

REPORT DOCUMENTATION PAGE			Form Approved OMB NO. 0704-0188	
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1. REPORT DATE (DD-MM-YYYY)		2. REPORT TYPE		3. DATES COVERED (From - To)
		Technical Report		-
4. TITLE AND SUBTITLE			5a. CONTRACT NUMBER	
Optimal natural frames			W911NF-06-1-0325	
			5b. GRANT NUMBER	
			5c. PROGRAM ELEMENT NUMBER	
			611102	
6. AUTHORS			5d. PROJECT NUMBER	
Eric Justh, P. S. Krishnaprasad				
			5e. TASK NUMBER	
			5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAMES AND ADDRESSES			8. PERFORMING ORGANIZATION REPORT NUMBER	
University of Maryland - College Park Office of Research Administration & Advancement University of Maryland, College Park College Park, MD 20742 -5141				
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)			10. SPONSOR/MONITOR'S ACRONYM(S) ARO	
U.S. Army Research Office P.O. Box 12211 Research Triangle Park, NC 27709-2211			11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
			51005-NS.14	
12. DISTRIBUTION AVAILABILITY STATEMENT				
Approved for Public Release; Distribution Unlimited				
13. SUPPLEMENTARY NOTES				
The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.				
14. ABSTRACT				
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15. SUBJECT TERMS				
natural frames, optimal control on Lie groups, Lie-Poisson reduction, nonlinear Schrödinger equation, vortex filament				
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	15. NUMBER OF PAGES
a. REPORT	b. ABSTRACT	c. THIS PAGE	UU	19a. NAME OF RESPONSIBLE PERSON
UU	UU	UU	UU	Perinkulam Krishnaprasad
				19b. TELEPHONE NUMBER
				301-405-6843

## Report Title

Optimal natural frames

### ABSTRACT

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# Optimal natural frames

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Problems of optimal control on Lie groups are of broad interest and application dating back to the early days of geometric control theory. We study a class of such problems defined on the special Euclidean group and demonstrate by appealing to reduction methods that the extremals in these problems admit special structure associated to the nonlinear Schrödinger equation.

**Keywords:** natural frames, optimal control on Lie groups, Lie-Poisson reduction, nonlinear Schrödinger equation, vortex filament

## 1. Introduction

In this paper, we discuss a class of optimal control problems on Lie groups, focusing on the special Euclidean groups  $SE(n)$ . The models we discuss arise in the dynamics and control of unmanned aerial vehicles (Justh & Krishnaprasad 2002, 2004, 2005), in continuum models in biophysics (Wiggins 2001) and in fluid mechanics (Newton 2001). In his Ph.D. thesis (Baillieul 1975, 1978), John Baillieul investigated optimal control problems on matrix Lie groups using the Maximum Principle of Pontryagin and methods from the calculus of variations, emphasizing in Chapter 3 of the thesis the details of the special case of  $SO(3)$ . The present paper is somewhat in the spirit of that chapter, seeking explicit solutions while finding common patterns in a wider class of problems.

In Section 2 of this paper we set up a notation explaining the natural framing of twice differentiable curves in  $\mathbb{R}^3$ . In Section 3 we discuss optimal control problems with symmetry on a Lie group and state the reduction of the Maximum Principle to the dual of the Lie algebra. With this as the starting point we show, with the help of examples, that in  $SE(n)$  the extremals are stationary solutions to a nonlinear Schrödinger equation. Connections to vortex filament equations that yield singular solutions in fluid mechanics (Newton 2001, Ricca 1996) are also indicated. In the work of Langer and Perline (1991, 1996) connections between the vortex filament equation and the nonlinear Schrödinger equation are discussed using natural frames and generalizations.

## 2. Natural frame representation of curves

Given a curve  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$  which is at least twice continuously differentiable, we can associate with it a natural Frenet frame  $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$  which evolves according

to (Bishop 1975)

$$\begin{aligned}
\gamma' &= \mathbf{T}, \\
\mathbf{T}' &= k_1 \mathbf{M}_1 + k_2 \mathbf{M}_2, \\
\mathbf{M}'_1 &= -k_1 \mathbf{T}, \\
\mathbf{M}'_2 &= -k_2 \mathbf{T},
\end{aligned} \tag{2.1}$$

where prime denotes differentiation with respect to the arc length parameter  $s$ ,  $\mathbf{T}$  is the unit tangent vector to the curve,  $\mathbf{M}_1$  and  $\mathbf{M}_2$  are unit normal vectors, and  $[\mathbf{T} \ \mathbf{M}_1 \ \mathbf{M}_2] \in SO(3)$ . The natural curvatures  $k_1$  and  $k_2$  can be thought of as controls: given an initial position  $\gamma(0)$  and initial choice  $\mathbf{M}_1(0)$  and  $\mathbf{M}_2(0)$ , the functions  $k_1 : \mathbb{R} \rightarrow \mathbb{R}$  and  $k_2 : \mathbb{R} \rightarrow \mathbb{R}$  completely determine the curve.

The natural Frenet frame equations (2.1) can be written as a left-invariant system on  $SE(3)$  as

$$\dot{g} = g\xi, \tag{2.2}$$

with

$$\begin{aligned}
g &= \begin{bmatrix} \mathbf{T} & \mathbf{M}_1 & \mathbf{M}_2 & \gamma \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3), \\
\xi &= \begin{bmatrix} 0 & -k_1 & -k_2 & 1 \\ k_1 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in se(3).
\end{aligned} \tag{2.3}$$

Curves and natural Frenet frames have proved useful for designing coordinated vehicle trajectories for tasks such as pursuit (Justh & Krishnaprasad 2006; Reddy, Justh & Krishnaprasad 2006), formation flight (Justh & Krishnaprasad 2004, 2005), and boundary tracking (Zhang, Justh & Krishnaprasad 2004). They have also been applied to gather evidence for particular pursuit strategies in nature (Reddy, 2007).

