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An insight into space–time block codes using Hurwitz–Radon families of matrices ☆

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Abstract

It is shown that for four-transmitter systems, a family of four-by-four unit-rate complex quasi-orthogonal space–time block codes, where each entry equals a symbol variable up to a change of sign and/or complex conjugation, can be generated from any two independent codes via elementary operations. The two independent groups of codes in the family generally have different properties of diversity, but the codes in each group have the same diversity provided that the differential symbol constellation is symmetric. It is also shown that for four-transmitter systems, an eight-by-four unit-rate complex linear dispersion space–time block code can be constructed by using Hurwitz–Radon families of matrices of size eight such that diversity three is guaranteed even when all symbols are independently selected from any given constellation. This code is so far the only known unit-rate linear dispersion code that has diversity no less than three for four transmitters under any given constellation.

Keywords: Space–time block codes (STBC); Orthogonal STBC; Quasi-orthogonal STBC; Non-orthogonal STBC; Hurwitz–Radon families of matrices; Diversity analysis

1. Introduction

Design and analysis of space–time block codes (STBC) for multiple transmitting antennas have been an active field of research since the work by Alamouti [1] and that by Tarokh et al. [2]. STBC is aimed to exploit the channel diversity between multiple transmitters and multiple receivers to improve the rate of reliable data transmission and/or the performance of bit error rate. STBC is also useful for cooperative relays in wireless

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mobile networks [3–6], where STBC can be used effectively as if between multiple transmitters and a single receiver.

STBC is a mapping (applied at the transmitters) between a sequence of input symbols and multiple sequences of output symbols. The number N of the output sequences typically corresponds to the number of transmitters. The ratio of the length T of the output sequences over the length S of the input sequence is called the rate of the STBC (assuming that both the input symbol constellation and the output symbol constellation have the same dimension). The output of the STBC mapping can be denoted by a $T \times N$ matrix $C(\underline{x})$ where the $S \times 1$ vector $\underline{x}$ represents the input symbols. Assume that the channel following the N transmitters is frequency flat, and there are M receiving antennas at the end of the channel. Then, the received baseband signals at the destination over a time interval of T symbols can be represented by the $T \times M$ matrix Y:

$$Y = C(\underline{x})H + W, \tag{1}$$

where H is an $N \times M$ channel matrix whose entries may be assumed to be i.i.d. complex Gaussian random variables (Rayleigh fading), and W is a $T \times M$ noise matrix whose entries may also be assumed to be i.i.d. complex Gaussian random variables. With a coherent maximum likelihood decoder, the pairwise error rate (PER) $P(\underline{x} \to \underline{\tilde{x}})$ averaged over the channel fading distributions is upper bounded as follows [2]:

$$\mathbf{E}_H[P(\underline{x} \to \underline{\tilde{x}})] \leqslant \left(\prod_{j=1}^{r} v_j\right)^{-M} (E_s/4N_0)^{-rM}, \tag{2}$$

where E_s and $N_0/2$ are, respectively, the symbol energy and the variance of noise per dimension; the signal-to-noise ratio (SNR) may be defined as the ratio of E_s over $N_0/2$; r is the minimal rank of $C(\underline{x} - \underline{\tilde{x}})$ over all possible distinct pairs of the symbol sequences; v_j $(j = 1, \ldots, r)$ are the non-zero eigenvalues of $C(\underline{x} - \underline{\tilde{x}})^{\mathrm{H}} C(\underline{x} - \underline{\tilde{x}})$. To reduce the PER of a code, one must increase r and the minimum of $\prod_{j=1}^{r} v_j$. The value of r is called the diversity of the code, and the minimum of $\prod_{j=1}^{r} v_j$ determines a coding gain. Diversity and coding gain are among the key measures of a code.

A detailed review of STBC is available in [7,8]. For convenience, we will also refer to STBC simply as codes. The most attractive codes are perhaps the orthogonal codes [1], which allow the maximum likelihood (optimal) detection to be performed independently on each of the individual symbols. But the unit-rate orthogonal complex codes exist only for two transmitters [9]. For more than two transmitters, there are only fractional-rate orthogonal complex codes [9]. Upper bounds on the rate of orthogonal complex codes are explored in [10]. There are also quasi-orthogonal codes that allow the maximum likelihood detection to be performed independently on pairs of symbols [11] or even independently on each symbol as shown in [12]. But the quasi-orthogonal code given in [11] does not have a full diversity. Various improvements of quasi-orthogonal codes are further developed in [13–16]. In [16], it is shown that unit-rate quasi-orthogonal codes with maximal diversity products can be constructed by using a finite information symbol set on square and triangular lattices. There are also codes that are designed to maximize an orthogonality measure [17]. Numerous other codes can be found via [18–24] and the references therein.

The purpose of this paper is not to present a new code competing against existing ones. But rather, we reveal a structural insight into a class of linear dispersion codes whose properties are intrinsically governed by the Hurwitz–Radon (HR) families of matrices. We first explore the quasi-orthogonal codes of the type shown in [11,25–28]. It will be shown that all 4×4 quasi-orthogonal codes, where each entry of the code matrix is a symbol variable up to a sign change and/or complex conjugation, can be generated from any two independent codes by elementary operations. This result is a fundamental unification of all existing (as well as numerous previously unrevealed) 4×4 quasi-orthogonal codes in this category. If all symbols are selected independently from a common constellation (which will also be referred to as common constellation condition), the quasi-orthogonal codes may have diversity two. More precisely, half of the quasi-orthogonal codes always have diversity two under the common constellation condition, and the other half may have diversity either two or four under the common constellation condition. If the common constellation is an odd-numbered phase-shift-keying, half of the quasi-orthogonal codes actually have diversity four. As illustrated later, both the 2×2 orthogonal codes and the 4×4 quasi-orthogonal codes can be expressed as linear dispersion codes in terms of

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the HR families of matrices. This observation motivated us to explore 8×4 linear dispersion codes using the HR families of matrices of size eight. It will be shown that a class of 8×4 linear dispersion codes constructed with the HR families of matrices have diversity three even under the common constellation condition. To our knowledge, this is the first 8×4 unit-rate linear dispersion code that is guaranteed to have diversity three when all symbols are independently selected from any given constellation. This is an unique insight, which is unknown from the previous studies of linear dispersion codes [19,22,29]. A high diversity order regardless of symbol constellation is practically useful since it could reduce the physical layer complexity associated with constellation constraint.

In Section 2, we review the HR families of matrices. In Section 3, we show that all 4×4 quasi-orthogonal codes can be constructed by two independent codes, and their properties are discussed. In Section 4, we introduce a class of 8×4 linear dispersion codes using the HR matrices of size eight, and show that these codes have diversity three under any given constellation. The proof of the diversity three property is a lengthy part of this paper. We hope that interested readers will find the proof theoretically insightful as it reveals detailed structures in the problem. In Section 5, we provide a simulation example to illustrate the performance of the non-orthogonal 8×4 HR code.

1.1. Notations and terminologies

- Lower case letters are used for scalars.
- Upper case letters are used for matrices.
- Underlined lower case letters are used for vectors.
- In normal script, $*$ denotes an undetermined quantity. As superscript, * denotes complex conjugation.
- In normal script, j denotes $\sqrt{-1}$. In subscript, $_j$ denotes an integer.
- Kronecker product is denoted by $\otimes$ as defined later.
- $\doteq$ denotes "equal by definition".
- I_l is an $l \times l$ identity matrix.
- As superscripts, $^{\mathrm{T}}$ denotes transpose, and $^{\mathrm{H}}$ denotes conjugate transpose.
- $\Re$ denotes real part, and $\Im$ imaginary part.
- Unless specified otherwise, by "symbol", we mean a complex variable.
- Unless specified otherwise, two vectors of variables are said to be orthogonal only if they are orthogonal for all values of the variables.
- All other notations are defined the first time they are used.

2. HR matrices

2.1. General properties of HR matrices

Radon Theorem [30]: Within the space of $L \times L$ integer matrices, there is a family of p matrices $\{A_0, A_1, \ldots, A_{p-1}\}$ satisfying $A_0 = I_L$ (the $L \times L$ identity matrix) and:

- Property 1(a): $A_i A_i^{\mathrm{T}} = I_L$.
- Property 1(b): $A_i = -A_i^{\mathrm{T}}$ $(i > 0)$.
- Property 2(a): $A_i^{\mathrm{T}} A_j = -A_j^{\mathrm{T}} A_i$ $(i \neq j)$,

where the maximum value $p_{\max}$ of p is governed by L as follows. Let $L = 2^a b$ where b is odd, $a = 4c + d$ and $0 \leqslant d \leqslant 3$, then $p_{\max} = 8c + 2^d$.

A family of matrices defined above is called an HR family of matrices of size L. The following properties of the HR matrices follow readily from Properties 1 and 2(a):

- Property 2(b): $A_i A_j^{\mathrm{T}} = -A_j A_i^{\mathrm{T}}$ $(i \neq j)$.
- Property 3: For any real vector $\underline{v}$, $\underline{v}^{\mathrm{T}} A_i^{\mathrm{T}} A_j \underline{v} = \delta(i-j)|\underline{v}|^2$ where $\delta(x)$ is one when $x = 0$ and zero otherwise, and $|\underline{v}|$ is the norm of $\underline{v}$.

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- Property 4: Given the matrices $A_j, j = 1, \ldots, N$, within an HR family, we have

$$\left(\sum_{j=1}^{N} \alpha_j A_j\right)\left(\sum_{j=1}^{N} \alpha_j A_j\right)^{\mathrm{T}} = \sum_{j=1}^{N} \alpha_j^2 I.$$

An HR family of matrices of any size can be constructed from the following 2×2 matrices [30]:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

2.2. HR matrices of size two

For $L = 2$, an HR family is $\{I_2; R\}$. The Alamouti code can be expressed in terms of the 2×2 HR matrices, e.g.,

$$\begin{aligned} C(s_1, s_2) &= \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix} \\ &= \left[R\begin{pmatrix} x_1(1) \\ x_1(2) \end{pmatrix} + \mathrm{j}I_2\begin{pmatrix} x_2(1) \\ x_2(2) \end{pmatrix}, I_2\begin{pmatrix} x_1(1) \\ x_1(2) \end{pmatrix} + \mathrm{j}R\begin{pmatrix} x_2(1) \\ x_2(2) \end{pmatrix}\right] \\ &= \begin{pmatrix} x_1(2) + \mathrm{j}x_2(1) & x_1(1) + \mathrm{j}x_2(2) \\ -x_1(1) + \mathrm{j}x_2(2) & x_1(2) - \mathrm{j}x_2(1) \end{pmatrix}, \end{aligned} \tag{4}$$

where $s_1 = x_1(2) + \mathrm{j}x_2(1)$ and $s_2 = x_1(1) + \mathrm{j}x_2(2)$.

2.3. HR matrices of size four

For $L = 4$, an HR family consists of the following matrices:

$$Q_0 = I_4, \quad Q_1 = P \otimes R, \quad Q_2 = R \otimes I_2, \quad Q_3 = Q \otimes R, \tag{5}$$

where $\otimes$ is the Kronecker product, e.g.,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix}.$$

The HR families of size four are closely related to 4×4 quasi-orthogonal codes as discussed later.

The following theorem provides the complete set of HR families of matrices of size four. If the first matrix in each family is fixed to be identity, there are total $2 \times 2^3 = 16$ HR families of size four.

Theorem 2.1. *Any HR family of matrices of size four has either one of the following two possible forms*:

$$\Omega_1 \doteq \{Q_0; \pm Q_1; \pm Q_2; \pm Q_3\} \quad \textit{and} \quad \Omega_2 \doteq \{G_0; \pm G_1; \pm G_2; \pm G_3\},$$

where $G_0 = Q_0$, $G_1 = Q_1[Q \otimes Q]$, $G_2 = Q_2[-I_2 \otimes Q]$, *and* $G_3 = Q_3(Q \otimes I_2)$. *The* Q_i *matrices were defined previously*. (*Note that in this paper*, $\pm$ *in one place should be treated as independent of* $\pm$ *in another place unless specified otherwise*.)

Proof. The proof of this theorem requires an exhaustive but finite search, which is tedious but feasible. In the following, we provide an outline of the proof. The goal is to show that under Properties 1(a), 1(b) and 2, only Ω_1 and Ω_2 can be valid HR families.

Let us first search for all possible 4×4 HR matrices satisfying Properties 1(a) and 1(b). Under Property 1(a), each entry of an HR matrix F is either zero or ± 1 and each row of F has no more than one non-zero entry.

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Under Property 1(b), i.e., $F^{\mathrm{T}} = -F$ where $F \neq I_4$, we can write

$$F = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix},$$

where $B_{i,j}$ is a 2×2 block matrix. It follows that $B_{i,i}^{\mathrm{T}} = -B_{i,i}$ for $i = 1, 2$ and $B_{1,2}^{\mathrm{T}} = -B_{2,1}$. Therefore, $B_{1,1}$ and $B_{2,2}$ are both equal to either the 2×2 zero matrix or $\pm R$, where $B_{1,1}$ and $B_{2,2}$ may have the same or opposite signs. Namely, we have

$$\text{either} \quad \begin{cases} B_{1,1} = B_{2,2} = 0, \\ B_{1,2}^{\mathrm{T}} = -B_{2,1} \neq 0 \end{cases} \quad \text{or} \quad \begin{cases} B_{1,1} = \pm R, \\ B_{2,2} = \pm R, \\ B_{1,2} = B_{2,1} = 0. \end{cases} \tag{6}$$

It is easy to verify that if $B_{1,1} = B_{2,2} = 0$, the only possible choices of $B_{1,2}$ are $\pm I_2, \pm P, \pm R, \pm Q$.

By an exhaustive, finite and feasible search, one can further verify that any HR matrix F of size four has to be one of the matrices in Ω_1 or Ω_2. In other words, the above two sets Ω_1 and Ω_2 contain all possible 4×4 HR matrices satisfying Properties 1(a) and 1(b).

We next consider Property 2(a). It is straightforward to verify that Ω_1 and Ω_2 each satisfies Property 2(a). Namely, each of Ω_1 and Ω_2 (with a fixed set of $\pm$ signs) is a valid HR family. In order to prove that there is no other possible HR family, we need to show that for any i, j with $1 \leqslant i, j \leqslant 3$, Q_i and G_j cannot co-exist in one family. Indeed, it is straightforward to verify that $Q_i^{\mathrm{T}} G_j \neq -G_j^{\mathrm{T}} Q_i$, i.e., Property 2(a) is not satisfied by Q_i and G_j for any i, j with $1 \leqslant i, j \leqslant 3$. □

2.4. HR matrices of size eight

For $L = 8$, an HR family is determined by the following eight matrices [30]:

$$I_8; I_2 \otimes R \otimes I_2; I_2 \otimes P \otimes R; Q \otimes Q \otimes R; P \otimes Q \otimes R; R \otimes P \otimes Q; R \otimes P \otimes P; R \otimes Q \otimes I_2 \tag{7}$$

which is easy to verify.

