Advances in Coastal Modeling
edited by C. Lakhan

Chapter 2

Frequency Domain Wave Models in the Nearshore and Surf Zones

James M. Kaihatu

Ocean Dynamics and Prediction Branch, Oceanography Division (Code 7322)
Naval Research Laboratory, Stennis Space Center, MS 39529-5004

1. INTRODUCTION

In deep water ($kh >> 1$, where k is the wave number and h the water depth), second-order wave nonlinearity can be described as a small correction to the underlying linear wave. Perturbation expansions in wave steepness $\epsilon = ka$, where a is the wave amplitude, are used (Phillips, 1960), and at second-order only non-resonant (bound) waves are possible among triads of wave frequencies. Thus the interacting waves with the frequency-vector wave number combination $(\omega_1, \mathbf{k}_1)$ and $(\omega_2, \mathbf{k}_2)$ excite secondary waves at $(\omega_1 + \omega_2, \mathbf{k}_1 + \mathbf{k}_2)$, but these secondary wave amplitudes always remain small relative to the primary amplitudes. At the next order, resonant interaction occurs between quartets of waves, with the resultant slow energy exchange between the interacting waves.

In shallow water ($kh << 1$) waves become less dispersive and more collinear, and triads of waves at second-order begin to more closely satisfy the resonant conditions for wave interaction. The perturbation solutions of finite depth do not apply in the nearshore, since significant energy transfer occurs over much shorter distances ($O(10)$ wavelengths) than in deep water. The Ursell number $U_r = a/k^2h^3$ (Ursell, 1953) is the typical measure for the validity of these perturbation solutions, which are only applicable if $U_r << 1$. Though the resonant conditions between triads are only exactly satisfied in the collinear, non-dispersive limit, the nonlinearity inherent in shoaling waves in the nearshore is strong enough for significant energy transfer to occur at near-resonance (Bryant, 1973). Recourse is often made to the Boussinesq equations (Peregrine, 1967) for simulation of nonlinear energy transfer in shallow water, as they are valid for $U_r = O(1)$, where weak nonlinearity and weak dispersion are balanced.

1.1. The Frequency Domain

One model framework which has been used in simulating ocean wave propagation in the nearshore has involved the application of Fourier transforms to the dynamical equations governing the propagation. This transformation involves imposing the following constraint on the dependent variable of these equations (usually the free surface η)

$$\eta(x, y, t) = \sum_{n=1}^{N} \hat{\eta}_n(x, y) e^{-i\omega_n t} + c.c. \tag{1}$$

where ω_n is the nth frequency in the spectrum, N is the total number of frequency components in the spectrum, $\hat{\eta}_n$ is a complex Fourier amplitude and $c.c.$ denotes complex conjugate. Assumption of temporal periodicity is a natural application to ocean waves.

The frequency domain format allows explicit detail of nonlinear wave-wave interaction and wave transformation properties. Nonlinearities in the equations appear as products of amplitudes at discrete frequencies in the spectrum, which can then be investigated in detail. Since the resulting equations are

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE	2. REPORT TYPE	3. DATES COVERED
2003		**00-00-2003 to 00-00-2003**

4. TITLE AND SUBTITLE	5a. CONTRACT NUMBER
Frequency Domain Wave Models in the Nearshore and Surf Zones	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Naval Research Laboratory,Oceanographic Division,Stennis Space Center,MS,39529-5004**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			
unclassified	**unclassified**	**unclassified**	**Same as Report (SAR)**	**30**	

in terms of evolving amplitudes rather than the free surface, spatial resolution requirements are usually less restrictive than in time domain models. Overall computational time, however, is a function of the number of frequency components kept in the simulation, whereas (outside of ensuring sufficient resolution for the shortest waves) this is not germane for time domain models. Additionally, there is often a disconnect between properties of a time domain model and those of the corresponding frequency domain models. A good example is seen in Rygg (1988), who used a time domain model of the classical (shallow water) Boussinesq equations of Peregrine (1967) to simulate intermediate depth cases of laboratory wave propagation experiments successfully. Similar experiments with corresponding frequency domain models (Liu et al., 1985, as used by Kaihatu and Kirby, 1995) have proven less favorable.

2. CLASSICAL BOUSSINESQ MODELS IN THE FREQUENCY DOMAIN

The Boussinesq equations can be derived from either the Euler equations (Peregrine, 1967) or the boundary value problem for water waves (Mei, 1983). In shallow water, it is reasonable to assume that vertical velocities in the water column are much smaller than horizontal velocities. This imposes the following scales on the independent variables

$$(x', y') = \frac{(x, y)}{L}; \quad z' = \frac{z}{h_o}; \quad t' = \frac{\sqrt{gh_o}}{L}t \tag{2}$$

where L is a characteristic wavelength, h_o a characteristic water depth, and the primes denote dimensionless variables. These scales are then applied to the physical quantities

$$(u', v') = \frac{h_o}{a\sqrt{gh_o}}(u, v); \quad w' = \frac{h_o^2}{aL\sqrt{gh_o}}w \tag{3}$$

$$\eta' = \frac{\eta}{a}; \quad h' = \frac{h}{h_o}; \quad p' = \frac{p}{\rho g a} \tag{4}$$

where a is a characteristic amplitude, (u, v, w) are the water particle velocity components, p is the pressure and ρ is the fluid density. When substituted into the Euler equations, the following dimensionless parameters become evident

$$\mu^2 = (kh_o)^2; \quad \delta = \frac{a}{h_o} \tag{5}$$

which are measures of frequency dispersion and nonlinearity, respectively. The Boussinesq equations can be derived by assuming

$$O(\mu^2) \approx O(\delta) \ll O(1) \tag{6}$$

Using the scaled Euler equations, Peregrine (1967) derived the Boussinesq equations for a varying bathymetry

$$\eta_t + \nabla \cdot (h + \eta)\overline{\mathbf{u}} = O(\mu^4, \delta\mu^2, \delta^2) \tag{7}$$

$$\overline{\mathbf{u}}_t + \overline{\mathbf{u}} \cdot \nabla\overline{\mathbf{u}} + g\nabla\eta = \frac{h}{2}\nabla\left[\nabla \cdot (h\overline{\mathbf{u}}_t)\right] - \frac{h^2}{6}\nabla\left[\nabla \cdot \overline{\mathbf{u}}\right] + O(\mu^4, \delta\mu^2, \delta^2) \tag{8}$$

where $\overline{\mathbf{u}}$ is the depth-averaged velocity vector. The quadratic nonlinear terms in the equation above represent the lowest-order nonlinearity of $O(\delta)$. Application of Fourier series to these terms requires

special treatment (see Mei, 1983), and thus gives rise to the triadic cross-spectral energy transfer which is the manifestation of nonlinearity in the frequency domain. Freilich and Guza (1984) derived frequency domain models from the one-dimensional form of these equations. The first (the "consistent" model) can be written

$$A_{nx} + \frac{h_x}{4h}A_n - \frac{in^3k^3h^2}{6}A_n + \frac{3ink}{8h}\left(\sum_{l=1}^{n-1} A_l A_{n-l} + 2\sum_{l=1}^{N-n} A_l^* A_{n+l}\right) \tag{9}$$

where A_n are complex amplitudes of the free surface and asterisks denote complex conjugate. Freilich and Guza (1984) solved the equation in terms of coupled amplitude and phase equations rather than the complex amplitudes seen in equation (9). The second model (the "dispersive" model) was also derived from the standard Boussinesq equations, but does not contain the phase-shifting dispersive term (third term in equation (9)). Instead, the weak dispersion is incorporated through the use of the dispersion relation for the Boussinesq equations

$$\omega^2 = \frac{ghk^2}{1 + \frac{1}{3}(kh)^2} \tag{10}$$

One consequence of the use of this dispersion relation is that the wave number k_n is no longer a linear function of ω. Thus, the interacting amplitudes $(A_n, A_{n\pm l}, A_{\mp l})$, while resonant in frequency, are in *near* resonance in wave number. Freilich and Guza (1984) then compared both models to field data, using offshore wave spectra to initialize the model and ably demonstrating the utility of frequency domain models to nearshore wave propagation problems. Their comparisons of wave spectra showed that the dispersive shoaling model performed slightly better than the consistent model; however, both models clearly deviated from the data in the higher frequency range, where kh was no longer small.