An alternative framing of the curve  $\gamma$  (when  $\gamma$  is three times continuously differentiable and  $\gamma'' \neq 0$ ) is the Frenet-Serret frame

$$\begin{aligned}
\gamma' &= \mathbf{T}, \\
\mathbf{T}' &= \kappa \mathbf{N}, \\
\mathbf{N}' &= -\kappa \mathbf{T} + \tau \mathbf{B}, \\
\mathbf{B}' &= -\tau \mathbf{N},
\end{aligned} \tag{2.4}$$

where  $\kappa$  is the curvature,  $\tau$  is the torsion,  $\mathbf{N}$  is the unit normal vector, and  $\mathbf{B}$  is the unit binormal vector. The Frenet-Serret equations (2.4) can also be expressed as a left-invariant system on  $SE(3)$ , and there are formulas which relate  $\kappa$  and  $\tau$  to the natural curvatures  $k_1$  and  $k_2$ . For example,

$$\kappa^2 = k_1^2 + k_2^2, \quad \tau = \frac{d}{ds} [\arg(k_1, k_2)] = \frac{d}{ds} [\tan^{-1}(k_2/k_1)]. \tag{2.5}$$

Circular helical curves are easily described using the Frenet-Serret frame, because  $\kappa$  and  $\tau$  are constant. The corresponding natural curvatures  $k_1$  and  $k_2$  are

sinusoidal functions of  $s$  (in phase quadrature). A right circular helix with radius  $\rho$ , pitch  $2\pi r$ , and initial point  $\gamma(0) = (\rho, 0, 0)$  is described by

$$\gamma(s) = \begin{bmatrix} \rho \cos(s/\sqrt{\rho^2 + r^2}) \\ \rho \sin(s/\sqrt{\rho^2 + r^2}) \\ rs/\sqrt{\rho^2 + r^2} \end{bmatrix}, \quad (2.6)$$

so that

$$\mathbf{T}(s) = \gamma' = \frac{1}{\sqrt{\rho^2 + r^2}} \begin{bmatrix} -\rho \sin(s/\sqrt{\rho^2 + r^2}) \\ \rho \cos(s/\sqrt{\rho^2 + r^2}) \\ r \end{bmatrix}, \quad (2.7)$$

and corresponding explicit formulas for  $\mathbf{M}_1(s)$  and  $\mathbf{M}_2(s)$  can also be derived. The curvature and torsion are given by

$$\kappa = |\gamma''| = \rho/(\rho^2 + r^2) \quad (2.8)$$

and

$$\tau = \gamma' \cdot (\gamma'' \times \gamma''')/\kappa^2 = r/(\rho^2 + r^2). \quad (2.9)$$

### 3. Maximum Principle and Poisson Reduction

The use of the Maximum Principle and Lie-Poisson Reduction for left-invariant systems on finite dimensional Lie groups is discussed in (Krishnaprasad 1993). Here we summarize the main points, as they apply to the problems analyzed below. Given a controlled left-invariant system

$$\dot{g} = g\xi_u, \quad (3.1)$$

where  $g \in G$ , a Lie group, and  $\xi_u \in \mathfrak{g}$ , the corresponding Lie algebra, we assume that  $\xi_u$  is affine in the control vector  $u$ , i.e.,

$$\xi_u = \xi_0 + \sum_{i=1}^m u_i(t)\xi_i, \quad i = 0, 1, \dots, m. \quad (3.2)$$

In (3.2),  $u = (u_1, u_2, \dots, u_m) \in \mathbb{R}^m$ ,  $\xi_i \in \{X_1, X_2, \dots, X_n\}$ , a set of basis vectors for  $\mathfrak{g}$ , and  $m < n$  where  $n$  is the dimension of  $G$ . Thus, the system is underactuated. In particular, we assume that

$$\begin{aligned} \xi_0 &= \sigma_0 X_q \text{ for some } q \in \{m+1, \dots, n\}, \quad \sigma_0 \in \{-1, 0, 1\}, \\ \xi_i &= \sigma_i X_i, \quad \sigma_i \in \{-1, 1\}, \quad i = 1, \dots, m, \end{aligned} \quad (3.3)$$

and we take  $L$  to have the form

$$L(u) = \frac{1}{2} \sum_{i=1}^m u_i^2. \quad (3.4)$$

If  $\sigma_0 = 0$  then the system is drift-free; otherwise, the system has drift. We consider the fixed-endpoint problem

$$\min_u \int_0^T L(u(t)) dt \quad (3.5)$$

subject to  $g(0) = g_0$  and  $g(T) = g_T$ ,  $T > 0$ , and note that  $L$  given by (3.4) is clearly  $G$ -invariant (i.e.,  $L$  depends only on  $u$  and not on  $g$ ). Assuming that controls exist which will drive the system from  $g_0$  at  $t = 0$  to  $g_T$  at  $t = T$ , and restricting attention to regular extremals of the fixed endpoint problem, we define the pre-hamiltonian

$$H(g, p, u) = \langle p, T_e L_g \cdot \xi_u \rangle - L(u), \quad (3.6)$$

where  $p \in T_g^* G$  and  $T_e L_g$  denotes the tangent map of left translation by  $g$  on  $G$ . The Maximum Principle then states that for optimal  $u(\cdot)$ , the trajectory in  $G$  is a base integral curve of the canonical hamiltonian system on  $T^* G$  with hamiltonian

$$\mathbf{H}(g, p) = \sup_u H(g, p, u), \text{ for a.e. } t \in [0, T], \quad (3.7)$$

where  $\mathbf{H}$  is the hamiltonian. If  $H$  is differentiable with respect to  $u$ , we have

$$\left. \frac{\partial H}{\partial u_i}(g, p, u) \right|_{u_i = u_i^{opt}} = \left( \frac{\partial}{\partial u_i} \langle p, T_e L_g \cdot \xi_u \rangle - \frac{\partial L}{\partial u_i} \right) \Big|_{u_i = u_i^{opt}} = 0, \quad i = 1, \dots, m. \quad (3.8)$$

