Hyperfast Numerical Integration of Ocean Surface Wave Dynamics: Extensions to Higher Order

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LONG-TERM GOALS

1. The ***inverse scattering transform*** **(IST)** can be used for the ***time series analysis of laboratory and oceanic wave data***. The approach may be viewed as a ***generalization of linear Fourier analysis***.

2. IST is being applied to the study of *"**rogue, freak or giant" ocean waves***.

3. A third long-term goal is the ***development of fast algorithms for numerically integrating the space/time dynamics of both shallow-water and deep-water wave trains***.

OBJECTIVES

1. The objective of the present research program is the development of ***fast numerical multidimensional Fourier*** techniques applied to a wide range of ***wave modeling*** and ***data analysis*** problems.

2. Important progress made in the past year has been the ***development of new methods for extending the nonlinear Fourier approach to arbitrary order***. Thus one can now push toward the solution of the ***Euler*** and other ***higher order equations*** in a more systematic way that requires ***very little additional central processor time***.

APPROACH

It is well known that equations such as the KdV, the modified KdV, the Gardner and the Kadomtsev-Petviashvili equations are all integrable. Hyperfast models for these equations can be developed on a straightforward basis using methods discussed in the references [Osborne, 2003; Osborne, 2008a,b; Osborne, 2009]. I show here how these equations allow one to go to higher order in a kind of hierarchy which provides physically important wave equations containing all of the many nonlinear aspects of water waves. The impact on cpu requirements for a hyperfast model is however minimal, no matter how high the order, using the new methods discussed here.

We first consider the Kadomtsev-Petviashvili (KP) equation

$$\eta_t + c_o\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} + \gamma\ \partial_x^{-1}\eta_{yy} = 0 \tag{1}$$

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14. ABSTRACT

1. The inverse scattering transform (IST) can be used for the time series analysis of laboratory and oceanic wave data. The approach may be viewed as a generalization of linear Fourier analysis. 2. IST is being applied to the study of ?rogue, freak or giant? ocean waves. 3. A third long-term goal is the development of fast algorithms for numerically integrating the space/time dynamics of both shallow-water and deep-water wave trains.

15. SUBJECT TERMS

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The constant coefficients $c_o, \alpha, \beta, \gamma$ are given after eq. (8) below. Here $\eta(x,y,t)$ is the wave amplitude as a function of the two spatial variables, x, y and time, t. The KP equation (1) is a natural two-space-dimensional extension of the KdV equation. The periodic KP solutions include *directional spreading* in the wave field:

$$\eta(x,t) = 2\frac{\partial^2}{\partial x^2}\ln\theta(x,y,t\,|\mathbf{B},\boldsymbol{\phi}) \tag{2}$$

Here the generalized Fourier series has the form given in (4) below, where the phase has the *two dimensional* expression:

$$\mathbf{X}(x,y,t) = \mathbf{k}x + \mathbf{l}y - \boldsymbol{\omega}t + \boldsymbol{\phi} \tag{3}$$

The spatial terms include both the *x* and *y* coordinates, $\mathbf{k}x$ and $\mathbf{l}y$, which allows wave spreading to be taken into account. The KP equation is the first nonlinear step toward a directional sea state; KP is however limited to small directional spreading. Improving the directional spreading characteristics of the KP equation requires the addition of physically important corrections to the equation, as discussed below.

The ***generalized Fourier series***, $\theta(x,t\,|\mathbf{B},\boldsymbol{\phi})$, is given by the expression

$$\theta(x,y,t\,|\,\mathbf{B},\boldsymbol{\phi}) = \sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\cdots\sum_{m_N=-\infty}^{\infty} e^{i\sum_{n=1}^{N} m_n X_n - \frac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N} m_m m_n B_{mn}} \tag{4}$$

where $X_n = k_n x + l_n y - \omega_n t + \phi_n$. The function $\theta(x,y,t\,|\mathbf{B},\boldsymbol{\phi})$ is also called a ***Riemann theta function*** or ***multidimensional Fourier series***. Here **B** is the Riemann matrix (the “spectrum” of the solution), the vectors **k**, **l** constitute the usual wave numbers, the vector $\boldsymbol{\omega}$ contains the frequencies and the vector $\boldsymbol{\phi}$ forms the phases. The ***inverse problem*** associated with (2), (3) allows one to determine the Riemann matrix, wave numbers, frequencies and phases appropriate for solving the *Cauchy problem* for KP: Given the spatial variation of the solutions at $t = 0$, $\eta(x,y,0)$, compute the solution for all time, $\eta(x,y,t)$. This is a necessary step for the numerical simulations presented herein. The ***solitons, Stokes waves and sine waves lie on the diagonal of the Riemann matrix***; the ***off-diagonal terms contain the nonlinear interactions***.

