) along with the average value as the red line.

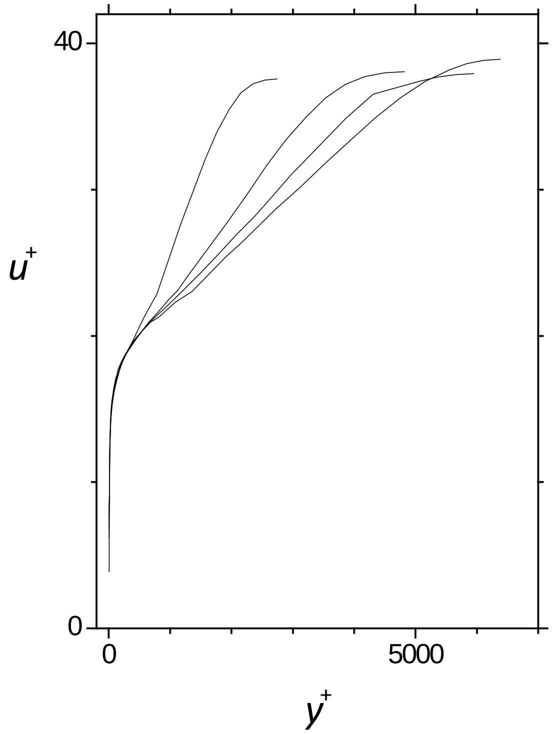

Figure 4a: The solid lines are the Bradshaw and Ferriss [13] profiles plotted in Plus units.

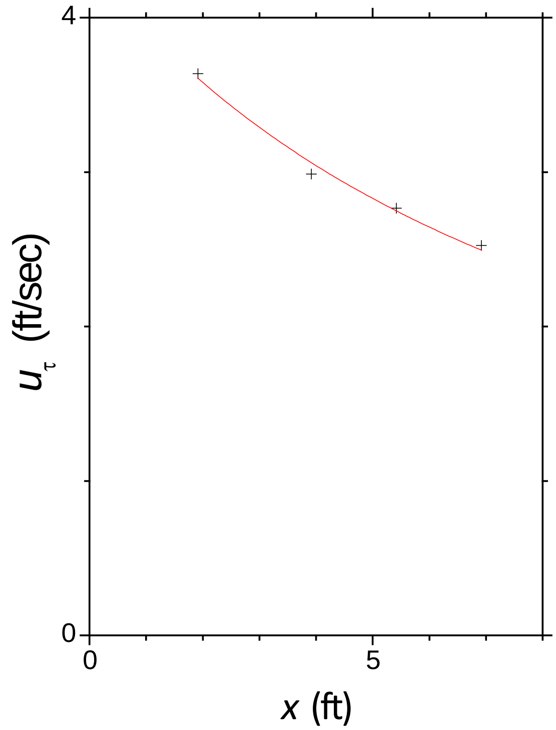

Figure 4b: Bradshaw and Ferriss [13] friction velocity (+) along with the $a/(x-x_0)$ fitted line.

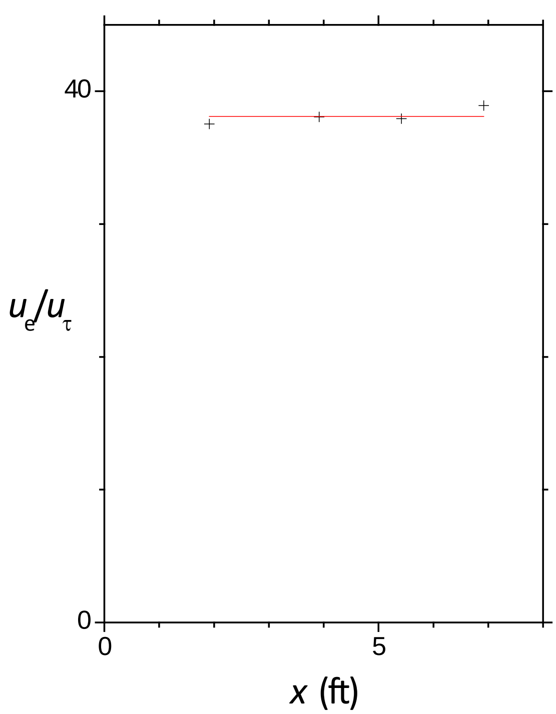

Figure 4c: Bradshaw and Ferriss [13] velocity ratio (+) along with the average value as the red line.