Defining  $\mu \in \mathfrak{g}^*$ , the linear dual of  $\mathfrak{g}$ , by  $\mu = T_e L_g^* \cdot p$ , and substituting (3.4) for  $L$ , (3.8) becomes

$$\begin{aligned} \left( \frac{\partial}{\partial u_i} \langle \mu, \xi_u \rangle - \frac{\partial L}{\partial u_i} \right) \Big|_{u_i = u_i^{opt}} &= \left\langle \mu, \frac{\partial}{\partial u_i} \left( \xi_0 + \sum_{i=1}^m u_i \xi_i \right) \Big|_{u_i = u_i^{opt}} \right\rangle - u_i^{opt} \\ &= \sum_{i=1}^m \langle \mu, \xi_i \rangle - u_i^{opt} \\ &= \sum_{i=1}^m \sigma_i \langle \mu, X_i \rangle - u_i^{opt} \\ &= \sigma_i \mu_i - u_i^{opt} = 0, \end{aligned} \quad (3.9)$$

where  $\mu$  is expressed in the dual basis  $\{X_1^b, X_2^b, \dots, X_n^b\}$  to that of  $\mathfrak{g}$ , as

$$\mu = \sum_{i=1}^n X_i^b \mu_i. \quad (3.10)$$

Thus, the optimal controls  $u_i^{opt}$  satisfy

$$u_i^{opt} = \sigma_i \mu_i, \quad i = 1, \dots, m. \quad (3.11)$$

Substituting the optimal controls back into the hamiltonian then gives

$$\begin{aligned} \mathbf{H}(g, p) &= \langle p, T_e L_g \cdot \xi_{u^{opt}} \rangle - L(u^{opt}) \\ &= \langle \mu, \xi_{u^{opt}} \rangle - L(u^{opt}) \\ &= \left\langle \mu, \xi_0 + \sum_{i=1}^m \sigma_i \mu_i \xi_i \right\rangle - \frac{1}{2} \sum_{i=1}^m \mu_i^2 \\ &= \sigma_0 \langle \mu, X_q \rangle + \sum_{i=1}^m \sigma_i^2 \langle \mu, X_i \mu_i \rangle - \frac{1}{2} \sum_{i=1}^m \mu_i^2 \\ &= \sigma_0 \mu_q + \frac{1}{2} \sum_{i=1}^m \mu_i^2. \end{aligned} \quad (3.12)$$

Clearly  $H$  is independent of  $g$ , and thus permits reduction. In fact, we are able to use Lie-Poisson reduction to take the original system on  $T^*G$  and reduce it to a system on  $\mathfrak{g}^*$ , with the reduced variables defined as  $\mu_1, \dots, \mu_n$ . The machinery of Lie-Poisson reduction (technical details may be found in Krishnaprasad, 1993) then allows us to write the reduced hamiltonian

$$h = \sigma_0 \mu_q + \frac{1}{2} \sum_{i=1}^m \mu_i^2, \quad (3.13)$$

along with the dynamics for  $\mu$ ,

$$\dot{\mu}_i = - \sum_{j=1}^n \sum_{k=1}^n \mu_k \Gamma_{ij}^k \frac{\partial h}{\partial \mu_j}, \quad (3.14)$$

where  $\Gamma_{ij}^k$  are the structure constants of  $\mathfrak{g}$ . We can express (3.14) in the form

$$\dot{\mu} = \Lambda(\mu) \nabla h = - \left( \sum_{k=1}^n \mu_k \begin{bmatrix} \Gamma_{11}^k & \Gamma_{12}^k & \cdots & \Gamma_{1n}^k \\ \Gamma_{21}^k & \Gamma_{22}^k & \cdots & \Gamma_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{n1}^k & \Gamma_{n2}^k & \cdots & \Gamma_{nn}^k \end{bmatrix} \right) \begin{bmatrix} \partial h / \partial \mu_1 \\ \partial h / \partial \mu_2 \\ \vdots \\ \partial h / \partial \mu_n \end{bmatrix}, \quad (3.15)$$

which turns out to be more illuminating.

**Remark:** We focus on extremal solutions, i.e., solutions which satisfy the necessary condition (3.8) for optimality. Because we have not considered (second-derivative-based) sufficient conditions, or analyzed conjugate points, we cannot assert that these solutions are truly optimal without doing more work. However, we shall continue to refer to the controls  $u_i^{opt}$  as the ‘‘optimal controls.’’

## 4. Optimal framing

We apply the Maximum Principle and Lie-Poisson reduction to fixed-endpoint optimal control problems on  $SO(3)$ ,  $SE(3)$ , and  $SE(4)$  (with implications for the corresponding problem on  $SE(n)$ ,  $n > 4$ ). For  $SO(3)$ , we obtain explicit equations for not only the optimal controls, but also the corresponding solutions. For  $SE(3)$ , we show that the optimal controls are stationary solutions to a nonlinear Schrödinger (NLS) equation. For  $SE(4)$ , the optimal controls are stationary solutions to a three-dimensional generalization of the stationary NLS (SNLS) equation, a pattern which also holds for  $SE(n)$ ,  $n > 4$ .

### (a) An optimal control problem on $SO(3)$

Consider the left-invariant system

$$\dot{g} = g \xi_u, \quad g \in SO(3), \quad \xi_u = X_1 u_1 - X_2 u_2, \quad (4.1)$$

where

$$X_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \quad (4.2)$$



Suppose that the endpoints for a fixed-endpoint optimal control problem are given by  $g(0) = g_0$ ,  $g(T) = g_T$ , and we seek (regular extremals which)

$$\min_{(u_1, u_2) \in \mathbb{R}^2} \int_0^T L(u_1(t), u_2(t)) dt, \quad (4.3)$$

where

$$L(u_1, u_2) = \frac{1}{2} (u_1^2 + u_2^2). \quad (4.4)$$

This problem clearly fits within the class of problems described in the previous section, with  $\sigma_0 = 0$  since there is no drift term in  $\xi_u$ . Because there is no drift, by the Lie Algebra Rank Condition (LARC) (Jurdevic, 1997), it is easily verified that the system is controllable, and therefore controls exist which will steer the system from  $g_0$  at  $t = 0$  to  $g_T$  at  $t = T$ . Thus, we may apply the Maximum Principle and Poisson reduction to obtain the reduced hamiltonian

$$h = \frac{1}{2} (\mu_1^2 + \mu_2^2), \quad (4.5)$$

and the reduced dynamics (3.14), where  $n = 3$ . The structure constants are easily found from

$$[X_1, X_2] = -X_3, \quad [X_1, X_3] = X_2, \quad [X_2, X_3] = -X_1, \quad (4.6)$$

to be

$$\Gamma_{12}^3 = -1, \quad \Gamma_{13}^2 = 1, \quad \Gamma_{23}^1 = -1, \quad (4.7)$$

with  $\Gamma_{ij}^k = -\Gamma_{ji}^k$  and  $\Gamma_{ii}^k = 0$  for all  $1 \leq i, j, k \leq 3$ . The reduced dynamics are thus

