Nonlinear Stochastic Markov Processes and Modeling Uncertainty in Populations

H.T. Banks and Shuhua Hu

Center for Research in Scientific Computation
Center for Quantitative Sciences in Biomedicine
North Carolina State University
Raleigh, NC 27695-8212

July 6, 2011

Abstract

We consider an alternative approach to the use of nonlinear stochastic Markov processes (which have a Fokker-Planck or Forward Kolmogorov representation for density) in modeling uncertainty in populations. These alternate formulations, which involve imposing probabilistic structures on a family of deterministic dynamical systems, are shown to yield pointwise equivalent population densities. Moreover, these alternate formulations lead to fast efficient calculations in inverse problems as well as in forward simulations. Here we derive a class of stochastic formulations for which such an alternate representation is readily found.

AMS subject classifications: 60J60, 60J22, 35Q84

Key Words: Nonlinear Markov processes, probabilistic structures on deterministic systems, uncertainty, Fokker-Planck, Forward Kolmogorov, pointwise equivalence.

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
06 JUL 2011		**00-00-2011 to 00-00-2011**

4. TITLE AND SUBTITLE	5a. CONTRACT NUMBER
Nonlinear Stochastic Markov Processes and Modeling Uncertainty in Populations	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **North Carolina State University,Center for Research in Scientific Computation,Department of Mathematics,Raleigh,NC,27695-8212**	8. PERFORMING ORGANIZATION REPORT NUMBER **CRSC-TR11-02**
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

We consider an alternative approach to the use of nonlinear stochastic Markov processes (which have a Fokker-Planck or Forward Kolmogorov representation for density) in modeling uncertainty in populations. These alternate formulations, which involve imposing probabilis- tic structures on a family of deterministic dynamical systems, are shown to yield pointwise equivalent population densities. Moreover, these alternate formulations lead to fast efficient calculations in inverse problems as well as in forward simulations. Here we derive a class of stochastic formulations for which such an alternate representation is readily found.

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**Same as Report (SAR)**	**31**	

1 Introduction and Motivation

In this paper we consider classes of nonlinear stochastic differential equations (SDE) with corresponding stochastic process density described by Fokker-Planck (FP) or Forward Kolmogorov (FK) equations with nonlinear drift (convective velocity or average transition rate) terms which dominate the (possibly nonlinear) diffusion (variability in transition rate) terms. We develop a general algorithmic approach for converting these computationally difficult nonlinear SDE to an *equivalent* (in a sense to be made precise below) probabilistic formulation which is much more amenable to fast (and parallel) computations. A major feature of our approach here is the bidirectional nature of our derivations. That is, we show how to transform from a given SDE to the corresponding equivalent probabilistic formulation, and from a given probabilistic formulation to the corresponding SDE. Our results are presented in two distinct steps. First, we show an equivalence for several classes of nonautonomous affine differential equations (in both scalar and system forms). Then we extend theses equivalences to rather general classes of nonlinear differential equations using invertible transformation techniques between nonlinear differential equations and the class of affine differential equations introduced in the first step. We illustrate the ideas with several different nonlinear examples including two important examples with growth or transition rates encountered frequently in modeling populations, label decay, tumor growth, etc. These are the popular Verhulst-Pearl logistic growth rate $g(x) = rx\left(1 - \frac{x}{\kappa}\right)$ and the general transition rates $g(x,t) = (a_0(t) - a_1(t)\ln x)x$ of which the standard Gompertz growth rates $g(x) = r\ln\left(\frac{\kappa}{x}\right)x = r(\ln\kappa - \ln x)x$ are special cases. Motivating ideas and a brief recap of results from previous efforts on population models are first summarized here and in the next section.

A fundamental modeling construction in many areas of science is the nonlinear Markov process as characterized by discrete or continuous time with discrete (e.g., Poisson) or continuous (e.g., diffusion) state processes [2, 22, 24, 27, 31, 33]. Here we focus on modeling with a general nonlinear Markov diffusion process with finite mean and variance. Mathematically this leads to a stochastic differential equation (SDE) of the form

$$dX(t) = g(X(t),t)dt + \sigma(X(t),t)dW(t),$$

which in turn can be investigated with Fokker-Planck (FP) or Forward Kolmogorov equations

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x}(g(x,t)u(x,t)) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(x,t)u(x,t)), \tag{1.1}$$

for the corresponding probability density u. These FP models are ubiquitous in mathematics and physics (e.g., particle transport, filtering), biology (population models), finance (e.g., Black-Scholes equations) among other areas of applications [2, 22, 27, 29]. In many of these applications one has what is commonly referred to as *convection dominated diffusion* which occurs when we have $g >> \sigma$. In this case the Fokker-Planck equations are notoriously difficult to solve especially when g depends on time. This is a serious drawback in forward simulations but can be untenable in inverse problem calculations [20].

In these applications one begins with a stochastic Markov diffusion process $X(t)$ with realizations $x(t)$ representing a structure variable (level) such as size (length, weight, volume, etc.), label intensity, chronological or physiological age, spatial location, etc., that changes according to a mean rate law $g(x,t)$ with mean variance $\sigma^2(x,t)$. Such models also arise in population biology with class structure [4, 5, 6, 7, 9, 10, 12, 14], complex nodal network models (in network security, social/insurgency networks, logistic and production networks) [3, 16], fluorescence intensity of labeled proliferating cell populations [17, 28] and general hyperbolic transport systems in random or uncertain environments.

To incorporate uncertainty or variability into structured deterministic dynamical models, several approaches have been considered in the literature. Of those of interest to us here, one involves a stationary probabilistic structure on a family of structured deterministic dynamical systems, while the other is constructed based on the assumption that movement from one structure level to another can be described by a stochastic diffusion process. As noted above, for computational purposes the latter can be represented mathematically by a Fokker-Planck equation. Even though these two formulations are conceptually quite different, in this paper we show for a class of examples (both linear and nonlinear differential equations) that they are equivalent in terms of corresponding probability density functions. Numerical methods for the probabilistic formulation are quite fast and lead to alternative methods for solution of the Fokker-Planck or Forward Kolmogorov equation associated with the stochastic diffusion process. Thus we establish that there are several classes of Fokker-Planck inverse problems (which are computationally intensive) that can be readily converted to inverse problems for probabilistic structures on deterministic systems which can be solved efficiently.

Because we are primarily interested in modeling transition uncertainty in this paper, for simplicity, we will not consider either sink or source terms in our formulations.

Unless otherwise indicated, a capital letter is used to denote a random variable throughout the presentation, and a corresponding small letter to denote its realization. We use $\mathcal{N}(\mu, \sigma^2)$ to denote a normal distribution with mean μ and variance σ^2, $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ for a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, and $\mathbb{E}(Z)$ for the expectation of random variable Z. We next give careful detailed formulations of the two approaches we consider.

1.1 Stochastic Formulation

A stochastic formulation can be motivated by recognizing that local factors (such as environmental or emotional fluctuations) can have a significant influence on the individual transition or transfer rates from one structure level to another. For example, in [5, 7, 9, 10, 20] the growth rate of two different marine species (mosquitofish and shrimp) are affected by several environmental factors such as temperature, dissolved oxygen level and salinity. In such examples, the stochastic formulation is constructed under the assumption that movement from one structure level (size in these examples) to another can be described by a stochastic diffusion process [2, 20, 24, 31]. Let $\{X(t) : t \geq 0\}$ be a Markov diffusion process with $X(t)$

representing structure level at time t (i.e., each process realization corresponds to the structure trajectory of an individual). Then $X(t)$ is described by the Ito stochastic differential equation (we refer to this equation as the *stochastic rate model* (SRM))

$$dX(t) = g(X(t), t)dt + \sigma(X(t), t)dW(t), \tag{1.2}$$

where $W(t)$ is the standard Wiener process [2, 24]. Here $g(x, t)$ denotes the average transition rate (the first moment of rate of change in structure level) of individuals with structure level x at time t, and is given by

$$g(x, t) = \lim_{\Delta t \to 0+} \frac{1}{\Delta t} \mathbb{E}\left\{\Delta X(t) | X(t) = x\right\}. \tag{1.3}$$

The function $\sigma(x, t)$ represents the variability in the transition rate of individuals (the second moment of rate of change in structure level) and is given by

$$\sigma^2(x, t) = \lim_{\Delta t \to 0+} \frac{1}{\Delta t} \mathbb{E}\left\{[\Delta X(t)]^2 | X(t) = x\right\}. \tag{1.4}$$

Hence, the transition process (growth in size in the marine examples cited above) of each individual is stochastic, and each individual changes structure level according to the stochastic rate model (1.2). Thus, for this formulation the transition uncertainty is introduced into the entire population by the stochastic transition of each individual. In addition, individuals with the same structure level at the same time have the same uncertainty in transition, and individuals also have the possibility of reducing their structure level during a transition period.

With this assumption on the transition process, we obtain the Fokker-Planck (FP) or forward Kolmogorov model for the population density u, which was carefully derived in [31] among numerous other places and subsequently studied in many references (e.g., [2, 20, 24]). The equation and appropriate boundary conditions are given by

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x, t) + \frac{\partial}{\partial x}(g(x, t)u(x, t)) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2(x, t)u(x, t)), \quad x \in (0, L), \quad t > 0,\\
&g(0, t)u(0, t) - \frac{1}{2}\frac{\partial}{\partial x}(\sigma^2(x, t)u(x, t))|_{x=0} = 0,\\
&g(L, t)u(L, t) - \frac{1}{2}\frac{\partial}{\partial x}(\sigma^2(x, t)u(x, t))|_{x=L} = 0,\\
&u(x, 0) = u_0(x).
\end{aligned} \tag{1.5}$$

Here L is the maximum structure level that individuals may attain in any given time period. Observe that the boundary conditions in (1.5) provide a conservation law for the FP model. Because both sink (death) and reproduction (birth) rates are assumed zero, the total number in the population is a constant given by $\int_0^L u_0(x)dx$. In addition, we observe that with the zero-flux boundary condition at zero (minimum structure level) one can equivalently set $X(t) = 0$ if $X(t) \leq 0$ for the stochastic transition model (1.2) in the sense that both are used to keep individuals in the system at the boundary levels. This means that if the structure

level of an individual is decreased to the minimum value, it remains in the system at that level with the possibility to once again increase its level.

As we have already mentioned, the above equations present formidable computational challenges when drift (or convection) g dominates. While this is certainly true for forward simulations, it is even more of a challenge in the inverse problems considered in [20]. Thus it is of great interest to explore alternative formulations that might lead to modeling representations which represent essentially the same transport process at the density/distribution level and that involve a much more efficient computational implementation. To this end in the next section we describe an alternative, but in a certain sense (to be made more precise below) equivalent, formulation for such dynamic processes with uncertainty in transition.