An HR family of matrices of size eight (or size integer-power-of-two no less than eight) have the following Properties 5 and 6:

- Property 5: Given distinct i, k, m, n, then $A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}} A_n$ is unitary and symmetric. And the eigenvalues of $A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}} A_n$ are ± 1 of differing signs.

Proof. It is easy to verify that $A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}} A_n$ is unitary. To prove the symmetry, we apply Property 2:

$$\begin{aligned} A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}} A_n &= A_m^{\mathrm{T}} A_j A_n^{\mathrm{T}} A_i \\ &= -A_m^{\mathrm{T}} A_n A_j^{\mathrm{T}} A_i \\ &= -A_n^{\mathrm{T}} A_m A_i^{\mathrm{T}} A_j \\ &= A_n^{\mathrm{T}} A_i A_m^{\mathrm{T}} A_j \\ &= (A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}} A_n)^{\mathrm{T}}. \end{aligned} \tag{8}$$

We now prove the eigenvalue property. The symmetry and orthogonality of $A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}} A_n$ tell us that each of its eigenvalues is $+1$ or -1. We only need to prove that the signs of these eigenvalues are always mixed. Suppose that the eigenvalues of $A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}} A_n$ are all equal to one or all equal to minus one. Then, the matrix is either I or $-I$, i.e.,

$$\begin{aligned} & A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}} A_n = \pm I_L \\ & \quad \Rightarrow A_i^{\mathrm{T}} A_n = \pm A_m^{\mathrm{T}} A_j \\ & \quad \Rightarrow A_n = \pm A_i A_m^{\mathrm{T}} A_j, \end{aligned} \tag{9}$$

where the signs in the above three equations are consistent with each other. Considering another member A_l in the HR family where l is distinct from m, n, i, j, we have

$$\begin{aligned} A_n A_l^{\mathrm{T}} &= \pm A_i A_m^{\mathrm{T}} A_j A_l^{\mathrm{T}} \\ &= \mp A_l A_i^{\mathrm{T}} A_m A_j^{\mathrm{T}} \\ &= \mp A_l A_m^{\mathrm{T}} A_j A_i^{\mathrm{T}} \\ &= \pm A_l A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}}, \end{aligned} \tag{10}$$

where the signs in the above equations are consistent with each other. From (9), we have

$$-A_l A_n^{\mathrm{T}} = \mp A_l A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}}. \tag{11}$$

Since $A_n A_l^{\mathrm{T}} = -A_l A_n^{\mathrm{T}}$, (10) and (11) imply $-A_l A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}} = A_l A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}}$ and hence $A_l A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}} = 0$. This contradicts the condition of non-zero eigenvalues. Therefore, all eigenvalues of $A_j^{\mathrm{T}} A_m A_i^{\mathrm{T}} A_n$ are ± 1 of differing signs. □

- Property 6: If $A_i^{\mathrm{T}} A_k A_m^{\mathrm{T}} A_n \pm A_j^{\mathrm{T}} A_t A_q^{\mathrm{T}} A_r = 0$ where all matrices are distinct HR matrices in one family, we have

  $$A_{i_1}^{\mathrm{T}} A_{k_1} A_{m_1}^{\mathrm{T}} A_{n_1} \pm (-1)^E A_{j_1}^{\mathrm{T}} A_{t_1} A_{q_1}^{\mathrm{T}} A_{r_1} = 0,$$

  where E is an integer and

  $$[A_{i_1}, A_{k_1}, A_{m_1}, A_{n_1}, A_{j_1}, A_{t_1}, A_{q_1}, A_{r_1}]$$

  is a matrix series generated by exchanging E pairs of matrices in the following matrix series:

  $$[A_i, A_k, A_m, A_n, A_j, A_t, A_q, A_r].$$

Proof. It suffices to prove that by switching any pair of matrices in $A_i^{\mathrm{T}} A_k A_m^{\mathrm{T}} A_n \pm A_j^{\mathrm{T}} A_t A_q^{\mathrm{T}} A_r = 0$, the $\pm$ sign changes to the $\mp$ sign.

First, we consider the two matrices exchanged are from either the first term $A_i^{\mathrm{T}} A_k A_m^{\mathrm{T}} A_n$ or the second term $A_j^{\mathrm{T}} A_t A_q^{\mathrm{T}} A_r$. Exchanging any two matrices A and B that have d other matrices in between is equivalent to the following process: (a) exchanging A with its next matrix repeatedly until A is right behind B and then (b) exchanging B with its preceding matrix repeatedly until B is in the original position of A. Step (a) undergoes $d+1$ exchanges of neighboring matrices, and step (b) undergoes d exchanges of neighboring matrices. By Property 2(a), the resulting expression has changed its sign $2d+1$ times, and therefore there is a net sign change.

We now need to prove the Property 6 for the case where a pair of matrices exchanged are "crossed" between the two terms. We first observe the following equivalent expressions:

$$A_i^{\mathrm{T}} A_k A_m^{\mathrm{T}} A_n \pm A_j^{\mathrm{T}} A_t A_q^{\mathrm{T}} A_r = 0, \tag{12}$$

$$\begin{aligned} &\Longleftrightarrow\ A_j^{\mathrm{T}} A_i A_i^{\mathrm{T}} A_k A_m^{\mathrm{T}} A_n \pm A_j^{\mathrm{T}} A_i A_j^{\mathrm{T}} A_t A_q^{\mathrm{T}} A_r = 0 \\ &\Longleftrightarrow\ A_j^{\mathrm{T}} A_k A_m^{\mathrm{T}} A_n \mp A_i^{\mathrm{T}} A_t A_q^{\mathrm{T}} A_r = 0, \end{aligned} \tag{13}$$

where we should notice that only the leading matrices A_i and A_j are actually exchanged when we move from (12) to (13). We now consider a matrix A (from the first term) that has d_A matrices proceeding it and another matrix B (from the second term) that has d_B matrices proceeding it. In order to exchange A and B, we can do the following steps: (a) move A to the front of the first term and move B to the front of the second term, (b) exchange A and B, and (c) move B to where A initially was and move A to where B initially was. We see that step (a) undergoes the sign change $d_A + d_B$ times, step (b) undergoes the sign change once, and step (c) undergoes the sign change $d_A + d_B$ times. Therefore, there is a net sign change. □

Given the HR family of size eight shown in (7), it can be verified that

$$A_1^{\mathrm{T}} A_6 A_7^{\mathrm{T}} A_0 = A_5^{\mathrm{T}} A_2 A_4^{\mathrm{T}} A_3. \tag{14}$$

(At this stage, it is not clear whether (14) holds under a more general condition.) Let i, k, m, n, j, t, q, r be a permutation of $[0, 1, \ldots, 7]$. Then, together with Properties 6 and 3, (14) implies the following properties:

- Property 7(a): $A_k^{\mathrm{T}} A_i A_m^{\mathrm{T}} A_n = \pm A_j^{\mathrm{T}} A_t A_q^{\mathrm{T}} A_r$ where the sign depends on the ordering of the indices.
- Property 7(b): $A_m^{\mathrm{T}} A_n = \mp A_k^{\mathrm{T}} A_i A_j^{\mathrm{T}} A_t A_q^{\mathrm{T}} A_r$ where the sign is consistent with Property 7(a).
- Property 7(c): For any real vector $\underline{v}$, $\underline{v}^{\mathrm{T}}(A_k^{\mathrm{T}} A_i A_j^{\mathrm{T}} A_t A_q^{\mathrm{T}} A_r)\underline{v} = 0$.

3. Quasi-orthogonal codes

We now present a complete family of 4×4 quasi-orthogonal codes (or code matrices) of Type I. A 4×4 quasi-orthogonal code matrix of Type I is defined as such that each entry in the matrix is an element from the symbol set $\{\pm s_{k_1}^{(*)}, \pm s_{k_2}^{(*)}, \pm s_{k_3}^{(*)}, \pm s_{k_4}^{(*)}\}$ and a pair of columns of the matrix is orthogonal to the other pair (and the two columns in each of the above pairs are not necessarily orthogonal to each other). Note that each $\pm$ is an independent plus or minus sign, and each superscript $^{(*)}$ denotes independently the presence or absence of complex conjugation. The above definition of quasi-orthogonal code of Type I was used in [11]. It is obvious that if $S(s_1, s_2, s_3, s_4)$ is a Type I 4×4 quasi-orthogonal matrix of the four symbols s_1, s_2, s_3, s_4, then numerous Type I quasi-orthogonal codes can be constructed by the following elementary operations:

$$C(s_1, s_2, s_3, s_4) = P_{\mathrm{r}} S(\pm s_{k_1}^{(*)}, \pm s_{k_2}^{(*)}, \pm s_{k_3}^{(*)}, \pm s_{k_4}^{(*)}) P_{\mathrm{c}}, \tag{15}$$

where (k_1, k_2, k_3, k_4) is a permutation of $(1, 2, 3, 4)$, P_{r} permutes the rows and/or reverses the signs of none or some rows, and P_{c} permutes the columns and/or reverses the signs of none or some columns. Note that the above statement is obvious because none of the operations P_{r}, P_{c}, $\pm$ and $*$ changes the quasi-orthogonality of (15). While the above statement is obvious, a number of Type I quasi-orthogonal code matrices have been introduced in the literature without mentioning the connections among them. For beginners, each of these codes appears to be a new one. Even for experts, it was unknown whether all existing codes of Type I are related to each other by (15). In this section, we will show that not all Type I codes are related to each other by (15), but, however, there are only two groups of Type I codes. The codes in each of the two groups are related to each other by (15), but no code from one group is related to any code from the other group.

One can also extend the family of quasi-orthogonal codes by allowing left or right multiplication of a diagonal matrix to (15), which is a simple extension of the Type I codes. There are also orthogonal or quasi-orthogonal codes where the entries of the code matrix are non-linear functions of the symbol vector $\{s_1, s_2, s_3, s_4\}$ and/or the symbol constellation is constrained [31]. In this paper, we will not consider any quasi-orthogonal codes other than the Type I quasi-orthogonal codes.

Two codes will be called independent of each other if they are not related to each other according to (15), or otherwise dependent on each other. It is clear that a complete set of quasi-orthogonal codes can be generated by all independent codes via (15). But we will show that the number of independent codes is two. Examples of such two independent codes are also given. All existing codes of this type will be explicitly expressed in terms of the two independent codes.

3.1. Independent quasi-orthogonal codes

We show next that there are only two independent 4×4 Type I quasi-orthogonal codes. But first, we have the following property about complex vectors:

Lemma 3.1. *Let $\underline{s} = \underline{s}_r + \mathrm{j}\underline{s}_i$ be a 4×1 complex vector in the four-dimensional complex space $\mathbb{C}^4$ where $\underline{s}_r$ is the real part and $\underline{s}_i$ is the imaginary part. Define the second 4×1 complex vector as $\underline{p} = M_r \underline{s}_r + \mathrm{j} M_i \underline{s}_i$ where M_r and M_i are unitary integer matrices. Then, $\underline{s}^{\mathrm{H}}\underline{p} = 0$ holds for all $\underline{s}$ in $\mathbb{C}^4$ if and only if (to be denoted by iff) a pair of elements in $\underline{s}$ is orthogonal to the corresponding pair in $\underline{p}$for all $\underline{s}$ in $\mathbb{C}^4$ and the other pair of elements in $\underline{s}$ is* *orthogonal to the other corresponding pair in* $\underline{p}$ *for all* $\underline{s}$ *in* $\mathbb{C}^4$. (*The pair-wise orthogonality is equivalent to that in Alamouti code* [1].)

Proof. Taking the real and imaginary parts of $\underline{s}^H \underline{p} = 0$ separately, we have $\underline{s}^H \underline{p} = 0$ iff $t_1 + t_2 = 0$ and $t_3 = 0$ where $t_1 = \underline{s}_r^T M_r \underline{s}_r$, $t_2 = \underline{s}_i^T M_i \underline{s}_i$, and $t_3 = \underline{s}_i^T M_r \underline{s}_r - \underline{s}_i^T M_i^T \underline{s}_r$.

Because of the independence between $\underline{s}_r$ and $\underline{s}_i$, $t_1 + t_2 = 0$ iff $t_1 = 0$ and $t_2 = 0$. Because both $\underline{s}_r$ and $\underline{s}_i$ are any real vectors, we have $t_1 = 0$ iff $M_r = -M_r^T$ and $t_2 = 0$ iff $M_i = -M_i^T$. Similarly, we have $t_3 = 0$ iff $M_r = M_i^T$.

The above implies that $\underline{s}^H \underline{p} = 0$ holds for all $\underline{s}$ in $\mathbb{C}^4$ iff $M_i = -M_r$ and $M_r = -M_r^T$.

With the above choice of M_r and M_i, $\underline{s}^H \underline{p} = 0$ is equivalent to $\underline{s}^H M_r \underline{s}^* = 0$. We next apply that $M_r = -M_r^T$ and each row of M_r has only one non-zero entry ± 1. Without loss of generality, we can assume that the (i_0, j_0) entry of M_r is ± 1, i.e., $M_r(i_0, j_0) = \pm 1$. Then $M_r(j_0, i_0) = -M_r(i_0, j_0)$ where $i_0 \neq j_0$ and all other elements in the i_0th and j_0th rows of M_r are zero. This property implies that the i_0th and j_0th elements in $\underline{s}$ cancel each other in the form $\pm(\underline{s}^*(i_0)\underline{s}^*(j_0) - \underline{s}^*(j_0)\underline{s}^*(i_0))$in $\underline{s}^H M_r \underline{s}^* = 0$. In other words, the i_0th and j_0th elements in $\underline{s}$ are orthogonal to the i_0th and j_0th elements in $\underline{p}$ for all $\underline{s}$ in $\mathbb{C}^4$. More explicitly, the i_0th and j_0th elements in $\underline{p}$ are $\pm \underline{s}^*(j_0)$ and $\mp \underline{s}^*(i_0)$.