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{\mu}_2 \\ \dot{\mu}_3 \end{bmatrix} = - \begin{bmatrix} 0 & -\mu_3 & \mu_2 \\ \mu_3 & 0 & -\mu_1 \\ -\mu_2 & \mu_1 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \mu_2 \mu_3 \\ -\mu_1 \mu_3 \\ 0 \end{bmatrix}. \quad (4.8)$$

It is clear that the reduced hamiltonian (4.5) is conserved, as is the Casimir

$$c = \frac{1}{2} (\mu_1^2 + \mu_2^2 + \mu_3^2). \quad (4.9)$$

Conservation of  $h$  and  $c$  imply that  $2(c - h) = \mu_3^2$  is also conserved, which is easily seen from (4.8). The optimal controls thus take the form

$$u_1(t) = \sqrt{2h} \cos(\omega t + \phi), \quad u_2(t) = -\sqrt{2h} \sin(\omega t + \phi), \quad (4.10)$$

where  $\omega \in \mathbb{R}$  and  $\phi \in S^1$  (i.e., the circle group).

It is possible to write down explicit solutions for the corresponding trajectories in  $SO(3)$  by identifying time  $t$  with arc length parameter  $s$ , noting that the sinusoidal optimal controls (4.10) are in phase quadrature, and interpreting the optimal controls as the natural curvatures for a circular helix in  $\mathbb{R}^3$ . If we assume that  $\omega > 0$  (the  $\omega < 0$  case may be treated similarly), we can use the formulas for a right circular helix from section 2 to derive

$$\tilde{g} = [ \tilde{\mathbf{x}} \quad \tilde{\mathbf{y}} \quad \tilde{\mathbf{z}} ], \quad (4.11)$$

with

$$\begin{aligned}
\tilde{\mathbf{x}} &= \omega_1 \begin{bmatrix} -\rho \sin(\omega_1 t) \\ \rho \cos(\omega_1 t) \\ r \end{bmatrix}, \\
\tilde{\mathbf{y}} &= -\omega_1 \sqrt{2h} \begin{bmatrix} \frac{\rho}{2(\omega_1 + \omega)} \cos(\omega_1 t + \omega t + \phi) + \frac{\rho}{2(\omega_1 - \omega)} \cos(\omega_1 t - \omega t - \phi) \\ \frac{\rho}{2(\omega_1 + \omega)} \sin(\omega_1 t + \omega t + \phi) + \frac{\rho}{2(\omega_1 - \omega)} \sin(\omega_1 t - \omega t - \phi) \\ \frac{r}{\omega} \sin(\omega t + \phi) \end{bmatrix}, \\
\tilde{\mathbf{z}} &= \omega_1 \sqrt{2h} \begin{bmatrix} \frac{\rho}{2(\omega_1 + \omega)} \sin(\omega_1 t + \omega t + \phi) - \frac{\rho}{2(\omega_1 - \omega)} \sin(\omega_1 t - \omega t - \phi) \\ -\frac{\rho}{2(\omega_1 + \omega)} \cos(\omega_1 t + \omega t + \phi) + \frac{\rho}{2(\omega_1 - \omega)} \cos(\omega_1 t - \omega t - \phi) \\ \frac{r}{\omega} \cos(\omega t + \phi) \end{bmatrix},
\end{aligned} \tag{4.12}$$

where  $\omega_1 = 1/\sqrt{\rho^2 + r^2}$ ,  $\rho/(\rho^2 + r^2) = \sqrt{2h}$  and  $r/(\rho^2 + r^2) = \omega$ . Thus, (4.11) and (4.12) can be written as functions of  $h$ ,  $\omega$ ,  $\phi$ , and  $t$  alone. The parameters  $h$ ,  $\omega$ , and  $\phi$  need to be chosen to satisfy  $\tilde{g}(T) = \tilde{g}(0)g_0^{-1}g_T$ .

(b) *An optimal control problem on  $SE(3)$*

Now consider the fixed endpoint problem  $\dot{g} = g\xi_u$ , where  $\xi_u$  is given by

$$\xi_u = X_4 + X_1 u_1 - X_2 u_2, \tag{4.13}$$

where

$$X_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.14}$$

and  $m = 2$ ,  $n = 3$ . The fixed endpoints in  $SE(3)$  are denoted by  $g(0) = g_0$  and  $g(T) = g_T$ ,  $L(u)$  is given by (4.4), and we note that  $\xi_u$  does have a drift term. We can view  $\dot{g} = g\xi_u$  as the natural Frenet frame system (2.1), where we identify the controls  $(u_1, -u_2)$  with the natural curvatures  $(k_1, k_2)$ .

**Remark:** We assume, but do not verify, that controls  $(u_1, u_2)$  exist which will take the system from  $g_0$  at  $t = 0$  to  $g_T$  at  $t = T$ . Because drift is present in this system, proving the existence of such controls is more complicated than in the  $SO(3)$  example above, even though the local strong accessibility condition is met (Jurdevic, 1997). For example, we must have

$$|(g_0^{-1}g_T)\mathbf{e}_4|^2 - 1 \leq T, \tag{4.15}$$

where  $\mathbf{e}_4 = [0 \ 0 \ 0 \ 1]^T$ , or it will be impossible to find any  $(u_1, u_2)$  which satisfies the endpoint conditions. In terms of curves, (4.15) simply states that the distance (in arc length) between the initial and final position (in  $\mathbb{R}^3$ ) must not exceed the time  $T$  (multiplied by unit speed).