1.2 An Alternative Probabilistic Formulation

To offer an alternative formulation to stochastic Markov processes for populations with uncertainty in transition rates, we turn to population models (labeled proliferating cells [17], mosquitofish [7, 14], shrimp [5, 9, 10]) where we have used structured dynamical models with uncertainty. The probabilistic formulation we present is motivated by the observation that in populations, intrinsic variables (genetic differences or non-lethal infections of some chronic disease) can have an effect on individual class transition rates. For example, in many marine species such as mosquitofish, females grow faster than males, which means that individuals with the same size may have different growth rates. In labeled cell populations where label intensity is the structure variable, label decay rates may vary across individuals during cell proliferation due to variable decay in cell cycle stages. The probabilistic formulation is constructed based on the assumption that each individual does change according to a deterministic rate model $\frac{dx}{dt}(t) = g(x,t)$ as posited in the Sinko-Streifer [32] formulation, but that different individuals may have different structure-dependent rates. Based on this underlying assumption, one partitions the entire population into (possibly a continuum of) subpopulations where individuals in each subpopulation have the same structure-dependent transition rates, and then assigns a probability distribution to this partition of possible transition rates in the population.

To be more precise here we can describe this construction in terms of *population growth models* where *size* (length in mosquitofish, weight in shrimp) is the structure variable. Then the growth process for individuals in a subpopulation with rate g is assumed to be described by the dynamics

$$\frac{dx(t;g)}{dt} = g(x(t;g),t), \qquad g \in \mathcal{G}, \tag{1.6}$$

where $\mathcal{G}$ is a collection of admissible rates. Model (1.6) combined with the probability distribution imposed on $\mathcal{G}$ will be called the *probabilistic rate distribution* (PRD) model in this paper. Hence, we can see that for the probabilistic formulation, the rate uncertainty is introduced into the entire population by the variability of transition (growth) rates among subpopulations. In the literature for size-structured population models, it is common to

assume that growth rate is a nonnegative function, that is, no loss in size occurs. However, individuals may experience loss in size due to disease or some other involuntary factors. Hence, we will permit these situations in this formulation, but for simplicity we assume that growth rate in each subpopulation is either a nonnegative function or a negative function, that is, the size of each individual may be either nondecreasing or decreasing continuously in its growth period.

With this assumption of a family of admissible transition rates and an associated probability distribution (i.e., the PRD model), one can obtain a generalization of the Sinko-Streifer model for densities as a function of time and structure variables. (This was called the growth rate distribution or (GRD) model as formulated and studied in [4, 6, 7, 12, 14].) The model, which here will be called the *class rate distribution-Sinko-Streifer* (CRDSS) model, consists of solving

$$\begin{aligned} &v_t(x,t;g) + (g(x,t)v(x,t;g))_x = 0, \qquad x \in (0,L), \quad t > 0, \\ &g(0,t)v(0,t;g) = 0 \text{ if } g \geq 0 \text{ or } g(L,t)v(L,t;g) = 0 \text{ if } g < 0, \\ &v(x,0;g) = v_0(x;g), \end{aligned} \tag{1.7}$$

for a given $g \in \mathcal{G}$ and then "summing" (with respect to the probability) the corresponding solutions over all $g \in \mathcal{G}$. If $v(x,t;g)$ is the population density of individuals with class structure value x at time t having transition rate g, the expectation of the total population density for class x at time t is given by

$$u(x,t) = \int_{g\in\mathcal{G}} v(x,t;g)d\mathcal{P}(g), \tag{1.8}$$

where $\mathcal{P}$ is a probability measure on $\mathcal{G}$. Hence the CRDSS model consists of the Sinko-Streifer equation (1.7) with (1.8) to compute the population density $u(x,t)$. Thus, this probabilistic formulation involves a *stationary probabilistic structure* on a family of *deterministic dynamical systems*, and $\mathcal{P}$ is the fundamental "parameter" that is to be estimated by either parametric or nonparametric methods (which depends on the prior information known about the form for $\mathcal{P}$). As detailed in [7, 12], this class rate distribution model is sufficiently rich to exhibit a number of phenomena of interest, for example, dispersion and development of two modes from one. Moreover, of paramount importance to us here, this formulation offers tremendous computational advantages in that it is what may be accurately termed *embarrassingly parallel* [14].

Observe that if all the subpopulations have nonnegative rate functions, then we need to set $g(L,t)v(L,t;g) = 0$ for each $g \in \mathcal{G}$ in order to provide a conservation law for the CRDSS model. Specifically if L denotes the maximum attainable class value of individuals in a life time, then it is reasonable to set $g(L,t) = 0$ (as commonly done in the literature). However, if we just consider the model in a short time period, then we may choose L sufficiently large so that $v(L,t)$ is negligible or zero if possible. We observe that if there exist some subpopulations whose rates are negative, then we can not provide a conservation law for these subpopulations as $g(0,t) < 0$. Hence, in this case, once the class value of an individual is decreased to below the minimum value, then that individual will be removed from the system. In other words,

we exclude those individuals whose class value go below the minimum size. This effectively serves as a sink for these subpopulations.

2 Summary of Previous Findings

We compare here the probabilistic formulation approach to incorporating the class rate uncertainty into a structured population model with the stochastic formulation of Section 1.1.

The discussions in Sections 1.1 and 1.2 indicate that these two formulations are *conceptually* quite different. One entails imposing a probabilistic structure on the set of possible transition rates permissible in the entire population while the other involves formulating transition as a stochastic diffusion process. However, the analysis in [9] reveals that in some cases the structure distribution (the probability density function of $X(t)$) obtained from the stochastic rate model is exactly the same as that obtained from the PRD model. For example, if we consider the two models

$$\begin{aligned} &\text{stochastic formulation: } dX(t) = b_0(X(t)+c_0)dt + \sqrt{2t}\sigma_0(X(t)+c_0)dW(t) \\ &\text{probabilistic formulation: } \frac{dx(t;b)}{dt} = (b-\sigma_0^2 t)(x(t;b)+c_0), \quad b\in\mathbb{R}, \\ &\qquad\qquad B\sim\mathcal{N}(b_0,\sigma_0^2), \end{aligned} \tag{2.1}$$

and assume their initial structure distributions are the same, then we obtain at each time t the same structure distribution from these two distinct formulations. Here b_0, σ_0 and c_0 are positive constants (for application purposes), and B is a normal random variable with b a realization of B. Moreover, by using the same analysis as in [9] we can show that if we compare

$$\begin{aligned} &\text{stochastic formulation: } dX(t) = (b_0+\sigma_0^2 t)(X(t)+c_0)dt + \sqrt{2t}\sigma_0(X(t)+c_0)dW(t) \\ &\text{probabilistic formulation: } \frac{dx(t;b)}{dt} = b(x(t;b)+c_0), \quad b\in\mathbb{R} \text{ with } B\sim\mathcal{N}(b_0,\sigma_0^2) \end{aligned} \tag{2.2}$$

with the same initial structure distributions, then we can also obtain at each time t the same structure distribution for these two formulations. In addition, we see that both the stochastic rate models and the probabilistic rate models in (2.1) and (2.2) reduce to the same deterministic growth model $\dot{x} = b_0(x+c_0)$ when there is no uncertainty or variability in rate (i.e., $\sigma_0 = 0$) even though both models in (2.2) *do not* satisfy the mean rate dynamics

$$\frac{d\mathbb{E}(X(t))}{dt} = b_0(\mathbb{E}(X(t))+c_0) \tag{2.3}$$

while both models in (2.1) *do*. This last observation was critical in the early efforts of [9, 10] which were derived under the additional constraint that (2.3) must hold. This was motivated by available shrimp data of longitudinal measurements of *average* shrimp weight (in gms),

i.e., an observation of $\bar{x}(t) = \mathbb{E}(X(t))$. In this earlier work it was found that an affine growth law $\dfrac{d\bar{x}(t)}{dt} = g(\bar{x}(t)) = b_0(\bar{x}(t) + c_0)$ yields a good fit to this data for early shrimp growth. This led to a search for equivalent mathematical representations which also satisfied this extra condition.

More specifically, one can prove that the formulations in (2.1) generate stochastic processes $X(t)$ which both satisfy the mean rate dynamics (2.3) and yield processes

$$X(t) = -c_0 + (X_0 + c_0)Y(t),$$

where

$$Y_{PRD}(t) = \exp(Bt - \tfrac{1}{2}\sigma_0^2 t^2), \text{ where } B \sim \mathcal{N}(b_0, \sigma_0^2), \tag{2.4}$$

$$Y_{SRM}(t) = \exp\left((b_0 t - \tfrac{1}{2}\sigma_0^2 t^2) + \sigma_0 \int_0^t \sqrt{2\tau} dW(\tau)\right). \tag{2.5}$$

Moreover it was shown that for each time t, both $Y_{PRD}(t)$ and $Y_{SRM}(t)$ are normally distributed with identical means and variances. Thus under the additional reasonable assumption (trivially true for non-random initial condition) that the random variables X_0 and each of $Y_{PRD}(t)$ and $Y_{SRM}(t)$ are independent we find that each of the stochastic processes derived from (2.1) possess at each time t the same distribution. That is, at each time t each of the processes $X(t)$ have the *same probability density*. Finally, the two stochastic processes are *NOT* the same. This can be seen immediately from (2.4) and (2.5), but also from a direct calculation of the covariances for Y_{PRD} and Y_{SRM} which we shall carry out below. In summary, while the two formulations of (2.1) generally lead to different processes, one can argue that they are *equivalent* in the sense that *they possess the same probability density at any time t*. For the subsequent discussions in this presentation, we shall refer to this as *pointwise equivalence in density*. This density must satisfy the corresponding Fokker-Planck or Forward Kolmogorov equation for the stochastic formulation in (2.1). Thus if one wishes to obtain a numerical solution of such a Fokker-Planck equation, one possibility is to consider the numerical solution of the equivalent but more readily solved CRDSS formulation of (2.1). For the particular forms of (2.1) and (2.2), this approach was demonstrated to be a computationally advantageous strategy in [11]. These findings lead to a *natural research question*: Are there general classes of Fokker-Planck equations that can be converted to an equivalent (in the distributional sense described above) CRDSS which can be efficiently solved numerically for the desired probability density function? A positive answer to this question is the primary focus of this paper and results are given in the next sections. In particular in the next section we develop general techniques to show equivalence for large classes of affine differential equations.

3 Equivalence between Probabilistic and Stochastic Formulations with Affine Dynamics

In this section, we turn to several class of examples with affine dynamics for which one can establish the desired equivalence between the probabilistic and stochastic formulations given above. The probabilistic formulations we consider here involve a finite-dimensional parameter family of structure rates of change. For example, if the probabilistic formulation is governed by scalar differential equations, then we assume that all the subsystems have the same functional form $g(x,t;\mathbf{b})$ for the structure rates of change but the values of parameters $\mathbf{b} = (b_0, b_1, \ldots, b_{n-1})^T$ vary across the system.

In establishing our results (and to discuss corresponding covariances below), the following relationship between normal distribution and log-normal distribution [21, page 109] is heavily used.

Lemma 3.1 *If $\ln Z \sim \mathcal{N}(\mu, \sigma^2)$, then Z is log-normally distributed, where its probability density function $f_Z(z)$ is defined by*

$$f_Z(z) = \frac{1}{z\sqrt{2\pi}\sigma} \exp\left(-\frac{(\ln z - \mu)^2}{2\sigma^2}\right),$$

and its mean and variance are given as follows

$$\mathbb{E}(Z) = \exp(\mu + \tfrac{1}{2}\sigma^2), \;\; Var(Z) = [\exp(\sigma^2) - 1]\exp(2\mu + \sigma^2).$$

In our subsequent arguments we shall also need the following basic result on the process generated by Ito integrals of Wiener processes that can be found in [27, Sec 4.3, Thm 4.11].

