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14. ABSTRACT

Several Topics in Stochastic Analysis are studied. A partial list is as follows: (i)Asymptotic Results for Near Critical Branching Processes, (ii)Large Deviation Properties of Weakly Interacting Processes, (iii)Multiscale Diffusion Approximations for Stochastic Networks in Heavy Traffic, (iv) Adaptive Ergodic Control of Markov Chains, (v)Controlled Stochastic Networks in Heavy Traffic, (vi)Exit Time and Invariant Measure Asymptotics for

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Some Topics in Stochastic Control: 52311-MA

Final Report, July 01, 2007 -- July 31, 2010.

ABSTRACT

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Small Noise Constrained Diffusions, (vi)Tug of War and Games Associated with the infinity-Laplacian, (vii) Large Deviations for Stochastic Flows of
Diffeomorphisms, (viii)Feller and Stability Properties of the Nonlinear
Filter, (ix) Particle filter methods for Atmospheric and Oceanic data assimilation
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List of papers submitted or published that acknowledge ARO support during this reporting period. List the papers, including journal references, in the following categories:

(a) Papers published in peer-reviewed journals (N/A for none)

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2. R. Atar and A. Budhiraja. On near optimal trajectories for a game associated with the infinity-Laplacian. To Appear in Prob. Th. Rel. Fields.
3. A. Budhiraja. Feller and Stability Properties of the Nonlinear Filter, The Oxford University Handbook of Nonlinear Filtering.
4. A. Budhiraja, P. Dupuis and V. Maroulas. Large Deviations for Stochastic Flows of Diffeomorphisms, Bernoulli, 16 (2010), no. 1, 91--113.
5. A. Budhiraja, P. Dupuis and V. Maroulas. Variational Representations for Continuous Time
Processes, To Appear in Annales de l'Institut Henri Poincare, Probabilites et Statistiques.
6. A. Budhiraja and D. Reinhold. Some asymptotic results for near critical branching processes,
Comm. on Stoch. Anal., 4 (2010), no. 1, 91--113.
7. A. Budhiraja, P.Dupuis and M. Fischer. Large deviation properties of weakly interacting processes via weak convergence methods, To Appear Ann. Prob.
8. A.Budhiraja and X. Liu. Multiscale Diffusion Approximations for Stochastic Networks in Heavy Tra±c. To Appear in Stoch. Proc. App.
9.A.Budhiraja and A.Ghosh. Controlled Stochastic Networks in Heavy Traffic: Convergence of
Value Functions, Under Revision for Ann. App. Prob.
10. E. T. Spiller, A. Budhiraja, K. Ide, C. Jones Modified particle filter methods for assimilating Lagrangian data into a point-vortex model. Physica D: Nonlinear Phenomena. 237 (2008), no. 10, 1498--1506

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(d) Manuscripts

1. A. Budhiraja, A. Ghosh and C. Lee. Ergodic Rate Control Problem for Single Class Queueing Networks, SIAM J. Cont. Opt., submitted.

2. A.Budhiraja, A. Shwartz and X.Liu. Action Time Sharing Policies for Ergodic Control of Markov Chains. Submitted to SIAM J. Cont. Opt.

3.A. Biswas and A.Budhiraja. Exit Time and Invariant Measure Asymptotics for Small Noise Constrained Di®usions, Submitted to Stoch. Proc. App.

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Patents Submitted

Patents Awarded

Awards

Graduate Students

NAME	PERCENT SUPPORTED
Chihoon Lee	0.10
Dominik Reinhold	0.10
Vaselios Maroulas	0.35
Xin Liu	0.35
Jiang Chen	0.10
FTE Equivalent:	**1.00**
Total Number:	**5**

Names of Post Doctorates

NAME	PERCENT_SUPPORTED
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Total Number:	

Names of Faculty Supported

NAME	PERCENT SUPPORTED	National Academy Member
Amarjit Budhiraja	0.20	No
FTE Equivalent:	**0.20**	
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Student Metrics
This section only applies to graduating undergraduates supported by this agreement in this reporting period

The number of undergraduates funded by this agreement who graduated during this period: ......

The number of undergraduates funded by this agreement who graduated during this period with a degree in science, mathematics, engineering, or technology fields: ......

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Names of Personnel receiving masters degrees

NAME
Total Number:

Names of personnel receiving PHDs

NAME
Chihoon Lee
Vaselios Maroulas
Total Number: **2**

Names of other research staff

NAME	PERCENT SUPPORTED
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Some Topics in Stochastic Control: 52311-MA Final Report, July 01, 2007 – July 31, 2010.

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October 14, 2010

A. Period of August 1, 2009–July 31, 2010.

Research conducted over this period culminated in the submission of five new research papers and revision of the two submissions made over the period 2008-2009. The latter have now been accepted for publication. A detailed description of the new submissions is provided below. Thesis advising of two Ph.D. students (Xin Liu and Dominik Reinhold) continued during this period. Two new Ph.D. students (Jiang Chen and Xuan Wang) began working on their dissertations. The topics of their study are outlined below. Invited presentations that were given during this period are also listed.

A.I Some Asymptotic Results for Near Critical Branching Processes [15]. Near critical single type Bienayme-Galton-Watson (BGW) processes are considered. It is shown that, under appropriate conditions, Yaglom distributions of suitably scaled BGW processes converge to that of the corresponding diffusion approximation. Convergences of stationary distributions for Q-processes and models with immigration to the corresponding distributions of the associated diffusion approximations are established as well. Although most of the work is concerned with the single type case, similar results for multitype settings can be obtained. As an illustration, convergence of Yaglom distributions of suitably scaled multitype subcritical BGW processes to that of the associated diffusion model is established. This work is part of the dissertation work of Mr. Dominik Reinhold.

A.II Large Deviation Properties of Weakly Interacting Processes via Weak Convergence Methods [16]. Collections of weakly interacting random processes have long been of interest in statistical physics, and more recently have appeared in problems of engineering and operations research. A simple but important example of such a collection is a group of particles, each of which evolves according to the solution of an Ito type stochastic differential equation (SDE). All

particles have the same functional form for the drift and diffusion coefficients. The coefficients of particle i are, as usual, allowed to depend on the current state of particle i, but also depend on the current empirical distribution of all particle locations. When the number of particles is large the contribution of any given particle to the empirical distribution is small, and in this sense the interaction between any two particles is considered *weak*. For various reasons, including model simplification and approximation, one may consider a functional law of large numbers (LLN) limit as the number of particles tends to infinity. The limit behavior of a single particle (under assumptions which guarantee that all particles are in some sense exchangeable) can be described by a two component Markov process. One component corresponds to the state of a typical particle, while the second corresponds to the limit of the empirical measures. Again using that all particles are exchangeable, under appropriate conditions one can show that the second component coincides with the distribution of the particle component. The limit process, which typically has an infinite dimensional state, is sometimes referred to as a nonlinear diffusion. Because the particle's own distribution appears in the state dynamics, the partial differential equations that characterize expected values and densities associated with this process are nonlinear, and hence the terminology. In this paper we consider the large deviation properties of the particle system as the number of particles tends to infinity. Thus the deviations we study are those of the empirical measure of the prelimit process from the distribution of the nonlinear diffusion. In the paper we develop an approach which is very different from the one taken in previous works. Unlike classical techniques, our proofs do not involve any time or space discretization of the system and no exponential probability estimates are invoked. The main ingredients in the proof are weak convergence methods for functional occupation measures and certain variational representation formulas. Our proofs cover models with degenerate noise and allow for interaction in both drift and diffusion terms. In fact the techniques are applicable to a wide range of model settings and an example of stochastic delay equations is considered to illustrate the possibilities. The starting point of our analysis is a variational representation for moments of nonnegative functionals of a Brownian motion. Using this representation, the proof of the large deviation principle reduces to the study of asymptotic properties of certain controlled versions of the original process. The key step in the proof is to characterize the weak limits of the control and controlled process as the large deviation parameter tends to its limit and under the same scaling that applies to the original process. More precisely, one needs to characterize the limit of the empirical measure of a large collection of controlled and weakly interacting processes. In the absence of control this characterization problem reduces to an LLN analysis of the original particle system, which has been studied extensively. Our main tools for the study of the controlled analogue are functional occupation measure methods. Indeed, these methods have been found to be quite useful for the study of averaging problems, but where the average is with respect to a time variable. In the problem studied here the measure valued processes of interest are obtained using averaging over particles rather than the time variable. Variational representations for Brownian motions and Poisson random measures[9, 10, 12] have proved to be extremely useful for the study of small noise large deviation problems and many recent papers (see references in [12]) have applied these results to a variety of infinite dimensional small noise systems. We expect the current work to be similarly a starting point for the study, using variational representations, of a rather different collection of large deviation problems, namely asymptotics of a large number of interacting particles.