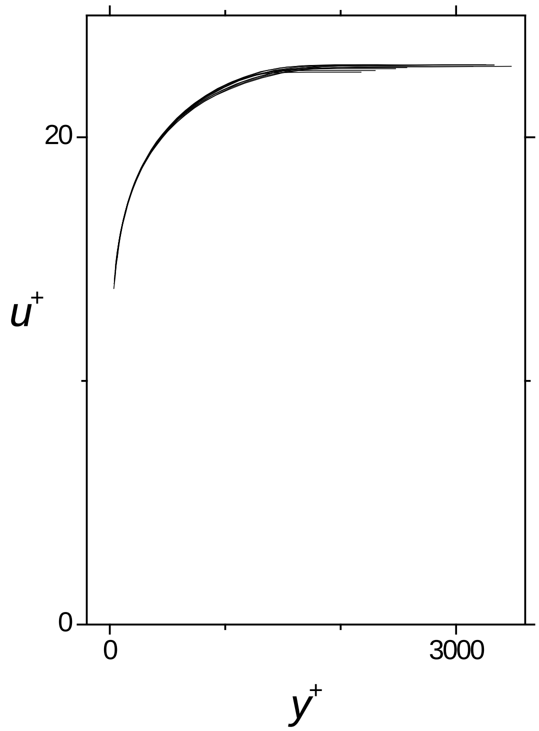

Figure 5a: The solid lines are twelve Jones, Marusic, and Perry [14] profiles plotted in Plus units.

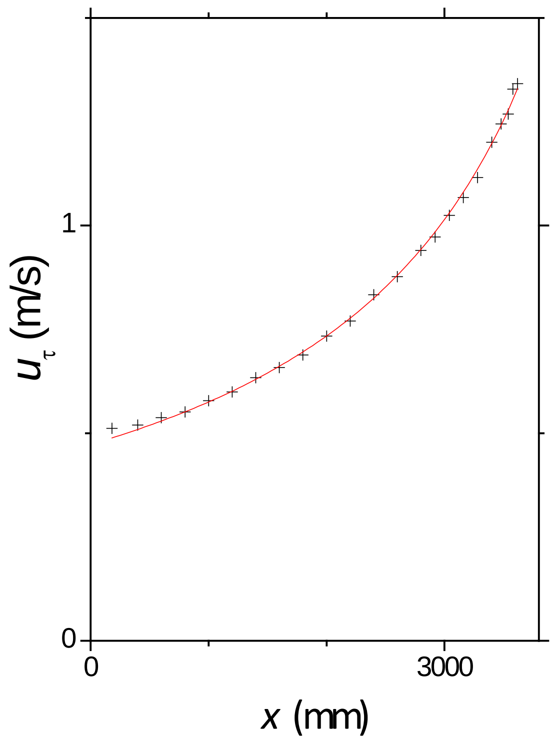

Figure 5b: Jones, Marusic, and Perry [14] friction velocity (+) along with the $a/(x-x_0)$ fitted line.

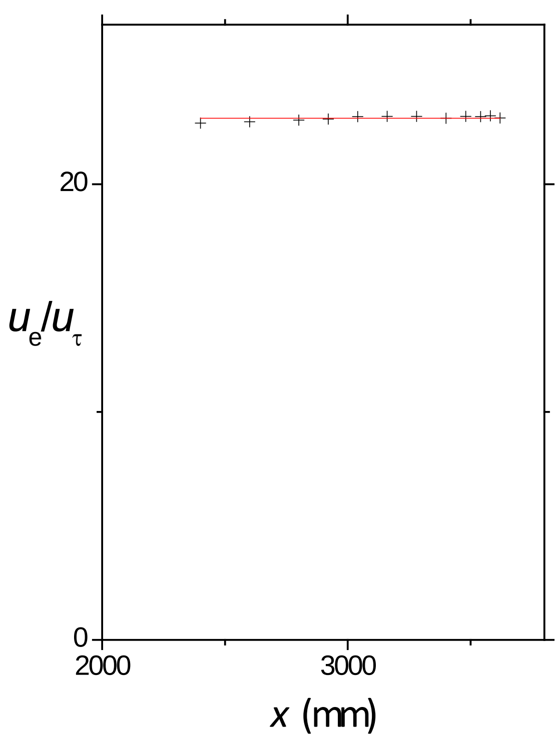

Figure 5c: Jones, Marusic, and Perry [14] velocity ratio (+) along with the average value as the red line.