Following the same reasoning, $\underline{s}^H \underline{p} = 0$ for all $\underline{s}$ in $\mathbb{C}^4$ iff the other two elements in $\underline{s}$ are also orthogonal (in the same way as described above) to the other two corresponding elements in $\underline{p}$ for all $\underline{s}$ in $\mathbb{C}^4$. □

From Lemma 3.1, the next theorem follows (which corrects a result shown in [32]):

Theorem 3.1. *Define a code set* $\mathbb{S}_Q$ *of* 4×4 *quasi-orthogonal codes for the symbol vector* $\underline{s} = (s_1, s_2, s_3, s_4)^T$, *where each element in a (normally full rank) code matrix has the form* $\pm s_k^{(*)}$ *and two of the four columns in each code are orthogonal to the other two over all* $\underline{s}$ *in* $\mathbb{C}^4$. *Then, the following two codes* $S_1(s_1, s_2, s_3, s_4)$ *and* $S_2(s_1, s_2, s_3, s_4)$:

$$S_1 = \begin{pmatrix} s_1 & -s_4 & s_2^* & -s_3^* \\ s_2 & s_3 & -s_1^* & -s_4^* \\ s_3 & -s_2 & -s_4^* & s_1^* \\ s_4 & s_1 & s_3^* & s_2^* \end{pmatrix}, \quad S_2 = \begin{pmatrix} s_1 & s_4 & s_2^* & -s_3^* \\ s_2 & s_3 & -s_1^* & s_4^* \\ s_3 & s_2 & -s_4^* & s_1^* \\ s_4 & s_1 & s_3^* & -s_2^* \end{pmatrix} \tag{16}$$

span all the codes in $\mathbb{S}_Q$ *via* (15), *and furthermore* $S_1(s_1, s_2, s_3, s_4)$ *and* $S_2(s_1, s_2, s_3, s_4)$ *are not related to each other via* (15).

Proof. We will say that a pair of codes are dependent of each other if they are related via (15), or otherwise independent of each other. Our proof is constructive in that we will construct a largest possible set $\mathbb{S}$ of independent codes. It is important to stress that permutations of rows and/or columns, permutations of symbol indices, change of sign to each row and/or column, and sign and/or conjugation changes to each symbol are all variations allowed by (15) among dependent codes. Our proof consists of several steps by which the above variations are eliminated from a largest possible set of independent codes. These steps will lead to two possibly independent codes S_1 and S_2. These two codes S_1 and S_2 are then finally verified to be independent.

Without loss of generality, we can fix the first column of each code matrix in $\mathbb{S}$ to be $[s_1, s_2, s_3, s_4]^T$. Furthermore, we can choose $\mathbb{S}$ in such a way that the first two columns of each code matrix are orthogonal to the last two columns. From Lemma 3.1, it follows that among all code matrices in $\mathbb{S}$, there are no more than the following two possible forms up to the variations defined by (15):

$$T_1 = \begin{pmatrix} s_1 & * & s_2^* & * \\ s_2 & * & -s_1^* & * \\ s_3 & * & -s_4^* & * \\ s_4 & * & s_3^* & * \end{pmatrix} \quad \text{or} \quad T_2 = \begin{pmatrix} s_1 & * & -s_2^* & * \\ s_2 & * & s_1^* & * \\ s_3 & * & -s_4^* & * \\ s_4 & * & s_3^* & * \end{pmatrix},$$

where $*$ (not in superscript) denotes a unspecified entry. Note that in each of T_1 and T_2, the first two elements of the first column are orthogonal to the first two elements of the third column for all $\underline{s}$ in $\mathbb{C}^4$, and the last two elements of first column are orthogonal to the last two elements of the third column for all $\underline{s}$ in $\mathbb{C}^4$. One can easily verify that no other possibility exists that is independent of T_1 and T_2.

Similarly, from T_1, one can generate no more than the following four possibilities in $\mathbb{S}$ up to (15):

$$T_{1,1} = \begin{pmatrix} s_1 & * & s_2^* & -s_3^* \\ s_2 & * & -s_1^* & -s_4^* \\ s_3 & * & -s_4^* & s_1^* \\ s_4 & * & s_3^* & s_2^* \end{pmatrix}, \quad T_{1,2} = \begin{pmatrix} s_1 & * & s_2^* & -s_3^* \\ s_2 & * & -s_1^* & s_4^* \\ s_3 & * & -s_4^* & s_1^* \\ s_4 & * & s_3^* & -s_2^* \end{pmatrix},$$

$$T_{1,3} = \begin{pmatrix} s_1 & * & s_2^* & -s_4^* \\ s_2 & * & -s_1^* & -s_3^* \\ s_3 & * & -s_4^* & s_2^* \\ s_4 & * & s_3^* & s_1^* \end{pmatrix}, \quad T_{1,4} = \begin{pmatrix} s_1 & * & s_2^* & -s_4^* \\ s_2 & * & -s_1^* & s_3^* \\ s_3 & * & -s_4^* & -s_2^* \\ s_4 & * & s_3^* & s_1^* \end{pmatrix}.$$

Furthermore, from T_2, one can generate no more than another four possibilities in $\mathbb{S}$ up to (15):

$$T_{2,1} = \begin{pmatrix} s_1 & * & -s_2^* & -s_3^* \\ s_2 & * & s_1^* & s_4^* \\ s_3 & * & -s_4^* & s_1^* \\ s_4 & * & s_3^* & -s_2^* \end{pmatrix}, \quad T_{2,2} = \begin{pmatrix} s_1 & * & -s_2^* & -s_3^* \\ s_2 & * & s_1^* & -s_4^* \\ s_3 & * & -s_4^* & s_1^* \\ s_4 & * & s_3^* & s_2^* \end{pmatrix},$$

$$T_{2,3} = \begin{pmatrix} s_1 & * & -s_2^* & -s_4^* \\ s_2 & * & s_1^* & -s_3^* \\ s_3 & * & -s_4^* & s_2^* \\ s_4 & * & s_3^* & s_1^* \end{pmatrix}, \quad T_{2,4} = \begin{pmatrix} s_1 & * & -s_2^* & -s_4^* \\ s_2 & * & s_1^* & s_3^* \\ s_3 & * & -s_4^* & -s_2^* \\ s_4 & * & s_3^* & s_1^* \end{pmatrix}.$$

By filling the second column of each of the above matrices (to satisfy the orthogonality condition), it follows that there are no more than the following eight possibilities in $\mathbb{S}$ up to (15):

$$T_{1,1} = \begin{pmatrix} s_1 & -s_4 & s_2^* & -s_3^* \\ s_2 & s_3 & -s_1^* & -s_4^* \\ s_3 & -s_2 & -s_4^* & s_1^* \\ s_4 & s_1 & s_3^* & s_2^* \end{pmatrix}, \quad T_{1,2} = \begin{pmatrix} s_1 & s_4 & s_2^* & -s_3^* \\ s_2 & s_3 & -s_1^* & s_4^* \\ s_3 & s_2 & -s_4^* & s_1^* \\ s_4 & s_1 & s_3^* & -s_2^* \end{pmatrix},$$

$$T_{1,3} = \begin{pmatrix} s_1 & s_3 & s_2^* & -s_4^* \\ s_2 & -s_4 & -s_1^* & -s_3^* \\ s_3 & s_1 & -s_4^* & s_2^* \\ s_4 & -s_2 & s_3^* & s_1^* \end{pmatrix}, \quad T_{1,4} = \begin{pmatrix} s_1 & s_3 & s_2^* & -s_4^* \\ s_2 & s_4 & -s_1^* & s_3^* \\ s_3 & -s_1 & -s_4^* & -s_2^* \\ s_4 & -s_2 & s_3^* & s_1^* \end{pmatrix},$$

$$T_{2,1} = \begin{pmatrix} s_1 & s_4 & -s_2^* & -s_3^* \\ s_2 & s_3 & s_1^* & s_4^* \\ s_3 & -s_2 & -s_4^* & s_1^* \\ s_4 & -s_1 & s_3^* & -s_2^* \end{pmatrix}, \quad T_{2,2} = \begin{pmatrix} s_1 & -s_4 & -s_2^* & -s_3^* \\ s_2 & s_3 & s_1^* & -s_4^* \\ s_3 & s_2 & -s_4^* & s_1^* \\ s_4 & -s_1 & s_3^* & s_2^* \end{pmatrix},$$

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$$T_{2,3} = \begin{pmatrix} s_1 & s_3 & -s_2^* & -s_4^* \\ s_2 & -s_4 & s_1^* & -s_3^* \\ s_3 & -s_1 & -s_4^* & s_2^* \\ s_4 & s_2 & s_3^* & s_1^* \end{pmatrix}, \quad T_{2,4} = \begin{pmatrix} s_1 & s_3 & -s_2^* & -s_4^* \\ s_2 & s_4 & s_1^* & s_3^* \\ s_3 & s_1 & -s_4^* & -s_2^* \\ s_4 & s_2 & s_3^* & s_1^* \end{pmatrix}.$$

If we let $S_1 = T_{1,1}$ and $S_2 = T_{1,2}$, then it can be verified that

$$T_{1,3}(\underline{s}) = P_{11} S_2(-s_1, s_2, s_4, s_3) P_{21}, \tag{17}$$

$$T_{1,4}(\underline{s}) = P_{12} S_1(s_3, s_2, s_4, s_1) P_{22}, \tag{18}$$

$$T_{2,1}(\underline{s}) = P_{13} S_1(s_4, s_2, s_3, s_1) P_{23}, \tag{19}$$

$$T_{2,2}(\underline{s}) = P_{14} S_2(-s_1, s_2, s_3, s_4) P_{24}, \tag{20}$$

$$T_{2,3}(\underline{s}) = P_{15} S_1(s_1, s_2, s_4, s_3) P_{25}, \tag{21}$$

$$T_{2,4}(\underline{s}) = P_{16} S_2(s_1, s_2, s_4, s_3) P_{26}, \tag{22}$$

where

$$P_{11} = P_{-(1,1),+(2,2),+(3,4),+(4,3)}, \tag{23}$$

$$P_{21} = P_{+(1,1),-(2,2),-(3,3),-(4,4)}, \tag{24}$$

$$P_{12} = P_{+(1,4),+(2,2),+(3,1),+(4,3)}, \tag{25}$$

$$P_{22} = P_{(1,1),+(2,2),-(3,4),-(4,3)}, \tag{26}$$

$$P_{13} = P_{+(1,4),+(2,2),+(3,3),+(4,1)}, \tag{27}$$

$$P_{23} = P_{+(1,1),+(2,2),-(3,4),-(4,3)}, \tag{28}$$

$$P_{14} = P_{-(1,1),+(2,2),+(4,3),+(3,4)}, \tag{29}$$

$$P_{24} = P_{+(1,1),+(2,2),+(3,3),-(4,4)}, \tag{30}$$

$$P_{15} = P_{+(1,1),+(2,2),+(4,3),+(3,4)}, \tag{31}$$

$$P_{25} = P_{+(1,1),-(2,2),-(3,3),+(4,4)}, \tag{32}$$

$$P_{16} = P_{+(1,1),+(2,2),+(4,3),+(3,4)}, \tag{33}$$

$$P_{26} = P_{+(1,1),+(2,2),-(3,3),+(4,4)} \tag{34}$$

and $P_{\pm(i_1,j_1),\pm(i_2,j_2),\pm(i_3,j_3),\pm(i_4,j_4)}$ denotes a matrix where the entries at $(i_1,j_1),(i_2,j_2),(i_3,j_3),(i_4,j_4)$ are ± 1 and all other entries are zero.

Therefore, there are no more than two independent codes in $\mathbb{S}$, and S_1 and S_2 are two possible independent codes.

To prove that S_1 and S_2 are indeed independent codes, we need to observe a property from (15). If all elements of $\underline{s}$ come from a common symmetric constellation (symmetric in terms of sign change and complex conjugation), then it is obvious from (15) that any two dependent codes $C_1(\underline{s})$ and $C_2(\underline{s})$ must satisfy the following identity:

$$\min_{\underline{s} \neq 0} \operatorname{rank}(C_1(\underline{s})) \equiv \min_{\underline{\bar{s}} \neq 0} \operatorname{rank}(C_2(\underline{\bar{s}})), \tag{35}$$

where $\bar{\underline{s}}$ is a function of $\underline{s}$, satisfying

$$\bar{\underline{s}} = (\pm s_{k_1}^{(*)}, \pm s_{k_2}^{(*)}, \pm s_{k_3}^{(*)}, \pm s_{k_4}^{(*)})^{\mathrm{T}}.$$

In the next subsection, we show that S_1 and S_2 do not satisfy the property (35). In fact, we will consider an equivalent situation where $\underline{s}$ is replaced by $\Delta\underline{s}$ and the constellation of $\Delta\underline{s}$ is symmetric with respect to sign change and complex conjugation. Since S_1 and S_2 do not satisfy the necessary condition (35) as required for any pair of dependent codes, S_1 and S_2 are independent. □

3.2. Diversity of quasi-orthogonal codes

The diversity of a code matrix $C(\underline{s})$ is the minimum rank of $C(\Delta\underline{s})$ over all possible $\Delta\underline{s} \neq 0$ where $\Delta\underline{s}$ is the difference between two symbol vectors. For the diversity analysis of quasi-orthogonal codes, we will assume that the constellation of $\Delta\underline{s}$ is symmetric with respect to sign and conjugation. Such a condition is common in practice. Because of the property (35) among dependent codes, to study the diversity of all the 4×4 quasi-orthogonal codes, it suffices to consider the diversity of $S_1(\underline{s})$ and $S_2(\underline{s})$.

The diversity of $S_i(\underline{s})$ is the minimum rank of $D_i(\Delta\underline{s})$ over all $\Delta\underline{s} \neq 0$ where $\Delta\underline{s} = (\Delta s_1, \Delta s_2, \Delta s_3, \Delta s_4)^{\mathrm{T}}$. Here,

$$D_1(\underline{s}) \doteq S_1(\underline{s})^{\mathrm{H}} S_1(\underline{s}) = \begin{bmatrix} a(\underline{s}) & \mathrm{j}b_1(\underline{s}) & 0 & 0 \\ -\mathrm{j}b_1(\underline{s}) & a(\underline{s}) & 0 & 0 \\ 0 & 0 & a(\underline{s}) & \mathrm{j}b_1(\underline{s}) \\ 0 & 0 & -\mathrm{j}b_1(\underline{s}) & a(\underline{s}) \end{bmatrix} \tag{36}$$

and

$$D_2(\underline{s}) \doteq S_2(\underline{s})^{\mathrm{H}} S_2(\underline{s}) = \begin{bmatrix} a(\underline{s}) & b_2(\underline{s}) & 0 & 0 \\ b_2(\underline{s}) & a(\underline{s}) & 0 & 0 \\ 0 & 0 & a(\underline{s}) & b_2(\underline{s}) \\ 0 & 0 & b_2(\underline{s}) & a(\underline{s}) \end{bmatrix}, \tag{37}$$

where

$$a(\underline{s}) = \sum_{k=1}^{4} |s_k|^2, \tag{38}$$

$$b_1(\underline{s}) = 2\Im(s_1 s_4^* + s_2^* s_3), \tag{39}$$

$$b_2(\underline{s}) = 2\Re(s_1^* s_4 + s_2^* s_3). \tag{40}$$

It is clear from (36) and (37) that the rank of $D_i(\Delta\underline{s})$ is either two or four as long as $\Delta\underline{s} \neq 0$. To determine the conditions under which $D_i(\Delta\underline{s})$ has the full rank (i.e., rank four), it is useful to examine its determinant. The determinants of $D_i(\underline{s})$ are given by $\det(D_i(\underline{s})) = (a(\underline{s}) - b_i(\underline{s}))^2 (a(\underline{s}) + b_i(\underline{s}))^2$ or equivalently

$$\det(D_1(\underline{s})) = [|\mathrm{j}s_1 - s_4|^2 + |s_2 - \mathrm{j}s_3|^2]^2 [|\mathrm{j}s_1 + s_4|^2 + |s_2 + \mathrm{j}s_3|^2]^2, \tag{41}$$

$$\det(D_2(\underline{s})) = [|s_1 + s_4|^2 + |s_2 + s_3|^2]^2 [|s_1 - s_4|^2 + |s_2 - s_3|^2]^2. \tag{42}$$

We show next that the conditions for a full rank $D_1(\Delta\underline{s})$ is generally different from that for a full rank $D_2(\Delta\underline{s})$ even if the constellation of $\Delta\underline{s}$ is symmetric. (Note that this was needed to complete the proof of Theorem 3.1.)