Why is the above approach useful for hyperfast numerical simulations? Because the Riemann theta function can be programmed as a ***fast theta function transform*** (FTFT), just as the Fourier transform can be programmed as a ***fast Fourier transform*** (FFT). Therefore the numerical integration of KP (1) can be evaluated at specific time points, necessary only for graphical purposes or for extracting useful (often statistical) properties of the sea surface. This contrasts to the FFT that must be evaluated at very small time steps when used for the numerical integration of a nonlinear partial differential equation. This is one reason why the higher order methods require considerable amounts of computer time.

WORK COMPLETED

The equation of interest herein is the so-called ***extended KP*** (***ExKP***) equation which has the following form:

$$\boxed{u_t + 6\beta u u_x + u_{xxx} + 3\sigma^2 \partial_x^{-1} u_{yy} = \frac{3}{2}\alpha^2 u^2 u_x + 3\alpha\sigma u_x \partial_x^{-1} u_y} \qquad (5)$$

Here α, β, σ are arbitrary constants and the field is in 2+1 dimensions: $u(x,y,t)$. Note that (5) consists of the KP equation (1) on the left hand side with a cubic term (or so-called Gardner term) together with an additional spreading term, both on the right hand side. *Thus the ExKP equation is superior to the KP equation because it extends the wave dynamics to higher waves via the cubic term and simultaneously improves the description of wave spreading.*

Now let us discuss the physics of ExKP. First notice that by setting the constant coefficients $\sigma = \alpha = 0$ we obtain the KdV equation in 1+1 dimensions:

$$u_t + 6\beta u u_x + u_{xxx} = 0$$

By setting $\beta = \sigma = 0$ we obtain the 1+1 modified KdV equation (***mKdV***):

$$u_t - \frac{3}{2}\alpha^2 u^2 u_x + u_{xxx} = 0$$

By setting $\sigma = 0$ we obtain the 1+1 ***Gardner equation***:

$$u_t + 6\beta u u_x + u_{xxx} - \frac{3}{2}\alpha^2 u^2 u_x = 0$$

By setting $\alpha = 0$ we obtain the 2+1 ***KP Equation***:

$$u_t + 6\beta u u_x + u_{xxx} + 3\sigma^2 \partial_x^{-1} u_{yy} = 0$$

An important result is the following ***Gardner transformation***:

$$\boxed{u = \beta v - \frac{1}{2}\alpha v_x - \frac{1}{4}\alpha^2 v^2 - \frac{1}{2}\sigma\alpha\partial_x^{-1} v_y} \qquad (6)$$

which maps ExKP (5) to the KP equation (1). For the physical case of $\beta = 1$ the above equation can also be referred to as an ***exact near-identity transformation***. From this point of view the inverse is:

$$v \approx u + \frac{1}{2}\alpha u_x + \frac{1}{4}\alpha^2 u^2 + \frac{1}{2}\sigma\alpha\partial_x^{-1} u_y \qquad (7)$$

This is obtained by using the leading order result in the higher order terms and then solving for *v*. While the latter result (7) is not exact, i.e. it does not *exactly* transform KP into ExKP, the equation does carry out this transformation to leading order. The important point to notice is that the Gardner transformation (6) given above ***is*** exact; the fact that it is an inverse transformation (indicated by the minus signs on the right hand side) suggests that higher order transformations of this type can lead to even higher-order interesting and physical wave equations, perhaps also leading to physically important equations at infinite order (e.g. the Euler equations, although this is still an open mathematical problem). Another important result is that (5) can be derived directly from the Euler equations by the same procedure that one uses to derive KP, an important physical verification of ExKP.