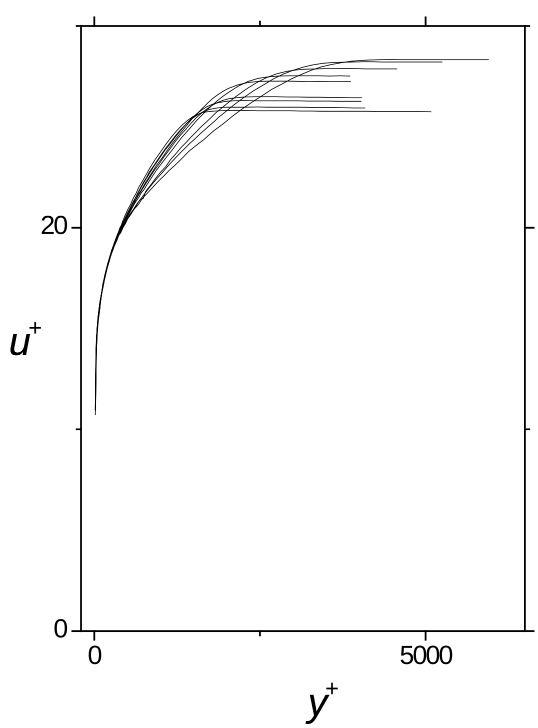

Figure 6a: The solid lines are the Smith and Smits [14] profiles plotted in Plus units.

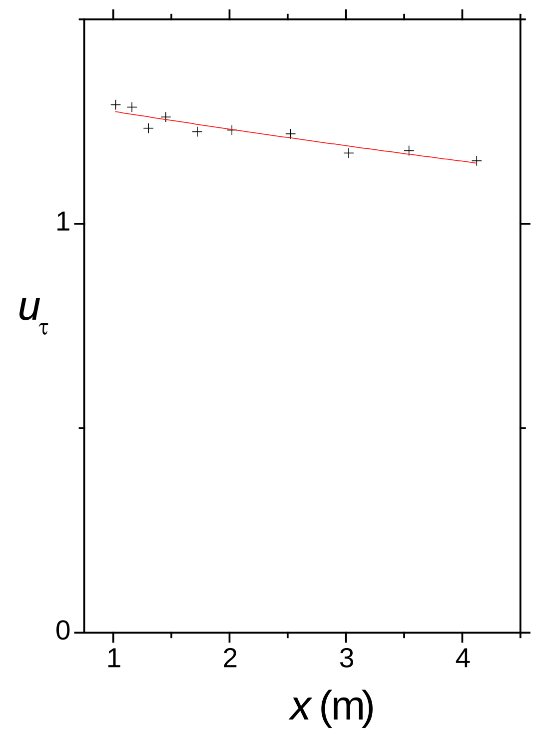

Figure 6b: Smith and Smit [14] friction velocity (+) along with the $a/(x-x_0)$ fitted line.

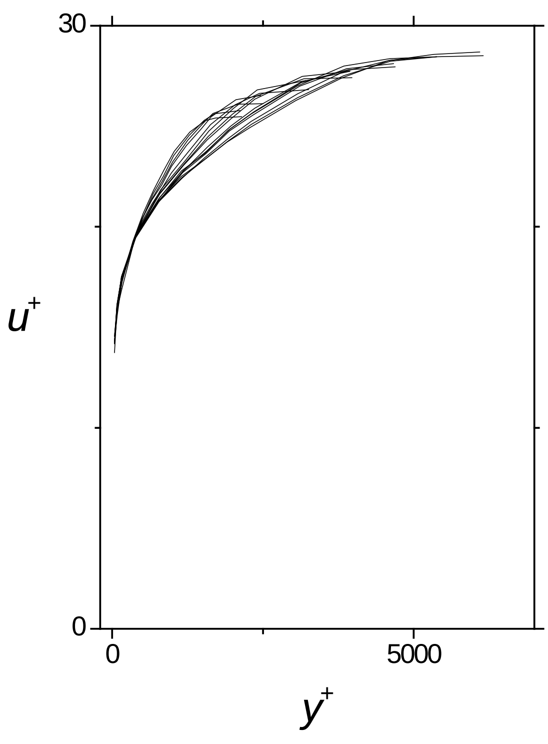

Figure 7a: The solid lines are the Wieghardt and Tillmann [15] profiles plotted in Plus units.