A.III Multiscale Diffusion Approximations for Stochastic Networks in Heavy Traffic

[**18**]. Jackson networks have been very well studied and extensively used for modeling and analysis in a variety of disciplines. The elegant distributional theory and asymptotic properties of Jackson networks break down when one attempts to incorporate some more realistic features of specific application settings into such a model. For example, when the distributions of the primitives are relaxed to be more general than i.i.d. exponential, or the rates of interarrivals and services are allowed to depend on the state of the system, the tractability of such a generalized Jackson network fails and thus one seeks suitable approximate models. One class of such approximations are diffusion models that can be rigorously justified when networks are operating in the heavy traffic regime, i.e., when the network capacity is roughly balanced with network load. Attractiveness of such approximations primarily lies in the fact that, analogous to the central limit theory, the limit model is described only using a few important parameters of the underlying networks and the complex distributional properties of the primitives are averaged out. The first general result in the study of such diffusion approximations is due to Reiman[27] who considered the case where the arrivals and services are mutually independent renewal processes with square integrable summands. Queueing networks in which rates of arrival and service processes depend on the current state of network arise in many application areas. In particular, Yamada[29] has shown that, under appropriate heavy traffic conditions, suitably scaled queue length processes for such networks converge weakly to a reflected diffusion process with drift and diffusion coefficients that depend on the state of the process.

In models considered in works of Reiman and Yamada, the underlying topology of the network is the same as that of a Jackson network. In particular, there is a fixed probability routing matrix P which governs the routing of jobs in the networks. In this work, we are interested in settings where the routing structure and arrival/service rates may change over time, according to an extraneous finite state Markov process. This Markov process can be interpreted as the random environment in which the system is operating. We begin by considering a situation where the system performance is analyzed on time scales which are much larger than typical interarrival/service times of jobs, which in turn are significantly larger than typical time intervals at which the state of the random environment changes. Such multi-scale models are motivated by the study of large computer networks where one is interested in the behavior of modeling traffic of files with moderate size over a long period of time, for a small subset of nodes in the system. Denote by $\mathcal{E}$ the collection of all nodes in the network and $\mathcal{E}_0 \subset \mathcal{E}$ the subset of nodes of interest. Here, the size of $\mathcal{E}_0$ is much smaller than that of $\mathcal{E}$. One is interested in building a model for traffic between nodes in $\mathcal{E}_0$ without taking a very precise account of the interactions of such nodes with those in $\mathcal{E} \setminus \mathcal{E}_0$. A node $e \in \mathcal{E}_0$ may receive files from a large number of nodes, most of which one does not want to explicitly account for in the reduced model. The point of view taken here is to model the effect of nodes in $\mathcal{E} \setminus \mathcal{E}_0$ at a node $e \in \mathcal{E}_0$ by a rapidly varying channel capacity (at e), which is modulated by a Markov process. If a large number of nodes in $\mathcal{E} \setminus \mathcal{E}_0$ are connected to e, one expects that the rate at which the channel capacity changes is much greater than the transmission rate of a typical file through e. Rapid changes in channel capacity lead to variations in processing rates and available routing options for files accepted at nodes in $\mathcal{E}_0$. Thus we propose a traffic model for nodes in $\mathcal{E}_0$ in term of a Jackson type network where the arrival/service rates and routing probability matrices vary randomly over time according to a finite state Markov process.

Goal of this work is to study diffusion approximations for such multiscale models. We show that, under appropriate conditions, the state of the system can be well approximated by that of a

reflected diffusion process with drift and diffusion coefficients depend on the equilibrium measure of the Markov process governing the random environment. A crucial requirement in typical heavy traffic approximation results is the traffic balance condition of the form $\lambda = [I - P]\alpha$. In the multi-scale setting considered here all these quantities in the above equation, i.e., P, λ, α, vary randomly over time according to the modulating Markov process. We find that if transition times of the Markov process are suitably fast, then the traffic balance condition can be relaxed to an average balance condition where the average is taken with respect to the equilibrium measure of the Markov process. Making this statement mathematically precise is one of the key contributions of this work.

In contrast to the scaling regime described above is the situation where the random environment changes very slowly relative to the rates of interarrival and service times. In such a regime, changes in the environment are revealed only when the system is viewed over long time intervals. Our second result in this work is to give a precise mathematical formulation and establish a suitable diffusion approximation for networks in such a scaling regime. For simplicity we restrict ourselves to a setting where the random environment affects only the arrival and service rates in the system while the routing probabilities stay the same (governed by a fixed non-random matrix P). We show that, with a suitable scaling, the state process for the network approaches a reflected diffusion with coefficients that are modulated by a Markov process that is independent of the driving Brownian motion.

The two scaling regimes considered above correspond to very different behaviors of the background environment process. In the first setting the environment changes at a much faster rate than the typical arrival/serive rates in the system, while in the second one the reverse is true. More generally, one can consider a setting where background variation of both types exists. Namely, there are two independent Markov processes governing the randomly varying environment, the first corresponds to fast changes while the second captures slow changes. We also establish an appropriate diffusion approximation result for such multi-scale queueing networks. This work is part of the dissertation work of Ms. Xin Liu.

A.IV Action Time Sharing Policies for Ergodic Control of Markov Chains [17]. Markov Decision processes are used extensively as the simplest models that involve both stochastic behavior and control. A common measure of performance is the long-time average (or ergodic) criterion. Given all relevant parameters, a typical goal is to find a simple (e.g. feedback, or deterministic stationary) policy that achieves the optimal value.

The goal of adaptive control is to obtain an optimal policy, when some relevant information concerning the behavior of the system is missing. The relevant information needs to be obtained while controls are chosen at each step. The classical approach is to design an algorithm which collects information, while at the same time choosing controls, in such a way that sufficient information is collected for making good control decisions, in the sense that the chosen controls "approach optimality over time." Existing results include general solutions for the case of countable state space, and specify an estimation and a control scheme. A different approach to this issue, including PAC criteria, can be found in the large literature on Reinforcement learning.

We are concerned with a more elementary question, namely: What are the basic controlled objects that determine the cost? Since the objective function is defined as a Cesaro limit, we can expect that a similar Cesaro definition of the choice of controls would suffice to determine

the cost. For the case of countable state and action spaces one fundamental result says that the control decisions can deviate from those dictated by the Markov policy q, and still produce the same long term average cost, as long as the conditional frequencies converge to the correct values. This flexibility is useful in many estimation and sampling applications. In the current work we are concerned with a setting where the state and action spaces are not (necessarily) countable. Our main objective is to formulate an appropriate definition for an Action-Time Sharing (ATS) policy for a given Markov control which, similar to the countable case, on the one hand leads to long term costs that are identical to those for the corresponding Markov control, while on the other hand allows for flexible implementation well suited for various estimation and adaptive control goals. We show that, under suitable stability, irreducibility and Feller continuity conditions, occupation measures for state and action sequences, under an ATS policy (suitably defined), converge a.s. to the same (deterministic) measure as under the corresponding Markov control. Such a result in particular shows that long term costs for a broad family of one stage cost functions, under the two control policies, coincide. ATS policies can be used to develop a variety of variance reduction schemes for ergodic control problems. Additionally, ATS policies provide much flexibility for sampling (namely using controls without regards to the ensuing cost), for example for the purpose of collecting information. This could be information which is related to the main optimization objective, but could also be other information which is of interest. We show that how ATS policies introduced in this work can be used for consistent estimation of unknown model parameters and also for adaptive control problems. This paper is part of the dissertation work of Ms. Xin Liu.

A.V Controlled Stochastic Networks in Heavy Traffic: Convergence of Value Functions [20]. As an approximation to control problems for critically-loaded stochastic networks, Harrison (1988) has formulated a stochastic control problem in which the state process is driven by a multi-dimensional Brownian motion along with an additive control that satisfies certain feasibility and non-negativity constraints. This control problem that is usually referred to as the Brownian Control Problem (BCP) has been one of the key developments in the heavy traffic theory of controlled stochastic processing networks (SPN). BCPs can be regarded as formal scaling limits for a broad range of scheduling and sequencing control problems for multiclass queuing networks. Finding optimal (or even near-optimal) control policies for such networks – which may have quite general non-Markovian primitives, multiple server capabilities and rather complex routing geometry – is prohibitive. In that regard BCPs that provide significantly more tractable approximate models are very useful. In this diffusion approximation approach to policy synthesis, one first finds an optimal (or near-optimal) control for the BCP which is then suitably interpreted to construct a scheduling policy for the underlying physical network. In recent years there have been many works that consider specific network models for which the associated BCP is explicitly solvable (i.e. an optimal control process can be written as a known function of the driving Brownian motions) and, by suitably adapting the solution to the underlying network, construct control policies that are asymptotically (in the heavy traffic limit) optimal. Although now there are several papers which establish a rigorous connection between a network control problem and its associated BCP by exploiting the explicit form of the solution of the latter, a systematic theory which justifies the use of BCPs as approximate models has been missing. In a recent work [21] it was shown that for a large family of Unitary Networks, with general interarrival and service times, probabilistic routing, and an infinite horizon discounted linear holding cost, the cost associated with any admissible control policy for the network is asymptotically, in the heavy traffic limit, bounded below by the