A table of several nonlinear wave equations is given below. I show both the lower and higher order equations, togther with the appropriate Gardner transformations. For convenience I show the Hirota transformation which carries each of the wave equations to its associated bilinear form, important for numerical modeling applications. The red boxes emphasize the ExKP equation and its associated Gardner transformation.

Lower Order Equation	***Hirota Transf.***	***Gardner Transf.***	***Higher Order Equation***	***Hirota Transf.***
KdV Equation: $u_t + 6\beta u u_x + u_{xxx} = 0$	$u = 2\partial_{xx} \ln\theta$	$u = -\frac{1}{2}\alpha v_x - \frac{1}{4}\alpha^2 v^2$	***1+1 Modified KdV:*** $u_t - \frac{3}{2}\alpha^2 u^2 u_x + u_{xxx} = 0$	$u = i\partial_x \ln\left(\frac{G}{F}\right)$
KdV Equation: $u_t + 6\beta u u_x + u_{xxx} = 0$	$u = 2\partial_{xx} \ln\theta$	$u = \beta v - \frac{1}{2}\alpha v_x - \frac{1}{4}\alpha^2 v^2$	***1+1 Gardner Equation:*** $u_t + 6\beta u u_x + u_{xxx} = \frac{3}{2}\alpha^2 u^2 u_x$	
2+1 KP Equation: $u_t + 6\beta u u_x + u_{xxx} + 3\sigma^2 \partial_x^{-1} u_{yy} = 0$	$u = 2\partial_{xx} \ln\theta$	$u = \beta v - \frac{1}{2}\alpha v_x - \frac{1}{4}\alpha^2 v^2 - \frac{1}{2}\sigma\alpha\partial_x^{-1} v_y$	***2+1 ExKP*** $u_t + 6\beta u u_x + u_{xxx} + 3\sigma^2 \partial_x^{-1} u_{yy} - \frac{3}{2}\alpha^2 u^2 u_x - 3\alpha\sigma u_x \partial_x^{-1} u_y = 0$	$u = \frac{2}{\alpha}\partial_x \ln\left(\frac{G}{F}\right)$

We see that the sequence of equations from KdV, mKdV, KP, 1+1 Gardner to 2+1 ExKP forms a kind of ***natural hierarchy of equations marching to higher order***. One leaps from KdV to 1+1 Gardner to ExKP, or from KP to ExKP. One is reminded of the use of near-identity transformations as introduced by Kodama to study higher order, asymptotically integrable equations. In the present case the Gardner transformations are however exact.

RESULTS

For physical and engineering purposes the ExKP equation can be written in the following dimensional form:

$$\eta_t + c_o\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} + \gamma\ \partial_x^{-1}\eta_{yy} = \alpha_1\eta^2\eta_x + \gamma_1\eta_x\partial_x^{-1}\eta_y \qquad (8)$$

Where

$$c_o = \sqrt{gh}\,, \qquad \alpha = \frac{3c_o}{2h}\,, \qquad \beta = \frac{c_o h^2}{6}\,, \qquad \gamma = \frac{c_o}{2}\,, \qquad \alpha_1 = \frac{15}{8}\frac{c_o}{h^2}\,, \qquad \gamma_1 = \sqrt{\frac{15}{8}\frac{c_o}{h}}$$

As usual g is the acceleration of gravity, h is the depth and c_o is the linear phase speed. Thus we have the KP equation with the addition of a cubic on the right hand side, together with an additional, *nonlinear* spreading term. ExKP substantially improves the KP equation because we now can have higher waves due to the cubic term, $\alpha_1 \eta^2 \eta_x$, and we improve the spreading characteristics with the nonlinear, non-local term $\gamma_1 \eta_x \partial_x^{-1} \eta_y$. Both of these are important for modeling purposes.

The ExKP equation is not only a substantial improvement for ***nonlinear surface wave dynamics***, but it also provides very important contributions necessary for ***internal wave dynamics***. It contains not only KdV type solitons, but soliton-hole pairs as in the modified KdV equation, the "fat" solitons of the Gardner equation, all together with directional spreading out to second order. Some of these are shown in Figs. 1 and 2. In Fig. 3 I show an example of a hole solution which forms due to the cubic Gardner term in (8).