We now compute the Lie-Poisson reduced equations. Using the standard basis for  $SE(3)$ , we compute the structure constants

$$\begin{aligned}\Gamma_{12}^3 &= -1, \quad \Gamma_{13}^2 = 1, \quad \Gamma_{23}^1 = -1, \\ \Gamma_{14}^5 &= \Gamma_{26}^4 = \Gamma_{35}^6 = 1, \\ \Gamma_{15}^4 &= \Gamma_{24}^6 = \Gamma_{36}^5 = -1,\end{aligned}\tag{4.16}$$

$\Gamma_{ji}^k = -\Gamma_{ij}^k$ , and  $\Gamma_{ij}^k = 0$  for all other  $1 < i, j, k < 6$ . The dynamics for  $\mu_j$ ,  $j = 1, \dots, 6$ , found from (3.14) are thus

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{\mu}_2 \\ \dot{\mu}_3 \\ \dot{\mu}_4 \\ \dot{\mu}_5 \\ \dot{\mu}_6 \end{bmatrix} = - \left[ \begin{array}{ccc|ccc} 0 & -\mu_3 & \mu_2 & \mu_5 & -\mu_4 & 0 \\ \mu_3 & 0 & -\mu_1 & -\mu_6 & 0 & \mu_4 \\ -\mu_2 & \mu_1 & 0 & 0 & \mu_6 & -\mu_5 \\ \hline -\mu_5 & \mu_6 & 0 & 0 & 0 & 0 \\ \mu_4 & 0 & -\mu_6 & 0 & 0 & 0 \\ 0 & -\mu_4 & \mu_5 & 0 & 0 & 0 \end{array} \right] \begin{bmatrix} \mu_1 \\ \mu_2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mu_3\mu_2 - \mu_5 \\ -\mu_3\mu_1 + \mu_6 \\ 0 \\ \mu_5\mu_1 - \mu_6\mu_2 \\ -\mu_4\mu_1 \\ \mu_4\mu_2 \end{bmatrix}.\tag{4.17}$$

Conserved quantities include the (reduced) hamiltonian,

$$h = \mu_4 + \frac{1}{2}(\mu_1^2 + \mu_2^2),\tag{4.18}$$

and the two Casimir functions

$$\begin{aligned}c_1 &= \frac{1}{2}(\mu_4^2 + \mu_5^2 + \mu_6^2), \\ c_2 &= \mu_1\mu_6 + \mu_2\mu_5 + \mu_3\mu_4.\end{aligned}\tag{4.19}$$

As in the optimal control problem on  $SO(3)$  described in the previous subsection,  $\mu_3$  is constant, as can be seen directly from (4.17).

(i) *Connection to the nonlinear Schrödinger equation*

Defining

$$\mathbf{a} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -\mu_5 \\ \mu_6 \end{bmatrix}, \quad \omega = \mu_3 = \text{constant},\tag{4.20}$$

we have

$$\begin{aligned}h &= \mu_4 + \frac{|\mathbf{a}|^2}{2}, \\ c_1 &= \frac{1}{2}(\mu_4^2 + |\mathbf{b}|^2), \\ c_2 &= \omega\mu_4 + \mathbf{b}^T J \mathbf{a},\end{aligned}\tag{4.21}$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.\tag{4.22}$$

The dynamics (4.17) become

$$\begin{aligned}\dot{\mathbf{a}} &= -\omega J\mathbf{a} + \mathbf{b}, \\ \dot{\mathbf{b}} &= \left(h - \frac{|\mathbf{a}|^2}{2}\right)\mathbf{a},\end{aligned}\quad (4.23)$$

which can be written in terms of  $\mathbf{a}$  alone as

$$\ddot{\mathbf{a}} + \omega J\dot{\mathbf{a}} - \left(h - \frac{|\mathbf{a}|^2}{2}\right)\mathbf{a} = 0. \quad (4.24)$$

We can now use a change of variables to eliminate the  $\dot{\mathbf{a}}$  term from (4.24). Defining  $\tilde{\mathbf{a}}$  by

$$\mathbf{a} = \exp\left(-\frac{1}{2}\omega Jt\right)\tilde{\mathbf{a}}, \quad (4.25)$$

we have

$$\dot{\mathbf{a}} = -\exp\left(-\frac{1}{2}\omega Jt\right)\frac{1}{2}\omega J\tilde{\mathbf{a}} + \exp\left(-\frac{1}{2}\omega Jt\right)\dot{\tilde{\mathbf{a}}}, \quad (4.26)$$

and

$$\ddot{\mathbf{a}} = \exp\left(-\frac{1}{2}\omega Jt\right)\frac{1}{4}\omega^2 J^2\tilde{\mathbf{a}} - \exp\left(-\frac{1}{2}\omega Jt\right)\omega J\dot{\tilde{\mathbf{a}}} + \exp\left(-\frac{1}{2}\omega Jt\right)\ddot{\tilde{\mathbf{a}}}. \quad (4.27)$$

Substituting into (4.24) and multiplying through by  $\exp\left(\frac{1}{2}\omega Jt\right)$ , we obtain

$$\begin{aligned}\frac{1}{4}\omega^2 J^2\tilde{\mathbf{a}} - \omega J\dot{\tilde{\mathbf{a}}} + \ddot{\tilde{\mathbf{a}}} + \omega J\left[-\frac{1}{2}\omega J\tilde{\mathbf{a}} + \dot{\tilde{\mathbf{a}}}\right] - \left(h - \frac{|\tilde{\mathbf{a}}|^2}{2}\right)\tilde{\mathbf{a}} \\ = \ddot{\tilde{\mathbf{a}}} + \frac{1}{4}\omega^2\tilde{\mathbf{a}} - \left(h - \frac{|\tilde{\mathbf{a}}|^2}{2}\right)\tilde{\mathbf{a}} = 0.\end{aligned}\quad (4.28)$$

Defining  $\tilde{h} = h - \frac{1}{4}\omega^2$ , we finally obtain

$$\ddot{\tilde{\mathbf{a}}} - \tilde{h}\tilde{\mathbf{a}} + \frac{1}{2}|\tilde{\mathbf{a}}|^2\tilde{\mathbf{a}} = 0. \quad (4.29)$$

Consider the (focusing cubic) nonlinear Schrödinger equation (NLS) (with an additional linear term),

$$-i\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2} - \tilde{h}\psi + \frac{1}{2}|\psi|^2\psi, \quad (4.30)$$

where  $\psi(t, x)$  is a complex function of time  $t$  and position  $x \in \mathbb{R}$ . Identifying the variable  $x$  in (4.30) with  $t$  in (4.29) and  $[\operatorname{Re}(\psi) \ \operatorname{Im}(\psi)]^T$  in (4.30) with  $\tilde{\mathbf{a}}$  in (4.29), we see that (4.29) can be viewed as a stationary NLS (SNLS).