value function of the BCP. This inequality, which provides a useful bound on the best achievable asymptotic performance for an admissible control policy (for this large family of models), was a key step in developing a rigorous general theory relating BCPs with SPN in heavy traffic. The current paper is devoted to the proof of the reverse inequality. The network model is required to satisfy assumptions made in [21]. In addition we impose a non-degeneracy condition, a condition on the underlying renewal processes regarding probabilities of deviations from the mean and regularity of a certain Skorohod map. Under these assumptions we prove that the value function of the BCP is bounded below by the heavy traffic limit (limsup) of the value functions of the network control problem. Combining this with the result obtained in [21] we obtain the main result of the paper. This theorem says that, under broad conditions, the value function of the network control problem converges to that of the BCP. This result provides, under general conditions, a rigorous basis for regarding BCPs as approximate models for critically loaded stochastic networks. Conditions imposed in this paper allow for a wide range of open multi-class queuing networks, parallel server networks and job shop type models. We note that our approach does not require the BCP to be explicitly solvable and the result covers many settings where explicit solutions are unavailable. Previous works noted earlier, that treat the setting of explicitly solvable BCP, do much more than establish convergence of value functions. In particular, these works give an explicit implementable control policy for the underlying network that is asymptotically optimal in the heavy traffic limit. In the generality treated in the current work, giving explicit recipes (eg. threshold type policies) is unfeasible, however the policy sequence that we construct, suggests a general approach for building near asymptotically optimal policies for the network given a near optimal control for the BCP. Obtaining near optimal controls for the BCP in general requires numerical approaches, study of which will be a topic for future work.

A.VI. Exit Time and Invariant Measure Asymptotics for Small Noise Constrained Diffusions [19]. Diffusions in polyhedral domains arise commonly as approximate models for stochastic processing networks in heavy traffic. In this work we consider a family of such constrained diffusions with a small parameter (denoted as ϵ) multiplying the diffusion coefficient. Goal of the work is the study of asymptotic properties of invariant measures and exit times from suitable domains, as $\varepsilon \to 0$. The classical reference for small noise asymptotics of diffusions in $\mathbb{R}^k$ is the book of Freidlin and Wentzell (1998). The basic object of study in this fundamental body of work is a collection of diffusion processes $\{X^\varepsilon\}_{\varepsilon>0}$, given as

$$dX^\varepsilon(t) = b(X^\varepsilon(t))dt + \varepsilon\sigma(X^\varepsilon(t))dW(t), \ X^\varepsilon(0) = x, \tag{1}$$

where W is a k dimensional standard Brownian motion and b, σ are suitable coefficients. The stochastic process $X^\varepsilon \equiv \{X^\varepsilon(t)\}_{0 \le t \le T}$, for each $T \ge 0$ can be regarded as a $\mathcal{C}_T = C([0,T] : \mathbb{R}^k)$ (space of continuous functions from $[0,T]$ to $\mathbb{R}^k$ with the uniform topology) valued random variable and under suitable conditions on b, σ one can show that, as $\varepsilon \to 0$, X^ε converges in probability to ξ which is the unique solution of the ordinary differential equation (ODE)

$$\dot{\xi} = b(\xi), \ \xi(0) = x. \tag{2}$$

One of the basic results in the field says that for each $T > 0$, as $\varepsilon \to 0$, X^ε satisfies a large deviation principle (LDP) in $\mathcal{C}_T$, uniformly in the initial condition x in any compact set K, with an appropriate rate function $I_T : \mathcal{C}_T \to [0, \infty]$. This result is a starting point for the study of numerous asymptotic questions for such small noise diffusions. In particular, when the underlying diffusions

have suitable stability properties, the above LDP plays a central role in the study of asymptotic properties of invariant measures and exit times from domains. The asymptotics are governed by the "quasi-potential" function V which is determined from the collection of rate functions $\{I_T : T > 0\}$.

Freidlin-Wentzell theory has been extended and refined in many different directions. One notable work is Day(1982) which studies asymptotics of solutions of Dirichlet problems associated with diffusions given by (1). To signify the dependence on the initial condition, denote the solution of (1) by X_x^ε. Let B be a bounded domain in $\mathbb{R}^k$ and K be an arbitrary compact subset of B. Under the assumption that all solutions of the ODE (2), with $x = \xi(0) \in B$, converge without leaving B, to a single linearly asymptotically stable critical point, Day shows that with suitable conditions on the coefficients, for all bounded measurable f

$$\sup_{x,y \in K} |\mathbb{E}\left(f(X_x^\varepsilon(\tau_x^\varepsilon))\right) - \mathbb{E}\left(f(X_y^\varepsilon(\tau_y^\varepsilon))\right)|$$

converges to 0 at an exponential rate. Here, $\tau_x^\varepsilon = \inf\{t : X_x^\varepsilon(t) \in B^c\}$. This property, usually referred to as "exponential leveling", says that although the exit time of the process from the domain approaches ∞, as $\varepsilon \to 0$, the moments of functionals of exit location, corresponding to distinct initial conditions, coalesce asymptotically, at an exponential rate. The key ingredient in the proof is the gradient estimate

$$\sup_{x \in K} |\nabla u^\varepsilon(x)| \leq c\varepsilon^{-1/2}, \tag{3}$$

where u^ε is the solution of the Dirichlet problem on B associated with the diffusion (1) with boundary data f.

The goal of the current work is to develop the Freidlin-Wentzell small noise theory for a family of constrained diffusions in polyhedral cones. Let $G \subset \mathbb{R}^k$ be convex polyhedral cone in $\mathbb{R}^k$ with the vertex at origin given as the intersection of half spaces $G_i, i = 1, 2, ..., N$. Let n_i be the unit vector associated with G_i via the relation

$$G_i = \{x \in \mathbb{R}^k : \langle x, n_i \rangle \geq 0\}.$$

We will denote the set $\{x \in \partial G : \langle x, n_i \rangle = 0\}$ by F_i. With each face F_i we associate a unit vector d_i such that $\langle d_i, n_i \rangle > 0$. This vector defines the *direction of constraint* associated with the face F_i. Roughly speaking such a process evolves infinitesimally as a diffusion in $\mathbb{R}^k$ and is instantaneously pushed back using the oblique reflection direction d_i upon reaching the face F_i. Formally, such a process, denoted once more as X_x^ε, can be represented as a solution of a stochastic integral equation of the form

$$X_x^\varepsilon(t) = \Gamma\left(x + \int_0^{\cdot} b(X^\varepsilon(s))ds + \varepsilon \int_0^{\cdot} \sigma(X^\varepsilon(s))dW(s)\right)(t), \tag{4}$$

where Γ is the Skorohod map taking trajectories with values in $\mathbb{R}^k$ to those with values in G, consistent with the constraint vectors $\{d_i, i = 1, \cdots N\}$. Under certain regularity assumptions on the Skorohod map and the usual Lipschitz conditions on the coefficients b and σ, our first result establishes a (locally uniform) LDP for X_x^ε, in $C([0,T] : G)$ for each $T > 0$. This result is the starting point for all exit time and invariant measure estimates obtained in this work.

Stability properties of constrained diffusions in polyhedral domains have been studied in several works. Let

$$\mathcal{C} = \{-\sum_{i=1}^{k} \alpha_i d_i : \alpha_i \geq 0; \, i \in \{1, ..., k\}\}.$$

The paper [4] shows that under regularity of the Skorohod map, uniform non-degeneracy of σ and Lipschitz coefficients, if for some $\delta > 0$

$$b(x) \in \mathcal{C}(\delta) = \{v \in \mathcal{C} \, : \, \text{dist}(v, \partial\mathcal{C}) \geq \delta\} \text{ for all } x \in G, \tag{5}$$

then the constrained diffusion X^ε is positive recurrent and consequently admits a unique invariant probability measure μ^ε. In the current work we study asymptotic properties of μ^ε, as $\varepsilon \to 0$.

We also consider asymptotic properties of exit times from a bounded domain $B \subset G$ that contains the origin. One important feature in this analysis is that the stability condition (5), in general, does not ensure that the trajectories of the associated deterministic dynamical system

$$\xi = \Gamma(x + \int_0^{\cdot} b(\xi(s))ds) \tag{6}$$

with initial condition in B will stay within the domain at all times. However, a weaker stability property holds, namely, one can find domains $B_0 \subset B$ such that all trajectories of (6) starting in B_0 stay within B at all times.

A significant part of this work is devoted to the proof of the exponential leveling property for constrained diffusions. We recall that the key ingredient in the proof of such a result for diffusions in $\mathbb{R}^k$ is the gradient estimate (3) for solutions of the associated Dirichlet problem. For diffusions in domains with corners and with oblique reflection fields that change discontinuously, there are no regularity (eg. C^1 solutions) results known for the associated partial differential equations(PDE). Our proof of the exponential leveling property is purely probabilistic and bypasses all PDE estimates. The main step in the proof is the construction of certain (uniform in ε) Lyapunov functions which are then used to construct a coupling of the processes X_x^ε, X_y^ε with explicit uniform estimates on exponential moments of time to coupling. The key ingredient in this coupling construction is a minorization condition on transition densities of reflected diffusions. We give some examples where such a minorization property holds. Obtaining general conditions under which this estimate on transition densities of reflected diffusions holds is a challenging open problem. The minorization property allows for the construction of a pseudo-atom, for each fixed $\varepsilon > 0$, using split chain ideas of Athreya-Ney-Nummelin. Coupling based on pseudo-atom constructions have been used by many authors for the study of a broad range of stability and control problems, however the current paper appears to be the first to bring these powerful techniques to bear for the study of exit time asymptotics of small noise Markov processes.