Lemma 3.2 *Let T be any positive constant. Then for a non-random function $f \in L^2(0,T)$, the Ito integrals $Q(t) = \int_0^t f(s)dW(s)$ for $0 < t \leq T$ yield a Gaussian stochastic process with pointwise distributions $\mathcal{N}\left(0, \int_0^t f^2(s)ds\right)$. Moreover, $Cov(Q(t), Q(t+\tau)) = \int_0^t f^2(s)ds$ for all $\tau \geq 0$.*

3.1 Scalar Differential Equations - Case I

In the first case we derive conditions under which the probabilistic and stochastic formulations generate stochastic processes with the same distributions (normal in the case the initial condition is a fixed constant) at each time t.

Probabilistic formulation The probabilistic formulation considered has the following form

$$\begin{aligned} &\frac{dx(t;\mathbf{b})}{dt} = \alpha(t)x(t;\mathbf{b}) + \gamma(t) + \mathbf{b}\cdot\boldsymbol{\varrho}(t), \quad \mathbf{b} = (b_0, b_1, \ldots, b_{n-1})^T \in \mathbb{R}^n \\ &B_j \sim \mathcal{N}(\mu_j, \sigma_j^2),\ j = 0, 1, 2, \ldots, n-1, \text{ which are mutually independent,} \end{aligned} \tag{3.1}$$

where α, γ and $\boldsymbol{\varrho} = (\varrho_0, \varrho_1, \ldots, \varrho_{n-1})^T$ are non-random functions of t, and $\mathbf{b}$ is chosen as realizations of $\mathbf{B} = (B_0, B_1, \ldots, B_{n-1})^T$. Hence, the dynamics of an individual with initial condition x_0 in a subsystem with its rates of change having parameter values $\mathbf{b}$ is described by the deterministic model

$$\frac{dx(t;\mathbf{b})}{dt} = \alpha(t)x(t;\mathbf{b}) + \gamma(t) + \mathbf{b}\cdot\boldsymbol{\varrho}(t), \quad x(0) = x_0.$$

Multiplying both sides of the above equation by the integrating factor $\exp\left(-\int_0^t \alpha(s)ds\right)$ we find that

$$\frac{d}{dt}\left[x\exp\left(-\int_0^t \alpha(s)ds\right)\right] = \gamma(t)\exp\left(-\int_0^t \alpha(s)ds\right) + \mathbf{b}\cdot\boldsymbol{\varrho}(t)\exp\left(-\int_0^t \alpha(s)ds\right).$$

We then integrate both sides of the above equation to obtain

$$\begin{aligned} x(t;x_0,\mathbf{b}) &= x_0\exp\left(\int_0^t \alpha(s)ds\right) + \int_0^t \gamma(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds \\ &\quad + \mathbf{b}\cdot\int_0^t \boldsymbol{\varrho}(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds. \end{aligned} \tag{3.2}$$

We assume that all the subsystems have the same probability density function for initial condition X_0, which is independent of B_j, $j = 0, 1, 2, \ldots, n-1$. Let $X(t) = x(t; X_0, \mathbf{B})$ and

$$Y(t) = \int_0^t \gamma(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds + \mathbf{B}\cdot\int_0^t \boldsymbol{\varrho}(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds.$$

Then we have that

$$X(t) = X_0\exp\left(\int_0^t \alpha(s)ds\right) + Y(t). \tag{3.3}$$

Note that $B_j \sim \mathcal{N}(\mu_j, \sigma_j^2)$, and B_j, $j = 0, 1, 2, \ldots, n-1$, are mutually independent. Hence, we find that for any fixed t, $Y(t)$ is normally distributed with mean defined by

$$\int_0^t (\gamma(s) + \boldsymbol{\mu}\cdot\boldsymbol{\varrho}(s))\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds, \tag{3.4}$$

where $\boldsymbol{\mu} = (\mu_0, \mu_1, \ldots, \mu_{n-1})^T$, and variance defined by

$$\sum_{j=0}^{n-1}\sigma_j^2\left[\int_0^t \varrho_j(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds\right]^2. \tag{3.5}$$

Stochastic formulation Based on the above discussions of the probabilistic formulation (3.1) we see that if all the individuals in the entire system have the same fixed initial condition x_0, then $X(t)$ is also normally distributed for any fixed time t. Based on this piece of information, the stochastic model is chosen to have the form

$$dX(t) = [\alpha(t)X(t) + \xi(t)]dt + \eta(t)dW(t), \quad X(0) = X_0, \tag{3.6}$$

where α, ξ and η are non-random functions of t, and X_0 is independent of $W(t)$. Multiplying both sides of the above equation by $\exp\left(-\int_0^t \alpha(s)ds\right)$ we find that

$$d\left[X(t)\exp\left(-\int_0^t \alpha(s)ds\right)\right] = \xi(t)\exp\left(-\int_0^t \alpha(s)ds\right)dt + \eta(t)\exp\left(-\int_0^t \alpha(s)ds\right)dW(t).$$

Integrating both sides of the above equation we obtain that

$$X(t) = X_0\exp\left(\int_0^t \alpha(s)ds\right) + Y(t), \tag{3.7}$$

where $Y(t)$ is defined by

$$Y(t) = \int_0^t \xi(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds + \exp\left(\int_0^t \alpha(\tau)d\tau\right)\int_0^t \eta(s)\exp\left(-\int_0^s \alpha(\tau)d\tau\right)dW(s).$$

By Lemma 3.2, for any fixed t we have that

$$\int_0^t \eta(s)\exp\left(-\int_0^s \alpha(\tau)d\tau\right)dW(s) \sim \mathcal{N}\left(0, \int_0^t \left[\eta(s)\exp\left(-\int_0^s \alpha(\tau)d\tau\right)\right]^2 ds\right).$$

Thus, we find that

$$Y(t) \sim \mathcal{N}\left(\int_0^t \xi(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds,\ \int_0^t \left[\eta(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)\right]^2 ds\right). \tag{3.8}$$

Equivalent conditions By (3.3), (3.4), (3.5), (3.7) and (3.8), we know that if functions ξ, η and ϱ_j, and constants μ_j, σ_j and n satisfy the following two equalities

$$\int_0^t \xi(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds = \int_0^t [\gamma(s) + \boldsymbol{\mu}\cdot\boldsymbol{\varrho}(s)]\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds \tag{3.9}$$

and

$$\int_0^t \left[\eta(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)\right]^2 ds = \sum_{j=0}^{n-1}\sigma_j^2\left[\int_0^t \varrho_j(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds\right]^2, \tag{3.10}$$

then the probabilistic formulation (3.1) and the stochastic formulation (3.6) yield stochastic processes that are pointwise equivalent in density.

Based on the equivalent conditions (3.9) and (3.10), we will derive the specific forms of the functional parameters and/or values of the parameters of the corresponding pointwise equivalent (stochastic/probabilistic) formulation in terms of the functional parameters and/or parameters of the known (probabilistic/stochastic) formulation.

3.1.1 Probabilistic Formulation to Stochastic Formulation

Here we assume that probabilistic formulation (3.1) is known, and we want to determine its corresponding stochastic formulation. In other words, we need to determine functions ξ and η in terms of functions γ and ϱ_j, and constants μ_j, σ_j and n. By (3.9), it is obvious that if function ξ is chosen to be

$$\xi(t) = \gamma(t) + \boldsymbol{\mu} \cdot \boldsymbol{\varrho}(t),$$

then (3.9) holds. Differentiating both sides of (3.10) with respect to t yields that

$$\begin{aligned}
&\eta^2(t) + 2\alpha(t) \sum_{j=0}^{n-1} \sigma_j^2 \left[\int_0^t \varrho_j(s) \exp\left(\int_s^t \alpha(\tau) d\tau \right) ds \right]^2 \\
&\quad = 2 \sum_{j=0}^{n-1} \sigma_j^2 \left[\varrho_j(t) + \alpha(t) \int_0^t \varrho_j(s) \exp\left(\int_s^t \alpha(\tau) d\tau \right) ds \right] \int_0^t \varrho_j(s) \exp\left(\int_s^t \alpha(\tau) d\tau \right) ds,
\end{aligned}$$

which can be simplified as follows

$$\eta^2(t) = 2 \sum_{j=0}^{n-1} \sigma_j^2 \varrho_j(t) \int_0^t \varrho_j(s) \exp\left(\int_s^t \alpha(\tau) d\tau \right) ds.$$

Hence, if $\boldsymbol{\varrho}$ is assumed to have the property that the right-hand-side of the above equation is nonnegative for any $t \geq 0$, then we can choose η to be

$$\eta(t) = \left[2 \sum_{j=0}^{n-1} \sigma_j^2 \varrho_j(t) \int_0^t \varrho_j(s) \exp\left(\int_s^t \alpha(\tau) d\tau \right) ds \right]^{\frac{1}{2}}$$

so that (3.10) holds.

3.1.2 Stochastic Formulation to Probabilistic Formulation

Next we assume that stochastic formulation (3.6) is known, and we wish to determine its corresponding probabilistic formulation. In other words, we need to determine functions γ and ρ_j, and constants μ_j, σ_j and n in terms of functions ξ and η. It is obvious that if we set

$$\gamma(t) + \boldsymbol{\mu} \cdot \boldsymbol{\varrho}(t) = \xi(t), \tag{3.11}$$

then (3.9) holds. By (3.11) and (3.10) we know that we have numerous different choices for the probabilistic formulation. Here we illustrate two of them.

Choice 1 One of the simple choices is to choose

$$n = 2, \quad \gamma \equiv 0, \quad \mu_0 = 1, \quad \mu_1 = 0, \quad \varrho_0(t) = \xi(t). \tag{3.12}$$

Then by the above equalities and (3.10) we have

$$\begin{aligned}&\sigma_1^2\left[\int_0^t \varrho_1(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds\right]^2\\&\quad=\int_0^t \eta^2(s)\exp\left(2\int_s^t \alpha(\tau)d\tau\right)ds-\sigma_0^2\left[\int_0^t \xi(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds\right]^2,\end{aligned}\tag{3.13}$$

which implies that we need to choose σ_0 sufficiently small such that its right-hand side is greater than 0. Now by (3.13) we have

$$\begin{aligned}&\sigma_1\int_0^t \varrho_1(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds\\&\quad=\left[\int_0^t \eta^2(s)\exp\left(2\int_s^t \alpha(\tau)d\tau\right)ds-\sigma_0^2\left(\int_0^t \xi(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

Differentiating both sides of the above equation with respect to t we obtain that

$$\begin{aligned}\sigma_1\varrho_1(t) &= \frac{d}{dt}\left[\int_0^t \eta^2(s)\exp\left(2\int_s^t \alpha(\tau)d\tau\right)ds-\sigma_0^2\left(\int_0^t \xi(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds\right)^2\right]^{\frac{1}{2}}\\&\quad-\alpha(t)\left[\int_0^t \eta^2(s)\exp\left(2\int_s^t \alpha(\tau)d\tau\right)ds-\sigma_0^2\left(\int_0^t \xi(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds\right)^2\right]^{\frac{1}{2}}.\end{aligned}$$

Hence, we can just assign any positive value for σ_1, and then use the above equality to determine function ϱ_1. Hence, one simple choice of the corresponding pointwise equivalent probabilistic formulation for (3.6) is as follows

$$\begin{aligned}&\frac{dx(t;b_0,b_1)}{dt}=\alpha(t)x(t;b_0,b_1)+b_0\xi(t)+b_1\varrho_1(t),\quad b_0,b_1\in\mathbb{R},\\&B_0\sim\mathcal{N}(1,\sigma_0^2),\quad B_1\sim\mathcal{N}(0,\sigma_1^2),\end{aligned}$$

where σ_0, σ_1 and ϱ_1 are chosen according to the above discussions.