Expression (41) would make Lemma 2.1 in [33] more complete and would also enrich Lemma 2.2 in the same paper. It follows from (41) that if and only if the set $\{\mathrm{j}\Delta s_1\}$ and the set $\{\pm\Delta s_4\}$ have no common elements except zero or the set $\{\Delta s_2\}$ and the set $\{\pm\mathrm{j}\Delta s_3\}$ have no common elements except zero, then $D_1(\Delta\underline{s})$ has full rank as long as $\Delta\underline{s} \neq 0$.

Expression (42) is given and well discussed in Lemma 2.1 in [33]. It follows from (42) that if and only if the set $\{\Delta s_1\}$ and the set $\{\pm\Delta s_4\}$ have no common elements except zero or the set $\{\Delta s_2\}$ and the set $\{\pm\Delta s_3\}$ have no

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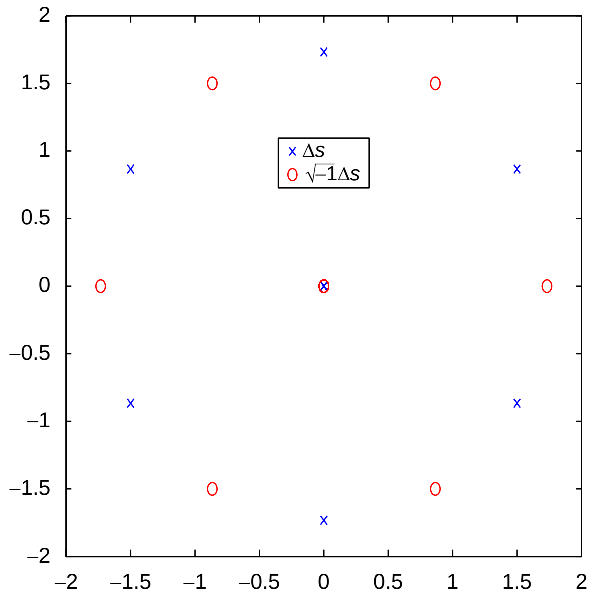

Fig. 1. The set of Δs_i (crosses) and the set of $\mathrm{j}\Delta s_i$ (circles) for 3-PSK. Note that the constellation of Δs_i is symmetric in terms of sign change and complex conjugation, which meets the condition for (35).

common elements except zero, then $D_2(\Delta \underline{s})$ has full rank as long as $\Delta \underline{s} \neq 0$. This is an observation also made in [13–16].

For example, if the constellation is odd-numbered phase-shift-keying as illustrated in Fig. 1, then the set $\{\mathrm{j}\Delta s_1\}$ and the set $\{\pm\Delta s_4\}$ have no common elements except zero and the set $\{\Delta s_2\}$ and the set $\{\pm \mathrm{j}\Delta s_3\}$ have no common elements except zero, and hence $D_1(\Delta \underline{s})$ has full rank under $\Delta \underline{s} \neq 0$. But for the same constellation, $D_2(\Delta \underline{s})$ does not have full rank under $\Delta \underline{s} \neq 0$.

3.3. Quasi-orthogonal codes in terms of HR matrices

We now illustrate that all quasi-orthogonal codes are linear dispersion codes by expressing the two independent codes S_1 and S_2 in terms of the HR matrices. Let the real and imaginary parts of each symbol s_k be expressed as $s_k = r_k + \mathrm{j}i_k$. It is not difficult to verify the following results. For the first code,

$$S_1 = \begin{pmatrix} r_1 & -r_4 & -r_2 & -r_3 \\ r_2 & r_3 & r_1 & -r_4 \\ r_3 & -r_2 & r_4 & r_1 \\ r_4 & r_1 & -r_3 & r_2 \end{pmatrix} + \mathrm{j} \begin{pmatrix} i_1 & -i_4 & i_2 & i_3 \\ i_2 & i_3 & -i_1 & i_4 \\ i_3 & -i_2 & -i_4 & -i_1 \\ i_4 & i_1 & i_3 & -i_2 \end{pmatrix}$$
$$= [Q_0\underline{r}_1, -Q_1\underline{r}_1, Q_3\underline{r}_1, -Q_2\underline{r}_1] + \mathrm{j}[Q_2\underline{i}_1, -Q_3\underline{i}_1, -Q_1\underline{i}_1, Q_0\underline{i}_1],$$

where $\underline{r}_1 = [r_1, \ldots, r_4]^{\mathrm{T}}$ and $\underline{i}_1 = [i_3, i_4, -i_1, -i_2]^{\mathrm{T}}$. For the second code,

$$S_2 = \begin{pmatrix} r_1 & r_4 & r_2 & -r_3 \\ r_2 & r_3 & -r_1 & r_4 \\ r_3 & r_2 & -r_4 & r_1 \\ r_4 & r_1 & r_3 & -r_2 \end{pmatrix} + \mathrm{j} \begin{pmatrix} i_1 & i_4 & -i_2 & i_3 \\ i_2 & i_3 & i_1 & -i_4 \\ i_3 & i_2 & i_4 & -i_1 \\ i_4 & i_1 & -i_3 & i_2 \end{pmatrix}$$
$$= [-Q_3K\underline{r}_2, Q_2\underline{r}_2, Q_0K\underline{r}_2, Q_1^{\mathrm{T}}\underline{r}_2] + \mathrm{j}[-Q_0K\underline{i}_2, Q_1^{\mathrm{T}}\underline{i}_2, Q_3K\underline{i}_2, Q_2\underline{i}_2], \tag{43}$$

where $\underline{r}_2 = [-r_2, -r_1, r_4, r_3]^{\mathrm{T}}$, $\underline{i}_2 = [i_1, -i_2, i_3, -i_4]^{\mathrm{T}}$, and $K = -I_2 \otimes Q$.

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Table 1
Expressions of existing quasi-orthogonal codes in terms of two independent quasi-orthogonal codes

Authors/reference	Original code	P_r	S_i	P_c
Papadias and Foschini [25]	$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2^* & -s_1^* & s_4^* & -s_3^* \\ s_3 & -s_4 & -s_1 & s_2 \\ s_4^* & s_3^* & -s_2^* & -s_1^* \end{pmatrix}$	$+(1,3)\quad +(2,1)$ $+(4,4)\quad +(3,2)$	$S_1 \begin{pmatrix} s_2^* \\ s_3 \\ s_1 \\ s_4^* \end{pmatrix}$	$+(1,1)\quad -(2,4)$ $-(3,2)\quad -(4,3)$
Hou et al. (20) in [26]	$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2 & s_1 & -s_4 & s_3 \\ -s_3^* & s_4^* & s_1^* & -s_2^* \\ -s_4^* & -s_3^* & s_2^* & s_1^* \end{pmatrix}$	$+(1,1)\quad +(2,4)$ $+(3,3)\quad +(4,2)$	$S_1 \begin{pmatrix} s_1 \\ -s_4^* \\ -s_3^* \\ -s_2 \end{pmatrix}$	$+(1,1)\quad +(2,2)$ $-(3,4)\quad +(4,3)$
Ran et al. (10) in [27]	$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_2^* & -s_1^* & -s_4^* & s_3^* \\ s_3^* & -s_4^* & -s_1^* & s_2^* \\ s_4 & s_3 & s_2 & s_1 \end{pmatrix}$	$+(1,1)\quad +(2,2)$ $+(3,3)\quad +(4,4)$	$S_2 \begin{pmatrix} s_1 \\ s_2^* \\ s_3^* \\ s_4 \end{pmatrix}$	$+(1,1)\quad +(2,4)$ $+(3,2)\quad -(4,3)$
Tirkkonen et al. (10) in [28]	$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ s_3 & s_4 & s_1 & s_2 \\ -s_4^* & s_3^* & -s_2^* & s_1^* \end{pmatrix}$	$+(1,1)\quad +(2,2)$ $+(3,4)\quad +(4,3)$	$S_2 \begin{pmatrix} s_1 \\ -s_2^* \\ -s_4^* \\ s_3 \end{pmatrix}$	$+(1,1)\quad -(2,3)$ $-(3,2)\quad +(4,4)$
Jafarkhani [11]	$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ -s_3^* & -s_4^* & s_1^* & s_2^* \\ s_4 & -s_3 & -s_2 & s_1 \end{pmatrix}$	$-(1,1)\quad +(2,2)$ $+(3,3)\quad +(4,4)$	$S_2 \begin{pmatrix} -s_1 \\ -s_2^* \\ -s_3^* \\ s_4 \end{pmatrix}$	$+(1,1)\quad -(2,4)$ $+(3,2)\quad -(4,3)$

A column vector of symbols is used instead of a row vector for a more compact form. The arrays of $\pm(i,j)$ in the third and fifth columns indicate the ± 1 entries in the integer matrices P_r and P_c. Other unspecified entries of P_r and P_c are zero.

3.4. *Previously published quasi-orthogonal codes*

The previously published 4×4 unit-rate quasi-orthogonal codes of Type I can all be expressed in terms of the above two independent codes via (15). Table 1 summarizes these connections.

4. Non-orthogonal HR codes

In this section, we present an 8×4 unit-rate code and prove that it has diversity no less than three under any constellation. Like the 4×4 quasi-orthogonal codes presented in the previous section, this code is also a linear dispersion code constructed from HR matrices. This 8×4 unit-rate code is

$$C(\underline{x}_1, \underline{x}_2) \doteq [\underbrace{A_0\underline{x}_1 + \mathrm{j}A_4\underline{x}_2}_{\underline{v}_1}, \underbrace{A_1\underline{x}_1 + \mathrm{j}A_5\underline{x}_2}_{\underline{v}_2}, \underbrace{A_2\underline{x}_1 + \mathrm{j}A_6\underline{x}_2}_{\underline{v}_3}, \underbrace{A_3\underline{x}_1 + \mathrm{j}A_7\underline{x}_2}_{\underline{v}_4}], \tag{44}$$

where $\underline{x}_1$ and $\underline{x}_2$ are two real-valued 8×1 symbol vectors, and A_i, $i = 0, \ldots, 7$, are 8×8 matrices from any single HR family of size eight satisfying (14). This code is motivated by the structure of a half-rate 4×4 orthogonal code for four transmitters. In fact, it is easy to verify that the code (44) is an orthogonal code if the 8×1 complex vector $\underline{x}_1 + \mathrm{j}\underline{x}_2$ is replaced by a 4×1 real vector and A_i, $i = 0, \ldots, 3$ are replaced by 4×4 HR matrices. This code has a very simple and appealing structure.

The above code, also referred to as non-orthogonal HR code, is a special form of a more general HR code introduced in [5], and is also a special form of the linear dispersion codes introduced in [29,34].

One specific form (example) of the non-orthogonal HR code follows from (44) with the HR family given in (7):

$$\begin{pmatrix} x_1(1)+jx_2(6) & x_1(3)+jx_2(7) & x_1(4)+jx_2(8) & x_1(2)+jx_2(5) \\ x_1(2)-jx_2(5) & x_1(4)-jx_2(8) & -x_1(3)+jx_2(7) & -x_1(1)+jx_2(6) \\ x_1(3)-jx_2(8) & -x_1(1)+jx_2(5) & x_1(2)+jx_2(6) & -x_1(4)-jx_2(7) \\ x_1(4)+jx_2(7) & -x_1(2)-jx_2(6) & -x_1(1)+jx_2(5) & x_1(3)-jx_2(8) \\ x_1(5)+jx_2(2) & x_1(7)-jx_2(3) & x_1(8)-jx_2(4) & -x_1(6)-jx_2(1) \\ x_1(6)-jx_2(1) & x_1(8)+jx_2(4) & -x_1(7)-jx_2(3) & x_1(5)-jx_2(2) \\ x_1(7)-jx_2(4) & -x_1(5)-jx_2(1) & x_1(6)-jx_2(2) & x_1(8)+jx_2(3) \\ x_1(8)+jx_2(3) & -x_1(6)+jx_2(2) & -x_1(5)-jx_2(1) & -x_1(7)+jx_2(4) \end{pmatrix}, \tag{45}$$

where $x_i(k)$ is the kth element of the real symbol vector $\underline{x}_i$.

From the theorem shown next, the code (44) is guaranteed to have diversity three even when all symbols are independently selected from any constellation (regardless of the design of the constellation). To our knowledge, for four-transmitters systems, the code (44) is the only known unit-rate linear dispersion code that is guaranteed to be of diversity (at least) three under any given constellation. This is a useful property in practice since any symbol constellation can be applied to this code while the diversity is guaranteed to be no less than three.

Theorem 4.1. *Given any* $\underline{x}_1+j\underline{x}_2\neq 0$, *the code* $C(\underline{x}_1,\underline{x}_2)$ *defined in* (44) *has a rank no less than three regardless of the constellation from which all symbols are independently selected.*

The rest of this section is to prove Theorem 4.1. Since the proof is quite lengthy, we divide the proof into several sections as explained next.

4.1. Outline of the proof of Theorem 4.1

The proof consists of a sequence of lemmas, and these lemmas are indexed as follows:

$$\text{Theorem 4.1}\begin{cases} \text{Lemma 4.1} \Leftarrow \text{Lemma 4.1.1} \\ \text{Lemma 4.2} \Leftarrow \text{Lemma 4.1.1} \\ \text{Lemma 4.3} \Leftarrow \text{Lemma 4.3.1} \Leftarrow \begin{cases} \text{Lemma 4.3.1.1} \Leftarrow (47), \\ (14). \end{cases} \end{cases}$$

All lemmas are stated below. The proofs are given in the subsequent subsections in the order shown above. Theorem 4.1 results immediately from the three main lemmas:

Lemma 4.1 (*Proof in Section 4.3*). *The minimum rank of* (44) *is no larger than three.*

Lemma 4.2 (*Proof in Section 4.4*). *Among any three column vectors in* (44), *at least two of them are independent.*

Lemma 4.3 (*Proof in Section 4.7*). *Given* (14), *if any three column vectors in* (44) *are linearly dependent, they are orthogonal to the fourth vector in* (44).