As a second consequence of our coupling constructions we show that difference of moments of an exit time functional with a sub-logarithmic growth corresponding to distinct initial conditions in B_0 is asymptotically bounded. Note that for typical unbounded functionals the associated moments will approach ∞, as $\varepsilon \to 0$ thus the result provides a rather non-trivial "leveling estimate". The third important consequence of our approach says that as initial conditions approach 0 at rate ε^2 the corresponding moments of exit time functionals with sub-logarithmic growth asymptotically

coalesce at an exponential rate. To the best of our knowledge, analogous estimates are not known even for the setting of small noise diffusions (with suitable stability properties) in $\mathbb{R}^k$.

These estimates are interesting for one dimensional models as well. Let X be a one dimensional reflected Brownian motion, starting from $x \in [0, 1)$, with drift $b \in (-\infty, 0)$ and variance $\sigma^2 \varepsilon^2 \in (0, \infty)$, given on some probability space $(\Omega, \mathcal{F}, \boldsymbol{P}_x^\varepsilon)$. I.e., $\boldsymbol{P}_x^\varepsilon$ a.e.,

$$X(t) = x + bt + \varepsilon\sigma B(t) - \inf_{0 \le s \le t}\{(x + bs + \varepsilon\sigma B(s)) \wedge 0\},\ t \ge 0$$

where B is a standard Brownian motion. Let $\tau = \inf\{t : X(t) = 1\}$. Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable map and let $\varphi_\varepsilon(x) = \mathbb{E}_x^\varepsilon \psi(\tau)$. Our results for this one dimensional setting says that if ψ has a sub-logarithmic growth then for some $c, \delta \in (0, \infty)$, as $\varepsilon \to 0$,

$$e^{\delta/\varepsilon}|\varphi_\varepsilon(c\varepsilon^2) - \varphi_\varepsilon(0)| \to 0. \tag{7}$$

If $\psi(t) = t$, $t \ge 0$, then φ_ε is the unique solution to the boundary value problem

$$\varepsilon^2\sigma^2\varphi_\varepsilon'' + b\varphi_\varepsilon' = -1,\ \varphi_\varepsilon'(0) = 0,\ \ \varphi_\varepsilon(1) = 0,$$

whose solution is given as $\varphi_\varepsilon(x) = \frac{\varepsilon^2\sigma^2}{b^2}[e^{\frac{-b}{\varepsilon^2\sigma^2}} - e^{\frac{-bx}{\varepsilon^2\sigma^2}}] + \frac{1}{b}(1 - x)$. It is easily verified that if $x_\varepsilon = c\varepsilon^2$, for some $c > 0$, then $|\varphi_\varepsilon(x_\varepsilon) - \varphi_\varepsilon(0)| \to 0$ at rate ε^2, as $\varepsilon \to 0$. In particular, the decay is not exponential. Furthermore, for $x \ne y$ in $[0, 1)$, $|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)| \to \infty$ as $\varepsilon \to 0$. For general ψ a similar ODE analysis is less tractable and thus the coupling techniques developed here give a powerful approach (especially in higher dimensions) to the study of such asymptotic estimates.

A.VIII. Dissertation Advising. Four Ph.D. students who were partially supported by the grant are currently working under the direction of the PI. Ms. Xin Liu (Topics in Diffusion Approximations for Multiscale queuing networks), Mr. Dominik Reinhold (Long time properties of Critical Branching Processes), Ms. Jiang Chen (Large Deviations for Stochastic Dynamical Systems) and Mr. Xuan Wang (Scaling limits of Random Graph Models).

A.IX. Invited Presentations.

- *A Stochastic Differential Game for the Inhomogeneous Infinity-Laplace Equation*, Institute of Mathematical Statistics, 73rd Annual Meeting, Aug 9-13, 2010, Gothenburg, Sweden.

- *Some Large Deviation Problems for Infinite Dimensional Stochastic Dynamical Systems*, Theory and Qualitative Behavior of Stochastic Dynamics, SAMSI, February 8-10, 2010.

- *Elliott-Kalton stochastic differential games associated with the infinity Laplacian*, Probability Seminars at University of Nice, Sophia-Antipolis July 2010, Department of Mathematics-National Taiwan University, May 2010.

- *Variational Representations and Large Deviations*, Institute of Math., Academia Sinica, Taiwan, May 2010.

B. Period of August 1, 2008–July 31, 2009.

Research conducted over this period culminated in the submission of two new research papers and revision of four of the five submissions made over the period 2007-2008. Three of the latter group have now appeared or have been accepted for publication. A detailed description of the new submissions is provided below. Thesis advising of two Ph.D. students (Xin Liu and Dominik Reinhold) was carried out during this period.

B.I. On Near Optimal Trajectories of Games Associated with the infinity-Laplacian [5]. Consider the equation

$$\begin{cases} -2\Delta_\infty u = h & \text{in } G, \\ \\ u = g & \text{on } \partial G, \end{cases} \tag{8}$$

where, for an integer $m \geq 2$, $G \subset \mathbb{R}^m$ is a bounded $\mathcal{C}^2$ domain, and $g \in \mathcal{C}(\partial G, \mathbb{R})$ and the functions $h \in \mathcal{C}(\bar{G}, \mathbb{R} \setminus \{0\})$ are given. The infinity-Laplacian is defined as

$$\Delta_\infty f = \frac{1}{|Df|^2} \sum_{i,j=1}^m D_i f \, D_{ij} f \, D_j f = \frac{Df'}{|Df|} \, D^2 f \, \frac{Df}{|Df|},$$

provided $Df \neq 0$, where for a $\mathcal{C}^2$ function f we denote by Df the gradient and by $D^2 f$ the Hessian matrix. This paper is motivated by recent work of Peres et. al. [26], where a discrete time random turn game, referred to as *Tug-of-War*, is developed in relation to (8). This game, parameterized by $\varepsilon > 0$, has the property that the vanishing-ε limit of the value function uniquely solves (8) in the viscosity sense. The stochastic differential equation (SDE)

$$dX_t = 2\bar{p}(X_t)dW_t + 2q(X_t)dt, \tag{9}$$

where

$$\bar{p} = \frac{Du}{|Du|}, \qquad q = \frac{1}{|Du|^2}(D^2 u \, Du - \Delta_\infty u \, Du), \tag{10}$$

is suggested in [26] as the game's dynamics in the vanishing-ε limit. The relation is rigorously established in examples, but only heuristically justified in general. Recently, in [3], a two-player zero-sum stochastic differential game (SDG) is considered, for which the value function uniquely solves (8) in the viscosity sense. The dynamics of the state process are given as

$$X_t = x + \int_0^t (A_s - B_s)dW_s + \int_0^t (C_s + D_s)(A_s + B_s)ds, \qquad t \in [0, \infty), \tag{11}$$

and (A, C) and (B, D) are control processes, chosen by the two players, taking values in $\mathcal{S}^{m-1} \times [0, \infty)$ where $\mathcal{S}^{m-1}$ is the unit sphere in $\mathbb{R}^m$. Defined in the Elliott-Kalton sense, the SDG of [3] is formulated in such a way that one of the players selects a strategy, and then the other selects a control process. The payoff functional of interest is an exit time cost of the form $\mathbf{E}[\int_0^\tau h(X_s)ds + g(X_\tau)]$, where $\tau = \inf\{t : X_t \notin G\}$ (with an appropriate convention regarding $\tau = \infty$). Some comments on how such a game arises from a discrete *Tug of War* formulation can be found in Section B.I.

The goal of this paper is to show that, with appropriate conditions, (9) can be rigorously interpreted as the optimal dynamics of the SDG. We will assume in this paper that the equation possesses a classical solution u i.e., $\mathcal{C}^2$ with non-vanishing gradient. Under this assumption we specify, for each $\delta > 0$, a δ-optimal strategy β^δ, and a control process Y^δ that is δ-optimal for play against β^δ, in terms of first and second derivatives of u. We then identify the limit law, as $\delta \to 0$, of the state process under (β^δ, Y^δ), as the solution X to the SDE (9), stopped when X hits the boundary ∂G.

The construction of near optimal strategy-control pairs, that is of interest in its own right, is based on an interpretation of (8) as the following Bellman-Isaacs type equation

$$\sup_{|b|=1, d\geq 0} \inf_{|a|=1, c\geq 0} \left\{ -\frac{1}{2}(a-b)'(D^2 u)(a-b) - (c+d)(a+b) \cdot Du \right\} = h.$$

In this form there is a natural way to construct strategy and control, by associating the supremum and infimum with the two players. The variables a, b, c and d selected by the players dictate the coefficients of the game's state process, and, we prove, the coefficients converge to those of equation (9) in the limit as the supremum and infimum are achieved. This convergence is then lifted to the convergence of the underlying state process in (11) to the diffusion (9).