In Fig. 4 I show the wave field for a sample preliminary run at a single instant of time for a simulation of the ExKP equation. I now discuss the numerical model. The basic procedure is shown in the flow chart of Fig. 5. One first chooses the desired directional spectrum (Pierson-Moskowitz or JONSWAP with an appropriate directional spreading function) using the linear Fourier transform. Then the Riemann spectrum (Riemann matrix, frequencies, phases) are computed on this basis. This process requires considerable space to explain and will be omitted from this short report for lack of space. The important ingredient is that we need to compute two sets of phases for the Riemann spectrum, a departure from the model for the KP equation. Two Riemann theta function spectra are then computed as a function of two-dimensional wave number and time: $F_{mn}(t) \equiv F(k_m, l_n, t)$, $G_{mn}(t) \equiv G(k_m, l_n, t)$. The space-time evolution of the theta functions is then computed by converting the spectra to space and time by a 2D FFT algorithm: $F(x,y,t)$, $G(x,y,t)$. Then the surface wave field is computed by the Hirota transformation:

$$u = \frac{2}{\lambda}\partial_x \ln\left(\frac{G}{F}\right)$$

And so we have the solution of the ExKP equation, which is about three orders of magnitude faster than the more traditional split-step algorithm!

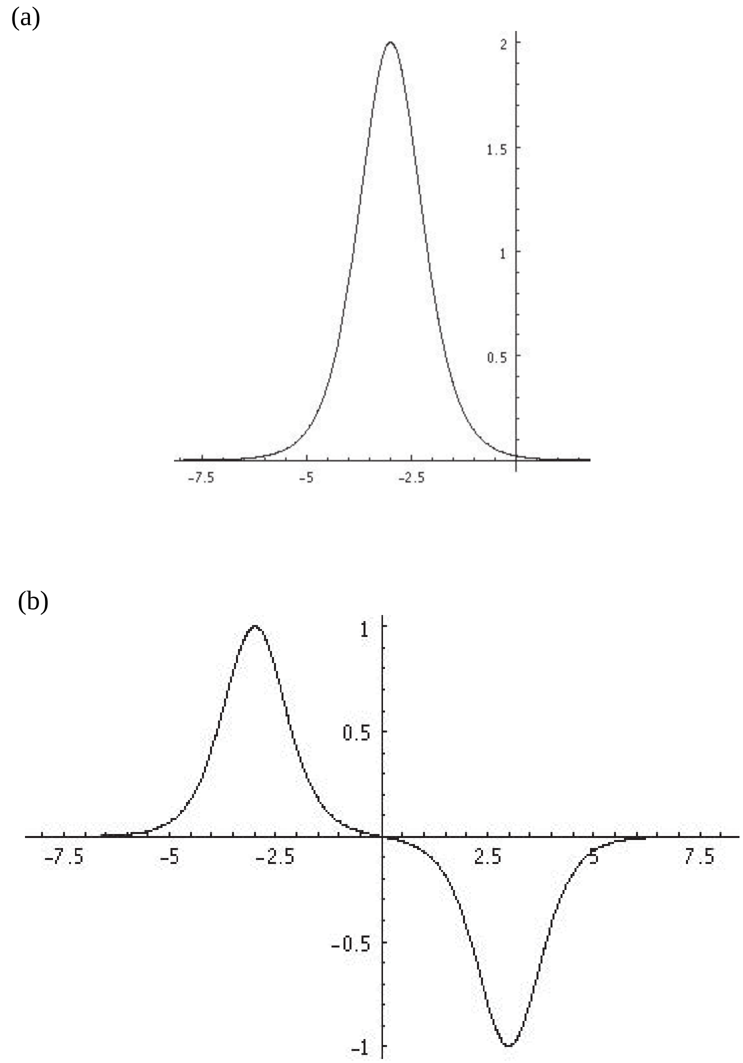

Figure 1. (a) Example of a single pulse soliton solution of the Kortweg-deVries equation. (b) Examples of single pulse solitons, both positive and negative (a hole), from the modified Kortweg-deVries equation.