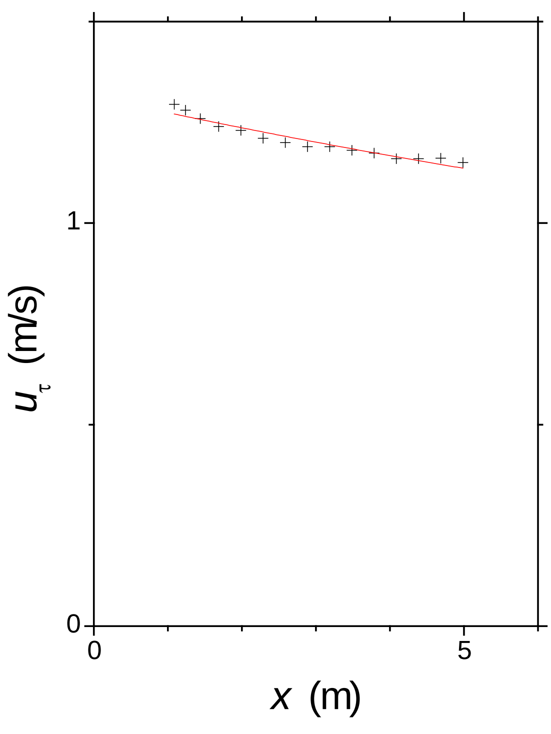

Figure 7b: Wieghardt and Tillmann [15] friction velocity (+) along with the $a/(x-x_0)$ fitted line.

5. DISCUSSION

The implications of the above results should be viewed in the context of two observations which stand out: 1) in spite of satisfying all of the requirements, the set of profiles given in Figs. 1a-7a clearly demonstrate that the Prandtl Plus scaling parameters do NOT, in general, result in similarity flow behavior (except at the near-wall), and 2) while the length scale $\delta \propto 1/u_\tau$ and the velocity scale $u_s \propto u_\tau$ have the correct *x*-behavior (Eq. 14), it is necessary to point out that only two of the geometries considered in Figs. 1-7 conform to the traditional sink flow/wedge flow geometry.

Consider the first observation, that is, the Prandtl Plus scaling parameters do NOT in general result in similarity flow behavior in spite of satisfying all of the conditions for similarity using the momentum balance type approach. First of all we must point out that the datasets chosen above were not picked at random. They were picked because the experiments in question were intentionally (or unintentionally) set up specifically to obtain similar profiles. In fact only a handful of wall-bounded turbulent boundary layer datasets fall into this category. In any case, given the examples above which do satisfy all of the constraints but do not show whole-profile similarity, it must be concluded that the scaling variables discovered by the momentum balance type approach are a necessary but not sufficient condition for velocity profile similarity.

The implications of this are rather perplexing. How is it possible to satisfy the conditions for similarity using the momentum balance type approach and yet not have similar profiles? One possibility is that there is some other factor that we are not capturing with the momentum balance type approach. Consider the results of Herring and Norbury [11] and Jones, Marusic, and Perry [14] shown in Figs. 2 and 5. It is apparent from Fig. 2a and Fig. 5a that there are stations along the flow direction that do in fact show similarity-like behavior over the whole boundary layer. It is important to note that these two datasets are the only ones studied here that actually corresponds to the sink flow/wedge flow geometry. Hence it is not unexpected that they show similarity behavior. What is unexpected is that the other datasets do not show similarity in spite of satisfying all the requirements. Thus it may be that there is some other factor that we are not capturing with the momentum balance type approach that explains why these sink flow results show similarity behavior and the other results do not.

The second related observation is also perplexing in that the scaling velocity under consideration behaves as $1/(x-x_0)$ but not all of the flow geometries are the traditional sink flow/wedge flow type geometry. This perplexing result may be due to a matter of definition. Traditional sink flows/wedge flows are associated with laminar flow between two converging planes which will have potential flows for which the outer velocity u_e will behave as $1/(x-x_0)$. Since the datasets above all obey the Rotta constraint and we have shown that the u_τ values in the flow direction behaves as $1/(x-x_0)$, then it follows that u_e will also behave as $1/(x-x_0)$, results we have confirmed but have not shown. Hence, not all flows for which the velocity u_e scales as $1/(x-x_0)$ corresponds to the traditional sink/wedge-type flow geometry.