B.II. Variational Representations for Continuous Time Processes [12]. In this paper we prove a variational representation for positive measurable functionals of a Poisson random measure and an infinite dimensional Brownian motion. These processes provide the driving noises for a wide range of important process models in continuous time, and thus we also obtain variational representations for these processes when a strong solution exists. The representations have a number of uses, the most important being to prove large deviation estimates.

The theory of large deviations is by now well understood in many settings, but there remain some situations where the topic is not as well developed. These are often settings where technical issues challenge standard approaches, and the problem of finding nearly optimal or even reasonably weak sufficient conditions is hindered as much by technique of proof as any other issue.

Variational representations of the sort developed in this paper have been shown to be particularly useful, when combined with weak convergence methods, for analyzing such systems. For example, Brownian motion representations have been used by many authors (see references in [12]) in the large deviation analysis of solutions to SPDEs in the small noise limit. The usefulness of the representations is in part due to the fact that they avoid certain discretization and/or approximation arguments, which can be cumbersome for complex systems. Another reason is that exponential tightness, a property that is often required by other approaches and which often leads to artificial conditions, is replaced by ordinary tightness for controlled processes with uniformly bounded control costs. (Although exponential tightness can a posteriori be obtained as consequence of the large deviation principle (LDP) and properties of the rate function). What is required for the weak convergence approach, beyond the variational representations, is that basic qualitative properties (existence, uniqueness and law of large number limits) can be demonstrated for certain controlled versions of the original process.

Previous work on variational representations has focused on either discrete time processes, functionals of finite dimensional Brownian motion, or various formulations of infinite dimensional Brownian motion. An important class of processes that were not covered are continuous time Markov processes with jumps, e.g., Levy processes. In this paper we eliminate this gap, and in fact give variational representations for functionals of a fairly general Poisson random measure (PRM) plus an independent infinite dimensional Brownian motion (BM), thereby covering many continuous time models.

In [30] Zhang has also proved a variational representation for functionals of a PRM. The representation in [30] is given in terms of certain predictable transformations on the canonical Poisson space. Existence of such transformations relies on solvability of certain nonlinear partial differential equations from the theory of mass transportation. This imposes restrictive conditions on the intensity measure (e.g., absolute continuity with respect to Lebesgue measure) of the PRM, and even the very elementary setting of a standard Poisson process is not covered. Additionally, use of such a representation for proving large deviation results for general continuous time models with jumps appears to be unclear.

In contrast, we impose very mild assumptions on the intensity measure (namely, it is a σ-finite measure on a locally compact space), and establish a representation, that is given in terms of a fixed PRM defined on an augmented space. A key question in formulating the representation for PRM is *"what form of controlled PRM is natural for purposes of representation?"* In the Brownian case there is little room for discussion, since control by shifting the mean is obviously very appealing. In [30] the control moves the atoms of the Poisson random measure through a rather complex nonlinear transformation. The fact that atoms are neither created nor destroyed is partly responsible for the fact that the representation does not cover the standard Poisson process. In the representation obtained in our work the control process enters as a censoring/thinning function in a very concrete fashion, which in turn allows for elementary weak convergence arguments in proofs of large deviation results.

As an application of the representation, we establish a general large deviation principle (LDP) for functionals of a PRM and an infinite dimensional BM. A similar LDP for functionals of an infinite dimensional BM [9] has been used in recent years by numerous authors to study small noise asymptotics for a variety of infinite dimensional stochastic dynamical models (see references in [12]). The LDP obtained in the current paper is expected to be similarly useful in the study of infinite dimensional stochastic models with jumps (e.g., SPDEs with jumps). We illustrate the use of the LDP in Section 4 via a simple finite dimensional jump-diffusion model. The goal is to simply show how the approach can be used and no attempt is made to obtain the best possible conditions.

B.III. Invited Presentations.

- *Variational Representations and Large Deviations*, 2009 Barrett Lectures at The University of Tennessee, April 17 - 18, 2009.

- *Variational Representations and Large Deviations*, 33rd Conference on Stochastic Processes and Their Applications, Berlin, 27-31 July, 2009.

- *Elliott-Kalton stochastic differential games associated with the infinity Laplacian*, Probability

Seminars at Stanford University, Univ. of Cal. at Berkeley, Columbia University, Carnegie Mellon University, Princeton University.

C. Period of July 1, 2007–July 31, 2008.

Research conducted over the period July 1, 2007 – July 31, 2008 culminated in five research papers that were submitted for publication. Four of the five have been accepted for publication while the fifth is currently under review. A detailed description of these works is given below. This period also included graduation of two Ph.D. students supported by the contract whose dissertation results are described briefly. Invited presentations that were given during this period are also listed.

C.I. A stochastic differential game for the inhomogeneous ∞-Laplace equation [3]. For an integer $m \geq 2$ let a bounded $\mathcal{C}^2$ domain $G \subset \mathbb{R}^m$, and functions $g \in \mathcal{C}(\partial G, \mathbb{R})$ and $h \in \mathcal{C}(\bar{G}, \mathbb{R} \setminus \{0\})$ be given. We study a two-player zero-sum stochastic differential game (SDG), defined in terms of an m-dimensional state process that is driven by a one-dimensional Brownian motion, played until the state exits the domain. The functions g and h serve as terminal, and, respectively, running payoffs. The players' controls enter in a diffusion coefficient and in an unbounded drift coefficient of the state process. The dynamics are degenerate in that it is possible for the players to completely switch off the Brownian motion. We show that the game has value, and characterize the value function as the unique viscosity solution u (uniqueness of solutions is known from [26]) of the equation

$$\begin{cases} -2\Delta_\infty u = h & \text{in } G, \\ \\ u = g & \text{on } \partial G. \end{cases} \tag{12}$$

Here Δ_∞ is the infinity-Laplacian defined as $\Delta_\infty f = (Df)'(D^2 f)(Df)/|Df|^2$, provided $Df \neq 0$, where for a $\mathcal{C}^2$ function f we denote by Df the gradient and by $D^2 f$ the Hessian matrix. Our work is motivated by a representation for u of Peres et. al. [26], as the limit, as $\varepsilon \to 0$, of the value function V^ε of a discrete time random turn game, referred to as *Tug-of-War*, in which ε is a parameter. The contribution of the current work is the identification of a game for which the value function is precisely equal to u.

The Tug-of-War game introduced in [26] is as follows. Fix $\varepsilon > 0$. Let a token be placed at $x \in G$, and set $X_0 = x$. At the k-th step of the game ($k \geq 1$), an independent toss of a fair coin determines which player takes the turn. The selected player is allowed to move the token from its current position $X_{k-1} \in G$ to a new position X_k in $\bar{G}$, in such a way that $|X_k - X_{k-1}| \leq \varepsilon$. The game ends at the first time K when $X_K \in \partial G$. The associated payoff is given by

$$\boldsymbol{E}\left[g(X_K) + \frac{\varepsilon^2}{4} \sum_{k=0}^{K-1} h(X_k) \right]. \tag{13}$$

Player I attempts to maximize the payoff and Player II's goal is to minimize it. It is shown in [26] that the value of the game, defined in a standard way and denoted $V^\varepsilon(x)$, exists, that V^ε converges uniformly to a function V referred to as the "continuum value function", and that V is the unique

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viscosity solution of (12) (these results are in fact also proved for the homogeneous case, and in generality greater than the scope of the current paper). The question of associating a game directly with the continuum value was posed and some basic technical challenges associated with it were discussed in [26].