(ii) *Special solutions of SNLS*

Among the solutions to (4.29) are two easily obtained classes: sinusoidal solutions and (Jacobi) elliptic function solutions. For the sinusoidal solutions, we have

$$\tilde{\mathbf{a}} = \alpha \begin{bmatrix} \cos(\tilde{\omega}t + \phi) \\ \sin(\tilde{\omega}t + \phi) \end{bmatrix}, \quad (4.31)$$

where  $\alpha = |\tilde{\mathbf{a}}|$  is a constant amplitude, and  $\phi$  is a constant phase. Then  $\alpha$  must satisfy

$$-\alpha\tilde{\omega}^2 - \tilde{h}\alpha + \frac{1}{2}\alpha^3 = 0, \quad (4.32)$$

or

$$\alpha = \sqrt{2(\tilde{\omega}^2 + \tilde{h}^2)}, \quad (4.33)$$

where  $\tilde{\omega}$ ,  $\tilde{h}$ , and  $\phi$  are determined by the specified endpoints  $g_0$  and  $g_T$  (assuming that there exist curves in  $SE(3)$  corresponding to these sinusoidal solutions which actually interpolate  $g_0$  and  $g_T$ ).

To describe the elliptic function solutions, we represent  $\tilde{\mathbf{a}}$  as

$$\tilde{\mathbf{a}} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \tilde{a}, \quad (4.34)$$

where  $\theta$  is constant and  $\tilde{a}$  is a scalar-valued function. For this class of solutions, (4.29) implies

$$\ddot{\tilde{a}} - \tilde{h}\tilde{a} + \frac{1}{2}\tilde{a}^3 = 0. \quad (4.35)$$

Equation (4.35) has solutions of the form

$$\tilde{a}(t) = (2\nu\sqrt{m})\text{cn}(\nu(t - \eta), m), \quad m = \frac{1}{2} \left( 1 + \frac{\tilde{h}}{\nu^2} \right), \quad (4.36)$$

where  $\eta$ ,  $\nu$ , and  $m$  are constant, and  $\nu$  satisfies  $\nu^2 \geq |\tilde{h}|$  (Davis, 1962). To connect back to the original optimization problem, these elliptic function solutions correspond to optimal controls for  $\dot{g} = g\xi_u$ , with  $\xi_u$  given by (4.13), of the form

$$\begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} = \mathbf{a} = \exp\left(-\frac{1}{2}\omega Jt\right) \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \tilde{a}, \quad (4.37)$$

where  $\tilde{a}$  given by (4.36),  $\tilde{h} = h - \frac{1}{4}\omega^2$ , and the constants  $\omega$ ,  $h$ ,  $\theta$ ,  $\eta$ , and  $\nu$  are determined by the endpoints  $g_0$  and  $g_T$  (if such constants exists). Of course, we could have written an analogous expression for the optimal  $u_1$  and  $u_2$  corresponding to the sinusoidal solutions, as well. The point is that we can work with the Lie-Poisson reduced equations to ultimately obtain explicit formulas (for the controls) for certain extremal solutions to the fixed-endpoint optimal control problem on  $SE(3)$ , and a key step is recognizing that the reduced equations can actually be massaged into the form of a stationary NLS equation.

(c) *An optimal control problem on  $SE(4)$*

It turns out that the techniques used above for the fixed-endpoint problem on  $SE(3)$  can be generalized to higher dimensions. We first use  $SE(4)$  for illustration, and then proceed to the general case. Consider the fixed endpoint problem  $\dot{g} = g\xi_u$  for  $g \in SE(4)$ , where

$$\xi_u = X_7 + X_1u_1 + X_2u_2 + X_3u_3, \quad (4.38)$$

with

$$\begin{aligned} X_1 &= \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, & X_2 &= \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ X_3 &= \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, & X_7 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \quad (4.39)$$

and

$$L(u) = \frac{1}{2} (u_1^2 + u_2^2 + u_3^2). \quad (4.40)$$

For a particular choice of basis for the Lie algebra  $se(4)$  (in particular,  $\{X_1, \dots, X_6\}$  correspond to rotation while  $\{X_7, \dots, X_{10}\}$  correspond to translation), we have

$$h = \mu_7 + \frac{1}{2} (\mu_1^2 + \mu_2^2 + \mu_3^2), \quad (4.41)$$

and the Lie-Poisson reduced equations are computed to be

$$\begin{bmatrix} \dot{\mu}_1 \\ \dot{\mu}_2 \\ \dot{\mu}_3 \\ \dot{\mu}_4 \\ \dot{\mu}_5 \\ \dot{\mu}_6 \\ \dot{\mu}_7 \\ \dot{\mu}_8 \\ \dot{\mu}_9 \\ \dot{\mu}_{10} \end{bmatrix} = -\Lambda(\mu) \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\mu_4\mu_2 - \mu_5\mu_3 - \mu_8 \\ \mu_4\mu_1 - \mu_6\mu_3 - \mu_9 \\ \mu_5\mu_1 + \mu_6\mu_2 - \mu_{10} \\ 0 \\ 0 \\ 0 \\ \mu_8\mu_1 + \mu_9\mu_2 + \mu_{10}\mu_3 \\ -\mu_7\mu_1 \\ -\mu_7\mu_2 \\ -\mu_7\mu_3 \end{bmatrix}, \quad (4.42)$$

where

$$\Lambda(\mu) = \left[ \begin{array}{cccccc|cccc} 0 & \mu_4 & \mu_5 & -\mu_2 & -\mu_3 & 0 & \mu_8 & -\mu_7 & 0 & 0 \\ -\mu_4 & 0 & \mu_6 & \mu_1 & 0 & -\mu_3 & \mu_9 & 0 & -\mu_7 & 0 \\ -\mu_5 & -\mu_6 & 0 & 0 & \mu_1 & \mu_2 & \mu_{10} & 0 & 0 & -\mu_7 \\ \mu_2 & -\mu_1 & 0 & 0 & \mu_6 & -\mu_5 & 0 & \mu_9 & -\mu_8 & 0 \\ \mu_3 & 0 & -\mu_1 & -\mu_6 & 0 & \mu_4 & 0 & \mu_{10} & 0 & -\mu_8 \\ 0 & \mu_3 & -\mu_2 & \mu_5 & -\mu_4 & 0 & 0 & 0 & \mu_{10} & -\mu_9 \\ \hline -\mu_8 & -\mu_9 & -\mu_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_7 & 0 & 0 & -\mu_9 & -\mu_{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_7 & 0 & \mu_8 & 0 & -\mu_{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_7 & 0 & \mu_8 & \mu_9 & 0 & 0 & 0 & 0 \end{array} \right]. \quad (4.43)$$