Two-dimensional frequency domain models of both the Boussinesq equations (7) and (8) and the Kadomtsev-Petviashvili (KP) equations (Kadomtsev and Petviashvili, 1970) were developed by Liu et al. (1985) in the form of parabolic models, which are formulated based on the assumption that the angle between the wave direction and the x-axis of the grid is small. Kirby (1990) developed angular spectrum models based on the Boussinesq equations of Peregrine (1967). Periodicity in both time and longshore direction was assumed, thus imposing resonant interaction among longshore wave number modes as well as frequency modes.

3. EXTENDED BOUSSINESQ MODELS IN THE FREQUENCY DOMAIN

One fundamental problem with frequency domain models of the classical Boussinesq equations is their lack of applicability in deeper water than that for which the shallow water theory is valid. Recent efforts, beginning with Witting (1984), have focused on improving the deep water behavior of Boussinesq models such that their linear properties (dispersion, shoaling, etc.) better mimic those of fully-dispersive linear theory for a wide range of water depths. McCowan and Blackman (1989), Madsen et al. (1991) and Nwogu (1993) represent some of the first attempts to improve time domain Boussinesq models in this regard; the resulting models were generally dubbed "extended" Boussinesq models because their linear properties were extendable to intermediate and deep water. Madsen et al. (1991) added terms to the classical Boussinesq momentum equation, multiplied by a free parameter, which would be zero in shallow water but have significant effect in deeper water. This free parameter was tuned via Padé approximations so that the dispersion relation of the equations would compare favorably to that of linear theory for a wide range of depths. Madsen and Sørensen (1992) extended the Madsen et al. (1991) model to include varying bathymetry. Madsen and Sørensen (1993)

investigated frequency domain formulations of the model of Madsen et al. (1991) for wave evolution over a flat bottom, and sloping-bottom extensions of this equation became the basis for further development in the frequency domain (Eldeberky and Battjes, 1996; Kofoed-Hansen and Rasmussen, 1998; Becq-Girard et al., 1999). The equations of Madsen et al. (1991), and their various nonlinear and dispersive enhancements, have been analyzed extensively by Schäffer and Madsen (1995), Madsen and Schäffer (1998) and Madsen and Schäffer (1999).

In contrast, but to the same end, Nwogu (1993) used the velocity variable at an arbitrary location in the water column (rather than the depth-averaged velocity as in the classical Boussinesq equations) as a basis for deriving extended Boussinesq equations from the inviscid Euler equations. The resulting equations contained higher-order terms in both the continuity and momentum equations, and are

$$\eta_t + \nabla \cdot [(h+\eta)\mathbf{u}_\alpha] + \nabla \cdot \left\{ \left(\frac{z_\alpha^2}{2} - \frac{h^2}{6} \right) h \nabla (\nabla \cdot \mathbf{u}_\alpha + \left(z_\alpha + \frac{h}{2} \right) h \nabla [\nabla \cdot (h\mathbf{u}_\alpha)] \right\}$$

$$= O(\mu^4, \delta\mu^2, \delta^2) \tag{11}$$

$$\mathbf{u}_{\alpha t} + g\nabla\eta + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + z_\alpha \left\{ \frac{z_\alpha}{2} \nabla(\nabla \cdot \mathbf{u}_{\alpha t}) + \nabla[\nabla \cdot (h\mathbf{u}_{\alpha t})] \right\} = O(\mu^4, \delta\mu^2, \delta^2) \tag{12}$$

where $\mathbf{u}_\alpha$ is the horizontal velocity at a location z_α in the water column. The dispersion relation of this set of equations is found by isolating the linear terms and substituting in a periodic, progressive wave, leading to

$$C^2 = \frac{\omega^2}{k^2} = gh \left[\frac{1 - \left(\alpha + \frac{1}{3} \right) (kh)^2}{1 - \alpha (kh)^2} \right] \tag{13}$$

where α is a free parameter related to z_α by

$$\alpha = \left(\frac{z_\alpha^2}{2h^2} + \frac{z_\alpha}{h} \right) \tag{14}$$

This free parameter α is then best-fit to that of fully-dispersive linear theory for a wide range of water depths. Nwogu (1993) determined that $\alpha = -0.390$ was the best-fit parameter value for the range $0 \leq h/L_o \leq 0.5$, where L_o is the deep water wavelength. This value of α corresponds to $z_\alpha = -0.522h$.

3.1. Frequency Domain Transformation of the Equations of Nwogu (1993): Linear Properties

Usually the first step undertaken in a frequency domain transformation is to combine the continuity and momentum equations into one via the use of first-order substitutions. Noting the extra dispersive terms in both the continuity equation (11) and momentum equation (12), Chen and Liu (1995) commented on the difficulty in determining a frequency domain form of the equations such that the linear dispersion relation (see equation (13)) would remain applicable to the resulting equation. Later, Kaihatu and Kirby (1998) determined a series of first-order substitutions which would lead to a set of equations retaining the original dispersion relation.

To illustrate the difficulty, we reduce equations (11) and (12) to their linear, one-dimensional form for a flat bottom

$$\eta_t + hu_{\alpha x} + \left(\alpha + \frac{1}{3} \right) h^3 u_{\alpha xxx} = 0 \tag{15}$$

$$u_{\alpha t} + g\eta_x + \alpha h^2 u_{xxt} = 0 \tag{16}$$

We follow the procedure of Liu et al. (1985) to formulate the frequency domain model. The first step involves combining the continuity (15) and momentum (16) equations. We make use of the following first-order relations

$$\eta_t = -hu_{\alpha x} \tag{17}$$

$$u_{\alpha t} = -g\eta_x \tag{18}$$

We then take the time derivative of equation (15), the x-derivative of equation (16), and combine the resulting equations. We then use equation (18) to eliminate u_α in favor of η. This results in

$$\eta_{tt} - gh\eta_{xx} + gh^3\eta_{xxxx} - g\left(\alpha + \frac{1}{3}\right)h^3\eta_{xxxx} = 0 \tag{19}$$

To obtain the linear dispersion relation, we substitute

$$\eta = Ae^{i(kx-\omega t)} \tag{20}$$

into equation (19) and obtain

$$\omega^2 = ghk^2\left[1 - \frac{1}{3}(kh)^2\right] \tag{21}$$

which is essentially the linear dispersion relation to weakly-dispersive Boussinesq theory to within a binomial expansion. The substitution sequence used to collapse the two equations did not retain the dispersion relation of the original equation. Schäffer and Madsen (1995) addressed this issue by applying differential operators of $O(\mu^2)$ (multiplied by free parameters) to the equations of Nwogu (1993) and then used a Padé [4, 4] expansion to determine the set of parameters which best fit the linear dispersion and shoaling characteristics from linear theory, with the results of Nwogu (1993) representing a subset of the parameters. In contrast, Kaihatu and Kirby (1998) used a different series of substitutions to retain the dispersion properties of the original equation; this is examined here. If we had taken the time derivative of equation (15), and then used equation (18) to replace u_α with η, we would have obtained

$$\eta_{tt} + hu_{\alpha xt} - g\left(\alpha + \frac{1}{3}\right)h^3\eta_{xxxx} = 0 \tag{22}$$

We then multiply equation (16) by h and substitute the time derivative of equation (17) to eliminate u_α. Substituting the result into equation (22) results in

$$\eta_{tt} - gh\eta_{xx} - \alpha h^2\eta_{xxtt} - gh^3\left(\alpha + \frac{1}{3}\right)\eta_{xxxx} = 0 \tag{23}$$

It can be shown that the linear dispersion relation of equation (23) is equation (13), the original dispersion relation of Nwogu (1993).