6. CONCLUSION

Prandtl's "Plus" scaling variables were examined as velocity profile scaling parameters using the momentum balance type approach to similarity. It was shown that certain datasets can be found for which the Plus scaling variables do in fact satisfy all of the requirements for similarity based on this approach. However, the same velocity profile datasets also confirm that Prandtl's Plus scaling variables do not always show whole profile similarity. Hence, we conclude that the scaling variables discovered by the momentum balance type approach are a necessary but not sufficient condition for velocity profile similarity.

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APPENDIX 1

In this section the relevant equations for the momentum balance type approach to velocity profile similarity are derived. Suppose that the *x*-axis is placed in the plane of the wall and that the *y*-axis is at right angles to this wall into the fluid layer. Furthermore, suppose only steady state solutions are considered. We start with the Reynolds decomposition into a mean (or average) component and a fluctuating component

$$\begin{aligned} \hat{u} &= u + \tilde{u}\,, \\ \hat{v} &= v + \tilde{v}\,, \\ \hat{p} &= p + \tilde{p}\,, \end{aligned} \tag{1.1}$$

where the average components are *u*, the *x*-velocity, *v*, the *y*-velocity, and *p*, the pressure. The fluctuating components are $\tilde{u}$, $\tilde{v}$, and $\tilde{p}$.

The equation for the mass conservation requires

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \text{ and } \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = 0. \tag{1.2}$$

For a two-dimensional, incompressible, turbulent boundary layer, we **ASSUME** the *x*-component of the momentum balance is given approximately by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial}{\partial x}\left\{\overline{\tilde{u}^2}\right\} + \frac{\partial}{\partial y}\left\{\overline{\tilde{u}\tilde{v}}\right\} \cong -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\frac{\partial^2 u}{\partial y^2} \tag{1.3}$$

where ρ is the density and ν is the kinematic viscosity. Next, we introduce the Reynolds stress transport equation given by

$$u\frac{\partial \overline{\tilde{u}\tilde{v}}}{\partial x} + v\frac{\partial \overline{\tilde{u}\tilde{v}}}{\partial y} - 2\overline{\tilde{u}\tilde{v}}\frac{\partial u}{\partial y} - v\frac{\partial^2 \overline{\tilde{u}\tilde{v}}}{\partial y^2} + \frac{\partial}{\partial y}\left(\overline{\tilde{u}\tilde{v}\tilde{v} + \frac{\tilde{P}\tilde{u}}{\rho}}\right) + 2v\overline{\frac{\partial \tilde{u}}{\partial y}\frac{\partial \tilde{v}}{\partial x}} - \overline{\frac{\tilde{P}}{\rho}\left(\frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x}\right)} = 0 . \tag{1.4}$$

In order to reduce these equations to dimensionless equations we start by defining a dimensionless independent variable

$$\eta \equiv \frac{y}{\delta(x)} \tag{1.5}$$

where the function $\delta(x)$ is an as yet unknown boundary layer thickness scaling parameter that is only a function of *x*. Furthermore, we define a stream function $\psi(x,y)$ in terms of a product of functions as

$$\psi(x,y) = \delta(x)u_s(x)f(x,\eta) \quad , \tag{1.6}$$

where $f(x,\eta)$ is a dimensionless function, and $u_s(x)$ is the as yet unknown scaling velocity. We **ASSUME** that the stream function satisfies the conditions

$$u = \frac{\partial \psi(x,y)}{\partial y}, \qquad v = -\frac{\partial \psi(x,y)}{\partial x} \quad . \tag{1.7}$$

This means that

$$v = -\frac{\partial \psi}{\partial x} = -\frac{d\{\delta u_s\}}{dx} f - \delta u_s \frac{\partial f}{\partial x} \tag{1.8}$$

where we have used the fact that

$$\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial x}\left\{\frac{y}{\delta(x)}\right\} = -\frac{\eta}{\delta}\frac{d\delta}{dx} \quad . \tag{1.9}$$