Defining

$$\mathbf{a} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}, \quad \mathbf{b} = - \begin{bmatrix} \mu_8 \\ \mu_9 \\ \mu_{10} \end{bmatrix}, \quad \Omega = \begin{bmatrix} \mu_4 \\ \mu_5 \\ \mu_6 \end{bmatrix}, \quad (4.44)$$

we have from (4.42)

$$\begin{aligned} \dot{\mathbf{a}} &= -\hat{\Omega}\mathbf{a} + \mathbf{b}, \\ \dot{\mathbf{b}} &= \left( h - \frac{|\mathbf{a}|^2}{2} \right) \mathbf{a}, \end{aligned} \quad (4.45)$$

where the optimal controls are given by

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{a}. \quad (4.46)$$

Note that the form of (4.45) is analogous to (4.23), except that now  $\mathbf{a}$  and  $\mathbf{b}$  are 3-dimensional vectors. Indeed, (4.45) leads to a generalization of the SNLS equation (see the next subsection). Among the solutions to (4.45) are sinusoidal and elliptic function solutions analogous to those obtained in the  $SE(3)$  example of the previous subsection.

(d) *An optimal control problem on  $SE(n)$*

More generally, we can consider the fixed endpoint problem  $\dot{g} = g\xi_u$  for  $g \in SE(n)$ , where  $se(n)$  is parameterized such that  $\{X_1, \dots, X_{n(n-1)/2}\}$  correspond to rotation and  $\{X_{[n(n-1)/2+1]}, \dots, X_{n(n+1)/2}\}$  correspond to the translation, and

$$\xi_u = X_{[n(n-1)/2+1]} + X_1 u_1 + X_2 u_2 + \dots + X_{n-1} u_{n-1}. \quad (4.47)$$

In particular,  $X_1, \dots, X_{n-1}$  take the following form: the  $[1, (1+k)]$  component of  $X_k$  is  $-1$ , the  $[(1+k), 1]$  component of  $X_k$  is  $1$ , and all other components of  $X_k$  are zero. (A particular form for all of the basis vectors for  $se(n)$  is chosen so that the patterns in the calculation are clear: the  $n = 4$  case is illustrative.) We consider the cost function

$$L(u) = \frac{1}{2} (u_1^2 + u_2^2 + \dots + u_{n-1}^2), \quad (4.48)$$

and we have

$$h = \mu_{[n(n-1)/2+1]} + \frac{1}{2} (\mu_1^2 + \mu_2^2 + \dots + \mu_{n-1}^2). \quad (4.49)$$

We define

$$\mathbf{a} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{n-1} \end{bmatrix}, \quad \mathbf{b} = - \begin{bmatrix} \mu_{[n(n-1)/2+2]} \\ \mu_{[n(n-1)/2+3]} \\ \vdots \\ \mu_{n(n+1)/2} \end{bmatrix}, \quad \hat{\Omega} = \hat{\Omega}(\mu_n, \dots, \mu_{n(n-1)/2}) \quad (4.50)$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are vectors of length  $n-1$  and  $\hat{\Omega}$  is an  $(n-1) \times (n-1)$  skew-symmetric matrix. We can verify that, indeed  $\hat{\Omega}$  is constant. We then have, analogously to the

calculation for  $SE(4)$ , that  $\mathbf{a}$  and  $\mathbf{b}$  obey (4.45) with optimal controls given by

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix} = \mathbf{a}. \quad (4.51)$$

Writing (4.45) In terms of  $\mathbf{a}$  alone, we have

$$\ddot{\mathbf{a}} + \hat{\Omega}\dot{\mathbf{a}} - \left( h - \frac{|\mathbf{a}|^2}{2} \right) \mathbf{a} = 0. \quad (4.52)$$

Defining  $\tilde{\mathbf{a}}$  by

$$\mathbf{a} = \exp\left(-\frac{1}{2}\hat{\Omega}t\right)\tilde{\mathbf{a}}, \quad (4.53)$$

we use the analogous calculation to the one for  $SE(3)$  to obtain

$$\ddot{\tilde{\mathbf{a}}} - \tilde{H}\tilde{\mathbf{a}} + \frac{1}{2}|\tilde{\mathbf{a}}|^2\tilde{\mathbf{a}} = 0, \quad (4.54)$$

where

$$\tilde{H} = h\mathbb{I}_{(n-1)\times(n-1)} - \frac{1}{4}\hat{\Omega}^2 \quad (4.55)$$

is a constant, symmetric matrix. Equation (4.54) can be considered a vector version of the stationary NLS, and among its solutions are classes of solutions analogous to the sinusoidal and elliptic function solutions identified in the analysis for  $SE(3)$ .

## 5. Physics

The nonlinear Schrödinger equation appears in the study of the DaRios-Betchov, or vortex filament equation (Betchov, 1965). Vortex filaments are persistent slender filamentary structures observed in three-dimensional fluid flows, and they exhibit self-induced motion (and shape change) due to the curvature distribution along the filament. The identification of vortex filament shapes and their stability has been studied extensively over the past half-century due to the aeronautical importance of vortices in fluid flow over airfoils.