The complicated substitution sequence required to retain the linear dispersion relation also affects the shoaling behavior of the frequency domain model. To examine this, we return to the derivation of equation (23), but retain bottom slope terms. Performing the same series of substitutions and neglecting h_{xx} and $(h_x)^2$ terms leads to

$$\eta_{tt} - g(h\eta_x)_x + 2\alpha hh_x\eta_{xtt} + \alpha h^2\eta_{xxtt} - gh^2(5\alpha + 2)h_x\eta_{xxx} - gh^3\left(\alpha + \frac{1}{3}\right)\eta_{xxxx} = 0 \tag{24}$$

Substituting equation (20) into equation (24) leads to

$$A_x + WA = 0 \tag{25}$$

where

$$W = \frac{Ek_x + Fh_x}{G} \tag{26}$$

$$E = gh + \omega^2 h^2 \alpha - 6gh^3 \left(\alpha + \frac{1}{3}\right) k^2 \tag{27}$$

$$F = gk + 2\alpha\omega^2 kh - gh^2(5\alpha + 2)k^3 \tag{28}$$

$$G = 2\left[gkh + \omega^2 h^2 \alpha k - 2gk^3 h^3 \left(\alpha + \frac{1}{3}\right)\right] \tag{29}$$

Though the derivation appears to be fairly straightforward, it will be shown that the linear shoaling term (equation (26)) compares very poorly to that of linear theory. Further analysis reveals that the balance between the η_{ttx} and $g(h\eta_{xx})_x$ terms governs the effectiveness of the wave shoaling relation. Kaihatu and Kirby (1998) addressed this by adding the following term to the equation

$$\beta(\eta_{tt} - gh\eta_{xx})_x = 0 \tag{30}$$

which is true at lowest order. In this equation β is a free parameter to be optimized. This changes equation (24) to

$$\eta_{tt} - g(h\eta_x)_x + (2\alpha + \beta)hh_x\eta_{xtt} + \alpha h^2\eta_{xxtt} - gh^2(5\alpha + 2 + \beta)h_x\eta_{xxx}$$
$$- gh^3\left(\alpha + \frac{1}{3}\right)\eta_{xxxx} = 0 \tag{31}$$

Carrying the calculation forward to the point of obtaining a shoaling relation results in a slight modification to the expression F in equation (28)

$$F = gk + (2\alpha + \beta)\omega^2 kh - gh^2(5\alpha + 2 + \beta)k^3 \tag{32}$$

Kaihatu and Kirby (1998) determined the free parameters α (for dispersion) and β (for shoaling) using a least squares optimization integrated as a function of h/L_o. Two sets of parameters were found. The first set optimized the shoaling while using the optimum α determined by Chen and Liu (1995). This set ($\alpha = -0.3855, \beta = -0.3540$) was known as the "dispersion optimized" (DO) set. The second parameter set was determined by finding the values of α and β which minimized the global error in (α, β) parameter space. This set ($\alpha = -0.4111, \beta = -0.3188$) was denoted the "dispersion and shoaling optimized" (DSO) parameter set. It is noted that the DSO parameter value of $\alpha = -0.4111$ is very close to Witting's (1984) optimum dispersion parameter value (found via Padé approximants) of $\alpha = -2/5$.

3.2. Frequency Domain Transformation of the Equations of Nwogu (1993): Nonlinear Parabolic Model

If nonlinearity and two-dimensionality had been retained when deriving equation (31), the result would have been (Kaihatu and Kirby, 1998)

$$\eta_{tt} - g\nabla\cdot(h\nabla\eta) - h\nabla\cdot[(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}_\alpha] + (2\alpha+\beta)h\nabla h\cdot\nabla\eta_{tt} + \alpha h^2\nabla^2\eta_{tt} + \nabla\cdot(\eta\mathbf{u}_\alpha)_t$$
$$- gh^2(5\alpha+2+\beta)\nabla h\cdot\nabla(\nabla^2\eta) - gh^3\left(\alpha+\frac{1}{3}\right)\nabla^2\nabla^2\eta = 0 \tag{33}$$

Complete elimination of the velocity $\mathbf{u}_\alpha$ requires assumption of time periodicity for both η and $\mathbf{u}_\alpha$. To eliminate $\mathbf{u}_\alpha$ from the nonlinear terms, we can use the time-periodic form of equation (18)

$$\hat{\mathbf{u}}_{\alpha n} = \frac{ig}{n\omega}\nabla\hat{\eta}_n \tag{34}$$

leading to the time-periodic equation for $\hat{\eta}_n$. To facilitate convenient numerical treatment, we make use of the parabolic approximation, first developed by Radder (1979) and Lozano and Liu (1980) for the linear mild-slope equation. We first make an explicit assumption that the waves are propagating forward

$$\hat{\eta}_n = A_n(x,y)e^{i\int k_n(x,y)dx} \tag{35}$$

The complex amplitudes $A_n(x,y)$ represent phase-like behavior in both x and y. The phase function is integrated only in x; this places all phase-like behavior in y in the complex amplitude A_n, while allowing explicit slow and fast variations in x. The consequence is that the angle between the incident wave and the x direction of the grid remains small in order to maintain slowly-varying wave-like behavior of A_n in the y direction.

Because of the third and fourth derivatives of η present in equation (33), terms proportional to $k_{nx}A_{nx}$, A_{nxx}, and other higher-order derivatives will be generated. We thus keep a higher degree of modulation in the y direction than in the x direction (following the ordering of Liu et al., 1985), leading to a parabolic evolution equation for A_n. However, because the phase function in equation (35) is integrated only in x, but k is a function of both x and y, a redefinition is required

$$A_n = a_n e^{i\int \bar{k}_n(x) - k_n(x,y)dx} \tag{36}$$

where $\bar{k}_n$ is a reference wave number that is the result of averaging along the y direction. With this redefinition, the model equation reads

$$2i\left[ghk_n + n^2\omega^2\alpha k_n h^2 - 2gh^3k_n^3\left(\alpha+\frac{1}{3}\right)\right]a_{nx}$$
$$- 2\left[ghk_n + n^2\omega^2\alpha k_n h^2 - 2gh^3k_n^3\left(\alpha+\frac{1}{3}\right)\right](\bar{k}_n - k_n)a_n$$
$$+ i[gk_n + n^2\omega^2(2\alpha+\beta)k_n h - gh^3k_n^3(5\alpha+2+\beta)]h_x a_n$$
$$+ i\left[gh + n^2\omega^2\alpha h^2 - 6gh^3k_n^2\left(\alpha+\frac{1}{3}\right)\right]k_{nx}a_n$$
$$+ \left[g + n^2\omega^2(2\alpha+\beta)h - gh^2k_n^2(5\alpha+2+\beta)\right]h_y a_{ny}$$
$$+ \left[gh + n^2\omega^2\alpha h^2 - 2gh^3k_n^2\left(\alpha+\frac{1}{3}\right)\right]a_{nyy}$$

$$-\frac{g}{4}\left(\sum_{l=1}^{n-1} Ra_l a_{n-l} e^{i\int(\bar{k}_l+\bar{k}_{n-l}-\bar{k}_n)dx} + 2\sum_{l=1}^{N-n} Sa_l^* a_{n+l} e^{i\int(\bar{k}_{n+l}-\bar{k}_l-\bar{k}_n)dx}\right) = 0 \tag{37}$$

where

$$R = \frac{gh}{l(n-l)\omega^2}[k_l k_{n-l}(k_l + k_{n-l})^2] + \frac{n^2 k_l k_{n-l}}{l(n-l)} + \frac{n^2\omega^2}{gh} \tag{38}$$

$$S = \frac{gh}{l(n+l)\omega^2}[k_l k_{n+l}(k_l - k_{n+l})^2] + \frac{n^2 k_l k_{n+l}}{l(n+l)} + \frac{n^2\omega^2}{gh} \tag{39}$$

This is the primary result from Kaihatu and Kirby (1998).