Choice 2 Another simple choice is to choose

$$n=1,\quad \gamma(t)=\xi(t),\quad \mu_0=0,\quad \sigma_0=1.$$

Thus, by the above equalities and (3.10) we find

$$\int_0^t \varrho_0(s)\exp\left(\int_s^t \alpha(\tau)d\tau\right)ds=\left(\int_0^t \eta^2(s)\exp\left(2\int_s^t \alpha(\tau)d\tau\right)ds\right)^{\frac{1}{2}}.\tag{3.14}$$

Differentiating both sides of the above equation with respect to t we have that

$$\begin{aligned}\varrho_0(t) &= \frac{d}{dt}\left[\left(\int_0^t \eta^2(s)\exp\left(2\int_s^t \alpha(\tau)d\tau\right)ds\right)^{\frac{1}{2}}\right] \\ &\quad -\alpha(t)\left[\int_0^t \eta^2(s)\exp\left(2\int_s^t \alpha(\tau)d\tau\right)\right]^{\frac{1}{2}}.\end{aligned} \tag{3.15}$$

Hence, another simple choice of the corresponding pointwise equivalent probabilistic formulation for (3.6) is as follows

$$\begin{aligned}&\frac{dx(t;b)}{dt} = \alpha(t)x(t;b) + \xi(t) + b\varrho_0(t), \quad b \in \mathbb{R}, \\ &B \sim \mathcal{N}(0,1),\end{aligned}$$

where ϱ_0 is defined by (3.15).

3.2 Scalar Differential Equations - Case II

In this section, we consider examples where the probabilistic and stochastic formulations lead to processes that have pointwise equivalent densities (either log-normal or shifted log-normal, depending on the value of the parameter (see details below), when the initial condition is a fixed constant).

Probabilistic formulation The probabilistic formulation considered has the form

$$\begin{aligned}&\frac{dx(t;\mathbf{b})}{dt} = (\mathbf{b}\cdot\boldsymbol{\rho}(t) + \gamma(t))(x(t;\mathbf{b}) + c), \quad \mathbf{b} = (b_0, b_1, \ldots, b_{n-1})^T \in \mathbb{R}^n \\ &B_j \sim \mathcal{N}(\mu_j, \sigma_j^2), j = 0, 1, 2, \ldots, n-1, \text{ which are mutually independent,}\end{aligned} \tag{3.16}$$

where $\mathbf{b}$ is a realization of $\mathbf{B} = (B_0, B_1, \ldots, B_{n-1})^T$, $\boldsymbol{\rho} = (\rho_0, \rho_1, \ldots, \rho_{n-1})^T$ is a non-random vector functions of t, γ is a non-random function of t, and c is a given constant. Hence, it is easy to see that the solution to the first equation of (3.16) with initial condition $x(0) = x_0$ is given by

$$x(t; x_0, \mathbf{b}) = -c + (x_0 + c)\exp\left(\mathbf{b}\cdot\int_0^t \boldsymbol{\rho}(s)ds + \int_0^t \gamma(s)ds\right). \tag{3.17}$$

Assume that all the subsystems have the same probability density function for initial condition X_0, which is independent of B_j, $j = 0, 1, 2, \ldots, n-1$. Let $X(t) = x(t; X_0, \mathbf{B})$ and

$$Y(t) = \mathbf{B}\cdot\int_0^t \boldsymbol{\rho}(s)ds + \int_0^t \gamma(s)ds.$$

Then we have

$$X(t) = -c + (X_0 + c)\exp(Y(t)). \tag{3.18}$$

Since $B_j \sim \mathcal{N}(\mu_j, \sigma_j^2)$, and B_j, $j = 0, 1, 2, \ldots, n-1$, are mutually independent, we find that

$$Y(t) \sim \mathcal{N}\left(\int_0^t (\boldsymbol{\mu} \cdot \boldsymbol{\rho}(s) + \gamma(s))\, ds,\ \sum_{j=0}^{n-1} \sigma_j^2 \left(\int_0^t \rho_j(s) ds \right)^2 \right). \tag{3.19}$$

Stochastic formulation By the above discussions on the probabilistic formulation (3.16), we see that if all the individuals in the entire system have the same fixed initial condition x_0, then for any fixed t, $X(t)$ has a log-normal distribution when $c = 0$, and a shifted log-normal distribution when c is nonzero. Based on this information, the stochastic model is chosen to have the form

$$dX(t) = \xi(t)(X(t) + c)dt + \eta(t)(X(t) + c)dW(t), \quad X(0) = X_0, \tag{3.20}$$

where both ξ and η are non-random functions of t, and X_0 is independent of $W(t)$. Let $h(x, t) = \ln(x + c)$. Then we have

$$h_t(x, t) = 0,\ h_x(x, t) = \frac{1}{x + c},\ h_{xx}(x, t) = -\frac{1}{(x + c)^2}.$$

Hence, by Ito's formula, we have

$$dh(X(t), t) = (\xi(t) - \tfrac{1}{2}\eta^2(t))dt + \eta(t)dW(t).$$

Integrating both sides we have

$$X(t) = -c + (X_0 + c)\exp(Y(t)). \tag{3.21}$$

where $Y(t) = \displaystyle\int_0^t (\xi(s) - \tfrac{1}{2}\eta^2(s))ds + \int_0^t \eta(s)dW(s)$. By Lemma 3.2, for any fixed t we have that

$$\int_0^t \eta(s)dW(s) \sim \mathcal{N}\left(0, \int_0^t \eta^2(s)ds\right).$$

Hence, we find that for any fixed t we have

$$Y(t) \sim \mathcal{N}\left(\int_0^t (\xi(s) - \tfrac{1}{2}\eta^2(s))ds,\ \int_0^t \eta^2(s)ds \right). \tag{3.22}$$

Equivalent conditions By (3.18), (3.19), (3.21), and (3.22), we see that if functions ξ, η and ρ_j, and constants μ_j, σ_j and n satisfy the following two equalities

$$\int_0^t (\xi(s) - \tfrac{1}{2}\eta^2(s))ds = \int_0^t (\boldsymbol{\mu} \cdot \boldsymbol{\rho}(s) + \gamma(s))\, ds \tag{3.23}$$

and

$$\int_0^t \eta^2(s)ds = \sum_{j=0}^{n-1} \sigma_j^2 \left(\int_0^t \rho_j(s)ds \right)^2, \tag{3.24}$$

then probabilistic formulation (3.16) and stochastic formulation (3.20) yield processes that are pointwise equivalent in density.

Based on the above equivalent conditions, we next derive the specific forms of the functional parameters and/or values of the parameters of the corresponding pointwise equivalent (stochastic/probabilistic) formulation in terms of the functional parameters and/or parameters of the known (probabilistic/stochastic) formulation.

3.2.1 Probabilistic Formulation to Stochastic Formulation

Here we assume that probabilistic formulation (3.16) is known, and we wish to determine its corresponding pointwise equivalent stochastic formulation. Differentiating both sides of (3.24) with respect to t we have

$$\eta^2(t) = 2\sum_{j=0}^{n-1} \sigma_j^2 \rho_j(t) \left(\int_0^t \rho_j(s) ds \right).$$

Hence, if we assume that $\rho_j(t)$, $j = 0, 1, 2, \ldots, n-1$ have the property that the right-hand-side of the above equation is nonnegative for any $t \geq 0$, then we can always find η such that (3.24) holds, and it is given by

$$\eta(t) = \left[2\sum_{j=0}^{n-1} \sigma_j^2 \rho_j(t) \left(\int_0^t \rho_j(s) ds \right) \right]^{\frac{1}{2}}. \tag{3.25}$$

By (3.23) and (3.24) we find that

$$\int_0^t \xi(s) ds = \int_0^t \left(\boldsymbol{\mu} \cdot \boldsymbol{\rho}(s) + \gamma(s) \right) ds + \frac{1}{2} \sum_{j=0}^{n-1} \sigma_j^2 \left(\int_0^t \rho_j(s) ds \right)^2.$$

Differentiating both sides of the above equation with respect to t yields

$$\xi(t) = \boldsymbol{\mu} \cdot \boldsymbol{\rho}(t) + \gamma(t) + \sum_{j=0}^{n-1} \sigma_j^2 \rho_j(t) \left(\int_0^t \rho_j(s) ds \right). \tag{3.26}$$

Thus, if ξ and η are chosen as those in (3.26) and (3.25), then both (3.23) and (3.24) hold.

3.2.2 Stochastic Formulation to Probabilistic Formulation

Now we assume that stochastic formulation (3.20) is known, and we wish to determine its corresponding probabilistic formulation. It is obvious that if we set

$$\boldsymbol{\mu} \cdot \boldsymbol{\rho}(t) + \gamma(t) = \xi(t) - \tfrac{1}{2}\eta^2(t), \tag{3.27}$$

then (3.23) holds. By (3.27) and (3.24) we know that we have lots of different choices for the probabilistic formulation. Here we illustrate two of them.

Choice 1 One simple choice is to choose

$$n = 2, \quad \gamma \equiv 0, \quad \mu_0 = 1, \quad \mu_1 = 0, \quad \rho_0(t) = \xi(t) - \tfrac{1}{2}\eta^2(t). \tag{3.28}$$

Then by the above equalities, (3.24) and (3.27) we obtain that

$$\sigma_1^2 \left(\int_0^t \rho_1(s) ds \right)^2 = \int_0^t \eta^2(s) ds - \sigma_0^2 \left(\int_0^t (\xi(s) - \tfrac{1}{2}\eta^2(s)) ds \right)^2, \tag{3.29}$$

which implies that we need to choose σ_0 sufficiently small such that its right-hand side is greater than 0. Now by (3.29) we have

$$\sigma_1 \int_0^t \rho_1(s) ds = \left[\int_0^t \eta^2(s) ds - \sigma_0^2 \left(\int_0^t (\xi(s) - \tfrac{1}{2}\eta^2(s)) ds \right)^2 \right]^{\frac{1}{2}}.$$

Differentiating both sides of above equation with respect to t we obtain that

$$\sigma_1 \rho_1(t) = \frac{d}{dt} \left[\int_0^t \eta^2(s) ds - \sigma_0^2 \left(\int_0^t (\xi(s) - \tfrac{1}{2}\eta^2(s)) ds \right)^2 \right]^{\frac{1}{2}}.$$

Hence, we can just assign any positive value for σ_1, and then use the above equality to determine function ρ_1. Thus, one simple choice of the corresponding pointwise equivalent probabilistic formulation for the stochastic formulation (3.20) is as follows

$$\begin{aligned} &\frac{dx(t; b_0, b_1)}{dt} = \left[b_0 \left(\xi(t) - \tfrac{1}{2}\eta^2(t) \right) + b_1 \rho_1(t) \right] (x(t; b_0, b_1) + c), \quad b_0, b_1 \in \mathbb{R}, \\ &B_0 \sim \mathcal{N}(1, \sigma_0^2), \quad B_1 \sim \mathcal{N}(0, \sigma_1^2), \end{aligned}$$

where σ_0, σ_1 and ϱ_1 are chosen according to the above discussions.