Vortex filaments can be modeled using curves and natural Frenet frames, where the vortex filament is modeled as a curve in three-dimensional space which does not change length, but which can move and change shape with time. In place of (2.1), we have the partial differential equation

$$\begin{aligned} \gamma_s &= \mathbf{T}, \\ \mathbf{T}_s &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \mathbf{M}_{1s} &= -k_1\mathbf{T}, \\ \mathbf{M}_{2s} &= -k_2\mathbf{T}, \end{aligned} \quad (5.1)$$

where  $\gamma$  and  $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$  are functions of both time  $t$  and arc-length parameter  $s$ . The DaRios-Betchov equation (or vortex filament equation) is

$$\gamma_t = \gamma_s \times \gamma_{ss}, \quad (5.2)$$



and it describes the evolution of the curve representing the vortex filament evolving in three-dimensional space. Note that (5.2) is length-preserving, i.e.,

$$\frac{d}{dt}|\gamma_s| = \frac{\gamma_s \cdot \gamma_{st}}{|\gamma_s|} = \frac{\gamma_s \cdot \gamma_{ts}}{|\gamma_s|} = \frac{\gamma_s \cdot (\gamma_{ss} \times \gamma_{ss} + \gamma_s \times \gamma_{sss})}{|\gamma_s|} = 0. \quad (5.3)$$

Equation (5.2) possesses even more structure: the Hasimoto transformation shows that the natural curvatures  $(k_1, k_2)$  in (5.1), evolving according to (5.2), are given by a nonlinear Schrödinger equation (Hasimoto, 1972). Analysis of the DaRios-Betchov equation from a geometric point of view, using natural frames and the nonlinear Schrödinger equation, can be found in the work of Langer and Perline (1991, 1996).

(a) *DaRios-Betchov equation and NLS*

Using the fact that mixed partial derivatives commute, i.e.,  $\gamma_{st} = \gamma_{ts}$  and  $\mathbf{T}_{ts} = \mathbf{T}_{st}$ , after some calculation we obtain

$$k_{2t} = \left( \frac{1}{2} (k_1^2 + k_2^2) - A(t) \right) k_1 + k_{1ss}. \quad (5.4)$$

where  $A(t)$  is a constant of integration. An analogous calculation gives

$$k_{1t} = \left( -\frac{1}{2} (k_1^2 + k_2^2) + A(t) \right) k_2 - k_{2ss}, \quad (5.5)$$

To show the connection between (5.4), (5.5) and the nonlinear Schrödinger equation, we let

$$\psi = k_1 + ik_2, \quad (5.6)$$

where  $i = \sqrt{-1}$ , so that

$$\begin{aligned} \psi_t &= k_{1t} + ik_{2t} = i(k_1 + ik_2)_{ss} + i \left[ \frac{1}{2} (k_1^2 + k_2^2) - A(t) \right] (k_1 + ik_2) \\ &= i \left[ \Delta + \left( \frac{1}{2} |\psi|^2 - A(t) \right) \right] \psi, \end{aligned} \quad (5.7)$$

where  $\Delta$  denotes the Laplacian operator with respect to  $s$ .

In fact, (5.7) can be simplified somewhat by introducing a change of variables based on a phase factor, as observed by Hasimoto (1972). We let

$$\tilde{\psi} = \exp \left( i \int_0^t A(\sigma) d\sigma \right) \psi, \quad (5.8)$$

so that

$$\begin{aligned} \tilde{\psi}_t &= iA(t) \exp \left( i \int_0^t A(\sigma) d\sigma \right) \psi + \exp \left( i \int_0^t A(\sigma) d\sigma \right) \psi_t \\ &= iA(t) \tilde{\psi} + \exp \left( i \int_0^t A(\sigma) d\sigma \right) \left[ i\Delta\psi + i \left( \frac{1}{2} |\psi|^2 - A(t) \right) \psi \right] \\ &= \exp \left( i \int_0^t A(\sigma) d\sigma \right) \left( i\Delta\psi + i \frac{1}{2} |\psi|^2 \psi \right) \\ &= i \left( \Delta + \frac{1}{2} |\tilde{\psi}|^2 \right) \tilde{\psi}. \end{aligned} \quad (5.9)$$

We have thus obtained

$$i\tilde{\psi}_t = - \left( \Delta + \frac{1}{2}|\tilde{\psi}|^2 \right) \tilde{\psi}, \quad (5.10)$$

the focusing cubic nonlinear Schrödinger equation (without the additional term involving  $A(t)$  in (5.7).

### (b) Steady vortex filaments

The NLS (5.7) thus describes how the natural curvatures of the vortex filament evolve with time. There are a variety of solutions to (5.7), including ones which move (i.e., which rotate about their axis or translate) without changing their form - such solutions are stationary solutions of (5.7), i.e., they satisfy the stationary NLS. One such class of solutions, whose stability properties have been extensively analyzed, are helical vortex filaments (Widnall, 1972). Others include circular rings, closed coils, planar sinusoidal curves, and curves described by elliptic functions (Kida, 1981).

Thus, the solutions to the fixed-endpoint optimal control problem on  $SE(3)$  in section 4.b are closely related to the steady vortex filament solutions, since both are given by the SNLS equation. There are cosmetic differences between (5.7) and (4.30), such as the role of fixed endpoint conditions versus the DaRios-Betchov equation for determining the constants which appear in the solutions. Nevertheless, there is clearly a close relationship in terms of the form of the solutions. A more subtle question is whether there is an underlying variational principle at work in both the optimal control and physics settings which explains the connection between these problems at a deeper level.

## 6. Conclusion

The present paper is inspired by earlier work of Baillieul on optimal control problems. We have shown that the equations governing extremals have interesting structure that appears to be previously unknown. Furthermore, the extremal solutions for a particular optimal control problem on  $SE(n)$  are associated with stationary solutions to the nonlinear Schrödinger equation (for  $n = 3$ ) and its higher-dimensional generalizations (for  $n \geq 4$ ). Connections to similar solutions which appear in the study of vortex filament equations have also been described. A generalization of the methods used here to multiple particles in  $SE(2)$  interacting through a fixed communication graph can be found in Justh & Krishnaprasad (2010).

## Acknowledgements

This research was supported in part by the Air Force Office of Scientific Research under AFOSR grant FA9550710446; the Army Research Office under ARO grant W911NF0610325; by the ODDR&E MURI2007 Program Grant N000140710734 (through the Office of Naval Research); and by the Office of Naval Research.

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