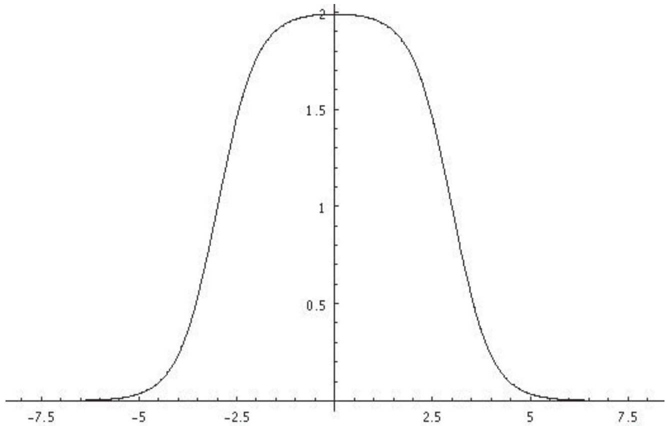

Figure 2. Example of a single "fat" pulse soliton from the Gardner equation. This solution is more appropriate for describing certain kinds of highly nonlinear internal waves.

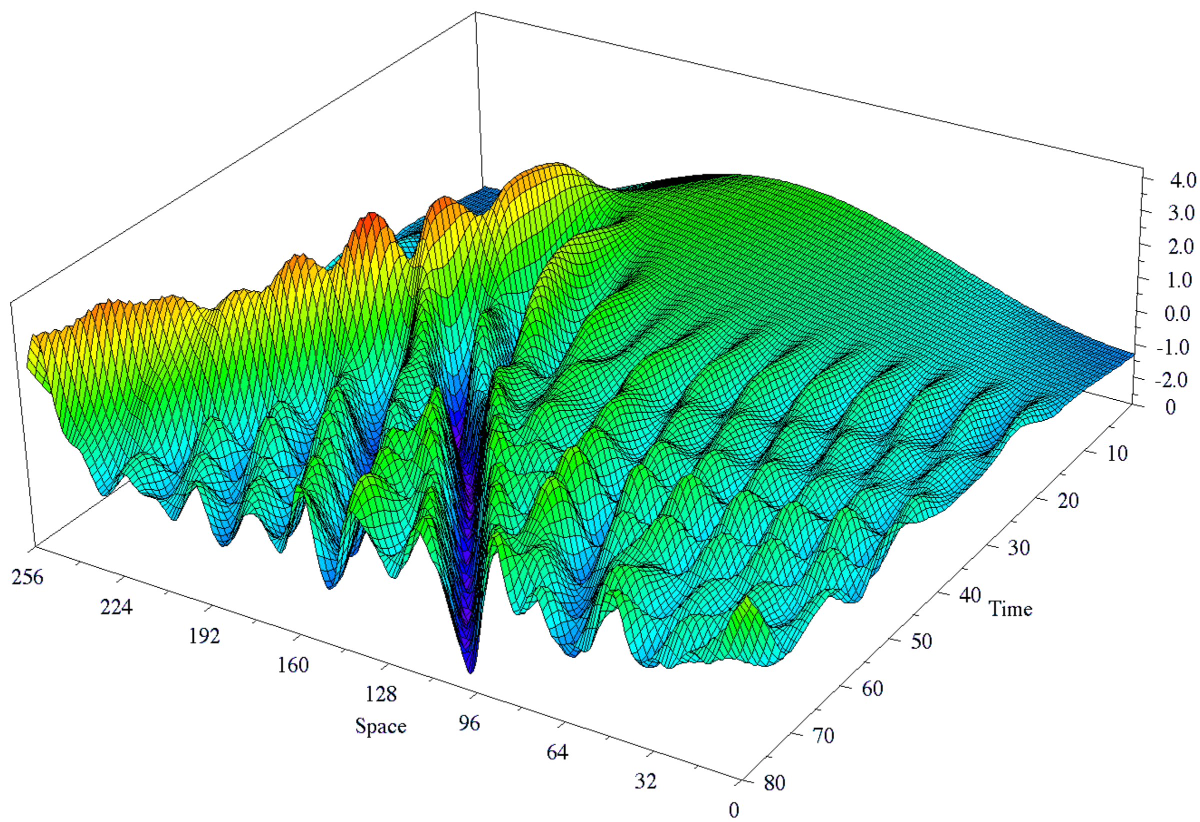

Figure 3. Evolution of a hole state from the Gardner equation. The hole is seen as a channel beginning near space coordinate 96 and time 80.