Choice 2 Another simple choice is to choose

$$n = 1, \quad \gamma(t) = \xi(t) - \tfrac{1}{2}\eta^2(t), \quad \mu_0 = 0, \quad \sigma_0 = 1.$$

Then by the above equalities and (3.24) we obtain that

$$\int_0^t \rho_0(s) ds = \left(\int_0^t \eta^2(s) ds \right)^{\frac{1}{2}}. \tag{3.30}$$

Hence, if we set

$$\rho_0(t) = \frac{d}{dt} \left[\left(\int_0^t \eta^2(s) ds \right)^{\frac{1}{2}} \right], \tag{3.31}$$

then (3.30) holds. Thus, another simple choice of the corresponding pointwise equivalent probabilistic formulation for the stochastic formulation (3.20) is as follows

$$\begin{aligned} &\frac{dx(t; b)}{dt} = \left(b\rho_0(t) + \xi(t) - \tfrac{1}{2}\eta^2(t) \right) (x(t; b) + c), \quad b \in \mathbb{R}, \\ &B \sim \mathcal{N}(0, 1), \end{aligned}$$

where function ρ_0 is defined by (3.31).

3.3 System of Differential Equations

In this section, we derive the conditions for the systems under which the probabilistic and stochastic formulations have the same distribution (multivariate normal in the case the initial condition is a constant vector) at each time t.

Probabilistic formulation The probabilistic formulation considered has the following form

$$\begin{aligned} &\frac{d\mathbf{x}(t;\mathbf{b})}{dt} = \mathcal{A}(t)\mathbf{x}(t;\mathbf{b}) + \boldsymbol{\gamma}(t) + \mathcal{H}(t)\mathbf{b}, \quad \mathbf{b} \in \mathbb{R}^n, \\ &\mathbf{B} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{B}}, \boldsymbol{\Sigma}_{\mathbf{B}}), \end{aligned} \tag{3.32}$$

where $\mathcal{A}$ is a non-random $m \times m$ matrix functions of t, $\boldsymbol{\gamma}$ is a non-random m vector functions of t, $\mathcal{H}$ is a non-random $m \times n$ matrix functions of t, and $\mathbf{b}$ is a realization of $\mathbf{B}$. The solution to the first equation of (3.32) with initial conditions $\mathbf{x}(0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t;\mathbf{x}_0,\mathbf{b}) = \boldsymbol{\Phi}(t)\mathbf{x}_0 + \boldsymbol{\Phi}(t)\left[\int_0^t \boldsymbol{\Phi}^{-1}(s)\boldsymbol{\gamma}(s)ds + \int_0^t \boldsymbol{\Phi}^{-1}(s)\mathcal{H}(s)\mathbf{b}ds\right]. \tag{3.33}$$

Here $\boldsymbol{\Phi}(t)$ is the solution of deterministic initial value problem

$$\frac{d\boldsymbol{\Phi}(t)}{dt} = \mathcal{A}(t)\boldsymbol{\Phi}(t), \quad \boldsymbol{\Phi}(0) = \mathcal{I}_m, \tag{3.34}$$

where $\mathcal{I}_m$ is $m \times m$ identity matrix. Assume that all the subsystems have the same probability density function for initial condition $\mathbf{X}_0$, which is independent of $\mathbf{B}$, and let $\mathbf{X}(t) = \mathbf{x}(t;\mathbf{X}_0,\mathbf{B})$ and

$$\mathbf{Y}(t) = \boldsymbol{\Phi}(t)\left[\int_0^t \boldsymbol{\Phi}^{-1}(s)\boldsymbol{\gamma}(s)ds + \int_0^t \boldsymbol{\Phi}^{-1}(s)\mathcal{H}(s)\mathbf{B}ds\right].$$

Then we have that

$$\mathbf{X}(t) \;=\; \boldsymbol{\Phi}(t)\mathbf{X}_0 + \mathbf{Y}(t). \tag{3.35}$$

Note that $\mathbf{B} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{B}}, \boldsymbol{\Sigma}_{\mathbf{B}})$. Hence, we find that for any fixed t we have

$$\mathbf{Y}(t) \sim \mathcal{N}\left(\boldsymbol{\Phi}(t)\int_0^t \boldsymbol{\Phi}^{-1}(s)\left(\boldsymbol{\gamma}(s) + \mathcal{H}(s)\boldsymbol{\mu}_{\mathbf{B}}\right)ds, \boldsymbol{\Phi}(t)\boldsymbol{\Pi}_p(t)(\boldsymbol{\Phi}(t))^T\right), \tag{3.36}$$

where

$$\boldsymbol{\Pi}_p(t) = \left[\int_0^t \boldsymbol{\Phi}^{-1}(s)\mathcal{H}(s)ds\right]\Sigma_{\mathbf{B}}\left[\int_0^t \boldsymbol{\Phi}^{-1}(s)\mathcal{H}(s)ds\right]^T.$$

Stochastic formulation By the above discussions on the probabilistic formulation (3.32), we see that if all the individuals in the entire system have the same fixed initial condition $\mathbf{x}_0$, then $\mathbf{X}(t)$ is multivariate normal distributed for any fixed time t. Based on this piece of information, the stochastic model is chosen to have the following form:

$$d\mathbf{X}(t) = [\mathcal{A}(t)\mathbf{X}(t) + \boldsymbol{\xi}(t)]dt + \mathcal{F}(t)d\mathbf{W}(t), \quad \mathbf{X}(0) = \mathbf{X}_0, \tag{3.37}$$

where $\boldsymbol{\xi}$ is a non-random m vector functions of t, $\mathcal{F}$ is a non-random $m \times l$ matrix function of t, and $\mathbf{W}(t) = (W_1(t), W_2(t), \ldots, W_l(t))^T$ is a l-vector standard Wiener process independent of initial vector $\mathbf{X}_0$. Then the solution to (3.37) is given by (e.g., see [26, Section 5.6])

$$\mathbf{X}(t) = \boldsymbol{\Phi}(t)\mathbf{X}_0 + \mathbf{Y}(t), \tag{3.38}$$

where $\mathbf{Y}(t)$ is defined by

$$\mathbf{Y}(t) = \boldsymbol{\Phi}(t)\left[\int_0^t \boldsymbol{\Phi}^{-1}(s)\boldsymbol{\xi}(s)ds + \int_0^t \boldsymbol{\Phi}^{-1}(s)\mathcal{F}(s)d\mathbf{W}(s)\right].$$

In addition, for any fixed t we have (e.g., see [26, Section 5.6])

$$\mathbf{Y}(t) \sim \mathcal{N}\left(\boldsymbol{\Phi}(t)\int_0^t \boldsymbol{\Phi}^{-1}(s)\boldsymbol{\xi}(s)ds,\ \boldsymbol{\Phi}(t)\boldsymbol{\Pi}_s(t)(\boldsymbol{\Phi}(t))^T\right), \tag{3.39}$$

where

$$\boldsymbol{\Pi}_s(t) = \int_0^t \boldsymbol{\Phi}^{-1}(s)\mathcal{F}(s)\left(\boldsymbol{\Phi}^{-1}(s)\mathcal{F}(s)\right)^T ds.$$

Equivalent conditions By (3.35), (3.36), (3.38) and (3.39), we know that if functions $\boldsymbol{\xi}$, $\mathcal{F}$, $\boldsymbol{\gamma}$, $\mathcal{H}$, $\boldsymbol{\mu}_{\mathbf{B}}$ and $\boldsymbol{\Sigma}_{\mathbf{B}}$ satisfy the following two equalities

$$\int_0^t \boldsymbol{\Phi}^{-1}(s)\left(\boldsymbol{\gamma}(s) + \mathcal{H}(s)\boldsymbol{\mu}_{\mathbf{B}}\right)ds = \int_0^t \boldsymbol{\Phi}^{-1}(s)\boldsymbol{\xi}(s)ds, \tag{3.40}$$

and $\boldsymbol{\Pi}_p(t) = \boldsymbol{\Pi}_s(t)$, that is,

$$\left[\int_0^t \boldsymbol{\Phi}^{-1}(s)\mathcal{H}(s)ds\right]\Sigma_{\mathbf{B}}\left[\int_0^t \boldsymbol{\Phi}^{-1}(s)\mathcal{H}(s)ds\right]^T = \int_0^t \boldsymbol{\Phi}^{-1}(s)\mathcal{F}(s)\left(\boldsymbol{\Phi}^{-1}(s)\mathcal{F}(s)\right)^T ds, \tag{3.41}$$

then probabilistic formulation (3.32) and stochastic formulation (3.37) are pointwise equivalent in density.

Next we will derive the specific forms of the functional parameters and/or values of the parameters of the corresponding pointwise equivalent (stochastic/probabilistic) formulation in terms of the functional parameters and/or parameters of the known (probabilistic/stochastic) formulation based on the equivalent conditions (3.40) and (3.41).

3.3.1 Probabilistic Formulation to Stochastic Formulation

Here we assume that probabilistic formulation (3.32) is known, and we want to determine its corresponding pointwise stochastic formulation. By (3.40), it is obvious that if function $\boldsymbol{\xi}$ is chosen to be

$$\boldsymbol{\xi}(t) = \boldsymbol{\gamma}(t) + \mathcal{H}(t)\boldsymbol{\mu}_{\mathbf{B}},$$

then (3.40) holds. Differentiating both sides of (3.41) with respect to t we find that

$$\mathcal{F}(t)\mathcal{F}^T(t) = \boldsymbol{\Phi}(t)\dot{\boldsymbol{\Pi}}_p(t)(\boldsymbol{\Phi}(t))^T. \tag{3.42}$$

Hence, if we assume $\boldsymbol{\Pi}_p$ have the property that $\boldsymbol{\Phi}(t)\dot{\boldsymbol{\Pi}}_p(t)(\boldsymbol{\Phi}(t))^T$ is a positive-semidefinite matrix for any $t \geq 0$, then we can always find $\mathcal{F}$ such that (3.42) holds.

