Rolling friction on a wheeled laboratory cart

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Abstract
A simple model is developed that predicts the coefficient of rolling friction for an undriven laboratory cart on a track that is approximately independent of the mass loaded onto the cart and of the angle of inclination of the track. The model includes both deformation of the wheels/track and frictional torque at the axles/bearings. The concept of rolling friction is contrasted with the static or kinetic friction that in general is also present, such as for a cylinder or ball rolling along a horizontal or inclined surface.

Introduction
Small rolling carts on metal tracks have become common in introductory physics laboratories at universities and secondary schools. Examples include PASCO’s ‘Collision Cart’, Vernier’s ‘Standard Cart’, PHYWE’s ‘Low Friction Cart’, and PSSC’s ‘Dynamics Cart’. Nevertheless, there exists confusion about how to simply and accurately model the resistance that such carts encounter when they move on level or inclined tracks. For example, two articles [1, 2] refer to the friction experienced by the carts as ‘kinetic’, which is incorrect because they roll not slide along the track. A more recent paper [3] does not specifically use the adjective ‘kinetic’, but nevertheless subscripts the coefficient of friction between the cart and track with the letter ‘k’ which presumably is an abbreviation for that adjective. Other authors hedge their bets by simply referring to the force opposing the motion of the carts as ‘friction’ without a qualifying adjective [4, 5] and the coefficient of friction by the unsubscripted symbol ‘μ’ [6], even though most introductory physics textbooks only denote friction as either static or kinetic. Part of the confusion is understandable, in that static or kinetic friction, under many circumstances, dominates and hence masks the rolling friction, which is always present and distinct from either of these other two kinds of friction for a rolling object. For example, if a string is wound around the axle of a rolling cylinder [7] or of a spool [8] and used to drive its motion, then static friction usually arises to prevent the object from slipping. (Depending on how it is pulled, static friction is not always needed [9], but usually it is.) However, static friction is not necessarily a ‘resistance’ to motion: in fact, it can point in (as opposed to against) the direction of motion [10]. In any case it does not dissipate any mechanical energy because its point of contact at the bottom of the rolling object has zero instantaneous velocity. (Zero displacement means zero work done. It is static after all.) Similarly, a cylinder rolling up or down an incline is driven by gravity, and a vehicle (such as a car or bicycle) accelerating along a level road is driven by a motor or by pedalling. In such cases, static friction again dominates (assuming the rolling is without slipping) and can point either forward or backward relative to the direction of motion [8]. If slipping does occur, then kinetic...
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friction will dominate, as is purposely exploited in some games such as bowling and billiards [11].

In contrast, if a cylinder or ball is freely rolled (i.e. without being driven by a string or motor) along a horizontal surface, it will eventually come to a stop (even if it does not strike any small bits of dirt or other obstacles). This slowing will arise even in the absence of air drag, as one can verify by rolling the object inside an evacuated bell jar. In the absence of slipping, the resistive force cannot be ordinary static friction, as explained in figure 1 of [12]; for example, if the friction force pointed backward (to translationally decelerate the object), then it would simultaneously rotationally accelerate the cylinder about its centre, which cannot be right! The object could partially stick to the surface and have to be ‘peeled’ away from it and thereby get slowed down [13], but that cannot be a complete explanation, as nonadhesive rolling objects also slow down. A simple but comprehensive model of rolling friction is developed in this article.

**General theory of rolling friction**

Rolling friction on the wheels of a cart can be modelled using the ideas of Krasner [12]. There are two contributions to the friction: deformation of the wheels and/or track, and the torque at the axles (or bearings) opposing the rotation of the wheels. The deformation will be represented here as a flattening of the wheels, although in general it can also arise from the cart creating a slight depression in (or bowing of) the track out of which it has to continuously roll.

Consider a cart of mass $m$ that is free rolling up an incline, as sketched in figure 1. The total frictional force $f$ on the cart from the track is assumed to oppose the direction of motion $\upsilon$ of the cart. Similarly, the total frictional torque $\tau$ at the four bearings opposes the direction of rotation $\omega$ of the wheels. Force components perpendicular to the incline must balance so that

$$N = mg \cos \theta$$  \hspace{1cm} (1)

where $N$ is the sum of the normal forces at all four wheels. (In general, $N$, $f$ and $\tau$ will not be equally divided among the four wheels.) Force components along the incline imply

$$mg \sin \theta + f = ma.$$  \hspace{1cm} (2)

The rotational form of Newton’s second law about point O in the enlarged view in figure 2 is

$$ND + \tau - fR = I \alpha$$  \hspace{1cm} (3)

where the angular acceleration of the wheels is $\alpha = a/R$ assuming they do not slip, and the sum of the moments of inertia of the four wheels is $I = \gamma MR^2$. Here $M$ is the total mass of the four wheels (each of radius $R$) and $\gamma$ is
a mass distribution factor, of the order of 0.5 if the wheels are approximately uniform discs. 
Solving equation (2) for $a$ and substituting it into equation (3) leads to

$$ND + τ = f R = γ MR (g \sin θ + f/m).$$  \hspace{1cm} (4)