The above three main lemmas are based on the following supporting lemmas:

Lemma 4.1.1 (*Proof in Section 4.2*). *Given distinct i, j, m, n and the* (*non-zero*) *real vectors* $\underline{x}_1$ *and* $\underline{x}_2$, *the equation* $A_i\underline{x}_1+jA_j\underline{x}_2=k(A_m\underline{x}_1+jA_n\underline{x}_2)$ *holds if and only if*

$$\begin{cases} k \doteq k_1+jk_2=\pm j, \\ (A_iA_m^{T}A_j+A_n)\underline{x}_2=0, \\ \underline{x}_1=-k_2A_i^{T}A_n\underline{x}_2, \end{cases} \tag{46}$$

where k_1 *and* k_2 *are real numbers. There is always a non-zero* $\underline{x}_2$ *that satisfies* $(A_iA_m^{T}A_j+A_n)\underline{x}_2=0$.

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Lemma 4.3.1 (*Proof in Section 4.6*). *Referring to* (44), $\underline{v}_2$ *is orthogonal to* $\underline{v}_3$ *and* $\underline{v}_4$ *if* (a) *the condition* (14) *holds and* (b) *there exist two complex scalars* $k \doteq k_1 + \mathrm{j}k_2$ *and* $t \doteq t_1 + \mathrm{j}t_2$ *with* $|t|^2 + |k|^2 \neq 0$ *and two real vectors* $\underline{x}_1$ *and* $\underline{x}_2$ *such that*

$$\underline{v}_1 = [\underline{v}_3, \underline{v}_4]\begin{pmatrix} k \\ t \end{pmatrix}. \tag{47}$$

Lemma 4.3.1.1 (*Proof in Section 4.5*). *The necessary and sufficient condition for* (47) *to hold is*

$$\begin{cases} M\underline{x}_1 = 0, \\ \underline{x}_2 = -(k_2^2 + t_2^2)^{-1}(A_6k_2 + A_7t_2)^{\mathrm{T}}(A_0 - A_2k_1 - A_3t_1)\underline{x}_1, \end{cases} \tag{48}$$

where

$$M = A_2k_2 + A_3t_2 + (k_2^2 + t_2^2)^{-1}(A_4 - A_6k_1 - A_7t_1)(A_6k_2 + A_7t_2)^{\mathrm{T}}(A_0 - A_2k_1 - A_3t_1). \tag{49}$$

One important property of M is

$$M^{\mathrm{T}}M = c_M I + N, \tag{50}$$

where $c_M > 0$, *N is symmetric and orthogonal, and the eigenvalues of N are* $\pm c_M$ *of differing signs. The exact content of* c_M *is given in* (68) *and the content of N is given in* (68) *and* (65). *Furthermore, M is singular if and only if*

$$\begin{cases} k_2^2 + t_2^2 = k_1^2 + t_1^2 + 1, \\ t_1t_2 + k_1k_2 = 0. \end{cases} \tag{51}$$

4.2. The proof of Lemma 4.1.1

Given the complex-valued equation $A_i\underline{x}_1 + \mathrm{j}A_j\underline{x}_2 = k(A_m\underline{x}_1 + \mathrm{j}A_n\underline{x}_2)$, there are two corresponding real-valued equations (real and imaginary parts):

$$(A_i - k_1A_m)\underline{x}_1 + k_2A_n\underline{x}_2 = 0, \tag{52}$$

$$(A_j - k_1A_n)\underline{x}_2 - k_2A_m\underline{x}_1 = 0. \tag{53}$$

Because $A_i - k_1A_m$ is an invertible matrix for distinct i and m, k_2 has to be non-zero. Otherwise, there is no non-zero solution for $\underline{x}_1$ or $\underline{x}_2$. From (53), we have

$$\underline{x}_1 = k_2^{-1}A_m^{\mathrm{T}}(A_j - k_1A_n)\underline{x}_2. \tag{54}$$

Taking (54) into (52) leads to

$$Z\underline{x}_2 = 0, \tag{55}$$

where $Z = (A_i - k_1A_m)k_2^{-1}A_m^{\mathrm{T}}(A_j - k_1A_n) + k_2A_n$. It is clear that Z must be singular, or otherwise there is no non-zero solution for $\underline{x}_2$.

With Properties 2 and 4 of the HR matrices, it is easy to verify that

$$Z^{\mathrm{T}}Z = [k_2^2 + (1 + k_1^2)^2k_2^{-2} + 2k_1^2]I_8 + 2A_j^{\mathrm{T}}A_mA_i^{\mathrm{T}}A_n. \tag{56}$$

It is known that Z is singular if and only if Z has at least one zero eigenvalue. Therefore, based on (56) and Property 5, the matrix Z is singular if and only if $k_2^2 + (1 + k_1^2)^2k_2^{-2} + 2k_1^2 = \pm 2$. This equation is equivalent to any of the following equations:

$$k_2^4 + (1 + k_1^2)^2 + 2k_1^2k_2^2 = \pm 2k_2^2,$$
$$k_2^4 + 2(k_1^2 + 1)k_2^2 + (1 + k_1^2)^2 = \pm 4k_2^2,$$
$$(1 + k_1^2 + k_2^2)^2 = \pm 4k_2^2.$$

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From the above, we see that the minus sign leads to no real solution. After dropping the minus sign, the above is equivalent to any of the following:

$$(1+k_1^2+k_2^2-2k_2)(1+k_1^2+k_2^2+2k_2)=0,$$

$$(k_1^2+(1-k_2)^2)(k_1^2+(1+k_2)^2)=0,$$

$$k_1=0 \quad \text{and} \quad k_2=\pm 1. \tag{57}$$

Therefore, $k=\pm \mathrm{j}$. Taking this back into (54) and (55) yields the sufficient and necessary conditions on $\underline{x}_2$ and $\underline{x}_1$ as shown in the lemma.

4.3. The proof of Lemma 4.1

It follows from Lemma 4.1.1 that there are always non-zero $\underline{x}_1$ and $\underline{x}_2$ such that any two columns from (44) are linearly dependent of each other. For example, if

$$\underline{x}_1=[-1,1,-1,1,-1,-1,-1,-1]^{\mathrm{T}},$$

$$\underline{x}_2=[-1,1,1,-1,-1,-1,1,1]^{\mathrm{T}}$$

then the codeword given in (44) has a rank no more than three.

4.4. The proof of Lemma 4.2

Based on Lemma 4.1.1 we can now prove that any three columns from (44) have a rank larger than one. Suppose that $A_i\underline{x}_1+\mathrm{j}A_j\underline{x}_2$ depends on each of $A_m\underline{x}_1+\mathrm{j}A_n\underline{x}_2$ and $A_t\underline{x}_1+\mathrm{j}A_r\underline{x}_2$. From Lemma 4.1.1, we have

$$\begin{cases} \underline{x}_1=-k_2A_i^{\mathrm{T}}A_n\underline{x}_2, \\ \underline{x}_1=-k_2'A_i^{\mathrm{T}}A_r\underline{x}_2 \end{cases} \tag{58}$$

which implies $A_i^{\mathrm{T}}(A_n\pm A_r)\underline{x}_2=0$. However, $A_i^{\mathrm{T}}(A_n\pm A_r)$ is an orthogonal matrix, which means that $\underline{x}_2=0$ and hence $\underline{x}_1=0$. Therefore, $A_i\underline{x}_1+\mathrm{j}A_j\underline{x}_2$ cannot depend on each of $A_m\underline{x}_1+\mathrm{j}A_n\underline{x}_2$ and $A_t\underline{x}_1+\mathrm{j}A_r\underline{x}_2$.

4.5. The proof of Lemma 4.3.1.1

Given the complex-valued equation (47), we equivalently have the following two real-valued equations (i.e., the real and imaginary parts):

$$A_0\underline{x}_1=A_2x_1k_1+A_3x_1t_1-A_6x_2k_2-A_7x_2t_2,$$

$$A_4\underline{x}_2=A_2x_1k_2+A_3x_1t_2+A_6x_2k_1+A_7x_2t_1$$

or equivalently

$$(A_0-A_2k_1-A_3t_1)\underline{x}_1=-(A_6k_2+A_7t_2)\underline{x}_2, \tag{59}$$

$$(A_4-A_6k_1-A_7t_1)\underline{x}_2=(A_2k_2+A_3t_2)\underline{x}_1. \tag{60}$$

Recalling Property 4, $(A_m-A_nk_1-A_lt_1)$ is always non-singular for distinct m, n and l, and $(A_mk_2+A_nt_2)$ is non-singular for distinct m and n unless $k_2^2+t_2^2=0$. So from (59) and (60), $k_2^2+t_2^2\neq 0$ unless both $\underline{x}_1$ and $\underline{x}_2$ are equal to zero. Also from (59) and (60), $\underline{x}_1=0$ if and only if $\underline{x}_2=0$.

From (59), we have

$$\underline{x}_2=-(k_2^2+t_2^2)^{-1}(A_6k_2+A_7t_2)^{\mathrm{T}}(A_0-A_2k_1-A_3t_1)\underline{x}_1. \tag{61}$$

Also, from (60), we have an equivalent form of $\underline{x}_2$:

$$\underline{x}_2=(1+k_1^2+t_1^2)^{-1}(A_4-A_6k_1-A_7t_1)^{\mathrm{T}}(A_2k_2+A_3t_2)\underline{x}_1. \tag{62}$$

Using (61) in (60) yields $M\underline{x}_1 = 0$ where

$$M = M_1 + M_2 \tag{63}$$

with $M_1 = A_2k_2 + A_3t_2$ and

$$M_2 = (k_2^2 + t_2^2)^{-1}(A_4 - A_6k_1 - A_7t_1)(A_6k_2 + A_7t_2)^{\mathrm{T}}(A_0 - A_2k_1 - A_3t_1).$$

Clearly, $\underline{x}_1 \neq 0$ if and only if M is singular. Summarizing the above, we have that (47) holds if and only if M is singular, $\underline{x}_1$ satisfies $M\underline{x}_1 = 0$, and $\underline{x}_2$ satisfies (61).

To reveal a property of $M^{\mathrm{T}}M$, we now apply Property 4 to obtain that $M_1M_1^{\mathrm{T}} = (k_2^2 + t_2^2)I_8$ and $M_2M_2^{\mathrm{T}} = (k_2^2 + t_2^2)^{-1}(1 + k_1^2 + t_1^2)^2 I_8$. Furthermore,

$$\begin{aligned} M_1^{\mathrm{T}}M_2 &= (k_2^2 + t_2^2)^{-1}(A_2^{\mathrm{T}}k_2 + A_3^{\mathrm{T}}t_2)(A_4A_6^{\mathrm{T}}k_2 + A_4A_7^{\mathrm{T}}t_2 - k_1k_2I_8 - A_6A_7^{\mathrm{T}}k_1t_2 - A_7A_6^{\mathrm{T}}t_1k_2 - t_1t_2I_8) \\ &\quad \times (A_0 - A_2k_1 - A_3t_1) \\ &= M_{c12,1} + M_{c12,2}, \end{aligned} \tag{64}$$

where

$$M_{c12,1} = (k_2^2 + t_2^2)^{-1}(A_2^{\mathrm{T}}k_2 + A_3^{\mathrm{T}}t_2)\underbrace{[A_4A_6^{\mathrm{T}}k_2 + A_4A_7^{\mathrm{T}}t_2 + A_6A_7^{\mathrm{T}}(-k_1t_2 + k_2t_1)]}_{\Delta_M}(A_0 - A_2k_1 - A_3t_1), \tag{65}$$

$$\begin{aligned} M_{c12,2} &= -(k_2^2 + t_2^2)^{-1}(A_2^{\mathrm{T}}k_2 + A_3^{\mathrm{T}}t_2)(t_1t_2 + k_1k_2)(A_0 - A_2k_1 - A_3t_1) \\ &= -(k_2^2 + t_2^2)^{-1}(t_1t_2 + k_1k_2)(A_2^{\mathrm{T}}k_2A_0 - k_1k_2I_8 - A_2^{\mathrm{T}}A_3t_1k_2 + A_3^{\mathrm{T}}t_2A_0 - A_3^{\mathrm{T}}t_2A_2k_1 - t_1t_2I_8). \end{aligned} \tag{66}$$

Using Property 2 and (66) yields

$$M_{c12,2} + M_{c12,2}^{\mathrm{T}} = 2(k_2^2 + t_2^2)^{-1}(t_1t_2 + k_2k_2)^2 I_8. \tag{67}$$

Therefore,

$$\begin{aligned} M^{\mathrm{T}}M &= M_1^{\mathrm{T}}M_1 + M_2^{\mathrm{T}}M_2 + M_{c12,1} + M_{c12,2} + M_{c12,1}^{\mathrm{T}} + M_{c12,2}^{\mathrm{T}} \\ &= [\underbrace{(k_2^2 + t_2^2) + (k_2^2 + t_2^2)^{-1}(1 + k_1^2 + t_1^2)^2 + 2(k_2^2 + t_2^2)^{-1}(k_1k_2 + t_1t_2)^2}_{c_M}]I_8 + \underbrace{M_{c12,1} + M_{c12,1}^{\mathrm{T}}}_{N}. \end{aligned} \tag{68}$$

It is straightforward to verify that

$$\begin{aligned} N &= M_{c12,1} + M_{c12,1}^{\mathrm{T}} \\ &= 2(k_2^2 + t_2^2)^{-1}\underbrace{[A_2^{\mathrm{T}}k_2\Delta_MA_0 + A_3^{\mathrm{T}}t_2\Delta_MA_0 - k_2A_2^{\mathrm{T}}\Delta_Mt_1A_3 - A_3^{\mathrm{T}}t_2\Delta_Mk_1A_2]}_{W_0}. \end{aligned} \tag{69}$$

We note that $\Delta_M^{\mathrm{T}}\Delta_M = c_\Delta I_8$ where $c_\Delta = t_2^2 + k_2^2 + (k_2t_1 - t_2k_1)^2$. It is also straightforward to verify that

$$W_0W_0^{\mathrm{T}} = c_\Delta^2 I_8. \tag{70}$$

We now need to prove $W_0 \neq \pm c_\Delta I_8$. Suppose $W_0 = \pm c_\Delta I_8$. Then,

$$A_2^{\mathrm{T}}k_2\Delta_MA_0 + A_3^{\mathrm{T}}t_2\Delta_MA_0 - k_2A_2^{\mathrm{T}}\Delta_Mt_1A_3 - A_3^{\mathrm{T}}t_2\Delta_Mk_1A_2 = \pm c_\Delta I_8$$

which can be rewritten as

$$k_2A_2^{\mathrm{T}}(A_0 - A_3t_1) = \pm f(\Delta_M)^{\mathrm{T}} - t_2A_3^{\mathrm{T}}(A_0 - k_1A_2), \tag{71}$$

where $f(\Delta_M) = A_4^{\mathrm{T}}A_6k_2 + A_4^{\mathrm{T}}A_7t_2 + A_6^{\mathrm{T}}A_7(-k_1t_2 + k_2t_1)$.