3.3.2 Stochastic Formulation to Probabilistic Formulation

Now we assume that stochastic formulation (3.37) is known, and we want to determine its corresponding pointwise equivalent probabilistic formulation. It is obvious if $\boldsymbol{\gamma}$, $\mathcal{H}$ and $\boldsymbol{\mu}_{\mathbf{B}}$ are chosen such that

$$\boldsymbol{\gamma}(t) + \mathcal{H}(t)\boldsymbol{\mu}_{\mathbf{B}} = \boldsymbol{\xi}(t),$$

then (3.40) holds. Note that $\mathcal{H}$ also needs to be properly chosen to satisfy (3.41). Hence, for simplicity, we choose

$$\boldsymbol{\gamma}(t) = \boldsymbol{\xi}(t), \quad \boldsymbol{\mu}_{\mathbf{B}} = \mathbf{0}. \tag{3.43}$$

Note that for any $t \geq 0$, $\Pi_s(t)$ is a positive-semidefinite matrix. Hence, there exists a $m \times m$ matrix function $\boldsymbol{\Lambda}(t)$ such that

$$\boldsymbol{\Pi}_s(t) = \boldsymbol{\Lambda}(t)(\boldsymbol{\Lambda}(t))^T.$$

Thus, if we choose

$$n = m, \quad \boldsymbol{\Sigma}_{\mathbf{B}} = \mathcal{I}_m, \quad \mathcal{H}(t) = \boldsymbol{\Phi}(t)\dot{\boldsymbol{\Lambda}}(t), \tag{3.44}$$

then (3.41) holds. Therefore, one of the simple choices of the corresponding pointwise equivalent probabilistic formulation for the stochastic formulation (3.37) is as follows

$$\begin{aligned} &\frac{d\mathbf{x}(t;\mathbf{b})}{dt} = \mathcal{A}(t)\mathbf{x}(t;\mathbf{b}) + \boldsymbol{\xi}(t) + \boldsymbol{\Phi}(t)\dot{\boldsymbol{\Lambda}}(t)\mathbf{b}, \quad \mathbf{b} \in \mathbb{R}^m, \\ &\mathbf{B} \sim \mathcal{N}(\mathbf{0}, \mathcal{I}_m). \end{aligned} \tag{3.45}$$

4 Computing Covariances

As we have already stated, the procedures above lead to pointwise equivalent in density systems that are <u>not</u> described by the same stochastic processes. We revisit the examples

of (2.1) to illustrate this by comparing covariances of the two processes. We proceed to use the Lemmas 3.1 and 3.2 to find the covariance function of the stochastic processes $Y_{PRD}(t)$ in the probabilistic formulation (2.4) and $Y_{SRM}(t)$ in the stochastic formulations (2.5).

Probabilistic formulation: Note that for this case Y_{PRD} is given by

$$Y_{PRD}(t) = \exp(Bt - \tfrac{1}{2}\sigma_0^2 t^2), \text{ where } B \sim \mathcal{N}(b_0, \sigma_0^2).$$

Hence, by Lemma 3.1 we find immediately

$$\mathbb{E}(Y_{PRD}(t)) = \exp(b_0 t). \tag{4.1}$$

Then using Lemma 3.1 and (4.1) we find the covariance function for the process $\{Y(t)\} = \{Y_{PRD}(t)\}$ given by

$$\begin{aligned}
\text{Cov}(Y(t), Y(s)) &= \mathbb{E}(Y(t)Y(s)) - \mathbb{E}(Y(t))\mathbb{E}(Y(s)) \\
&= \mathbb{E}\left\{\exp\left(B(t+s) - \tfrac{1}{2}\sigma_0^2(t^2+s^2)\right)\right\} - \exp(b_0(t+s)) \\
&= \exp\left(b_0(t+s) - \tfrac{1}{2}\sigma_0^2(t^2+s^2) + \tfrac{1}{2}\sigma_0^2(t+s)^2\right) - \exp(b_0(t+s)) \\
&= \exp\left(b_0(t+s) + st\sigma_0^2\right) - \exp(b_0(t+s)) \\
&= \exp(b_0(t+s))\left[\exp\left(st\sigma_0^2\right) - 1\right].
\end{aligned}$$

Stochastic formulation: In this case we have

$$Y_{SRM}(t) = \exp\left((b_0 t - \tfrac{1}{2}\sigma_0^2 t^2) + \sigma_0 \int_0^t \sqrt{2\tau} dW(\tau)\right).$$

Let $Q(t) = \sigma_0 \int_0^t \sqrt{2\tau} dW(\tau)$. Then by Lemma 3.2, we have that $\{Q(t)\}$ is a Gaussian process with zero mean and covariance function given by

$$\text{Cov}(Q(t), Q(s)) = \sigma_0^2 \min\{t^2, s^2\}. \tag{4.2}$$

Using Lemma 3.1 and (4.2) we find that

$$\mathbb{E}(Y_{SRM}(t)) = \exp(b_0 t). \tag{4.3}$$

By Lemma 3.2 we know that $\{Q(t)\}$ is a Gaussian process. Hence, $Q(t)+Q(s)$ has a Gaussian distribution with zero mean and variance defined by

$$\begin{aligned}
\text{Var}(Q(t)+Q(s)) &= \text{Var}(Q(t)) + \text{Var}(Q(s)) + 2\text{Cov}(Q(t), Q(s)) \\
&= \sigma_0^2\left(t^2 + s^2 + 2\min\{t^2, s^2\}\right).
\end{aligned} \tag{4.4}$$

Now we use Lemma 3.1, along with equations (4.3) and (4.4) to find the covariance function of $\{Y(t)\} = \{Y_{SRM}(t)\}$.

$$\begin{aligned}
&\mathrm{Cov}(Y(t), Y(s)) \\
&\quad = \mathbb{E}(Y(t)Y(s)) - \mathbb{E}(Y(t))\mathbb{E}(Y(s)) \\
&\quad = \mathbb{E}\left\{\exp\left(b_0(t+s) - \tfrac{1}{2}\sigma_0^2(t^2+s^2) + Q(t) + Q(s)\right)\right\} - \exp(b_0(t+s)) \\
&\quad = \exp\left(b_0(t+s) - \tfrac{1}{2}\sigma_0^2(t^2+s^2) + \tfrac{1}{2}\sigma_0^2\left(t^2+s^2+2\min\{t^2,s^2\}\right)\right) - \exp(b_0(t+s)) \\
&\quad = \exp\left(b_0(t+s) + \sigma_0^2\min\{t^2,s^2\}\right) - \exp(b_0(t+s)) \\
&\quad = \exp(b_0(t+s))\left[\exp\left(\sigma_0^2\min\{t^2,s^2\}\right) - 1\right].
\end{aligned}$$

Thus we see that $\{Y_{PRD}(t)\}$ and $\{Y_{SRM}(t)\}$ have different covariance functions and hence are *not* the same stochastic process.

5 Equivalence between Probabilistic and Stochastic Formulations with Nonlinear Dynamics

In this section we turn to nonlinear stochastic differential equations which can be shown equivalent to a PRD formulation with nonlinear dynamics. In summary of our results to date, based on the discussions in Sections 3.1 and 3.2, we see that we can find the corresponding pointwise equivalent probabilistic formulation for two types of scalar stochastic differential equations:

$$dX(t) = [\alpha(t)X(t) + \xi(t)]dt + \eta(t)dW(t),$$

and

$$dX(t) = \xi(t)(X(t) + c)dt + \eta(t)(X(t) + c)dW(t),$$

where ξ, η, and α are all deterministic functions of t, and c is a given constant. In addition, we can find the corresponding pointwise equivalent probabilistic formulation for the following linear system of stochastic differential equations (see Section 3.3)

$$d\mathbf{X}(t) = [\mathcal{A}(t)\mathbf{X}(t) + \boldsymbol{\xi}(t)]dt + \mathcal{F}(t)d\mathbf{W}(t),$$

where $\mathcal{A}$ is a non-random $m \times m$ matrix functions of t, $\boldsymbol{\xi}$ is a non-random m vector functions of t, $\mathcal{F}$ is a non-random $m \times l$ matrix function of t, and $\mathbf{W}(t) = (W_1(t), W_2(t), \ldots, W_l(t))^T$ is a l-vector standard Wiener process. Hence, if a nonlinear stochastic differential equation (or system of stochastic differential equations) can be reduced to one of the above forms by some invertible transformation, then one can find its corresponding probabilistic formulation. The same thing is true for the probabilistic formulation.

5.1 Reducible Nonlinear Stochastic Differential Equations

First we will consider several special cases of nonlinear stochastic differential equations that can be reduced to linear stochastic differential equations after some invertible transformation. Proofs are given in [24, Section 4.1].

Theorem 5.1 *Consider the stochastic differential equation*

$$dX(t) = g(X(t),t)dt + \sigma(X(t),t)dW(t), \tag{5.1}$$

where g and σ are non-random functions of x and t. If the equality

$$\frac{\partial}{\partial x}\left\{\sigma(x,t)\left[\frac{1}{\sigma^2(x,t)}\frac{\partial\sigma}{\partial t}(x,t) - \frac{\partial}{\partial x}\left(\frac{g}{\sigma}\right)(x,t) + \frac{1}{2}\frac{\partial^2\sigma}{\partial x^2}(x,t)\right]\right\} = 0$$

holds, then the nonzero deterministic function $\bar{\sigma}(t)$ can be determined from

$$\bar{\sigma}'(t) = \bar{\sigma}(t)\sigma(x,t)\left[\frac{1}{\sigma^2(x,t)}\frac{\partial\sigma}{\partial t}(x,t) - \frac{\partial}{\partial x}\left(\frac{g}{\sigma}\right)(x,t) + \frac{1}{2}\frac{\partial^2\sigma}{\partial x^2}(x,t)\right],$$

and some smooth invertible function $h(x,t)$ can be computed from

$$\frac{\partial h}{\partial x}(x,t) = \frac{\bar{\sigma}(t)}{\sigma(x,t)}.$$

Moreover, (5.1) *can be reduced to the linear stochastic differential equation*

$$dZ(t) = \bar{g}(t)dt + \bar{\sigma}(t)dW(t),$$

where $Z(t) = h(X(t),t)$ and the deterministic function $\bar{g}(t)$ can be computed from

$$\bar{g}(t) = \frac{\partial h}{\partial t}(x,t) + \frac{\partial h}{\partial x}(x,t)g(x,t) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(x,t)\sigma^2(x,t).$$

Theorem 5.2 *The autonomous stochastic differential equation*

$$dX(t) = g(X(t))dt + \sigma(X(t))dW(t)$$

can be reduced to the linear stochastic differential equation

$$dZ(t) = (\lambda_0 + \lambda_1 Z(t))dt + (\nu_0 + \nu_1 Z(t))dW(t)$$

if and only if

$$\psi'(x) = 0 \text{ or } \left(\frac{(\sigma\psi')'}{\psi'}\right)'(x) = 0. \tag{5.2}$$

Here g and σ are non-random functions of x, λ_0, λ_1, ν_0 and ν_1 are some constants, $\psi(x) = \frac{g(x)}{\sigma(x)} - \frac{1}{2}\sigma'(x)$, and $Z(t) = h(X(t))$, where h is some invertible transformation.

If the latter part of (5.2) *is satisfied, then we set* $\nu_1 = -\dfrac{(\sigma\psi')'}{\psi'}$. *If* $\nu_1 \neq 0$, *then we can choose*

$$h(x) = c\exp\left(\nu_1 \int_a^x \frac{1}{\sigma(\tau)}d\tau\right),$$

where c *is some constant. If* $\nu_1 = 0$, *then we can choose*

$$h(x) = \nu_0 \int_a^x \frac{1}{\sigma(\tau)}d\tau + c.$$

Remark 5.3 *For a general nonlinear system of stochastic differential equations, it is difficult to obtain the explicit form for the invertible transformation (as it strongly depends on the specific form of the system). Hence, we do not pursue this effort in this paper.*

5.2 Examples

We next use several examples to illustrate this transformation method to find the corresponding equivalent probabilistic/stochastic formulations for stochastic/probabilistic formulations.