Kaihatu (1994) noted that the ambiguity which affected the substitution process in the linear terms also appears in the formulation of the nonlinear terms, to the effect that several different nonlinear coefficient sets are possible. Unlike the linear terms, however, there are no corresponding analytical metrics for determining which set of coefficients have the most desirable properties, outside of comparisons to nonlinear permanent form solutions. The general complexity of the Boussinesq equations of Nwogu (1993), particularly in retaining shoaling and dispersive properties during transformation into the frequency domain, has motivated investigations into developing simpler forms of the equations. This is explored in the next section.

3.3. Frequency Domain Transformation of the Equations of Chen and Liu (1995)

Chen and Liu (1995) and Kaihatu and Kirby (1994) investigated using the extended Boussinesq equations of Nwogu (1993) in the form of velocity potential (rather than velocity) and free surface elevation. The resulting equations would be an analogue to the Boussinesq equations of Wu (1981), which were a (ϕ, η) form of the classical Boussinesq equations of Peregrine (1967). The use of velocity potential as a dependent variable simplifies the treatment of the extended Boussinesq equations considerably, since ϕ is a scalar quantity.

Chen and Liu (1995) began with the boundary value problem for water waves, with surface boundary conditions scaled for shallow water waves and expanded to $O(\delta, \mu^2)$. The derivation is similar to that seen in Mei (1983), except that the velocity potential is taken at an unknown level in the water column rather than averaged over depth. Kaihatu and Kirby (1994) expanded the equations to $O(\delta, \mu^2, \delta\mu^2)$, thereby including dispersive effects in the nonlinear terms. The resulting equations are

$$\eta_t + \nabla\cdot\left[(h+\eta)\nabla\phi_\alpha\right] + \nabla\cdot\left[h\nabla\left\{z_\alpha\nabla\cdot(h\nabla\phi_\alpha) + \frac{z_\alpha^2}{2}\nabla^2\phi_\alpha\right\}\right.$$
$$\left. + \frac{h^2}{2}\nabla[\nabla\cdot(h\nabla\phi_\alpha)] - \frac{h^3}{6}\nabla\nabla^2\phi_\alpha\right] + \Upsilon = 0 \tag{40}$$

$$\phi_{\alpha t} + g\eta + \frac{1}{2}(\nabla\phi_\alpha)^2 + \left[z_\alpha\nabla\cdot(h\nabla\phi_{\alpha t}) + \frac{z_\alpha^2}{2}\nabla^2\phi_{\alpha t}\right] + \Lambda = 0 \tag{41}$$

where ϕ_α is the velocity potential at z_α. The terms Υ and Λ represent terms of $O(\delta\mu^2)$, which are zero in Chen and Liu (1995). In Kaihatu and Kirby (1994) they are

$$\Upsilon = \nabla\cdot(\eta\nabla\phi_\alpha) + \nabla\cdot(\eta\nabla[z_\alpha\nabla\cdot(h\nabla\phi_\alpha)])$$

$$+\eta\nabla\left(\frac{z_\alpha^2}{2}\nabla^2\phi_\alpha\right) \tag{42}$$

$$\Lambda = -\eta\nabla\cdot(h\nabla\phi_{\alpha t}) + \nabla\phi_\alpha\cdot[\nabla z_\alpha\nabla\cdot(h\nabla\phi_\alpha)] + z_\alpha(\nabla\phi_\alpha\cdot\nabla)(\nabla\cdot(h\nabla\phi_\alpha))$$
$$+\nabla\phi_\alpha\cdot\left(z_\alpha\nabla z_\alpha\nabla^2\phi_\alpha\right) + \frac{1}{2}\nabla\phi_\alpha z_\alpha^2\cdot\nabla(\nabla^2\phi_\alpha) + \frac{1}{2}\nabla\cdot(h\nabla\phi_\alpha) \tag{43}$$

Temporal periodicity was assumed for the velocity potential ϕ_α and the following equation developed for the amplitudes of the velocity potential b_n (Chen and Liu, 1995, although expressed somewhat differently)

$$\left[h+\frac{\omega_n^2h^2\alpha}{g}-2\left(\alpha+\frac{1}{3}\right)h^3k_n^2\right]b_{nyy}+2i\left[k_n\left(h+\frac{\omega_n^2h^2\alpha}{g}\right)-2k_n^3h^3\left(\alpha+\frac{1}{3}\right)\right]b_{nx}$$
$$+\left[1+\frac{\omega_n^2z_\alpha}{g}-(1+5\alpha+\sqrt{1+2\alpha})k_n^2h^2\right]h_yb_{ny}$$
$$+i\left\{k_n\left(1+\frac{\omega^2z_\alpha}{g}\right)-[k_n^3h^2(1+5\alpha+\sqrt{1+2\alpha})]\right\}h_xb_n$$
$$-2\left[k_n\left(h+\frac{\omega_n^2h^2\alpha}{g}\right)-2k_n^3h^3\left(\alpha+\frac{1}{3}\right)\right](\bar{k}_n-k_n)b_n$$
$$+i\left[h+\frac{\omega_n^2h^2\alpha}{g}-6\left(\alpha+\frac{1}{3}\right)h^3k_n^2\right]k_{nx}b_n$$
$$-\frac{i\omega}{4g}\left(\sum_{l=1}^{n-1}\tilde{R}b_lb_{n-l}e^{i\int(\bar{k}_l+\bar{k}_{n-l}-\bar{k}_n)dx}+2\sum_{l=1}^{N-n}\tilde{S}b_l^*b_{n+l}e^{i\int(\bar{k}_{n+l}-\bar{k}_l-\bar{k}_n)dx}\right) \tag{44}$$

where

$$\tilde{R}=lk_{n-l}^2+2nk_lk_{n-l}+(n-l)k_l^2-\alpha h^2\left(lk_l^3k_{n-l}+nk_l^2+(n-l)k_lk_{n-l}^3\right) \tag{45}$$

$$\tilde{S}=(n+l)k_l^2-2nk_lk_{n+l}-lk_{n+l}^2+\alpha h^2\left((n+l)k_lk_{n+l}^3-nk_l^2k_{n+l}^2-lk_{n+l}k_l^3\right) \tag{46}$$

From the second-order dynamic free surface boundary condition, the nonlinear relation between the amplitudes of ϕ_α and those of the free surface η can be derived

$$a_n=\frac{i\omega_n}{g}\alpha h^2b_{nyy}+\frac{i\omega_nz_\alpha}{g}h_yb_{ny}-\frac{2\omega_n\alpha h^2k_n}{g}b_{nx}-\frac{\omega_nz_\alpha k_n}{g}h_xb_n$$
$$-\frac{\omega_n\alpha h^2}{g}k_{nx}b_n+\frac{i\omega_n}{g}[1-\alpha k_n^2h^2-2\alpha h^2k_n(\bar{k}_n-k_n)]b_n$$
$$+\frac{1}{4g}\left(\sum_{l=1}^{n-1}\tilde{R}'b_lb_{n-l}e^{i\int(\bar{k}_l+\bar{k}_{n-l}-\bar{k}_n)dx}-2\sum_{l=1}^{N-n}\tilde{S}'b_l^*b_{n+l}e^{i\int(\bar{k}_{n+l}-\bar{k}_l-\bar{k}_n)dx}\right) \tag{47}$$