Note that $f(\Delta_M)$ is antisymmetric, and $f(\Delta_M)^{\mathrm{T}}f(\Delta_M) = c_\Delta I_8$. For A_k where $k \neq (4,6,7)$, $A_k^{\mathrm{T}}\Delta_M = f(\Delta_M)A_k^{\mathrm{T}}$. From (71), we have

$$(k_2^2 + t_2^2)A_0 = (A_2k_2 + A_3t_2)\{\pm f^{\mathrm{T}}(\Delta_M) - A_3^{\mathrm{T}}A_2(k_2t_1 - k_1t_2)\}. \tag{72}$$

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We now multiply (72) by A_3^{T} to yield

$$\begin{aligned}-(k_2^2+t_2^2)A_3^{\mathrm{T}}A_0 &= -(A_3^{\mathrm{T}}A_2k_2+t_2I)\{\pm f^{\mathrm{T}}(\Delta_M)-A_3^{\mathrm{T}}A_2(k_2t_1-k_1t_2)\}\\ &= \pm[k_2^2A_3^{\mathrm{T}}A_2A_4^{\mathrm{T}}A_6+k_2t_2A_3^{\mathrm{T}}A_2A_4^{\mathrm{T}}A_7+k_2(k_2t_1-k_1t_2)A_3^{\mathrm{T}}A_2A_6^{\mathrm{T}}A_7]\\ &\quad \pm t_2f(\Delta_M)+(-1)k_2(k_2t_1-k_1t_2)I_8+A_3^{\mathrm{T}}A_2(k_2t_1-k_1t_2)t_2. \end{aligned} \tag{73}$$

Applying Properties 2 and 5, we add Eq. (73) to its transposed version to yield

$$k_2\left(\pm\underbrace{(k_2A_3^{\mathrm{T}}A_2A_4^{\mathrm{T}}A_6+t_2A_3^{\mathrm{T}}A_2A_4^{\mathrm{T}}A_7+(k_2t_1-k_1t_2)A_3^{\mathrm{T}}A_2A_6^{\mathrm{T}}A_7)}_{F_0}+(-1)(k_2t_1-k_1t_2)I_8\right)=0. \tag{74}$$

Recall $k_2^2+t_2^2\neq 0$. From the definition of F_0 shown in (74), it is easy to verify that $F_0F_0^{\mathrm{T}}=[k_2^2+t_2^2+(k_2t_1-t_1k_2)^2]I_8\neq(k_2t_1-t_1k_2)^2$. Therefore, (74) implies $k_2=0$. Similarly, we can multiply (72) by A_2^{T}, and then follow the same analysis as shown above to conclude that $t_2=0$. Therefore, we have proven by contradiction that $W_0\neq\pm c_\Delta I_8$.

Since $M^{\mathrm{T}}M=c_MI_8+N$ where N has the eigenvalues $\pm 2(k_2^2+t_2^2)^{-1}c_\Delta$, M is singular if and only if

$$c_M=2(k_2^2+t_2^2)^{-1}c_\Delta \tag{75}$$

which is equivalent to

$$[(k_2^2+t_2^2)^2-(k_1^2+t_1^2+1)^2]^2+4(t_1t_2+k_1k_2)^2=0$$

and hence (51).

4.6. *Proof of Lemma 4.3.1*

Given the definitions of $\underline{v}_2$ and $\underline{v}_3$ shown in (44), it follows that

$$\begin{aligned}\underline{v}_2^{\mathrm{H}}\underline{v}_3 &= (\underline{x}_1^{\mathrm{T}}A_1^{\mathrm{T}}A_2\underline{x}_1+\underline{x}_2^{\mathrm{T}}A_5^{\mathrm{T}}A_6\underline{x}_2)+(\underline{x}_1^{\mathrm{T}}A_1^{\mathrm{T}}A_6\underline{x}_2-\underline{x}_2^{\mathrm{T}}A_5^{\mathrm{T}}A_2\underline{x}_1)\mathrm{j}\\ &= (\underline{x}_1^{\mathrm{T}}A_1^{\mathrm{T}}A_6\underline{x}_2-\underline{x}_2^{\mathrm{T}}A_5^{\mathrm{T}}A_2\underline{x}_1)\mathrm{j}. \end{aligned} \tag{76}$$

Using (61) and Property 3, the first term (ignoring j) in (76) becomes

$$\begin{aligned}\underline{x}_1^{\mathrm{T}}A_1^{\mathrm{T}}A_6\underline{x}_2 &= -(k_2^2+t_2^2)^{-1}\underline{x}_1^{\mathrm{T}}A_1^{\mathrm{T}}A_6(A_6k_2+A_7t_2)^{\mathrm{T}}(A_0-A_2k_1-A_3t_1)\underline{x}_1\\ &= -(k_2^2+t_2^2)^{-1}\underline{x}_1^{\mathrm{T}}(k_2A_1^{\mathrm{T}}A_0-k_1k_2A_1^{\mathrm{T}}A_2-k_2t_1A_1^{\mathrm{T}}A_3+t_2A_1^{\mathrm{T}}A_6A_7^{\mathrm{T}}A_0\\ &\quad -t_2k_1A_1^{\mathrm{T}}A_6A_7^{\mathrm{T}}A_2-t_2t_1A_1^{\mathrm{T}}A_6A_7^{\mathrm{T}}A_3)\underline{x}_1\\ &= -(k_2^2+t_2^2)^{-1}\underline{x}_1^{\mathrm{T}}\left(t_2A_1^{\mathrm{T}}A_6A_7^{\mathrm{T}}A_0-t_2k_1A_1^{\mathrm{T}}A_6A_7^{\mathrm{T}}A_2-t_2t_1A_1^{\mathrm{T}}A_6A_7^{\mathrm{T}}A_3\right)\underline{x}_1. \end{aligned} \tag{77}$$

Taking (62) into the second term in (76), we can similarly show that

$$\begin{aligned}\underline{x}_1^{\mathrm{T}}A_2^{\mathrm{T}}A_5\underline{x}_2 &= (1+k_1^2+t_1^2)^{-1}\underline{x}_1^{\mathrm{T}}A_2^{\mathrm{T}}A_5(A_4-A_6k_1-A_7t_1)^{\mathrm{T}}(A_2k_2+A_3t_2)\underline{x}_1\\ &= (1+k_1^2+t_1^2)^{-1}t_2\underline{x}_1^{\mathrm{T}}(A_2^{\mathrm{T}}A_5A_4^{\mathrm{T}}A_3-k_1A_2^{\mathrm{T}}A_5A_6^{\mathrm{T}}A_3-t_1A_2^{\mathrm{T}}A_5A_7^{\mathrm{T}}A_3)\underline{x}_1. \end{aligned} \tag{78}$$

Based on (14) and Property 2, we have $A_1^{\mathrm{T}}A_6A_7^{\mathrm{T}}A_0=A_5^{\mathrm{T}}A_2A_4^{\mathrm{T}}A_3=-A_2^{\mathrm{T}}A_5A_4^{\mathrm{T}}A_3$. Also, recall $1+k_1^2+t_1^2=k_2^2+t_2^2$. Therefore,

$$\begin{aligned}\underline{v}_2^{\mathrm{H}}\underline{v}_3 &= (k_2^2+t_2^2)^{-1}\underline{x}_1^{\mathrm{T}}(-t_2A_1^{\mathrm{T}}A_6A_7^{\mathrm{T}}A_0+t_2k_1A_1^{\mathrm{T}}A_6A_7^{\mathrm{T}}A_2+t_2t_1A_1^{\mathrm{T}}A_6A_7^{\mathrm{T}}A_3\\ &\quad -t_2A_2^{\mathrm{T}}A_5A_4^{\mathrm{T}}A_3+k_1t_2A_2^{\mathrm{T}}A_5A_6^{\mathrm{T}}A_3+t_1t_2A_2^{\mathrm{T}}A_5A_7^{\mathrm{T}}A_3)\underline{x}_1\mathrm{j}\\ &= (k_2^2+t_2^2)^{-1}t_2\underline{x}_1^{\mathrm{T}}\underbrace{(k_1A_1^{\mathrm{T}}A_6A_7^{\mathrm{T}}A_2+t_1A_1^{\mathrm{T}}A_6A_7^{\mathrm{T}}A_3+k_1A_2^{\mathrm{T}}A_5A_6^{\mathrm{T}}A_3+t_1A_2^{\mathrm{T}}A_5A_7^{\mathrm{T}}A_3)}_{M_N}\underline{x}_1\mathrm{j}. \end{aligned} \tag{79}$$

In order to prove $\underline{v}_2^{\mathrm{H}}\underline{v}_3 = 0$, it remains to prove $\underline{x}_1^{\mathrm{T}} M_N \underline{x}_1 = 0$. By Lemma 4.3.1.1, $M\underline{x}_1 = 0$ and $M^{\mathrm{T}}M = c_M I_8 + N$. From the property of N, it follows that range($c_M I_8 + N$) and range($c_M I_8 - N$) are orthogonal complement of each other. Therefore, the solution space of $M\underline{x}_1 = 0$ is given by the range of $M_0 \doteq c_M I_8 - N$, i.e., $\underline{x}_1 = M_0\,\underline{v}$ for any real vector $\underline{v}$. Then, $\underline{x}_1^{\mathrm{T}} M_N \underline{x}_1 = 0$ if and only if $\underline{v}^{\mathrm{T}} M_0^{\mathrm{T}} M_N M_0\,\underline{v} = 0$ for any $\underline{v}$. The proof of $\underline{v}^{\mathrm{T}} M_0^{\mathrm{T}} M_N M_0\,\underline{v} = 0$ is straightforward but very lengthy, the details of which are given in the Appendix.

To prove $\underline{v}_2^{\mathrm{H}}\underline{v}_4 = 0$, we need to exchange A_2 with A_3 and A_6 with A_7 in the proof for $\underline{v}_2^{\mathrm{H}}\underline{v}_3 = 0$. With Property 6, (14) still holds after the double exchanges. So, the proof of $\underline{v}_2^{\mathrm{H}}\underline{v}_4 = 0$ is basically identical to the proof of $\underline{v}_2^{\mathrm{H}}\underline{v}_3 = 0$.

4.7. *Proof of Lemma 4.3*

Each vector in $[\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4]$ depends on two out of eight HR matrices satisfying (14). Exchanging any two vectors in $[\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4]$ is equivalent to exchanging two pairs of HR matrices. With Property 6, condition (14) continues to hold under any even number of exchanges of HR matrices. Therefore, following the same proof as for Lemma 4.3.1, if any three vectors in $[\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4]$ are linearly dependent of each other, they must be orthogonal to the fourth vector.

5. Further remarks on the non-orthogonal HR codes

5.1. *Full diversity non-orthogonal HR codes*

Like the quasi-orthogonal codes, the non-orthogonal HR codes can also be made full diversity by introducing proper diversity in symbol constellations. While the best method to achieve full diversity of the non-orthogonal HR codes is still an open problem, we give one method here to achieve full diversity. Consider the codeword as in (45). Let each complex element of $\underline{x}_1(i) + \mathrm{j}\underline{x}_2(i)$, $i = 1, 2, 3, 4$, be from the constellation set of $\exp[\mathrm{j}\pi(m/4 + 1/6)]$, $m = 1, 3, 5, 7$, and each complex element of $\underline{x}_1(i) + \mathrm{j}\underline{x}_2(i)$, $i = 5, 6, 7, 8$, be from another constellation set $\exp(\mathrm{j}\pi m/4)$, $m = 1, 3, 5, 7$. Through exhaustive search, it has been verified that the rank of the four column code matrix is always four.

5.2. *Simulation*

To illustrate the performance of the non-orthogonal HR codes, we show a simulation example. For a system of single receiver and four transmitters, each block of received data can be expressed as

$$\underline{y} = C(\underline{x})\,\underline{h} + \underline{n}, \tag{80}$$

where we assume

- The entries of the fading vector $\underline{h}$ are i.i.d. Gaussian distributed with unit variance.
- The entries of the noise vector $\underline{n}$ are i.i.d. Gaussian distributed of variance $\sigma_m^2 \times 4 \times 10^{-0.1\mathrm{SNR}}$ where SNR is the dB value of the ratio of the transmitted power over the noise variance. Here, σ_m^2 is dependent on modulations. For example, $\sigma_m^2 = 2$ for 4-QAM (four symbols on the corners of a square of side equal to 2), $\sigma_m^2 = 1$ for QPSK (four symbols uniformly spaced on a circle of unit radius), and $\sigma_m^2 = 10$ for 16-QAM. The factor 4 is due to four transmitters.
- The code matrix $C(\underline{x})$ is given by (45).

An alternative form of (80) is as follows:

$$\begin{pmatrix} \Re(\underline{y}) \\ \Im(\underline{y}) \end{pmatrix} = H \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} + \begin{pmatrix} \Re(\underline{n}) \\ \Im(\underline{n}) \end{pmatrix}, \tag{81}$$

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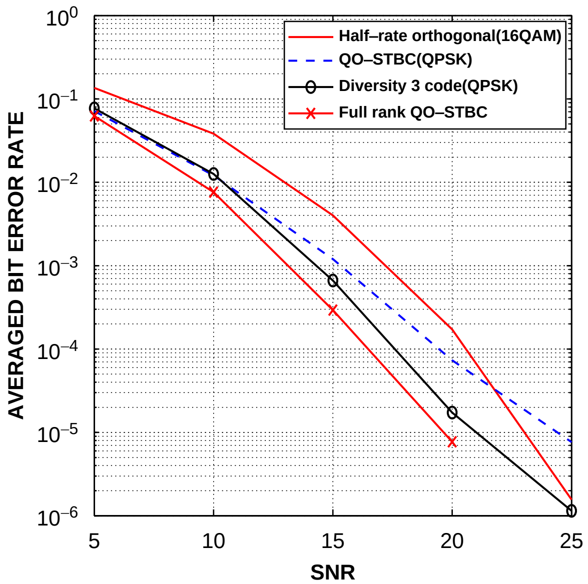

Fig. 2. Averaged bit error rate versus SNR for four space–time block codes. The data rate for all codes is 2 bits/s/Hz. A system of four transmitters and one receiver is considered.

where

$$H = \begin{pmatrix} \sum_{i=0}^{3} A_i \Re(h_i) & \sum_{i=0}^{3} A_{i+4} \Im(h_i) \\ \sum_{i=0}^{3} A_i \Im(h_i) & \sum_{i=0}^{3} A_{i+4} \Re(h_i) \end{pmatrix}. \tag{82}$$

We used (81) with a sphere decoding algorithm [35–37] to detect $\underline{x}$. For each realization of $\underline{x}$, we chose an independent realization of $\underline{h}$ and $\underline{n}$.