Our approach to the question above is via a SDG formulation. To motivate the form of the SDG we start with the Tug-of-War game and present some formal calculations. Let $\{\xi_k, k \in \mathbb{N}\}$ be a sequence of i.i.d. random variables on some probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ with $\boldsymbol{P}(\xi_k = 1) = \boldsymbol{P}(\xi_k = -1) = 1/2$, interpreted as the sequence of coin tosses. Let $\{\mathcal{F}_k\}_{k \geq 0}$ be a filtration of $\mathcal{F}$ to which $\{\xi_k\}$ is adapted and such that $\{\xi_{k+1}, \xi_{k+2} \ldots\}$ is independent of $\mathcal{F}_k$ for every $k \geq 0$. Let $\{a_k\}$, $\{b_k\}$ be $\{\mathcal{F}_k\}$-predictable sequences of random variables with values in $\overline{\mathbb{B}_\varepsilon(0)} = \{x \in \mathbb{R}^m : |x| \leq \varepsilon\}$. These sequences correspond to control actions of Player I and II; that is, a_k [resp., b_k] is the displacement exercised by Player I [resp., Player II] if it wins the k-th coin toss. Associating the event $\{\xi_k = 1\}$ with Player I winning the k-th toss, one can write the following representation for the position of the token, starting from initial state x. For $j \in \mathbb{N}$,

$$X_j = x + \sum_{k=1}^{j} \left[a_k \frac{1 + \xi_k}{2} + b_k \frac{1 - \xi_k}{2} \right] = \sum_{k=1}^{j} \frac{a_k - b_k}{2} \xi_k + \sum_{k=1}^{j} \frac{a_k + b_k}{2}.$$

We shall refer to $\{X_j\}$ as the 'state process'. This representation, in which turns are not taken at random but both players select an action at each step, and the noise enters in the dynamics, is more convenient for the development that follows. Let $\varepsilon = 1/\sqrt{n}$ and rescale the control processes by defining, for $t \geq 0$, $A_t^n = \sqrt{n} \, a_{[nt]}$, $B_t^n = \sqrt{n} \, b_{[nt]}$. Consider the continuous time state process $X_t^n = X_{[nt]}$, and define $\{W_t^n\}_{t \geq 0}$ by setting $W_0^n = 0$ and using the relation

$$W_t^n = W_{(k-1)/n}^n + \left(t - \frac{k-1}{n} \right) \sqrt{n} \, \xi_k, \qquad t \in \left(\frac{k-1}{n}, \frac{k}{n} \right], \ k \in \mathbb{N}.$$

Then we have

$$X_t^n = x + \frac{1}{2} \int_0^t (A_s^n - B_s^n) dW_s^n + \frac{1}{2} \int_0^t \sqrt{n}(A_s^n + B_s^n) ds. \tag{14}$$

Note that W^n converges weakly to a standard Brownian motion, and since $|A_t^n| \vee |B_t^n| \leq 1$, the second term on the right hand side of (14) forms a tight sequence. Thus it is easy to guess a substitute for it in the continuous game. Interpretation of the asymptotics of the third term is more subtle, and is a key element of the formulation. One possible approach is to replace the factor $\sqrt{n}$ by a large quantity that is dynamically controlled by the two players. This point of view motivates one to consider the identity

$$-2\Delta_\infty f = \sup_{|b|=1, \, d \geq 0} \ \inf_{|a|=1, \, c \geq 0} \left\{ -\frac{1}{2}(a-b)' \left(D^2 f \right) (a-b) - (c+d)(a+b) \cdot Df \right\},$$
$$f \in \mathcal{C}^2, \ Df \neq 0, \tag{15}$$

for the following reason. Let $\mathcal{H} = \mathcal{S}^{m-1} \times [0, \infty)$ where $\mathcal{S}^{m-1}$ is the unit sphere in $\mathbb{R}^m$. The expression in curly brackets is equal to $\mathcal{L}^{a,b,c,d} f(x)$, where for $(a, c), (b, d) \in \mathcal{H}$, $\mathcal{L}^{a,b,c,d}$ is the controlled generator associated with the process

$$X_t = x + \int_0^t (A_s - B_s) dW_s + \int_0^t (C_s + D_s)(A_s + B_s) ds, \qquad t \in [0, \infty), \tag{16}$$

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and (A, C) and (B, D) are control processes taking values in $\mathcal{H}$. Since Δ_∞ is related to (14) via the Tug-of-War, and $\mathcal{L}^{a,b,c,d}$ to (16), identity (15) suggests to regard (16) as a formal limit of (14). Thus the SDG we consider has (16) as a state process, where the controls (A, C) and (B, D) are chosen by the two playes. Finally, the payoff functional, as a formal limit of (13), and accounting for the extra factor of 1/2 in (14), is given as $\boldsymbol{E}[\int_0^\tau h(X_s)ds + g(X_\tau)]$, where $\tau = \inf\{t : X_t \notin G\}$ (with an appropriate convention regarding $\tau = \infty$). We show that the zero sum SDG, where player I chooses (A, C) and seeks to maximize the payoff whereas player II chooses (B, D) aiming to minimize the payoff, has a value. Furthermore the value function is the unique viscosity solution of (12). Thus our result provides a game theoretic interpretation of the continuum value of [26] and also gives a stochastic representation for the solution of the nonlinear degenerate elliptic PDE (12).

C.II. Ergodic Rate Control Problem for Single Class Queueing Networks [14]. We study Jackson networks with state dependent and dynamically controlled arrival and service rates. The network consists of K service stations, each of which has an associated infinite capacity buffer. Arrivals of jobs can be from outside the system and/or from internal routing. Upon completion of service at a station, the customer is routed to one of the other service stations (or exits the system) according to a probabilistic routing matrix. Network is assumed to be in heavy traffic in an appropriate sense. Roughly speaking, one considers a sequence of networks with identical topology, parametrized by $n \in \mathbb{N}$. Instantaneous arrival and service rates in the n-th network are queue length dependent and are of order $\mathcal{O}(n)$. Additionally, a system manager can exercise service rate controls of order $\mathcal{O}(n^{1/2})$. The heavy traffic assumption says that traffic intensity at each station is of the order $1 - \mathcal{O}(1/n^{1/2})$. In an uncontrolled setting, diffusion approximations of suitably scaled queue length processes for such networks have been studied in [29]. The limit stochastic process is a reflected diffusion in the nonnegative orthant. Dependence of arrival and service rates on queue length processes leads to drift and diffusion coefficients for the limit model that are state dependent. Existence and uniqueness of solutions to such reflected stochastic differential equations (SDE) follows from the classical theory and well understood regularity properties of the Skorohod map.

For the controlled setting considered here, a formal analysis along similar lines leads to controlled reflected diffusions with control entering linearly in the drift coefficient. Goal of this work is to use such a formal diffusion approximation in order to obtain provably asymptotically optimal rate control policies for the underlying physical networks. We are concerned with an average long term cost per unit time criterion. Cost function is a sum of two terms; the first term measures the cost for holding customers in the buffer, while the second term is the cost for exercising control. The holding cost is allowed to have a polynomial growth (as a function of queue lengths) while the control cost is assumed to be linear. Since the cost criterion involves an infinite time horizon, additional stability conditions are needed to ensure the finiteness of the cost. In our work these conditions are manifested in the definition of the class of admissible controls. The definition in particular ensures that the controls are "uniformly stabilizing", viz. all polynomial moments of the queue length process are bounded uniformly in time, the scaling parameter and the choice of admissible control sequence. This stability property is a key ingredient in our proofs.

Our main result relates the value function (i.e. the optimum value of the cost) of the queueing

rate control problem with the value function associated with an ergodic cost problem for the formal controlled diffusion limit. Cost function for the diffusion model is analogous to that for the physical networks. It is shown that, under certain conditions, the value function for the n-th queueing network converges to that for the diffusion model, as $n \to \infty$. The theorem also shows that using an appropriately chosen ε-optimal feedback control for the diffusion model, one can construct an asymptotically ε-optimal rate control policy for the physical queueing network. Thus the theorem describes the precise sense in which the diffusion model, which is originally justified only formally, is a rigorous asymptotic approximation for the queueing model. Rate control problems with an ergodic cost criterion for single class open queueing networks in heavy traffic have been studied in several papers. Most works concern a one dimensional problem (i.e. a single server queue) and provide explicit expressions/recipes for asymptotically optimal policies. The paper [25] studies a general K dimensional Jackson type network with finite buffers. In this setting, due to compactness of the state space, all stability issues become trivial. Furthermore, convergence of value functions is established under the additional assumption that ε-optimal continuous feedback controls exist for the limit diffusion model– but no details on when such an assumption holds are provided.

One of the main steps in the proof of our main result is establishing existence of an ε-optimal continuous feedback control for the limit diffusion model. Existence of optimal feedback controls for the class of ergodic diffusion control problems studied here has been established in [7]. Our proof starts from such an optimal control b^* and constructs a sequence $\{b_n\}$ of continuous feedback controls in a manner such that an appropriately chosen measure of the set $\{b_n \neq b^*\}$ converges to 0 as $n \to \infty$. We show that, as long as initial distributions converge, the solution X^n of the reflected SDE that uses b_n as the feedback control converges weakly to X which solves the equation with b_n replaced with b^*. Proof of this result relies on certain representations and estimates for transition probability densities of reflected Brownian motions that are of independent interest. Once weak convergence of X^n to X is established, existence of an ε-optimal continuous feedback control is an immediate consequence of the linearity of the control cost. Next, an ε-optimal continuous feedback control b^ε, obtained from above result is used to define a rate control policy for the n-th network, for each $n \in \mathbb{N}$. The associated costs are shown to converge to the cost associated with b^ε in the diffusion control problem. As a consequence of this result one obtains that the value function for the limit diffusion model is an upper bound for any limit point of the sequence of value functions for the controlled queueing networks. We then establish the reverse inequality. Combining these bounds we have the main result of the paper. Proofs of these bounds use functional occupation measure methods developed in [25]. The main idea is to represent the cost associated with the n-th network in terms of integrals with respect to occupation measures associated with various stochastic processes that constitute the dynamical description of the network. Using stability estimates of this paper one can establish tightness of these occupation measures and then classical martingale characterization methods can be applied to identify the limit points in an appropriate manner.