where

$$\tilde{R}' = k_l k_{n-l} \tag{48}$$

$$\tilde{S}' = k_l k_{n+l} \tag{49}$$

The extension to $O(\delta\mu^2)$ results in the nonlinear coefficients (Kaihatu and Kirby, 1994)

$$\tilde{\tilde{R}} = \tilde{R} - \alpha h^2 \left((n-l)k_l^4 + (2n-l)k_l^3 k_{n-l} + (n+l)k_l k_{n-l}^3 + l k_{n-l}^4 \right)$$
$$- nh^2 \left(\frac{(n-l)^2 + l(n-l) + l^2}{l(n-l)} \right) k_l^2 k_{n-l}^2 \tag{50}$$

$$\tilde{\tilde{S}} = \tilde{S} + \alpha h^2 \left(l k_{n+l}^4 + (2n+l)k_l^3 k_{n+l} + (n-l)k_l k_{n+l}^3 - (n+l)k_l^4 \right)$$
$$+ nh^2 \left(\frac{(n+l)^2 - l(n+l) + l^2}{l(n+l)} \right) k_l^2 k_{n+l}^2 \tag{51}$$

$$\tilde{\tilde{R}}' = \tilde{R}' - \alpha h^2 \left[k_l k_{n-l}^3 + k_l^3 k_{n-l} \right] - h^2 \left[\frac{(n-l)^2 + l(n-l) + l^2}{l(n-l)} \right] k_l^2 k_{n-l}^2 \tag{52}$$

$$\tilde{\tilde{S}}' = \tilde{S}' - \alpha h^2 \left[k_l k_{n+l}^3 + k_l^3 k_{n+l} \right] - h^2 \left[\frac{(n+l)^2 - l(n+l) + l^2}{l(n+l)} \right] k_l^2 k_{n+l}^2 \tag{53}$$

The $O(\delta\mu^2)$ terms generally tend to induce a slower growth in the generated higher harmonics in shoaling wave applications. Truncation of the extended Boussinesq model to $O(\mu^2)$ (with all nonlinearity retained) was performed by Wei et al. (1995). We note that the ambiguities which arose in the previous frequency domain treatment of the equations of Nwogu (1993) are not present here.

3.4. Model Evaluation

We noted earlier that a primary motivation for development of the extended Boussinesq models was to impart linear properties which mimicked those of fully-dispersive linear theory for a wide range of water depths, thus removing a substantial obstacle to general model application. In this section we examine how well the resulting frequency domain models capture wave shoaling, having insured (via the substitution process) that linear dispersion is well represented. Additionally, because the resulting models are parabolic, we will also examine the wide-angle behavior of the equations.

3.4.1. Wave Shoaling

As mentioned previously, one of the consequences of frequency domain transformation of the equations of Nwogu (1993) has been the lack of any guarantee that the characteristics of the original equations survive the transformation process. Great care had to be exercised in the combination of the equations such that the advantageous linear dispersion properties could be maintained. Additional steps were required for handling shoaling. In contrast, the equations of Chen and Liu (1995) were relatively simple to transform into the frequency domain, particularly with respect to retaining the dispersion relation of the equations of Nwogu (1993).

Madsen and Sørensen (1992) noted that the most reliable measure of the effectiveness of the shoaling mechanism was the shoaling gradient

$$\frac{|A|_x}{|A|} = -\frac{h_x}{h}\gamma \tag{54}$$

where γ is the shoaling gradient. While deviation of the shoaling gradient from linear theory may exaggerate that seen in the shoaling model itself, particularly in intermediate water depth (Chen and Liu, 1995), it is a convenient measure since integration is not required. The resulting expressions for γ for the shoaling models compared herein can be found in the original publications. Kaihatu and Kirby (1998), as mentioned previously, used two sets of parameters: the DO parameters ($\alpha = -0.3855$, $\beta = -0.3540$) and the DSO parameters ($\alpha = -0.4111$, $\beta = -0.3188$). To demonstrate the effect of the β term, we also include the ($\alpha = -0.3855$, $\beta = 0$) case; this is what would result if the ambiguity in the substitution process detailed earlier had not been realized. The shoaling mechanisms of Madsen and Sørensen (1992) and Chen and Liu (1995) will be used with the optimized free parameters determined by the authors in the original publications. We note that the model of Kaihatu and Kirby (1994) shares the same shoaling characteristics as that of Chen and Liu (1995).

As a benchmark, the shoaling gradient from fully-dispersive linear theory is used (Madsen and Sørensen, 1992)

$$\gamma = \frac{G'\left(1 + \frac{1}{2}G'(1 - \cosh 2kh)\right)}{(1+G')^2} \tag{55}$$

where

$$G' = \frac{2kh}{\sinh 2kh} \tag{56}$$

Figure 1 shows a comparison between the different shoaling mechanisms and that of linear theory. Of the five shoaling mechanisms, those of Madsen and Sørensen (1992) and Kaihatu and Kirby (1998) with DSO parameters compare the best, with a slight improvement yielded by the free parameterization of the latter model. In contrast, the shoaling mechanisms of Chen and Liu (1995) and Kaihatu and Kirby (1998) with $\beta = 0$ compare poorly, particularly the latter model. The model of Kaihatu and Kirby (1998) may represent the limit of optimum shoaling performance possible with two free parameters. Schäffer and Madsen (1995), using a Padé [4, 4] approximation and more free parameters, developed an expression for γ which exhibited virtually no deviation from linear theory for the complete range of water depths.

3.4.2. Wide-Angle Behavior of the Parabolic Equations

One consequence of the parabolic approximation is the assumption that the angle between the x-coordinate of the grid and the incident wave direction remains small. Methods for ameliorating this problem have generally taken the form of retention of higher-order derivative terms with coefficients which can be determined by Padé approximations (Booij, 1981; Kirby, 1986a) or by rational approximations (Kirby, 1986b), and have been shown to work well for parabolic approximations of the mild-slope equation (Berkhoff, 1972; Smith and Sprinks, 1975). The suitability for parabolic approximations of the Boussinesq equations, however, have not been established except by comparisons with data (for example, Kirby, 1990). In this section we analyze the effectiveness of the parabolic approximation used in the development of the frequency domain extended Boussinesq models. We note that the models of Chen and Liu (1995) and Kaihatu and Kirby (1994; 1998) reduce to the same basic parabolic form in the linear limit.

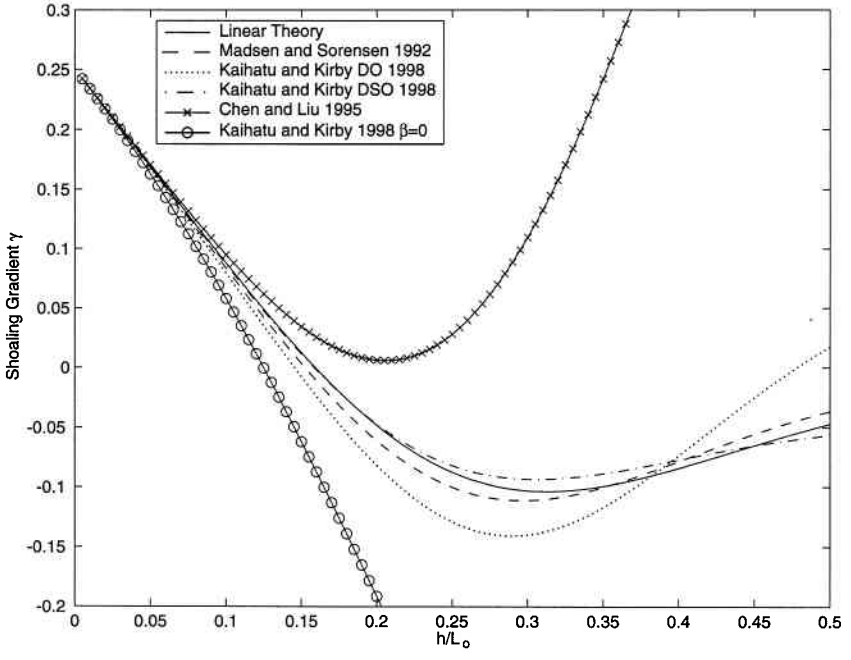

Figure 1. Comparison of shoaling mechanisms from various extended Boussinesq models to that of linear theory.