Hence the stream-wise velocity becomes

$$u = \frac{\partial \psi}{\partial y} = u_s f' \tag{1.10}$$

where the prime indicates differentiation with respect to η and where

$$\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial y}\left\{\frac{y}{\delta(x)}\right\} = \frac{1}{\delta} \quad . \tag{1.11}$$

Substituting the above variable property similarity variables into the *x*-component of the momentum equation (Eq. 1.3), starting on the left-hand side, we have

$$u\frac{\partial u}{\partial x} = u_s \frac{du_s}{dx} f'^2 + u_s^2 f' \frac{\partial f'}{\partial x}. \tag{1.12}$$

The next term is

$$v\frac{\partial u}{\partial y} = -\frac{u_s}{\delta}\frac{d\{\delta u_s\}}{dx} ff'' - u_s^2 f'' \frac{\partial f}{\partial x} \quad . \tag{1.13}$$

Combining these terms

$$\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right\} = u_s \frac{du_s}{dx} f'^2 - \frac{u_s}{\delta}\frac{d\{\delta u_s\}}{dx} ff'' + u_s^2 f' \frac{\partial f'}{\partial x} - \delta u_s f'' \frac{\partial f}{\partial x}. \tag{1.14}$$

The next step is to transform the viscous component in Eq. 1.3 given by

$$\nu \frac{\partial^2 u}{\partial y^2} = \nu \frac{u_s}{\delta^2} f''' \quad . \tag{1.15}$$

ASSUME that the pressure in the boundary layer is constant and equal to the pressure at the boundary layer edge and that the Euler equation given by

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = u_e \frac{du_e}{dx} \tag{1.16}$$

holds where $u_e(x)$ is the velocity at the boundary edge.

ASSUME that the stress terms can be separated into the product of an *x*-dependent functional and an η-dependent functional as

$$\frac{\overline{\tilde{u}\tilde{v}}}{uv(x)} = g_{12}(\eta) \quad , \quad \frac{\overline{\tilde{u}^2}}{uu(x)} = g_{11}(\eta) \quad . \tag{1.17}$$

The Reynolds stress terms reduce to

$$\frac{\partial}{\partial x}\left\{\overline{\tilde{u}^2}\right\}+\frac{\partial}{\partial y}\left\{\overline{\tilde{u}\tilde{v}}\right\} = -\frac{uu(x)}{\delta}\frac{d\delta}{dx}\eta g_{11}'+\frac{duu(x)}{dx}g_{11}+\frac{uv(x)}{\delta}g_{12}' \quad . \tag{1.18}$$

Combining the transformed terms, we get the *x*-component of the momentum balance as

$$u_s\frac{du_s}{dx}f'^2-\frac{u_s}{\delta}\frac{d\{\delta u_s\}}{dx}ff''+u_s^2f'\frac{\partial f'}{\partial x}-\delta u_s f''\frac{\partial f}{\partial x}+ \tag{1.19}$$

$$-\frac{uu(x)}{\delta}\frac{d\delta}{dx}\eta g_{11}'+\frac{duu(x)}{dx}g_{11}+\frac{uv(x)}{\delta}g_{12}' = u_e\frac{du_e}{dx}+\nu\frac{u_s}{\delta^2}f''' \quad .$$

For similarity, we will ASSUME that the terms involving $\partial f'/\partial x$ and $\partial f/\partial x$ are negligible. The next step is the key to the problem of finding similarity restrictions using the momentum balance approach. Specifically, it is apparent from examination of Eq. 1.19 is that similar solutions will only be possible if the variable groupings with an *x*-dependence change in the identical manner. Equivalently, we can divide through by one of the x-groupings and similarity would then require that the *x*-grouping ratios be constants. To this end we divide Eq. 1.19 through by $\frac{u_s}{\delta}\frac{d\{\delta u_s\}}{dx}$. The transformed dimensionless *x*-component of the momentum balance then reduces to

$$-(1/\alpha)f'''-ff''-\frac{\Lambda}{1-\Lambda}f'^2-\tau_1\eta g_{11}'+\tau_2 g_{11}+\tau_3 g_{12}' = \phi \tag{1.20}$$