Stability estimates obtained in this work are also useful for the study of the uncontrolled setting. We show that, under appropriate conditions, stationary distributions of scaled queue length processes (which are strong Markov processes when there are no controls) converge to the unique stationary distribution of the limit reflecting diffusion model. For the setting where the arrival and service rates are constant, such a result has been proved in [13]. To our knowledge the current paper is the first one to treat a setting with state dependent rates.

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C.III. Large Deviations for Stochastic Flows of Diffeomorphisms [11]. Stochastic flows of diffeomorphisms have been a subject of much research. In this paper, we are interested in an important subclass of such flows, namely the Brownian flows of diffeomorphisms. Our goal is to study small noise asymptotics, specifically, the large deviation principle (LDP) for such flows.

Elementary examples of Brownian flows are those constructed by solving finite dimensional Itô stochastic differential equations. More precisely, suppose $b, f_i, i = 1, \ldots, m$ are functions from $\mathbb{R}^d \times [0, T]$ to $\mathbb{R}^d$ that are continuous in (x, t) and $(k + 1)$–times continuously differentiable (with uniformly bounded derivatives) in x. Let $\beta_1, \ldots \beta_m$ be independent standard real Brownian motions on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$. Then for each $s \in [0, T]$ and $x \in \mathbb{R}^d$, there is a unique continuous $\{\mathcal{F}_t\}$–adapted, $\mathbb{R}^d$–valued process $\phi_{s,t}(x)$, $s \leq t \leq T$, satisfying

$$\phi_{s,t}(x) = x + \int_s^t b(\phi_{s,r}(x), r) dr + \sum_{i=1}^m \int_s^t f_i(\phi_{s,r}(x), r) d\beta_i(r). \tag{17}$$

By choosing a suitable modification, $\{\phi_{s,t}, 0 \leq s \leq t \leq T\}$ defines a Brownian flow of $\mathbb{C}^k$–diffeomorphisms. In particular, denoting by G^k the topological group of $\mathbb{C}^k$–diffeomorphisms, one has that $\phi \equiv \{\phi_{0,t}, 0 \leq t \leq T\}$ is a random variable with values in the Polish space $\hat{W}_k = C([0, T] : G^k)$. For $\varepsilon \in (0, \infty)$, when f_i is replaced by εf_i in (17), we write the corresponding flow as ϕ^ε. Large deviations for ϕ^ε in $\hat{W}_k$, as $\varepsilon \to 0$, have been studied in several works.

As is well known, not all Brownian flows can be expressed as in (17) and in general one needs infinitely many Brownian motions to obtain a stochastic differential equation (SDE) representation for the flow. Indeed typical space-time stochastic models with a realistic correlation structure in the spatial parameter naturally lead to a formulation with infinitely many Brownian motions (one such example, coming from problems in image analysis, is studied in this work). Thus, following Kunita's notation for stochastic integration with respect to semi-martingales with a spatial parameter, the study of general Brownian flows of $\mathbb{C}^k$–diffeomorphisms leads to SDEs of the form

$$d\phi_{s,t}(x) = F(\phi_{s,t}(x), dt), \ \phi_{s,s}(x) = x, \ 0 \leq s \leq t \leq T, \ x \in \mathbb{R}^d, \tag{18}$$

where $F(x, t)$ is a $\mathbb{C}^{k+1}$–Brownian motion. Note that such an F can be regarded as a random variable with values in the Polish space $W_k = C([0, T] : \mathbb{C}^{k+1}(\mathbb{R}^d))$, where $\mathbb{C}^{k+1}(\mathbb{R}^d)$ is the space of $(k+1)$ times continuously differentiable functions from $\mathbb{R}^d$ to $\mathbb{R}^d$. Representations of such Brownian motions in terms of infinitely many independent standard real Brownian motions is well known. Indeed, one can represent F as

$$F(x, t) \doteq \int_0^t b(x, r) dr + \sum_{i=1}^\infty \int_0^t f_i(x, r) d\beta_i(r), \ (x, t) \in \mathbb{R}^d \times [0, T], \tag{19}$$

where $\{\beta_i\}_{i=1}^\infty$ is an infinite sequence of i.i.d. real Brownian motions and b, f_i are suitable functions from $\mathbb{R}^d \times [0, T]$ to $\mathbb{R}^d$.

Letting $a(x, y, t) = \sum_{i=1}^\infty f_i(x, t) f_i'(y, t)$ for $x, y \in \mathbb{R}^d$, $t \in [0, T]$, the functions (a, b) are referred to as the *local characteristics* of the Brownian motion F. When equation (18) is driven by the

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Brownian motion F^ε with local characteristics $(\varepsilon a, b)$, we will denote the corresponding solution by ϕ^ε. In this work we establish a large deviation principle for $(\phi^\varepsilon, F^\varepsilon)$ in $\hat{W}_{k-1} \times W_{k-1}$. Note that the LDP is established in a larger space than the one in which $(\phi^\varepsilon, F^\varepsilon)$ take values (namely, $\hat{W}_k \times W_k$). This is consistent with results in literature, which consider stochastic flows driven by only finitely many real Brownian motions. The main technical difficulty in establishing the LDP in $\hat{W}_k \times W_k$ is the proof of tightness of certain controlled processes, when $k-1$ is replaced by k.

As noted above, the stochastic dynamical systems considered in this work are driven by an infinite dimensional Brownian motion. A broadly applicable approach to the study of large deviations for such systems, based on variational representations for functionals of infinite dimensional Brownian motions, has been developed in [9, 10]. Several authors have adopted this approach to analyze the large deviation properties of a variety of models, including stochastic PDEs with random dynamic boundary conditions, stochastic Navier–Stokes equations and infinite dimensional SDEs with non–Lipschitz coefficients. The approach is a particularly attractive alternative to standard discretization/approximation methods when the state spaces are non–standard function spaces, such as the space of diffeomorphisms used in the present paper.

The proof of our main result proceeds by verification of a general sufficient condition obtained in [10] (see Assumption 2 and Theorem 6 therein). The verification of this condition essentially translates into establishing weak convergence of certain stochastic flows defined via controlled analogues of the original model. These weak convergence proofs proceed by first establishing convergence for $N-$point motions of the flow and then using Sobolev and Rellich-Kondrachov embedding theorems to argue tightness and convergence as flows. The key point here is that the estimates needed in the proofs are precisely those that have been developed in literature for general qualitative analysis (e.g. existence, uniqueness) of the uncontrolled versions of the flows. Unlike past approaches taken for the study of (finite dimensional) stochastic flows, the proof of the LDP does not require any exponential probability estimates or discretization/approximation of the original model.

We next study an application of these results to a problem in image analysis. Stochastic diffeomorphic flows have been suggested for modeling *prior* statistical distributions on the space of possible images/targets of interest in the study of nonlinear inverse problems in this field. Along with a data model, noise corrupted observations with such a prior distribution can then be used to compute a *posterior* distribution on this space, the "mode" of which yields an estimate of the true image underlying the observations. Motivated by such a Bayesian procedure a variational approach to this image matching problem has been suggested and analyzed in [22]. In this work we develop a rigorous asymptotic theory that relates standard stochastic Bayesian formulations of this problem, in the small noise limit, with the deterministic variational approach taken in [22].

C.IV. Feller and Stability Properties of the Nonlinear Filter [8]. The classical model of nonlinear filtering consists of a pair of stochastic processes $(X_t, Y_t)_{t \geq 0}$ where (X_t) is called the signal process and (Y_t) the observation process. The signal is taken to be a Markov process with values in some Polish space $\mathbb{S}$ and the observations are given via the relation:

$$Y_t = \int_0^t h(X_s) ds + W_t, \tag{20}$$

where (W_t) is a standard d dimensional Brownian motion independent of (X_t) and h, referred to as the observation function, is a map from $\mathbb{S} \to \mathbb{R}^d$. Nonlinear filtering concerns with the study of the measure valued process (Π_t) which is the conditional distribution of X_t given $\mathcal{Y}_t = \sigma\{Y_s : 0 \le s \le t\}$. This measure valued process is called the nonlinear filter. In a very influential paper, Kunita[24] has shown, using uniqueness of solutions to Kushner-Stratonovich equations, that in the above filtering model if the signal is Feller-Markov with a compact Hausdorff state space $\mathbb{S}$ then the nonlinear filter is also a Feller-Markov process with state space $\mathcal{P}(\mathbb{S})$, where $\mathcal{P}(\mathbb{S})$ is the space of all probability measures on $\mathbb{S}$. Furthermore, [24] provides a proof of the statement that if the signal has a unique invariant measure μ then, under a minor additional condition, the filter has a unique invariant measure. Recently, Baxendale *et al* [6] have found a gap in the proof of this fundamental result. In this survey article we revisit this work of Kunita and several other works that extend or make use of [24], in light of the discovery made in [6].