Numerical treatment of parabolic wave models generally involves the Crank-Nicholson discretization scheme, which is second-order accurate in both horizontal directions. A tridiagonal matrix is formed at each x location and is usually solved using a Thomas algorithm. The highest order derivative of A which would allow this treatment is A_{xyy} (Kirby, 1986a), which was not retained in our previous development. Expanding the time-periodic form of equation (33) into its horizontal components, neglecting nonlinear and bottom slope terms, substituting in equation (35) and retaining the A_{xyy} term that results yields

$$2i\left[ghk+\omega^2kh^2-2g(kh)^3\left(\alpha+\frac{1}{3}\right)\right]A_x+\left[gh+\omega^2h^2\alpha-2gh^3k^2\left(\alpha+\frac{1}{3}\right)\right]A_{yy}$$

$$+4igh^3k\left(\alpha+\frac{1}{3}\right)A_{xyy}=0 \tag{57}$$

We note that many terms were truncated in the substitution process used to derive equation (57), specifically third and fourth derivatives with respect to y. Additionally, similar derivatives with respect to x are represented only in the differentiation of the phase function, generating terms proportional to k^4. To ascertain the effect this truncation has on the accuracy of wave propagation, we need to transform the equation back into $\hat{\eta}$ by

$$A=\hat{\eta}e^{-ikx} \tag{58}$$

leading to

$$2i\left[ghk+\omega^2kh^2\alpha-2gk^3h^3\left(\alpha+\frac{1}{3}\right)\right]\hat{\eta}_x+2k\left[ghk+\omega^2kh^2\alpha-2gk^3h^3\left(\alpha+\frac{1}{3}\right)\right]\hat{\eta}$$

$$+\left[gh+\omega^2h^2\alpha-2gh^3k^2\left(\alpha+\frac{1}{3}\right)\right]\hat{\eta}_{yy}+4igh^3k\left(\alpha+\frac{1}{3}\right)\hat{\eta}_{xyy}$$

$$+4k^2gh^3\left(\alpha+\frac{1}{3}\right)\hat{\eta}_{yy}=0 \tag{59}$$

Making a final substitution

$$\hat{\eta}=ae^{i(k_xx+k_yy)} \tag{60}$$

where k_x and k_y denote wave number vector components in the x and y directions, respectively, we can obtain an expression for k_y in terms of k and k_x

$$k_y=\sqrt{\frac{2(k-k_x)\left[ghk+\omega^2kh^2\alpha-2gk^3h^3\left(\alpha+\frac{1}{3}\right)\right]}{gh\left[4h^2(k^2-kk_x)\left(\alpha+\frac{1}{3}\right)+\left[1+\frac{\omega^2h\alpha}{g}-2k^2h^2\left(\alpha+\frac{1}{3}\right)\right]\right]}} \tag{61}$$

The neglect of the A_{xyy} term would have the effect of zeroing out the kk_x term in the denominator of equation (61). The baseline for comparison is the circle

$$k_y=\sqrt{k^2-k_x^2} \tag{62}$$

which was derived by substitution of equation (60) into the Helmholtz equation. Additionally, we also compare to the small-angle parabolic approximations used in the mild-slope equation (Kirby, 1986b)

$$k_y=\sqrt{2(k^2-kk_x)} \tag{63}$$

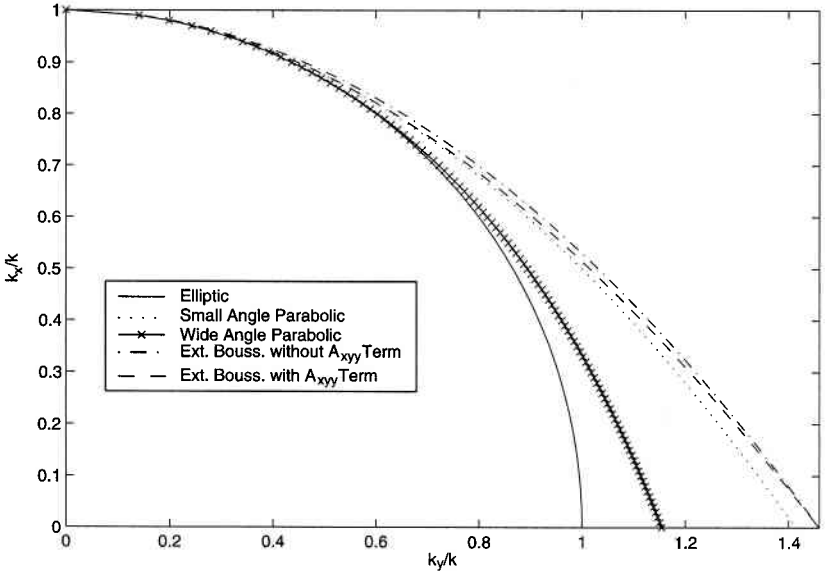

Figure 2. Analysis of the parabolic approximations used for various models. Relation from elliptic model is benchmark.

and the wide angle expression from the Padé approximation of Kirby (1986a)

$$k_y = \sqrt{\frac{4k^2\left(1-\frac{k_x}{k}\right)}{\left(3-\frac{k_x}{k}\right)}} \tag{64}$$

Figure 2 shows a comparison of the above expressions relating k_x to k_y. It appears that the parabolic models of the extended Boussinesq equation have slightly worse characteristics at oblique angles than the small-angle approximation of the mild-slope equation, with the A_{xyy} term imparting no appreciable improvement. This could be due to the considerable amount of information contained in the $\nabla^2\nabla^2\eta$ terms (among others) discarded when the form (equation (35) is substituted and the parabolic approximation made. Optimizations similar to Kirby's (1986a; 1986b) developments with the mild-slope equation could be implemented here.

4. NONLINEAR MILD-SLOPE EQUATION MODELS

An alternative approach to developing nonlinear frequency domain models involves incorporating nonlinear effects into models already equipped with fully-dispersive transformation characteristics. The mild-slope equation (Berkhoff, 1972; Smith and Sprinks, 1975) simulates wave refraction, shoaling and diffraction over mildly-varying bathymetry; its applicability has been greatly increased with the advent of the parabolic approximation (Radder, 1979; Lozano and Liu, 1980) encountered in earlier sections.

Bryant (1973, 1974) first studied the efficacy of developing fully-dispersive models with second-order nonlinear characteristics. He developed a spatially-periodic solution to the truncated Laplace boundary value problem (thereby retaining full frequency dispersion) and compared numerical evaluations of this solution to those from various forms of the Korteweg-deVries (KdV)equation. Furthermore, Bryant (1974) also demonstrated that three harmonic amplitudes of his spatially-periodic solution matched those of third-order Stokes waves, as did the nonlinear dispersion characteristics.

Keller (1988) derived coupled nonlinear equations derived from the shallow water equations, the Boussinesq equations and the Euler equations, all truncated to two harmonics. He showed that, in the shallow water limit, the equations reduced to identical forms.

Agnon et al. (1993) developed a unidirectional shoaling model based on a nonlinear extension of fully-dispersive linear shoaling. The linear part of the resulting model contained fully-dispersive shoaling, and triad interactions between wave components described the nonlinear evolution. This was later extended to two-dimensional propagation by Kaihatu and Kirby (1995), Tang and Ouelette (1997) (both parabolic models) and Agnon and Sheremet (1997) (hyperbolic model). The paper by Kaihatu and Kirby (1995) also detailed the inclusion of a surf zone dissipation mechanism. Tang and Ouelette (1997) extended the model of Kaihatu and Kirby (1995) by including diffraction and bottom slope effects in the nonlinear terms of their model.