In our simulation, we compared the four different codes: the quasi-orthogonal code [11], and the full rank quasi-orthogonal code with the constellation rotation given in [14,16], the non-orthogonal code (44), and the half-rate complex orthogonal code [9]. To ensure the same bit rate, we used QPSK for the first two codes, 4-QAM for the third code, and 16-QAM for the fourth code.

The performances of the four different codes are compared in Fig. 2. The non-orthogonal HR code (referred to as diversity 3 code in the figure) shows a good performance in a medium range of SNR, i.e., better than the original quasi-orthogonal code and even the half-rate orthogonal code. The half-rate orthogonal code performs well at very high SNR because of its full diversity. But the full diversity quasi-orthogonal code performs the best among the four codes compared.

It is important to remember that the code (44) has diversity no less than three for any constellation while the full diversity quasi-orthogonal code needs to be readjusted for each different constellation.

6. Conclusion

In this paper, we have investigated STBC that have strong connections with the HR families of matrices. The key contributions are Theorems 3.1 and 4.1. Theorem 3.1 states that the Type I family of all published as well as unpublished 4×4 unit-rate quasi-orthogonal codes are simply variations from two independent codes shown in (16) via (15). Theorem 4.1 states that the unit-rate code (44) for four transmitters has a rank no less than three under any given constellation. To our knowledge, the code (44) is the only known unit-rate linear dispersion code that is guaranteed to have diversity (at least) three for four transmitters. This is a useful advantage since it could reduce the physical layer complexity associated with constellation constraint. It remains a challenge to discover whether or not there exists a linear dispersion code that guarantees a higher diversity than the code (44) for four transmitters over all possible constellations. It is our hope that the in-depth analysis shown in this paper will motivate and help this pursuit.

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Appendix A. Proof of $\underline{v}^{\mathrm{T}} M_0^{\mathrm{T}} M_N M_0 \underline{v} = 0$

The proof is relatively lengthy. We will repeatedly apply Properties 1–7 of the HR matrices as well as condition (14). To help the presentation of the proof, we will use the sum table to be introduced next.

A.1. Introduction of the sum table

For example, to describe the sum $\alpha_1 A_1^{\mathrm{T}} A_2 A_3^{\mathrm{T}} A_4 + \alpha_2 A_5^{\mathrm{T}} A_6 A_7^{\mathrm{T}} A_3$, we use the following sum table:

coefficients	products of matrices
α_1	[1234]
α_2	[5673]

(83)

where each group of four integers from $(0, 1, 2, 3, 4, 5, 6, 7)$ corresponds to a product of four matrices, i.e.,

$$\pm[ijmn] \doteq \pm A_i^{\mathrm{T}} A_j A_m^{\mathrm{T}} A_n. \quad (84)$$

To describe $\beta_1 A_2^{\mathrm{T}} A_4 (\alpha_1 A_1^{\mathrm{T}} A_2 A_3^{\mathrm{T}} A_4 + \alpha_2 A_5^{\mathrm{T}} A_6 A_7^{\mathrm{T}} A_3)$ as another example, we will use

coefficients	Products of matrices
$\beta_1 \alpha_1$	[24][1234]
$\beta_1 \alpha_2$	[24][5673]

(85)

From Properties 1(a) and 2(a), it is easy to verify that [241234] = [13]. Because of $A_1^{\mathrm{T}} A_6 A_7^{\mathrm{T}} A_0 = A_5^{\mathrm{T}} A_2 A_4^{\mathrm{T}} A_3 = A_2^{\mathrm{T}} A_4 A_5^{\mathrm{T}} A_3$, we can write [245673] = [245367] = [1670][67] = [01]. Therefore, we can also express

$$\beta_1 A_2^{\mathrm{T}} A_4 (\alpha_1 A_1^{\mathrm{T}} A_2 A_3^{\mathrm{T}} A_4 + \alpha_2 A_5^{\mathrm{T}} A_6 A_7^{\mathrm{T}} A_3)$$

as

coefficients	Products of matrices and simplification
$\beta_1 \alpha_1$	[24][1234] = [13]
$\beta_1 \alpha_2$	[24][5673] = [245367] = [1670][67] = [01]

(86)

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A.2. Main body of the proof

From the definition of M_0, it is straightforward to verify that

$$\begin{aligned} M_0 &= c_M I_8 - N \\ &= 2(k_2^2+t_2^2)I_8 - 2k_2(k_2^2+t_2^2)^{-1}A_2^{\mathrm{T}}[A_4A_6^{\mathrm{T}}k_2 + A_4A_7^{\mathrm{T}}t_2 + A_6A_7^{\mathrm{T}}(-k_1t_2+k_2t_1)](A_0-A_3t_1) \\ &\quad - 2(k_2^2+t_2^2)^{-1}t_2A_3^{\mathrm{T}}[A_4A_6^{\mathrm{T}}k_2 + A_4A_7^{\mathrm{T}}t_2 + A_6A_7^{\mathrm{T}}(-k_1t_2+k_2t_1)](A_0-k_1A_2). \end{aligned} \tag{87}$$

With $c_1 \doteq k_2^2+t_2^2$ and $c_2 \doteq k_1t_2+k_2t_1$, we can write

$M_0 =$ (88)

coefficients	products of matrices
$2c_1$	I_8
$-2k_2^2c_1^{-1}$	[2460]
$-2k_2t_2c_1^{-1}$	[2470]
$-2k_2c_2c_1^{-1}$	[2670]
$2k_2^2t_1c_1^{-1}$	[2463]
$2k_2t_2t_1c_1^{-1}$	[2473]
$2k_2t_1c_2c_1^{-1}$	[2673]
$-2k_2t_2c_1^{-1}$	[3460]
$-2t_2t_2c_1^{-1}$	[3470]
$-2c_2t_2c_1^{-1}$	[3670]
$2k_1k_2t_2c_1^{-1}$	$[3462] = -[2463]$
$2k_1t_2t_2c_1^{-1}$	$[3472] = -[2473]$
$2k_1c_2t_2c_1^{-1}$	$[3672] = -[2673]$

Therefore,

$M_0 =$ (89)

coefficients	products of matrices
$2c_1$	I_8
$-2k_2^2c_1^{-1}$	[2460]
$-2k_2t_2c_1^{-1}$	[2470]
$-2k_2c_2c_1^{-1}$	[2670]
$2k_2^2t_1c_1^{-1} - sk_1k_2t_2c_1^{-1} = 2c_2k_2c_1^{-1}$	[2463]
$2k_2t_2t_1c_1^{-1} - 2k_1t_2t_2c_1^{-1} = 2c_2t_2c_1^{-1}$	[2473]
$2k_2t_1c_2c_1^{-1} - 2k_1c_2t_2c_1^{-1} = 2c_2^2c_1^{-1}$	[2673]
$-2k_2t_2c_1^{-1}$	[3460]
$-2t_2t_2c_1^{-1}$	[3470]
$-2c_2t_2c_1^{-1}$	[3670]

From (79) we have

$$M_N = k_1 A_1^{\mathrm{T}} A_6 A_7^{\mathrm{T}} A_2 + t_1 A_1^{\mathrm{T}} A_6 A_7^{\mathrm{T}} A_3 + k_1 A_2^{\mathrm{T}} A_5 A_6^{\mathrm{T}} A_3 + t_1 A_2^{\mathrm{T}} A_5 A_7^{\mathrm{T}} A_3$$
$$= k_1[1672] + t_1[1673] + k_1[2563] + t_1[2573]. \quad (90)$$

To compute the product $M_0 M_N$ (where M_0 is symmetric), we have

$M_0 \times k_1[1672] =$

coefficients	products of matrices and simplification	group index in (95)
$2k_1c_1$	$I_8[1672] = [1672]$	1
$-2k_1k_2^2c_1^{-1}$	$[2460][1672] = [4017]$	2
$-2k_1k_2t_2c_1^{-1}$	$[2470][1672] = -[4016]$	3
$-2k_1k_2c_2c_1^{-1}$	$[2670][1672] = -[01]$	4
$2k_1c_2k_2c_1^{-1}$	$[2463][1672] = [4317] = [6025]$ because $[1670] = [5243] \Leftrightarrow [5620] = [1743]$	5
$2k_1c_2t_2c_1^{-1}$	$[2473][1672] = -[4316] = [7025]$ because $[1670] = [5243] \Leftrightarrow [5270] = [1643]$	6
$2k_1c_2^2c_1^{-1}$	$[2673][1672] = -[31]$	7
$-2k_1k_2t_2c_1^{-1}$	$[3460][1672] = -[3460][5043] = [65]$	8
$-2k_1t_2t_2c_1^{-1}$	$[3470][1672] = -[3470][5043] = -[57]$	9
$-2k_1c_2t_2c_1^{-1}$	$[3670][1672] = -[3012]$	10

(91)

$M_0 \times t_1[1673] =$

coefficients	products of matrices and simplification	group index in (95)
$2t_1c_1$	$I_8[1673] = [1673]$	11
$-2t_1k_2^2c_1^{-1}$	$[2460][1673] = -[2460][5240] = [56]$	8
$-2t_1k_2t_2c_1^{-1}$	$[2470][1673] = -[2470][5240] - [75]$	9
$-2t_1k_2c_2c_1^{-1}$	$[2670][1673] = -[2013] = [3012]$	10
$2t_1c_2k_2c_1^{-1}$	$[2463][1673] = [2417] = -[6053]$ because $[1670] = [5243] \Leftrightarrow [5630] = [1247]$	12
$2t_1c_2t_2c_1^{-1}$	$[2473][1673] = -[2416] = -[7053]$ because $[1670] = [5243] \Leftrightarrow [5703] = [1624]$	13
$2t_1c_2^2c_1^{-1}$	$[2673][1673] = [21]$	14
$-2t_1k_2t_2c_1^{-1}$	$[3460][1673] = [4017]$	2
$-2t_1t_2t_2c_1^{-1}$	$[3470][1673] = -[4016]$	3
$-2t_1c_2t_2c_1^{-1}$	$[3670][1673] = -[01]$	4

(92)

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$M_0 \times k_1[2563] =$

coefficients	products of matrices and simplification	group index in (95)
$2k_1c_1$	$I_8[2563] = [2563]$	2
$-2k_1k_2^2c_1^{-1}$	$[2460][2563] = -[4053]$	1
$-2k_1k_2t_2c_1^{-1}$	$[2470][2563] = [2470][1407] = -[12]$	14
$-2k_1k_2c_2c_1^{-1}$	$[2670][2563] = [7053]$	13
$2k_1c_2k_2c_1^{-1}$	$[2463][2563] = [45]$	15
$2k_1c_2t_2c_1^{-1}$	$[2473][2563] = -[4756] = -[3012]$ because $[1670] = [5243] \Leftrightarrow [1230] = [5647]$	10
$2k_1c_2^2c_1^{-1}$	$[2673][2563] = -[75]$	9
$-2k_1k_2t_2c_1^{-1}$	$[3460][2563] = -[4025]$	11
$-2k_1t_2t_2c_1^{-1}$	$[3470][2563] = [3470][1407] = [13]$	7
$-2k_1c_2t_2c_1^{-1}$	$[3670][2563] = [7025]$	6

(93)

$M_0 \times t_1[2573] =$ (94)

coefficients	products of matrices and simplification	group index in (95)
$2t_1c_1$	$I_8[2573] = [2573]$	3
$-2t_1k_2^2c_1^{-1}$	$[2460][2573] = -[2460][1406] = -[21]$	14
$-2t_1k_2t_2c_1^{-1}$	$[2470][2573] = -[4053]$	1
$-2t_1k_2c_2c_1^{-1}$	$[2670][2573] = -[6053]$	12
$2t_1c_2k_2c_1^{-1}$	$[2463][2573] = -[4657] = [3012]$	10
$2t_1c_2t_2c_1^{-1}$	$[2473][2573] = [45]$	15
$2t_1c_2^2c_1^{-1}$	$[2673][2573] = [65]$	8
$-2t_1k_2t_2c_1^{-1}$	$[3460][2573] = -[3460][1406] = [13]$	7
$-2t_1t_2t_2c_1^{-1}$	$[3470][2573] = -[4025]$	11
$-2t_1c_2t_2c_1^{-1}$	$[3670][2573] = -[6025]$	5

The following table shows how the common terms in the previous four tables are combined.