We first consider the Feller property and existence of invariant measures. It is shown that these properties hold in a much greater generality than the one originally considered in [24]. We then turn to the study of ergodicity of the nonlinear filter. We recall the gap pointed out by Baxendale *et al* in [6]. The main issue is the equality of certain $\sigma-$fields claimed in [24]. To illustrate the difficulty, consider a probability space $(\Omega_0, \mathcal{F}_0, P_0)$ with sub- σ- fields $\mathcal{F}^*$, $(\mathcal{G}_n^*)_{n \ge 0}$, such that all the σ fields are P_0 complete and $\{\mathcal{G}_n^*\}_{n \ge 0}$ is a decreasing family of σ fields such that $\bigcap_{n \ge 0} \mathcal{G}_n^*$ is P_0 trivial. Then the equality claimed in [24] is analogous to the statement that $\mathcal{H}^* \doteq \bigcap_{n \ge 0} (\mathcal{F}^* \vee \mathcal{G}_n^*) = \mathcal{F}^*$. This equality holds in many circumstances – for instance when $\mathcal{F}^*$ is independent of $\mathcal{G}_0^*$ – however, it is false in general. Clearly $\mathcal{H}^* \supseteq \mathcal{F}^*$, but one can construct examples that show that the reverse inclusion can fail. In this work we give some sufficient conditions under which the equality of the $\sigma-$ fields stated in [24] is true. These sufficient conditions are phrased in terms of certain stability properties of the nonlinear filter, eg. finite memory property or asymptotic stability. Previous works by several authors on asymptotic stability of the filter for a variety of filtering models, then identify a rich class of filtering problems for which the uniqueness of invariant measures for the nonlinear filter holds. We in fact find that the equality of $\sigma-$ fields stated in [24] is equivalent to several asymptotic properties of the nonlinear filter, such as, uniqueness of invariant measure for the nonlinear filter, uniqueness of invariant measure for the signal-filter pair, asymptotic stability and finite memory of the filter.

C.V. Modified particle filter methods for assimilating Lagrangian data into a point-vortex model [28].

The issues surrounding the assimilation of data into models have recently gained prominence in almost all areas of geophysics. The quantity and quality of data has increased dramatically with ever-improving observational technologies. At the same time, our ability to run extensive simulations on ever-faster computers has enhanced the use of models. Data assimilation is the problem of estimating the optimal prediction that combines model output, with its attendant uncertainty, and observations, which will also contain errors. Most of the traditional techniques of data assimilation are based on (linear) control theory and optimization. Their adaptation to highly nonlinear fluid flows, as in atmospheric and oceanic dynamics, presents many mathematical challenges. We focus here on one particular context in which such nonlinear behavior occurs, namely the trajectory

(Lagrangian) motion in point vortex flow.We take an approach, based on the statistical method of particle filtering, which is not based on any linearization and hence accounts well for nonlinear effects. Lagrangian observations refer to sequences of position measurements along trajectories of (Lagrangian) tracers. Such data are abundant in geophysical systems. In contrast to the assimilation of Eulerian data, assimilation of Lagrangian data poses several complications. The reason is that most numerical models for geophysical systems are solved on a fixed grid in space or as spectral model and do not relate to the Lagrangian observations directly in terms of the model variables. Thus the conventional approach for assimilating Lagrangian data is to transform them into the Eulerian velocity data. A new approach for assimilating Lagrangian data into the fluid dynamical systems, the so-called Langrangian data assimilation (LaDA) method, was introduced by Ide et al[23]. Although the LaDA method has been shown to be more efficient than conventional approaches, its formulation, which is based on either the EKF or the Ensemble Kalman Filter, approximates the model uncertainty only up to second order since it uses linearization schemes. When model nonlinearity is strong and the observation period is long, this approximation may become invalid and filter divergence can occur The particle filter (PF), in contrast, is an approach in which a full distribution of the model uncertainty is attained by a large number of particles. In other words, the resulting distributions are not approximated by Gaussians in the process of particle filtering. Highly nonlinear effects in the flow can therefore be accommodated without causing breakdown of the filter. In this paper, we implement the PF with the LaDA into the point vortex model and investigate the filter performance. Owing to small dimensionality and yet complex nonlinear (Lagrangian) dynamics, vortex models are a natural paradigm for the investigation of data assimilation system with Lagrangian or Eulerian observations . Several improved PFs with LaDA tailored to the point vortex model are developed and the PF results with the EKF results are compared. We introduce the idea of occasional backtracking for the PF to overcome suspected divergence and describe three specific reinitialization schemes implemented with backtracking. We conduct numerical experiments and compare the performance of several PFs and the EKF applied to the point-vortex system. We find that the backtracking particle filters (BPF) introduced in this paper and applied to the two-point vortex system are an effective assimilation tool for this complex nonlinear system. The cloud expanding BPF and the directed doubling BPF both achieved lowest failure rates (for nearly every case) in the experiment testing dependence of initial condition. In the experiment testing dependence on observation period, all particle filter methods outperformed the EKF. Typically, for a given application, any data assimilation method requires some application-specific tuning to perform well. The ideas we introduced here for particle filters in the context of the point-vortex model – specifically backtracking to a previous observation time, reinitializing the filter, doubling the number of particles, and running assimilation forward – are effective at overcoming filter divergence, and general enough to be applied to other nonlinear physical systems.

C.VI. Dissertation abstract of Ph.D. student V. Maroulas. Large deviations theory concerns with study of precise asymptotics governing the decay rate of probabilities of rare events. A classical area of large deviations is the FreidlinWentzell (FW) theory that deals with path probability asymptotics for small noise Stochastic Dynamical Systems (SDS). For finite dimensional SDS, FW theory has been very well studied. The goal of the present work is to develop a systematic framework for the study of FW asymptotics for infinite dimensional SDS. Our first result is

a general LDP for a broad family of functionals of an infinite dimensional small noise Brownian motion (BM). Depending on the application, the driving infinite dimensional BM may be given as a spacetime white noise, a Hilbert space valued BM or a cylindrical BM. We provide sufficient conditions for LDP to hold for all such different model settings. As a first application of these results we study FW LDP for a class of stochastic reaction diffusion equations. The model that we consider has been widely studied by several authors. Two main assumptions imposed in all previous studies are the boundedness of the diffusion coefficient and a certain geometric condition on the underlying domain. These restrictive conditions are needed in proofs of certain exponential probability estimates that form the basis of classical proofs of LDPs. Our proofs instead rely on some basic qualitative properties, eg. existence, uniqueness, tightness, of certain controlled analogues of the original systems. As a result, we are able to relax the two restrictive requirements described above. As a second application we study large deviation properties of certain stochastic diffeomorphic flows driven by an infinite sequence of i.i.d. standard real BMs. LDP for small noise finite dimensional flows has been studied by several authors. Typical space time stochastic models with a realistic correlation structure in the spatial parameter naturally leads to infinite dimensional flows. We establish a LDP for such flows in the small noise limit. We also apply our result to a Bayesian formulation of an image analysis problem. An approximate maximum likelihood property is shown for the solution of an optimal image matching problem that involves the large deviation rate function.

C.VII. Dissertation abstract of Ph.D. student C. Lee. Stochastic processing networks arise commonly from applications in computers, telecommunications, and large manufacturing systems. Study of stability and control for such networks is an active and important area of research. In general the networks are too complex for direct analysis and therefore one seeks tractable approximate models. Heavy traffic limit theory yields one of the most useful collection of such approximate models. Typical results in the theory say that, when the network processing resources are roughly balanced with the system load, one can approximate such systems by suitable diffusion processes that are constrained to live within certain polyhedral domains (e.g., positive orthants). Stability and control problems for such diffusion models are easier to analyze and, once these are resolved, one can then infer stability properties and construct good control policies for the original physical networks. In my dissertation we consider three related problems concerning stability and long time control for such networks and their diffusion approximations. In the first part of the dissertation we establish results on long time asymptotic properties, in particular geometric ergodicity, for limit diffusion models obtained from heavy traffic analysis of stochastic networks. The results address the rate of convergence to steady state, moment estimates for steady state, uniform in time moment estimates for the process and central limit type results for time averages of such processes. In the second part of the dissertation we consider invariant distributions of an important subclass of stochastic networks, namely the generalized Jackson networks (GJN). It is shown that, under natural stability and heavy traffic conditions, the invariant distributions of GJN converge to unique invariant probability distribution of the corresponding constrained diffusion model. The result leads to natural methodologies for approximation and simulation of steady state behavior of such networks. In the final part of the dissertation we consider a rate control problem for stochastic processing networks with an ergodic cost criterion. It is shown that value functions and near

optimal controls for limit diffusion models serve as good approximations for the same quantities for certain physical networks that are heavily loaded.

C.VIII. Invited Presentations.

- *Diffusion Approximations for Controlled Stochastic Networks*, Stochastic Networks Conference, Ecolé Normale Superieure, Paris, June 23-28, 2008.

- *Elliott-Kalton Stochastic Differential Games Associated with the Infinity Laplacian*, Seminar on Stochastic Processes, Delaware, April 3-5, 2008.

- *Singular Control with State Constraints*, Recent Advances in Probability, Indian Statistical Institute, Kolkata, Decemeber 11 - 15, 2007.

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