Wavelet Shrinkage and Denoising

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WAVELET SHRINKAGE AND DENOISING

- Introduction to Wavelet Shrinkage and Denoising
- Procedure for wavelet shrinkage
- Why Wavelet Shrinkage Works
- Two methods of Wavelet Shrinkage

Introduction:

- An image is often corrupted by noise in its acquisition and transmission stage.
- Noises are normally created when scanning images to produce digital images, recording a voice to an audio file, and even transmitting digital image often produce Noise.
- This noise can be random or white noise with no coherence or coherent noise introduced by device mechanism

Introduction

- Wavelet shrinkage is a signal denoising technique based on the idea of thresholding the wavelet coefficients.
- Denoising is the process of removing noise from a signal.
- Wavelet coefficients having small absolute values are considered to encode very fine details of the signal.
- Wavelet shrinkage denoising should not be confused with smoothing , Whereas smoothing removes high frequencies and retains low ones, denoising attempts to remove whatever noise is present and retain whatever signal is present regardless of the signal's frequency content .

Noisy Image

Denoised Image

Threshold rules

There are two types of rules that we are ging to use in this talk.

Hard thresholding rule – Let $A = \{x : |x| > \lambda\}$

$$H_\lambda(x) = x \cdot \chi_A(x)$$

where χ_A is the characteristic set function.

Soft thresholding rule –

$$T_\lambda(x) = \mathrm{sgn}(x) \cdot \max(|x| - \lambda, 0)$$

Procedure for wavelet Shrinkage.

The wavelet shrinkage method involves the following steps:

1. Apply wavelet transform to the signal.
2. Obtain a threshold Value that minimizes the Mean Square Error. Then using the Soft threshold function we remove (zero out) the coefficients that are smaller than the threshold.
3. Reconstruct the signal (apply the inverse of the Wavelet transform).

Why does Wavelet Shrinkage work?

Suppose we take one iteration of wavelet transformation to a noisy signal to get: $w = WX$ (t) = WS (t) + WN (t).

WX(t) is the signal that contains the lowpass portion of transformation in the first half and the highpass portion transformation in the second half.

The highpass portion of the signal is sparse since most of the energy is stored in the lowpass portion of the transformation.

However, *WN* (t) is the Gaussian white noise, when we apply an Orthogonal Matrix *W* to a Gaussian white noise signal; it returns a Gaussian white noise signal with the same variance.

So, the noise level for *WN* (t) is the same as the noise level for *N* (t). This tells us that the highpass portion of *WX* (t) is comprised primarily of noise!

The observed noisy data will have the form

$$X(t) = S(t) + N(t)$$

Where $S(t)$ is the true signal and $N(t)$ is the noise as functions of time t (for sampled values).

We calculate the k iterations of the wavelet transformation of $X(t)$ to obtain $Y = [l \mid h]$, where l is the lowpass (approximation) portion and h is the highpass (details) Portion of Y.

Apply the soft threshold rule to h to either shrink or zero the coefficients in h. Let this new highpass portion be h'.

Form a modified transform $Y' = [l \mid h']$ by rejoining l with h'. Finally, compute k iterations of the inverse wavelet transformation of Y' to obtain a denoised estimate S' of S.

Measuring the Effectiveness of Denoising method.

We can use the mean squared error to determine if the shrinkage method does a good job of denoising the given signal.

The **mean squared error** or **MSE** is defined as

$$E\left(\| S - \hat{S} \|^2 \right) = \sum_{k=1}^{N} (s_k - \hat{s}_k)^2$$

We will construct the thresholds (tolerance) values that minimizes the MSE value.

Two methods of wavelet shrinkage:

- ✓ Goal: To determine a threshold (or tolerence) value λ that minimizes **MSE** (mean squared Error)
- ✓ There are two methods that we will discuss for choosing λ that is the visushrink and sureshrink.
- ✓ If the image is sparse we use visushrink method to select otherwise we use sureshrink

Sure-shrinkage

[**S**tein's **U**nbiased **R**isk **E**stimator]

Theorem: Suppose the N-vector w is formed by w = z + e, where e = (e_1,…,e_n) and each e_k is normally distributed with mean theta and variance 1. Let $\hat{z}$ be the estimator formed by $\hat{z}$ = w + g(w) where the coordinate functions g_k: R^N arrow R of the vector-valued function g: $R^N \rightarrow R^N$ are differentiable except at a finite number of points. We obtain

$$E(\|\hat{z} - z\|^2) = E(N + \|g(w)^2\| + 2\sum_{k=1}^{N} \frac{\partial}{\partial \omega_k} g_k(w))$$

To get the threshold function we simplify right hand side of equation, then take the minimum value to obtain the threshold

$$f(\lambda) = N + \|g(w)\|^2 + 2\sum_{k=1}^{N} \frac{\partial}{\partial \omega_k} g_k(w)$$

To get the threshold function we simplify right hand side of equation, then take the minimum value to obtain the threshold λ^{sure}. Put

$$f(\lambda) = N + \left\| g(\omega) \right\|^2 + 2\sum_{k=1}^{N} \frac{\partial}{\partial \omega_k} g_k(\omega)$$

By considering different cases for λ we get

$$f(\lambda) = N + \sum_{k=1}^{N} \min(\lambda^2, \omega_k{}^2) - 2\#\{k : |\omega_k| < \lambda\}$$

Putting ω_l in place of λ and simplifying we obtain

$$f(|\omega_{l+1}|) = f(|\omega_l|) + (N-l)(\omega_{l+1}^2 - \omega_l^2) \quad \text{for } l \geq 1$$

$$f(|\omega_1|) = N + N|\omega_1|^2$$

Finally, we obtain the threshold values using

$$\lambda^{sure} = \min\{f(|\omega_1|), f(|\omega_2|), \ldots\ldots, f(|\omega_N|)\}$$

Visu-shrinkage

Donoho and Johnston proved the following result:

let $S \in R^N$ and given $\boldsymbol{X = S + N}$, where $\boldsymbol{N}$ is the white Gaussian noise, with noise level $\boldsymbol{\sigma}$. Let $T_\lambda(t)$ be a soft threshold function with $\lambda = \sigma\sqrt{2\ln(N)}$.
If $\hat{S}$ is a vector formed by applying $T_\lambda(t)$ to *S*, then

$$E\left(\| s - \hat{s} \|^2\right) \le (2\ln(N) + 1)(\sigma^2 + \sum_{k=1}^{N} \min(s_k{}^2, \sigma^2))$$

We call the tolerance value $\lambda = \sigma\sqrt{2\ln(N)}$ to be universal threshold and donate it by λ^{univ}.

For λ^{univ} and any S the soft threshold rule produces a mean square error that is bounded above by a constant times the noise level square plus the ideal mean square error.

Notice that the threshold value depends on signal size and the noise level **σ**. In practice, we do not know the value of **σ**.

To estimate the noise level **σ**, we will make use of a result by Frank R Hampel that showed the Median Absolute Deviation $MAD(X) = |X - Median(X)|$ converges to **0.6745σ** as the sample size goes to infinity.

To estimate noise level σ we use:

$$\hat{\sigma} = MAD(h)/0.6745$$

Where ***h*** is the highpass portion of the transformation 1st iteration.