Group index	Final Sum
1	$2k_1c_1[1672]$
2	$2k_1c_1[2563]$
3	$2t_1c_1[2573]$
4	0
5	0
6	0
7	$-2k_1c_2^2c_1^{-1}[31] + (-2k_1t_2t_2c_1^{-1})(-[13]) + (-2t_1k_2t_2c_1^{-1})[13]$
8	$-2k_1k_2t_2c_1^{-1}[65] + -2t_1k_2^2c_1^{-1}[56] + 2t_1c_2^2c_1^{-1}[65]$
9	$2k_1t_2t_2c_1^{-1}[57] + 2t_1k_2t_2c_1^{-1}[75] + -2k_1c_2^2c_1^{-1}[75]$
10	0
11	$2t_1c_1[1673]$
12	0
13	0
14	$2t_1c_2^2c_1^{-1}[21] + 2k_1k_2t_2c_1^{-1}[12] + 2t_1k_2^2c_1^{-1}[21]$
15	0

(95)

Therefore,

$M_0M_N =$

coefficients	products of matrices
$2k_1c_1$	[1672]
$2k_1c_1$	[2563]
$2t_1c_1$	[2573]
$2t_1c_1$	[1673]
$2k_1c_2^2c_1^{-1} + 2k_1t_2^2c_1^{-1} - 2t_1k_2t_2c_1^{-1}$	[13]
$2t_1c_2^2c_1^{-1} + 2t_1k_2^2c_1^{-1} - 2k_1k_2t_2c_1^{-1}$	[65]
$2k_1c_2^2c_1^{-1} + 2k_1t_2^2c_1^{-1} - 2t_1k_2t_2c_1^{-1}$	[57]
$-2t_1c_2^2c_1^{-1} - 2t_1k_2^2c_1^{-1} + 2k_1k_2t_2c_1^{-1}$	[12]

(96)

We now need to multiply each term of M_0M_N, i.e., (96), by M_0 from right. Note that $\underline{v}^{\mathrm{T}}(A_k^{\mathrm{T}}A_iA_j^{\mathrm{T}}A_tA_q^{\mathrm{T}}A_r)\underline{v} = 0$ and $\underline{v}^{\mathrm{T}}(A_k^{\mathrm{T}}A_i)\underline{v} = 0$ for any vector $\underline{v}$ and distinct indices. So, we will use $[kijtqr]\sim 0$ and $[ki]\sim 0$ to denote that the corresponding terms become zero after being multiplied by $\underline{v}$ from left and right. We will use

$$E_1 \doteq 2k_1c_2^2c_1^{-1} + 2k_1t_2^2c_1^{-1} - 2t_1k_2t_2c_1^{-1}, \tag{97}$$

$$E_2 \doteq 2t_1c_2^2c_1^{-1} + 2t_1k_2^2c_1^{-1} - 2k_1k_2t_2c_1^{-1}, \tag{98}$$

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$2k_1c_1[1672] \times M_o =$ (99)

coefficients	products of matrices and simplification	group index
$2k_1c_12c_1$	$[1672]I_8 = [1672]$	1
$-2k_1c_12k_2^2c_1^{-1}$	$[1672][2460] = [1740]$	2
$-2k_1c_12k_2t_2c_1^{-1}$	$[1672][2470] = -[1640]$	3
$-2k_1c_12k_2c_2c_1^{-1}$	$[1672][2670] = -[10] \sim 0$	
$2k_1c_12c_2k_2c_1^{-1}$	$[1672][2463] = [1743]$	4
$2k_1c_12c_2t_2c_1^{-1}$	$[1672][2473] = -[1643]$	5
$2k_1c_12c_2^2c_1^{-1}$	$[1672][2673] = -[13] \sim 0$	
$-2k_1c_12k_2t_2c_1^{-1}$	$[1672][3460] = [172340] \sim 0$	
$-2k_1c_12t_2t_2c_1^{-1}$	$[1672][3470] = -[162340] \sim 0$	
$-2k_1c_12c_2t_2c_1^{-1}$	$[1672][3670] = -[1230]$	6

$2k_1c_1[2563] \times M_o =$ (100)

coefficients	products of matrices and simplification	group index
$2k_1c_12c_1$	$[2563]I_8 = [2563]$	2
$-2k_1c_12k_2^2c_1^{-1}$	$[2563][2460] = -[5340]$	1
$-2k_1c_12k_2t_2c_1^{-1}$	$[2563][2470] = -[563470] \sim 0$	
$-2k_1c_12k_2c_2c_1^{-1}$	$[2563][2670] = [5370]$	7
$2k_1c_12c_2k_2c_1^{-1}$	$[2563][2463] = [54] \sim 0$	
$2k_1c_12c_2t_2c_1^{-1}$	$[2563][2473] = -[5647]$	6
$2k_1c_12c_2^2c_1^{-1}$	$[2563][2673] = [57] \sim 0$	
$-2k_1c_12k_2t_2c_1^{-1}$	$[2563][3460] = -[2540]$	8
$-2k_1c_12t_2t_2c_1^{-1}$	$[2563][3470] = [256470] \sim 0$	
$-2k_1c_12c_2t_2c_1^{-1}$	$[2563][3670] = [2570]$	5

$2t_1c_1[2573] \times M_o =$ (101)

coefficients	products of matrices and simplification	group index
$2t_1c_12c_1$	$[2573]I_8 = [2573]$	3
$-2t_1c_12k_2^2c_1^{-1}$	$[2573][2460] = -[573460] \sim 0$	
$-2t_1c_12k_2t_2c_1^{-1}$	$[2573][2470] = -[5340]$	1
$-2t_1c_12k_2c_2c_1^{-1}$	$[2573][2670] = -[5360]$	9
$2t_1c_12c_2k_2c_1^{-1}$	$[2573][2463] = -[5746]$	6
$2t_1c_12c_2t_2c_1^{-1}$	$[2573][2473] = [54] \sim 0$	
$2t_1c_12c_2^2c_1^{-1}$	$[2573][2673] = [56] \sim 0$	
$-2t_1c_12k_2t_2c_1^{-1}$	$[2573][3460] = [257460] \sim 0$	
$-2t_1c_12t_2t_2c_1^{-1}$	$[2573][3470] = -[2540]$	8
$-2t_1c_12c_2t_2c_1^{-1}$	$[2573][3670] = -[2560]$	4

$2t_1c_1[1673] \times M_o =$ (102)

coefficients	products of matrices simplification	group index
$2t_1c_12c_1$	$[1673]I_8 = [1673]$	8
$-2t_1c_12k_2^2c_1^{-1}$	$[1673][2460] = [173240] \sim 0$	
$-2t_1c_12k_2t_2c_1^{-1}$	$[1673][2470] = -[163240] \sim 0$	
$-2t_1c_12k_2c_2c_1^{-1}$	$[1673][2670] = -[1320]$	6
$2t_1c_12c_2k_2c_1^{-1}$	$[1673][2463] = [1724]$	9
$2t_1c_12c_2t_2c_1^{-1}$	$[1673][2473] = -[1624]$	7
$2t_1c_12c_2^2c_1^{-1}$	$[1673][2673] = [12] \sim 0$	
$-2t_1c_12k_2t_2c_1^{-1}$	$[1673][3460] = [1740]$	2
$-2t_1c_12t_2t_2c_1^{-1}$	$[1673][3470] = -[1640]$	3
$-2t_1c_12c_2t_2c_1^{-1}$	$[1673][3670] = -[10] \sim 0$	

$E_1[13] \times M_o =$ (103)

coefficients	products of matrices and simplification	group index
$E_1 2c_1$	$[13]I_8 = [13] \sim 0$	
$-E_1 2k_2^2 c_1^{-1}$	$[13][2460] = [132460] \sim 0$	
$-E_1 2k_2 t_2 c_1^{-1}$	$[13][2470] = [132470] \sim 0$	
$-E_1 2k_2 c_2 c_1^{-1}$	$[13][2670] = [132670] \sim 0$	
$E_1 2c_2 k_2 c_1^{-1}$	$[13][2463] = -[1246]$	7
$E_1 2c_2 t_2 c_1^{-1}$	$[13][2473] = -[1247]$	9
$E_1 2c_2^2 c_1^{-1}$	$[13][2673] = -[1267]$	1
$-E_1 2k_2 t_2 c_1^{-1}$	$[13][3460] = [1460]$	3
$-E_1 2t_2 t_2 c_1^{-1}$	$[13][3470] = [1470]$	2
$-E_1 2c_2 t_2 c_1^{-1}$	$[13][3670] = [1670]$	10

$E_1[57] \times M_o =$ (104)

coefficients	products of matrices and simplification	group index
$E_1 2c_1$	$[57]I_8 = [57] \sim 0$	
$-E_1 2k_2^2 c_1^{-1}$	$[57][2460] = [572460] \sim 0$	
$-E_1 2k_2 t_2 c_1^{-1}$	$[57][2470] = [5240]$	8
$-E_1 2k_2 c_2 c_1^{-1}$	$[57][2670] = [5260]$	4
$E_1 2c_2 k_2 c_1^{-1}$	$[57][2463] = [572463] \sim 0$	
$E_1 2c_2 t_2 c_1^{-1}$	$[57][2473] = [5243]$	10
$E_1 2c_2^2 c_1^{-1}$	$[57][2673] = [5263]$	2
$-E_1 2k_2 t_2 c_1^{-1}$	$[57][3460] = [573460] \sim 0$	
$-E_1 2t_2 t_2 c_1^{-1}$	$[57][3470] = [5340]$	1
$-E_1 2c_2 t_2 c_1^{-1}$	$[57][3670] = [5360]$	9

$-E_2[12] \times M_o =$ (105)

coefficients	products of matrices and simplification	group index
$-E_2 2c_1$	$[12]I_8 = [12] \sim 0$	
$E_2 2k_2^2 c_1^{-1}$	$[12][2460] = [1460]$	3
$E_2 2k_2 t_2 c_1^{-1}$	$[12][2470] = [1470]$	2
$E_2 2k_2 c_2 c_1^{-1}$	$[12][2670] = [1670]$	10
$-E_2 2c_2 k_2 c_1^{-1}$	$[12][2463] = [1463]$	5
$-E_2 2c_2 t_2 c_1^{-1}$	$[12][2473] = [1473]$	4
$-E_2 2c_2^2 c_1^{-1}$	$[12][2673] = [1673]$	8
$E_2 2k_2 t_2 c_1^{-1}$	$[12][3460] = [123460] \sim 0$	
$E_2 2t_2 t_2 c_1^{-1}$	$[12][3470] = [123470] \sim 0$	
$E_2 2c_2 t_2 c_1^{-1}$	$[12][3670] = [123670] \sim 0$	

$E_2[65] \times M_o =$ (106)

coefficients	products of matrices	group index
$E_2 2c_1$	$[65]I_8 = [65] \sim 0$	
$-E_2 2k_2^2 c_1^{-1}$	$[65][2460] = -[5240]$	8
$-E_2 2k_2 t_2 c_1^{-1}$	$[65][2470] = [652470] \sim 0$	
$-E_2 2k_2 c_2 c_1^{-1}$	$[65][2670] = [5270]$	5
$E_2 2c_2 k_2 c_1^{-1}$	$[65][2463] = -[5243]$	10
$E_2 2c_2 t_2 c_1^{-1}$	$[65][2473] = [652473] \sim 0$	
$E_2 2c_2^2 c_1^{-1}$	$[65][2673] = [5273]$	3
$-E_2 2k_2 t_2 c_1^{-1}$	$[65][3460] = -[5340]$	1
$-E_2 2t_2 t_2 c_1^{-1}$	$[65][3470] = [653470] \sim 0$	
$-E_2 2c_2 t_2 c_1^{-1}$	$[65][3670] = [5370]$	7

There are 10 distinct groups of common terms in the previous eight tables. All the common terms cancel each other as shown by the following tables.

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The 1st group:

coefficients	product of matrices
$2k_1c_12c_1$	$[1672]$
$-2k_1c_12k_2^2c_1^{-1}$	$-[5340] = -[1672]$ because $[1670] = [5243] \Leftrightarrow [1672] = -[5043]$
$-2t_1c_12k_2t_2c_1^{-1}$	$-[5340]$
$E_12c_2^2c_1^{-1}$	$-[1267] = -[1672]$
$-E_12t_2t_2c_1^{-1}$	$[5340]$
$-E_22k_2t_2c_1^{-1}$	$-[5340]$

$= 0$ (107)

The 2nd group:

coefficients	product of matrices
$-2k_1c_12k_2^2c_1^{-1}$	$[1740]$
$2k_1c_12c_1$	$[2563] = -[1740]$ because $[1670] = [5243] \Leftrightarrow [1470] = -[5263]$
$-2t_1c_12k_2t_2c_1^{-1}$	$[1740]$
$-E_12t_2t_2c_1^{-1}$	$[1470] = -[1740]$
$E_12c_2^2c_1^{-1}$	$[5263] = [1740]$
$E_22k_2t_2c_1^{-1}$	$[1470]$

$= 0$ (108)

The 3rd group:

coefficients	product of matrices
$-2k_1c_12k_2t_2c_1^{-1}$	$-[1640]$
$2t_1c_12c_1$	$[2573] = [1640]$ because $[1670] = [5243] \Leftrightarrow [1640] = -[5273]$
$-2t_1c_12t_2t_2c_1^{-1}$	$-[1640]$
$-E_12k_2t_2c_1^{-1}$	$[1460] = -[1640]$
$E_22k_2^2c_1^{-1}$	$[1460]$
$E_22c_2^2c_1^{-1}$	$[5273] = -[1640]$

$= 0\,.$ (109)

The 4th group of terms in the sum is

coefficients	product of matrices
$2k_1c_12c_2k_2c_1^{-1}$	[1743]
$-2t_1c_12c_2t_2c_1^{-1}$	$-[2560] = -[1743]$ because $[1670] = [5243] \Leftrightarrow [1473] = [5260]$
$-E_12k_2c_2c_1^{-1}$	$[5260] = -[1743]$
$-E_22c_2t_2c_1^{-1}$	$[1473] = -[1743]$

$= 0\,.$ (110)

The 5th group:

coefficients	product of matrices
$2k_1c_12c_2t_2c_1^{-1}$	$-[1643]$
$-2k_1c_12c_2t_2c_1^{-1}$	$[2570] = -[1643]$ because $[1670] = [5243] \Leftrightarrow [1643] = [5270]$
$-E_22c_2k_2c_1^{-1}$	$[1463] = -[1643]$
$-E_22k_2c_2c_1^{-1}$	$[5270] = [1643]$

$= 0\,.$ (111)

The 6th group:

coefficients	product of matrices
$-2k_1c_12c_2t_2c_1^{-1}$	$-[1230]$
$2k_1c_12c_2t_2c_1^{-1}$	$-[5647] = -[1230]$ because $[1670] = [5243] \Leftrightarrow [5674] = [1203]$
$2t_1c_12c_2k_2c_1^{-1}$	$-[5746] = [1230]$
$-2t_1c_12k_2c_2c_1^{-1}$	$-[1320] = [1230]$

$= 0\,.$ (112)

The 7th group:

coefficients	product of matrices
$-2k_1c_12k_2c_2c_1^{-1}$	[5370]
$2t_1c_12c_2t_2c_1^{-1}$	$-[1624] = -[5370]$ because $[1670] = [5243] \Leftrightarrow [5370] = [1246]$
$E_12c_2k_2c_1^{-1}$	$-[1246] = -[5370]$
$-E_22c_2t_2c_1^{-1}$	[5370]

$= 0\,.$ (113)

The 8th group:

coefficients	product of matrices
$-2k_1c_12k_2t_2c_1^{-1}$	$-[2540]$
$-2t_1c_12t_2t_2c_1^{-1}$	$-[2540]$
$2t_1c_12c_1$	$[1673]=[2540]$ because $[1670]=[5243] \Leftrightarrow [1673]=-[5240]$
$-E_12k_2t_2c_1^{-1}$	$[5240]=-[2540]$
$-E_22c_2^2c_1^{-1}$	$[1673]$
$-E_22k_2^2c_1^{-1}$	$-[5240]$

$= 0$ (114)

The 9th group:

coefficients	product of matrices
$-2t_1c_12k_2c_2c_1^{-1}$	$-[5360]$
$2t_1c_12c_2k_2c_1^{-1}$	$[1724]=-[5360]$ because $[1670]=[5243] \Leftrightarrow [1274]=[5603]$
$E_12c_2t_2c_1^{-1}$	$-[1247]=[5360]$
$-E_12c_2t_2c_1^{-1}$	$[5360]$

$= 0$ (115)

The 10th group:

coefficients	product of matrices
$-E_12c_2t_2c_1^{-1}$	$[1670]$
$E_12c_2t_2c_1^{-1}$	$[5243]=[1670]$
$E_22k_2c_2c_1^{-1}$	$[1670]$
$E_22c_2k_2c_1^{-1}$	$-[5243]=-[1670]$

$= 0$ (116)

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