Now suppose we can relate the frictional force on the cart to the normal force on the entire cart via a coefficient of friction $μ$ (labelled without subscripts for the moment),

$$f = μN = μmg \cos θ$$  \hspace{1cm} (5)

using equation (1) to get the second equality. 
Similarly suppose that the frictional torque on the axle can be related to the normal component of the gravitational load on them from the body of the cart without the wheels, $(m - M)g \cos θ$, so that

$$τ = kr(m - M)g \cos θ$$  \hspace{1cm} (6)

where $r$ is the radius of the axes and $k$ is a coefficient of frictional torque (dimensionless) coefficient of frictional torque. It is assumed that $μ$ and $k$ are independent of speed. 
In support of that assumption, measurements of a ball on an incline [14] indicate that friction will angularly decelerate the object as it rolls up the ramp; the friction in this case is static not rolling [8]. We see that $μ$ is not a constant independent of $θ$, but instead its value automatically adjusts (up to a maximum value of $μ$ if slipping is not to occur). It is more conventional to write $f_1 = μN = mg \sin θ/(1 + γ)$ from equation (5) and not refer directly to $μ$ at all. Substituting this expression for $f_1$ into equation (2) leads to the familiar result $a = g \sin θ/(1 + γ)$ for the translational acceleration of a rolling object on an inclined plane.

As a specific example, consider a standard PASCO Collision Cart (model number ME-9454). Then equation (7) can be numerically approximated as follows. The wheels are hard plastic discs so that $γ ≈ \frac{1}{2}$. I dismantled a cart and measured the plastic part of a wheel to have a mass of $1.74 \pm 0.02$ g and a bearing to have a mass of $0.72 \pm 0.01$ g. Adding these together and multiplying by 4 gives the mass of all the rotating parts as $M = 9.84 \pm 0.12$ g. (Only a portion of each bearing rotates, so this value is actually an upper limit on $M$.) The total mass of a cart with its wheels is $m = 0.5015 \pm 0.0004$ kg. Consequently $M/m \approx 0.02 < 1$ and equation (7) becomes

$$μ ≈ \frac{D}{R} + krR - γ \frac{M}{m} \tan θ.$$

Experimentally it was found that $μ = 0.0065 \pm 0.0002$ by rolling a cart along a level track and fitting a straight line to the speed squared as a
function of position, both measured using a motion
detector [18]. (The same value of \( \mu \) was found
for a variety of different cart speeds, up to at
least 0.7 m s\(^{-1} \).) However, for values of \( \theta \) up
to 5\(^\circ \) (which is about the largest track tilt one
would realistically use for a free rolling cart),
\( \gamma(M/m)\tan \theta \approx 0.0009 \) and thus the final term
in equation (8) only makes a small contribution to
the overall sum. Neglecting it results in

\[
\mu_r \approx \frac{D + kr}{R}
\]  

(9)

where a subscript ‘\( r \)’ has been added to denote
that this is now the coefficient of rolling friction.
This final expression is independent of the angle
of inclination, justifying the experimental method
mentioned above in which the coefficient was
measured on a horizontal track. It is also
independent of the mass loading of a cart.
Equation (9) is a simple, dimensionally correct
sum of contributions due to contact deformation
and axle friction.

It is left as an exercise for the reader (or her
students) to redo the analysis presented here for
the case of a cart free rolling down (rather than up)
an incline. The directions of the arrows for \( \nu, \omega, \tau \)
and \( f \) need to be reversed in figures 1 and 2, and
point C needs to be moved to the downhill (rather
than the uphill) side of the wheel, a distance \( D \)
to the left of point O. Equation (7) then becomes

\[
\mu = \frac{\frac{D}{R} + k\frac{1 - \frac{D}{R}}{1 + \frac{\gamma M}{R}} \tan \theta}{1 + \frac{\gamma M}{R}}
\]  

(10)

which differs only in the last sign in the numerator.
(That difference can be rationalized simply as a
change in the sign of \( \theta \) to transform an uphill
to a downhill inclination.) But that last term is
dropped in deriving equation (9) and hence that
final expression for \( \mu_r \) is valid for both uphill and
downhill motions of a cart.

Conclusions
A universal equation (7) has been found for the
ratio of the effective frictional force to the normal
force on an object freely rolling along a horizontal
or inclined track. If the object is a standard lab
cart, this ratio is equal to the coefficient of rolling
friction \( \mu_r \) for realistic angles of inclination. The
fundamental reason that rolling friction dominates
the static friction in this case is that the wheels

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(which are the rotating components) make up only
a few per cent of the total mass of the cart.

On the other hand, if the object is a single
rotating object such as a cylinder or ball, then there
can exist both a significant static frictional force of
magnitude \( f_s = mg \sin \theta/(1 + \gamma^{-1}) \) and rolling
frictional force \( f_r = mg \cos \theta(D/R)/(1 + \gamma) \),
according to equations (5) and (7). Their ratio is

\[
\frac{f_r}{f_s} = \gamma \frac{R}{D} \tan \theta.
\]  

(11)

Normally \( D \) is much less than \( R \) (see figure 2)
and the static friction dominates unless the track is
horizontal or nearly so (i.e. \( \theta \approx 0 \)). However, both
are actually present, as illustrated in figure 10.20
of [19]. Rolling friction manifests itself as a
loss of mechanical energy as a ball descends a
ramp, because there is no such dissipation
under the idealized conditions usually discussed
in the classroom. In particular, if the familiar
demonstration of a marble on a loop-the-loop track
is performed [20], one quickly discovers that the
marble has to be released from a height well above
the idealized prediction of 2.7 times the radius of
the loop. (Much closer agreement with idealized
theory is obtained if the demonstration is instead
performed using an interrupted pendulum [21],
since air drag is much weaker than rolling friction.)

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