To explain some of the subtle points outlined later in this section, we briefly outline the development of the nonlinear mild-slope equation described by Kaihatu and Kirby (1995). We start from the boundary value problem for water waves, with free surface conditions expanded to second-order in wave amplitude. We first assume that the solution can be expressed as a superposition

$$\phi(x,y,z,t) = \sum_{n=1}^{N} \tilde{f}_n(z)\hat{\phi}_n(x,y)e^{-i\omega_n t} + c.c. \tag{65}$$

where $\hat{\phi}_n$ is complex, and

$$\tilde{f}_n = \frac{\cosh k_n(h+z)}{\cosh k_n h} \tag{66}$$

where the dispersion relation is that of linear theory

$$\omega_n^2 = g k_n \tanh k_n h \tag{67}$$

We then make use of Green's Second Identity on the variables $\tilde{f}_n$ and $\hat{\phi}_n$, and then use resonant triad interaction theory to create a time-periodic evolution equation for $\hat{\phi}_n$. We then assume a propagating wave

$$\hat{\phi}_n = -\frac{ig}{\omega_n} A_n e^{i\int k_n dx} \tag{68}$$

We substitute equation (68) and its conjugate into the time-periodic equation for $\hat{\phi}_n$, and employ the parabolic approximation to justify neglecting $\partial^2 A_n/\partial x^2$ terms in the resulting equation. We then make use of a phase function redefinition similar to that done for the extended Boussinesq models in earlier sections. This results in

$$\begin{aligned} &2i(kCC_g)_n a_{nx} - 2(kCC_g)_n(\bar{k}_n - k_n)a_n + i(kCC_g)_{nx} a_n + [(CC_g)_n(a_n)_y]_y \\ &= \frac{1}{4}\left(\sum_{l=1}^{n-1} Y a_l a_{n-l} e^{i\int(\bar{k}_l + \bar{k}_{n-l} - \bar{k}_n)dx} + 2\sum_{l=1}^{N-n} Z a_l^* a_{n+l} e^{i\int(\bar{k}_{n+l} - \bar{k}_l - \bar{k}_n)dx}\right) \end{aligned} \tag{69}$$

where

$$\begin{aligned} Y = &\frac{g}{\omega_l \omega_{n-l}}[\omega_n^2 k_l k_{n-l} + (k_l + k_{n-l})(\omega_{n-l} k_l + \omega_l k_{n-l})\omega_n] \\ &- \frac{\omega_n^2}{g}(\omega_l^2 + \omega_l \omega_{n-l} + \omega_{n-l}^2) \end{aligned} \tag{70}$$

$$\begin{aligned} Z = &\frac{g}{\omega_l \omega_{n+l}}[\omega_n^2 k_l k_{n+l} + (k_{n+l} - k_l)(\omega_{n+l} k_l + \omega_l k_{n+l})\omega_n] \\ &- \frac{\omega_n^2}{g}(\omega_l^2 - \omega_l \omega_{n+l} + \omega_{n+l}^2) \end{aligned} \tag{71}$$

Equation (69) comprises the model of Kaihatu and Kirby (1995).

Equation (68) is derived from the first-order dynamic free surface boundary condition

$$\phi_t + g\eta = 0; \qquad z = 0 \tag{72}$$

and is a transformation from amplitudes of velocity potential to those of the free-surface elevation. This transformation is first-order, and does not include the nonlinear terms inherent in the dynamic free surface boundary condition. Eldeberky and Madsen (1999) determined that the neglect of these second-order terms had the effect of underpredicting the nonlinear energy transfer, particularly with respect to the superharmonic energy transfer. They used successive approximations to invert the second-order dynamic free surface boundary condition and eliminate the velocity potential, resulting

in an evolution equation that could be conveniently solved for the free surface amplitudes alone. The linear terms are the same as those of earlier models, but the nonlinear coefficients are

$$Y' = Y - \frac{g}{\omega_l \omega_{n-l}} \Gamma^+ \omega_n k_l k_{n-l} + \frac{\Gamma^+ \omega_n^2}{g} (\omega_l^2 + \omega_l \omega_{n-l} + \omega_{n-l}^2) \tag{73}$$

$$Z' = Z - \frac{g}{\omega_l \omega_{n+l}} \Gamma^- \omega_n k_l k_{n+l} - \frac{\Gamma^- \omega_n^2}{g} (\omega_l^2 - \omega_l \omega_{n+l} + \omega_{n+l}^2) \tag{74}$$

where

$$\Gamma^+ = \frac{2(k_l + k_{n-l} - k_n) C_{gn}}{\omega_n} \tag{75}$$

$$\Gamma^- = \frac{2(k_{n+l} - k_l - k_n) C_{gn}}{\omega_n} \tag{76}$$

Eldeberky and Madsen (1999) demonstrated that the above terms contributed substantially to the superharmonic energy transfer. We note here that Tang and Ouelette (1997) also took the second-order terms in this transformation into account, though in a manner different than Eldeberky and Madsen (1999).

Kaihatu (2001) took a different approach and derived the required correction to make the resulting equations consistent. This correction was derived from the second-order dynamic free surface boundary condition, and is applied whenever the free surface is required

$$\tilde{a}_n = a_n + \frac{1}{4g} \left(\sum_{l=1}^{n-1} Y'' a_l a_{n-l} e^{i \int (\bar{k}_l + \bar{k}_{n-l} - \bar{k}_n) dx} + 2 \sum_{l=1}^{N-n} Z'' a_l^* a_{n+l} e^{i \int (\bar{k}_{n+l} - \bar{k}_l - \bar{k}_n) dx} \right) \tag{77}$$

where $\tilde{a}_n$ is the total amplitude of the free surface to second-order, a_n is the solution to equation (69), and

$$Y'' = \omega_l^2 + \omega_l \omega_{n-l} + \omega_{n-l}^2 - g^2 \frac{k_l k_{n-l}}{\omega_l \omega_{n-l}} \tag{78}$$

$$Z'' = \omega_l^2 - \omega_l \omega_{n+l} + \omega_{n+l}^2 - g^2 \frac{k_l k_{n+l}}{\omega_l \omega_{n+l}} \tag{79}$$

Kaihatu (2001) also investigated the effect of this correction by comparing permanent form solutions of the one-dimensional version of equation (69), with and without the correction (equation (77)), to third-order Stokes theory. While the correction did not noticeably enhance phase speed comparisons to the theory, it did improve comparisons of the respective free surfaces. Additionally, Kaihatu (2001) also extended the parabolic model to include wide-angle propagation terms, which notably improved model performance when compared to laboratory measurements of waves propagating over a shoal (Chawla et al., 1998). These wide-angle propagation terms are essentially those of Kirby (1986a) and therefore share the same accuracy in oblique-angle propagation as equation (64).

5. BREAKING WAVE MODELS

For general utility in solving nearshore wave propagation problems, consideration of wave breaking and surf zone decay is required. Due to the nature of the evolution equations, such breaking and decay descriptions must necessarily be statistical. Two formulations are generally used for this application; those of Battjes and Janssen (1978) and Thornton and Guza (1983), though several others are extant (for example, Dally, 1990; Battjes and Groenendijk, 2000).