where

$$\alpha = \frac{\delta^2}{\nu}\frac{du_s}{dx}+\frac{\delta u_s}{\nu}\frac{d\delta}{dx} \quad , \quad \Lambda = -\frac{\delta}{u_s}\frac{du_s/dx}{d\delta/dx} \quad , \quad \phi = -\frac{u_e}{u_s}\frac{du_e/dx}{du_s/dx}\frac{\Lambda}{1-\Lambda} \quad , \tag{1.21}$$

$$\tau_1=\frac{uu(x)}{u_s^2(1+\lambda)} \quad , \quad \tau_2=\frac{duu(x)/dx}{u_s\,du_s/dx+\left(u_s^2/\delta\right)d\delta/dx} \quad \text{, and } \tau_3=\frac{uv(x)}{\delta u_s\,du_s/dx+u_s^2\,d\delta/dx} \quad .$$

The factors δ, u_s, $uv(x)$, and $uu(x)$ appearing in Eq. 1.20 are as yet unknown parameters.

In a similar fashion we can transform the Reynolds Stress Transport equation given by Eq. 1.4. Substituting the above similarity variables into the Reynolds stress transport equation, starting on the left-hand side, we have

$$u\frac{\partial\overline{\tilde{u}\tilde{v}}}{\partial x} = u_s f'\frac{\partial\{uv(x)g_{12}(\eta)\}}{\partial x} = -uv(x)\frac{u_s}{\delta}\frac{d\delta}{dx}\eta f'g_{12}'+u_s\frac{\partial uv(x)}{\partial x}f'g_{12} \quad . \tag{1.22}$$

The next term is

$$v\frac{\partial\overline{\tilde{u}\tilde{v}}}{\partial y} = -uv(x)\frac{1}{\delta}\frac{d\{\delta u_s\}}{dx}fg_{12}'-uv(x)u_s\frac{\partial f}{\partial x}g_{12}' \quad . \tag{1.23}$$

The first term on the right-hand side, the Production term, is

(1.24)

$$2\overline{\tilde{u}\tilde{v}}\frac{\partial u}{\partial y} \quad = \quad 2\frac{u_s uv(x)}{\delta}g_{12} f'' \quad .$$

The next step is to transform the viscous-like component in Eq. 1.4 given by

$$\nu\frac{\partial^2 \overline{\tilde{u}\tilde{v}}}{\partial y^2} \quad = \quad \nu\frac{uv(x)}{\delta^2}g_{12}'' \quad . \tag{1.25}$$

Combining the transformed terms, we get the stress balance as

$$-uv(x)\frac{du_s}{dx} f g_{12}' - uv(x)\frac{u_s}{\delta}\frac{d\delta}{dx} f g_{12}' + u_s \frac{\partial uv(x)}{\partial x} f' g_{12} - uv(x)\frac{u_s}{\delta}\frac{d\delta}{dx}\eta f' g_{12}' + \tag{1.26}$$

$$2\frac{u_s uv(x)}{\delta} f'' g_{12} + \nu\frac{uv(x)}{\delta^2}g_{12}'' + \left\{3\, additional\ terms\right\} \quad = \quad 0 \quad ,$$

where the three additional terms are not written out expressly because they cannot be written in terms of $\overline{\tilde{u}\tilde{v}}$. This is not to say that these terms can be neglected. In fact the opposite is true; the three additional terms in Eq. 1.26 include the numerically significant energy dissipation rate and the velocity-pressure gradient terms. However, for the purposes of obtaining similarity scaling information for δ and u_s, these three additional terms would require additional assumptions and the identification of additional unknown parameters. Hence, for the purposes herein, we will leave the terms unused.

The last step is to insure that any of the variable groupings with an *x*-dependence are constant. In order to make the stress equation dimensionless we divide through by $\frac{u_s uv(x)}{\delta}$ to get the transformed Reynolds stress balance as

$$-\kappa f g_{12}' - \varepsilon\eta f' g_{12}' + \tau_4 f' g_{12} - \varepsilon f g_{12}' + 2 g_{12} f'' + \chi g_{12}'' + \left\{3\, additional\ terms\right\} = 0 \quad . \tag{1.27}$$

where

$$\chi \quad = \quad \frac{\nu}{\delta u_s} \quad , \quad \varepsilon \quad = \quad \frac{d\delta}{dx} \quad , \quad \text{and} \quad \kappa \quad = \quad \frac{\delta}{u_s}\frac{du_s}{dx} \quad . \tag{1.28}$$