Example 5.1 *(Exponential Modulated Growth in the Drift)*
In this example, we use the transformation method to find the equivalent probabilistic formulation for the following nonlinear stochastic differential equation

$$dX = \left[1 - \frac{1}{2}\exp(-2X)\right]dt + \exp(-X)dW.$$

Note here that $g(x) = 1 - \dfrac{1}{2}\exp(-2x)$. Let $Z = \exp(X)$. Then by Ito's formula we find that

$$\begin{aligned} dZ &= \exp(X)\left\{\left[1 - \frac{1}{2}\exp(-2X)\right]dt + \exp(-X)dW\right\} + \frac{1}{2}\exp(X)\exp(-2X)dt \\ &= Zdt + dW. \end{aligned}$$

By the discussions in Section 3.1.2 we find the equivalent probabilistic formulation for the above linear stochastic differential equation is

$$\frac{dz(t;b)}{dt} = z(t;b) + b\rho(t),\ b \in \mathbb{R};\ B \sim \mathcal{N}(0,1),$$

where $\rho(t) = \dfrac{\exp(2t)}{\sqrt{2[\exp(2t)-1]}} - \sqrt{\dfrac{\exp(2t)-1}{2}}$. Let $x = \ln(z)$. Then we have

$$\frac{dx}{dt} = \frac{1}{z}(z + b\rho(t)) = 1 + b\rho(t)\exp(-x).$$

Thus, the following two formulations

$$dX = \left[1 - \frac{1}{2}\exp(-2X)\right] dt + \exp(-X)dW$$

and

$$\begin{array}{l} \dfrac{dx(t;b)}{dt} = 1 + b\left[\dfrac{\exp(2t)}{\sqrt{2[\exp(2t)-1]}} - \sqrt{\dfrac{\exp(2t)-1}{2}}\right]\exp(-x(t;b)), \quad b \in \mathbb{R}, \\ B \sim \mathcal{N}(0,1) \end{array}$$

are pointwise equivalent in density.

Example 5.2 *(Logistics Growth Dynamics with Uncertainty)*
We begin with logistic growth in the probabilistic formulation and derive its equivalent stochastic formulation. Consider the deterministic logistic equation

$$\frac{dx}{dt} = bx\left(1 - \frac{x}{\kappa}\right),\ x(0) = x_0, \tag{5.3}$$

where b is some constant representing the intrinsic growth rate, and κ is a given constant representing the carrying capacity. Let $z = \dfrac{1}{x}$. Then it is easy to find that

$$\frac{dz}{dt} = -b\left(z - \frac{1}{\kappa}\right).$$

By the discussions in Section 3.2.1, we know that for probabilistic formulation

$$\frac{dz(t;b)}{dt} = -b\left(z(t;b) - \frac{1}{\kappa}\right),\ b \in \mathbb{R};\ B \sim \mathcal{N}(\mu_0, \sigma_0^2),$$

its equivalent form of stochastic formulation is given by

$$dZ(t) = (-\mu_0 + \sigma_0^2 t)\left(Z - \frac{1}{\kappa}\right)dt + \sqrt{2t}\sigma_0\left(Z - \frac{1}{\kappa}\right)dW(t).$$

Let $X(t) = \dfrac{1}{Z(t)}$. Then by Ito's formula we find that

$$\begin{aligned} dX(t) &= -\frac{1}{Z^2(t)}\left[(-\mu_0 + \sigma_0^2 t)\left(Z - \frac{1}{\kappa}\right)dt + \sqrt{2t}\sigma_0\left(Z - \frac{1}{\kappa}\right)dW(t)\right] \\ &\quad + \frac{1}{Z^3(t)}\left[\sqrt{2t}\sigma_0\left(Z - \frac{1}{\kappa}\right)\right]^2 dt \\ &= X\left[(\mu_0 - \sigma_0^2 t)\left(1 - \frac{X}{\kappa}\right) + 2t\sigma_0^2\left(1 - \frac{X}{\kappa}\right)^2\right]dt - \sqrt{2t}\sigma_0 X\left(1 - \frac{X}{\kappa}\right)dW(t). \end{aligned}$$

Thus, the probabilistic formulation

$$\frac{dx(t;b)}{dt} = bx(t;b)\left(1 - \frac{x(t;b)}{\kappa}\right), \ b \in \mathbb{R}; \ B \sim \mathcal{N}(\mu_0, \sigma_0^2) \tag{5.4}$$

and the stochastic formulation

$$dX(t) = X\left[(\mu_0 - \sigma_0^2 t)\left(1 - \frac{X}{\kappa}\right) + 2t\sigma_0^2\left(1 - \frac{X}{\kappa}\right)^2\right]dt - \sqrt{2t}\sigma_0 X\left(1 - \frac{X}{\kappa}\right)dW(t) \tag{5.5}$$

are pointwise equivalent in density. Figure 1 depicts the probability density function $p(x,t)$ at different times t for the probabilistic formulation (5.4) and the stochastic formulation (5.5) with $\kappa = 100$, $x_0 = 10$, $\mu_0 = 1$ and $\sigma_0 = 0.1$, where $p(x,t)$ is obtained by simulating 10^5 sample paths for each formulation. Here for the probabilistic formulation (5.4), we

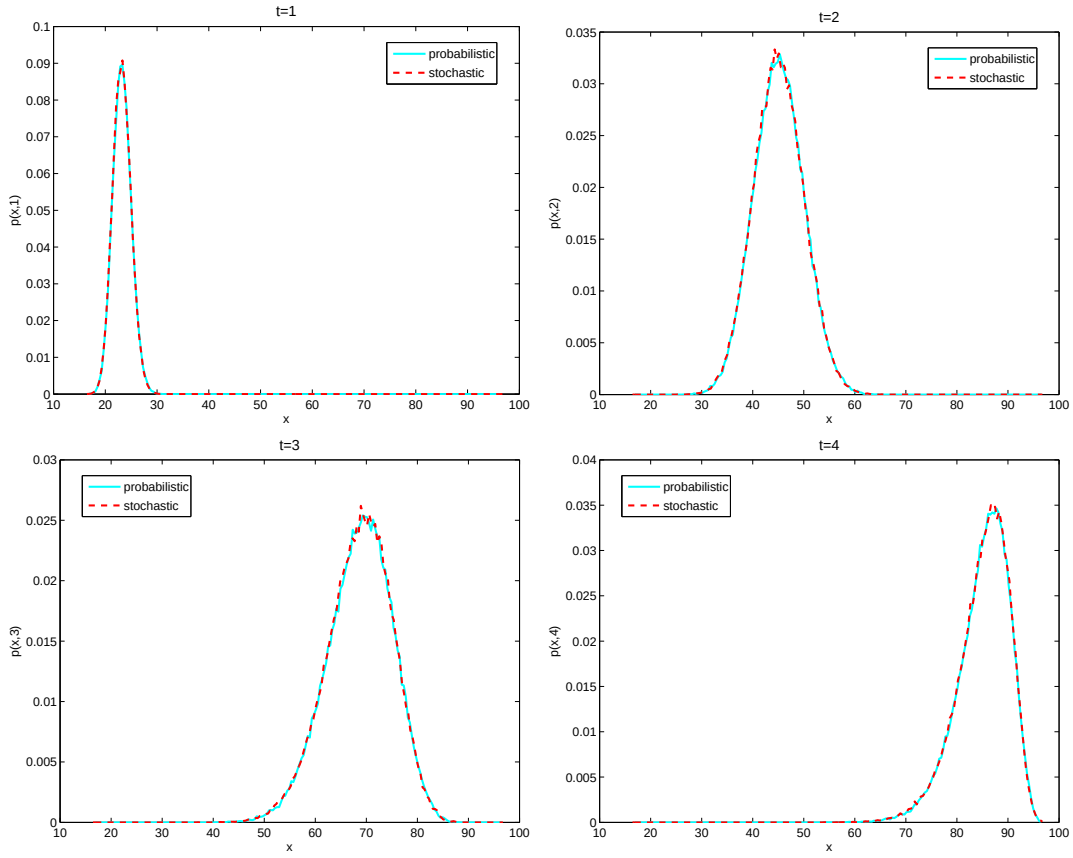

Figure 1: Probability density function $p(x,t)$ are obtained by simulating 10^5 sample paths for probabilistic formulation (5.4) and stochastic formulation (5.5) at $t = 1$, 2, 3 and 4, where $\Delta t = 0.004$ is used in (5.6), and $T = 4$.

analytically solve each deterministic differential equation (5.3) with b being a realization of B, which follows a normal distribution $\mathcal{N}(1, 0.01)$. This solution is given by

$$x(t;b) = \frac{x_0\kappa\exp(bt)}{\kappa + x_0(\exp(bt) - 1)}.$$

Thus implementation of the probabilistic formulation (PRD) is extremely rapid (essentially a function evaluation). This is not true for the SRM and hence the PRD formulation requires much less (orders of magnitude) implementation time than does the SRM. Even in examples where both methods require numerical integration, the PRD is highly preferred to the SRM in implementation of inverse problems (e.g., see [11, 14, 20]). We use an Euler explicit method (crude but sufficient for our purposes here since our main objective is to demonstrate our theoretical equivalence results with a numerical example) to numerically approximate the sample paths for the stochastic differential equation (5.5). Let T denote the final time, and m be the number of mesh points interval. Then the mesh time points are given by $t_k = k\Delta t, k = 0, 1, 2, \ldots, m$, where $\Delta t = T/m$. Denote by X^k the numerical solution for $X(t_k)$; then we have the following numerical scheme:

$$\begin{aligned} X^{k+1} &= X^k + X^k \left[(\mu_0 - \sigma_0^2 t_k) \left(1 - \frac{X^k}{\kappa} \right) + 2t_k \sigma_0^2 \left(1 - \frac{X^k}{\kappa} \right)^2 \right] \Delta t \\ &\quad -\sqrt{2t_k}\sigma_0 X^k \left(1 - \frac{X^k}{\kappa} \right) \mathcal{E}_k, \ k = 1, 2, \ldots, m-1. \end{aligned} \tag{5.6}$$

where $\mathcal{E}_k$ is a random variable following a normal distribution $\mathcal{N}(0, \Delta t)$. From Figure 1 we see that we obtain the same probability density function for the probabilistic formulation (5.4) and the stochastic formulation (5.5), which nicely illustrates our earlier theoretical results.