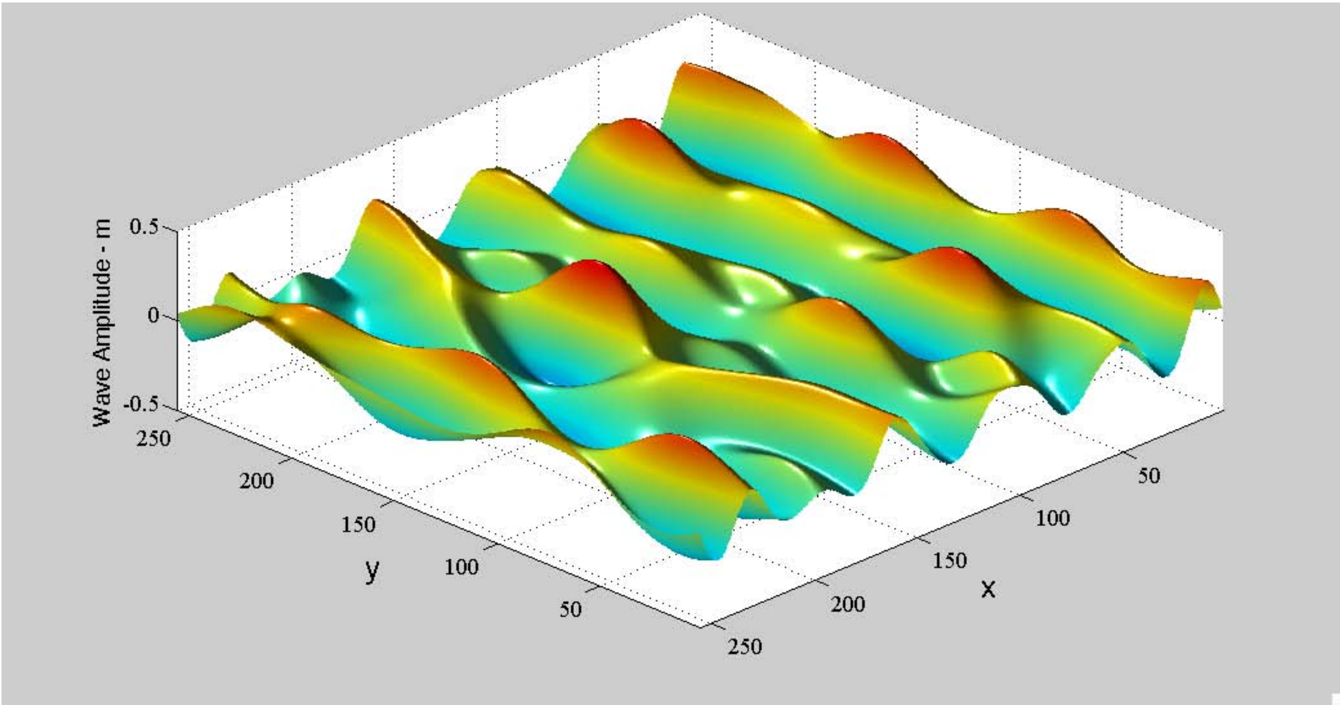

Figure 4. Example of a wave simulation with the ExKP equation.

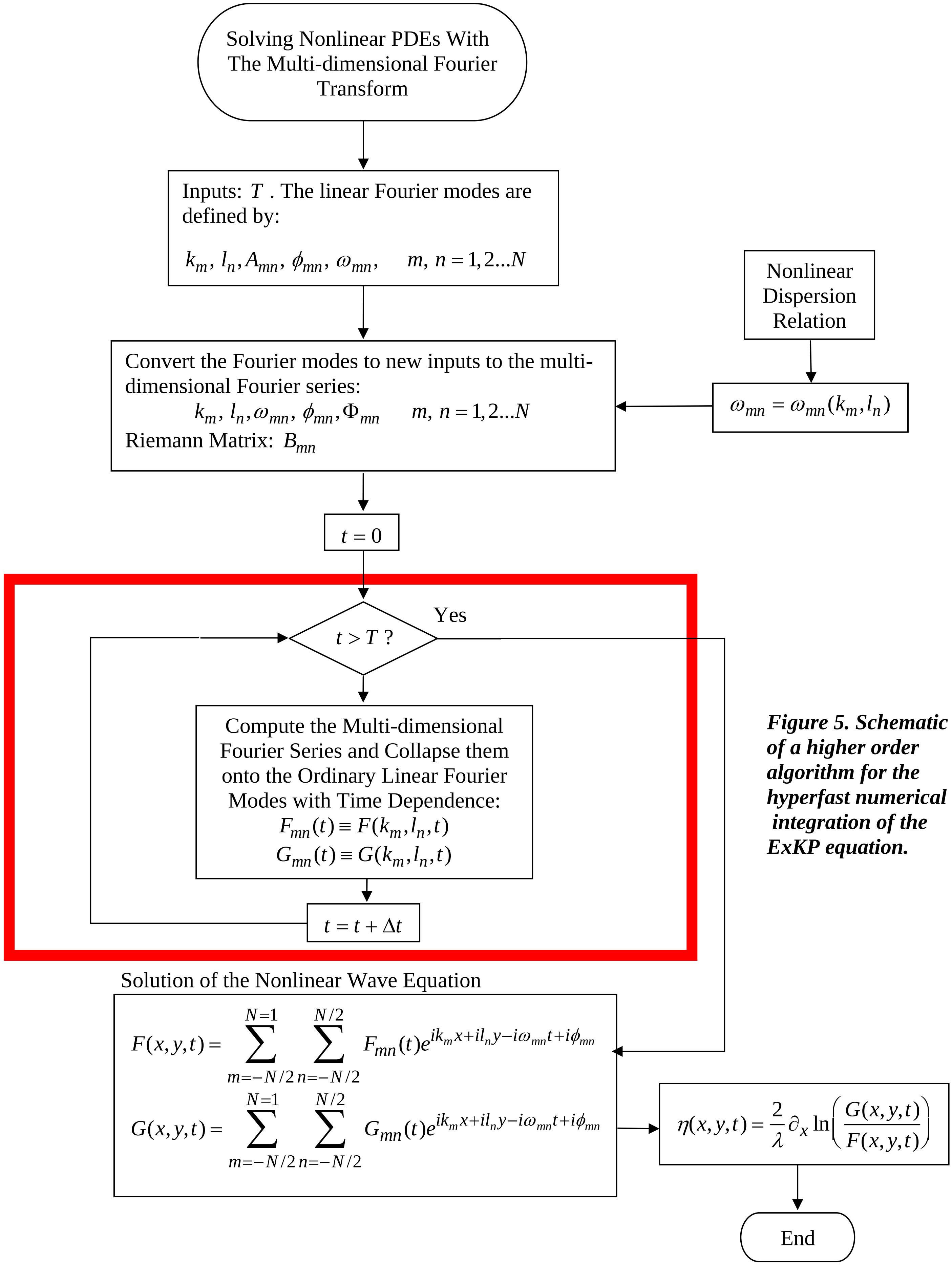

Figure 5. Schematic of a higher order algorithm for the hyperfast numerical integration of the ExKP equation.

TRANSITIONS

Transitions expected are related to the use of the codes as guidance to ships and unmanned, unteathered vehicles as the kind of environment in which one resides and for the real time sampling of the environment, including the acoustic environment.

RELATED PROJECTS

An intimate relationship between our results and other projects exists because the sea surface provides a major forcing input to many kinds of offshore activities, including the dynamics of floating and drilling vessels, barges, risers and tethered vehicles. The present work leads to a nonlinear representation of the sea surface forcing and vessel response for shallow water waves.

REFERENCES

Osborne, A. R., Nonlinear Ocean Waves and the Inverse Scattering Transform, in *Scattering*, edited by R. Pike and P. Sabatier, Academic Press, 2001.

Osborne, A. R., Numerical Computation of 2+1 Dimensional Shallow-Water Waves with Riemann Theta Functions, submitted, 2008.

Osborne, A. R., Fast Computation of Shallow-Water Wave Dynamics with Riemann Theta Functions, submitted, 2008.

Osborne, A. R., *Nonlinear Ocean Waves*, a monograph on multidimensional Fourier methods, to appear 2009.