Example 5.3 *(Gompertz Growth in the Drift)*
In this example, we consider the following nonlinear stochastic differential equation

$$dX(t) = [a_0(t) - a_1(t) \ln(X(t))] X(t)dt + \sqrt{2d_0(t)}X(t)dW(t), \tag{5.7}$$

where a_0, a_1 and d_0 are some deterministic functions of t, and a_1 and d_0 are assumed to be positive. This equation is a stochastic version of the generalized Gompertz model $\dot{x} = (a_0(t) - a_1(t) \ln x)x$, which has been extensively used in biological and medical research to describe population dynamics such as tumor growth in humans and animals either with or without treatment (e.g., [1, 23] and the references therein) as well in cell proliferations models [18]. Next we use the transformation method to find an equivalent probabilistic formulation for (5.7). Let $Z = \ln(X)$. Then by Ito's formula and (5.7) we have

$$\begin{aligned} dZ(t) &= \frac{1}{X(t)} \left\{ [a_0(t) - a_1(t) \ln(X(t))] X(t)dt + \sqrt{2d_0(t)}X(t)dW(t) \right\} \\ &\quad -\frac{1}{2X^2(t)} \left[2d_0(t)X^2(t) \right] dt \\ &= [-a_1(t)Z(t) + (a_0(t) - d_0(t))] dt + \sqrt{2d_0(t)}dW(t). \end{aligned}$$

By the discussions in Section 3.1.2 we find that a pointwise equivalent probabilistic formulation for the above linear stochastic differential equation is given by

$$\begin{aligned} &\frac{dz(t;b)}{dt} = -a_1(t)z(t;b) + a_0(t) - d_0(t) + b\rho(t), \quad b \in \mathbb{R}, \\ &B \sim \mathcal{N}(0, 1). \end{aligned} \tag{5.8}$$

Here b is a realization of B, and ρ is given by

$$\rho(t) = \frac{d}{dt}\left(\sqrt{\varphi(t)}\right) + a_1(t)\sqrt{\varphi(t)}.$$

where

$$\varphi(t) = \int_0^t 2d_0(s)\exp\left(-2\int_s^t a_1(\tau)d\tau\right)ds.$$

Let $x = \exp(z)$. Then by (5.8) we find

$$\begin{aligned}\frac{dx(t;b)}{dt} &= \exp(z(t;b))\left[-a_1(t)z(t;b) + a_0(t) - d_0(t) + b\rho(t)\right] \\ &= x(t;b)\left[-a_1(t)\ln(x(t;b)) + a_0(t) - d_0(t) + b\rho(t)\right].\end{aligned}$$

Thus, the following two formulations

$$dX(t) = \left[a_0(t) - a_1(t)\ln(X(t))\right]X(t)dt + \sqrt{2d_0(t)}X(t)dW(t)$$

and

$$\frac{dx(t;b)}{dt} = x(t;b)\left[-a_1(t)\ln(x(t;b)) + a_0(t) - d_0(t) + b\rho(t)\right], \quad b \in \mathbb{R}$$
$$B \sim \mathcal{N}(0,1)$$

are pointwise equivalent in density.

6 Concluding Remarks

In summary, we have derived several classes of examples with affine dynamics for which we can establish pointwise equivalence in density for the corresponding probabilistic and stochastic formulations. We then argue that a large class of nonlinear SDE can be reduced by invertible transformation to one of the affine cases. We presented several examples of nonlinear SDE arising frequently in applications that can be transformed to the more readily computed probabilistic formulation. It has been well documented (e.g., see [20, 25] and references therein) that difficulties arise in numerically solving Fokker-Planck equations such as (1.1) when the drift g dominates the diffusion σ^2. This motivated our efforts and the results here lead to alternative methods that can be fast and efficient in numerically solving (1.1) by employing its pointwise equivalent in density probabilistic formulation. Future efforts include investigation of the approximate equivalent probabilistic formulations for non-reducible nonlinear stochastic differential equations by the linearization method as well investigation of applications of our methodology to the control of systems with uncertainty.

Acknowledgment

This research was supported in part by grant number R01AI071915-07 from the National Institute of Allergy and Infectious Diseases and in part by the Air Force Office of Scientific

Research under grant number FA9550-09-1-0226. The authors are also grateful to several referees whose comments and questions led to improvements in the presentation of this paper.

References

[1] G. Albano, V. Giorno, P. Roman-Roman, and F. Torres-Ruiz, Inferring the effect of therapy on tumors showing stochastic Gompertzian growth, *Journal of Theoretical Biology*, **276** (2011), 67–77.

[2] L.J.S. Allen, *An Introduction to Stochastic Processes with Applications to Biology*, Prentice Hall, New Jersey, 2003.

[3] P. Bai, H.T. Banks, S. Dediu, A. Y. Govan, M. Last, A. Loyd, H. K. Nguyen, M. S. Olufsen, G. Rempala, and B. D. Slenning, Stochastic and deterministic models for agricultural production networks, CRSC-TR07-06, February, 2007; *Math. Biosci. and Engr.*, **4** (2007), 373–402.

[4] H.T. Banks and K.L. Bihari, Modelling and estimating uncertainty in parameter estimation, *Inverse Problems*, **17** (2001), 95–111.

[5] H.T. Banks, V.A. Bokil, S. Hu, A.K. Dhar, R.A. Bullis, C.L. Browdy and F.C.T. Allnutt, Modeling shrimp biomass and viral infection for production of biological countermeasures, CRSC-TR05-45, NCSU, December, 2005; *Mathematical Biosciences and Engineering*, **3** (2006), 635–660.

[6] H.T. Banks, D.M. Bortz, G.A. Pinter and L.K. Potter, Modeling and imaging techniques with potential for application in bioterrorism, CRSC-TR03-02, NCSU, January, 2003; Chapter 6 in *Bioterrorism: Mathematical Modeling Applications in Homeland Security*, (H.T. Banks and C. Castillo-Chavez, eds.) Frontiers in Applied Math, **FR28**, SIAM (2003), Philadelphia, PA, 129–154.

[7] H.T. Banks, L.W. Botsford, F. Kappel and C. Wang, Modeling and estimation in size structured population models, LCDS-CCS Report 87-13, Brown University; *Proceedings 2nd Course on Mathematical Ecology*, (Trieste, December 8-12, 1986) World Press, Singapore, 1988, 521–541.

[8] H.T. Banks and J.L. Davis, Quantifying uncertainty in the estimation of probability distributions, CRSC-TR07-21, December, 2007; *Math. Biosci. Engr.*, **5** (2008), 647–667.

[9] H.T. Banks, J.L. Davis, S.L. Ernstberger, S. Hu, E. Artimovich, A.K. Dhar and C.L. Browdy, A comparison of probabilistic and stochastic formulations in modeling growth uncertainty and variability, CRSC-TR08-03, NCSU, February, 2008; *Journal of Biological Dynamics*, **3** (2009) 130–148.

[10] H.T. Banks, J.L. Davis, S.L. Ernstberger, S. Hu, E. Artimovich and A.K. Dhar, Experimental design and estimation of growth rate distributions in size-structured shrimp populations, CRSC-TR08-20, NCSU, November, 2008; *Inverse Problems,* **25** (2009), 095003(28pp), Sept.

[11] H.T. Banks, J.L. Davis and S. Hu, A computational comparison of alternatives to including uncertainty in structured population models, CRSC-TR09-14, June, 2009; in *Three Decades of Progress in Systems and Control*, X. Hu, U. Jonsson, B. Wahlberg, B. Ghosh (Eds.), Springer, 2010.

[12] H.T. Banks and B.G. Fitzpatrick, Estimation of growth rate distributions in size structured population models, *Quarterly of Applied Mathematics*, **49** (1991), 215-235.

[13] H.T. Banks and S. Hu, Nonlinear stochastic markov processes and modeling uncertainty in populations, CRSC-TR11-02, Center for Research in Scientific Computation, North Carolina State University, January, 2011.

[14] H.T. Banks, B.G. Fitzpatrick, L.K. Potter and Y. Zhang, Estimation of probability distributions for individual parameters using aggregate population data, CRSC-TR98-6, NCSU, January, 1998; in *Stochastic Analysis, Control, Optimization and Applications*, (Edited by W. McEneaney, G. Yin and Q. Zhang), Birkhauser, Boston, 1989, 353-371.

[15] H.T. Banks, P.M. Kareiva and L. Zia, Analyzing field studies of insect dispersal using two dimensional transport equations, *Environmental Entomology*, **17** (1988), 815-820.

[16] H.T. Banks, K.L. Rehm and K.L. Sutton, Dynamic social network models incorporating stochasticity and delays, CRSC-TR09-11, May, 2009; *Quarterly Applied Math.*, **68** (2010), 783–802.

[17] H.T. Banks, K.L. Sutton, W.C. Thompson, G. Bocharov, D. Roose, T. Schenkel and A. Meyerhans, Estimation of cell proliferation dynamics using CFSE data, Center for Research in Scientific Computation Technical Report CRSC-TR09-17, NCSU, August, 2009; *Bull. Math. Biol.*, **73** (2011), 116–150; DOI: 10.1007/s11538-010-9524-5.

[18] H.T. Banks, K.L. Sutton, W.C. Thompson, G. Bocharov, M. Doumic, T. Schenkel, J. Argilaguet, S. Giest, C. Peligero and A. Meyerhans, A new model for the estimation of cell proliferation dynamics using CFSE data, CRSC-TR11-05, N.C. State University, Raleigh, NC, March, 2011; *J. Immunological Methods,* submitted.

[19] H.T. Banks and H.T. Tran, *Mathematical and Experimental Modeling of Physical and Biological Processes*, CRC Press, Boca Raton, 2009.

[20] H.T. Banks, H.T. Tran and D.E. Woodward, Estimation of variable coefficients in the Fokker-Planck equations using moving node finite elements, *SIAM J. Numer. Anal.*, **30** (1993), 1574–1602.

[21] G. Casella and R.L. Berger, *Statistical Inference*, Duxbury, California, 2002.

[22] S.N. Ethier and T.G. Kurtz, *Markov Processes: Characterization and Convergence*, Wiley Series in Probability and Statistics, New York, 1986.

[23] L. Ferrante, S. Bompadre, L. Possati, and L. Leone, Parameter estimation in a Gompertzian stochastic model for tumor growth, *Biometrics*, **56** (2000), 1076–1081.

[24] T.C. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York, 1988.

[25] G.W. Harrison, Numerical solution of the Fokker-Planck equation using moving finite elements, *Numerical Methods for Partial Differential Equations*, **4** (1988), 219–232.

[26] I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Second Edition, Springer, New York, 1991.

[27] F. Klebaner, *Introduction to Stochastic Calculus with Applications*, 2nd ed., Imperial College Press, London, 2006.

[28] T. Luzyanina, D. Roose, T. Schenkel, M. Sester, S. Ehl, A. Meyerhans and G. Bocharov, Numerical modelling of label-structured cell population growth using CFSE distribution data, *Theoretical Biology and Medical Modelling*, **4** (2007), 1–26.

[29] J.A.J. Metz and E.O. Diekmann, *The Dynamics of Physiologically Structured Populations,* Lecture Notes in Biomathematics. Vol. **68**, Springer, Heidelberg, 1986.

[30] S. Michelson, K. Ito, H.T. Tran and J.T. Leith, Stochastic models for subpopulation emergence in heterogeneous tumors, *Bulletin of mathematical Biology*, **51** (1989), 731-747.

[31] A. Okubo, *Diffusion and Ecological Problems: Mathematical Models*, Biomathematics, **10** (1980), Springer-Verlag, Berlin.

[32] J. Sinko and W. Streifer, A new model for age-size structure of a population, *Ecology,* **48** (1967), 910–918.

[33] T.T. Soong, *Random Differential Equations in Science and Engineering*, Academic Press, New York and London, 1973.