Battjes and Janssen (1978) assumed that the probability distribution of breaking waves could be described as a Rayleigh distribution, where the percentage of breaking waves at a particular location is related to the area under the truncated probability distribution. This percentage Q_b was determined by solution of the following implicit relation

$$\frac{1-Q_b}{\ln Q_b} = -\left(\frac{H_{rms}}{H_{max}}\right)^2 \tag{80}$$

where H_{rms} is the root-mean-square wave height and H_{max} the maximum wave height in the distribution. Framing the energy dissipation from breaking waves as an energy balance

$$\left(\overline{E}\sqrt{gh}\right)_x = -\epsilon_b \tag{81}$$

where $\overline{E}$ is wave energy, Battjes and Janssen (1978) determined an expression for the energy dissipation ϵ_b

$$\epsilon_b = \frac{1}{4} Q_b \overline{f} \rho g H_{max}^2 \tag{82}$$

where $\overline{f}$ is the average frequency of the spectrum. The maximum wave height H_{max} is determined by

$$H_{max} = \frac{0.88}{\overline{k}} \tanh\left(\frac{\tilde{\gamma}\overline{k}h}{0.88}\right) \tag{83}$$

where $\overline{k}$ is the average wave number. An expression for $\tilde{\gamma}$ based on the deep water wave steepness was found by Battjes and Stive (1985).

Thornton and Guza (1983) extended the Battjes and Janssen (1978) model by accounting for the transformation of the wave height probability distribution through the surf zone. They hypothesized a distribution of breaking waves as being a weighted Rayleigh distribution with tunable parameters. The energy dissipation for a single bore was then integrated through this probability distribution to obtain

$$\epsilon_b = \frac{3\sqrt{\pi}}{4\sqrt{gh}} \frac{B^3 \overline{f} H_{rms}^5}{\overline{\gamma}^4 h^5} \tag{84}$$

where B and $\overline{\gamma}$ are free parameters. Reasonable fit to field data was found with $\overline{\gamma} = 0.42$ and $B = 1.3 - 1.7$ (Thornton and Guza, 1983). Mase and Kirby (1992) incorporated this dissipation mechanism into their "hybrid KdV" shoaling model, which used full linear shoaling with shallow water nonlinearity. This particular study also detailed a laboratory experiment in which a spectrum of waves was allowed to shoal and break over a long sloping bottom. The unique feature of this experiment is that the energy at the peak of the spectrum is in intermediate water depth at the wave maker; this would serve as a severe test of shoaling models, and would invalidate those limited to

weak dispersion (for example, models based on the classical Boussinesq equations). This dissipation mechanism was also included in the linear spectral parabolic model of Chawla et al. (1998), as well as the nonlinear shoaling model of Kaihatu and Kirby (1995).

Both Battjes and Janssen (1978) and Thornton and Guza (1983) developed their dissipation mechanisms purely as lumped parameter models, dependent only on integrated properties of the spectrum with no other details. For use in phase-resolving frequency domain models such as those detailed in this study, some assumptions concerning the distribution of the dissipation over the frequency range must be made. The majority of nonlinear models of this type assume an equal weighting of dissipation across the entire frequency range (Eldeberky and Battjes, 1996; Eldeberky and Madsen, 1999); this assumption is also used in linear spectral models (Chawla et al. 1998). Alternatively, Mase and Kirby (1992), Kaihatu and Kirby (1995) and Kirby and Kaihatu (1996) used a distribution which assumes a frequency-squared weighting of the dissipation term, thus accounting for nonlinear transfer of energy from low to high frequencies due to triad interactions. Chen et al. (1997) demonstrated (using the model of Chen and Liu 1995) that, while the frequency-squared weighting did not affect the spectral shape significantly, it did offer greater accuracy in estimating skewness and asymmetry. Both quantities are indications of wave shape; for example, negative asymmetry corresponds to a forward-pitched shape of the wave field, redolent of surf zone waves.

The dissipation mechanisms are usually formulated in terms of energy $\overline{E}$. Some manipulation is required to implement these into complex-amplitude evolution equations. Mase and Kirby (1992) implemented the Thornton and Guza (1983) dissipation by starting with the conservation of energy flux equation with a damping term added

$$A_{nx} + \frac{C_{gnx}}{2C_{gn}} A_n + \tilde{\alpha} A_n = 0 \tag{85}$$

where $\tilde{\alpha}$ is the dissipation coefficient. Multiplying this equation by the conjugate amplitude, adding it to its conjugate equation, then summing over all components, we obtain

$$\sum_{n=1}^{N} \left(C_{gn} |A_n|^2 \right)_x = -2 \sum_{n=1}^{N} \tilde{\alpha}_n C_{gn} |A_n|^2 \tag{86}$$

Then, assuming shallow water and switching to an energy definition, we obtain

$$\left(\overline{E} \sqrt{gh} \right)_x = \rho g \sqrt{gh} \left(\sum_{n=1}^{N} \tilde{\alpha}_n |A_n|^2 \right) \tag{87}$$

where ρ is mass density of water. Equating this to the dissipation function in equation (84) yields

$$\sum_{n=1}^{N} \tilde{\alpha}_n |A_n|^2 = \frac{3\sqrt{\pi}}{4\sqrt{gh}} \frac{B^3 f_{peak} H_{rms}^5}{\gamma^2 h^3} = \tilde{\beta}(x) \sum_{n=1}^{N} |A_n|^2 \tag{88}$$

We now require a frequency distribution for $\tilde{\alpha}_n$. Mase and Kirby (1992) investigated the trends evidenced in a back-calculation of $\tilde{\alpha}_n$ from the data, and determined that a strong f_n^2 dependence for the dissipation existed, analogous to a frequency domain transformation of the Burgher's equation with viscous damping. A reasonable representation of this frequency dependence can be achieved by assuming the following form for $\tilde{\alpha}_n$

$$\tilde{\alpha}_n = \tilde{\alpha}_{n0} + \left(\frac{f_n}{f_{peak}} \right)^2 \tilde{\alpha}_{n1} \tag{89}$$

where

$$\tilde{\alpha}_{n0} = \tilde{F}\tilde{\beta}(x) \tag{90}$$

$$\tilde{\alpha}_{n1} = \left(\tilde{\beta}(x) - \tilde{\alpha}_{n0}\right) \frac{f_{peak}^2 \sum_{n=1}^{N} |A_n|^2}{\sum_{n=1}^{N} f_n^2 |A_n|^2} \tag{91}$$

This essentially splits the frequency dependence into a balance between one that is flat across the frequency range ($\tilde{\alpha}_{n0}$) and one that is weighted to the square of the frequency ($\tilde{\alpha}_{n1}$). This split is controlled by the parameter $\tilde{F}$. Mase and Kirby (1992) found that $\tilde{F} = 0.5$ seemed to work best for their experimental data, though Chen et al. (1997) noted that $\tilde{F} = 0$ worked best if the comparison at the shallowest gauge of the experiment ($h = .025$ m) was neglected.

Eldeberky and Battjes (1996) and Eldeberky and Madsen (1999) both used the dissipation of Battjes and Janssen (1978), and assumed that the dissipation was weighted equally across all frequencies. By realizing that amplitude changes are half those of changes in energy, they found that

$$\tilde{\alpha}_n = \frac{1}{2} \frac{\epsilon_b}{\sum_{n=1}^{N} C_{gn} |A_n|^2} \tag{92}$$

where the denominator represents the total energy flux in the spectrum. Eldeberky and Madsen (1999) show comparisons of their model (with the inclusion of the second-order relation between ϕ and η discussed earlier) with the data of Mase and Kirby (1992). While they showed improved comparisons of skewness relative to the original model of Kaihatu and Kirby (1995), they were unable to replicate the increase in negative asymmetry seen in the data. Kaihatu (2001), using the f_n^2 dissipation weighting detailed in Mase and Kirby (1992) as well as the second-order transformation between ϕ and η, demonstrated improved asymmetry predictions relative to those seen in Eldeberky and Madsen (1999) by using $\tilde{F} = 0$ in equation (90).

6. COMPARISONS TO DATA

6.1. Whalin (1971)

In this section we compare several of the described models to the experimental data of Whalin (1971), who conducted a laboratory study to investigate the limits of linear refractive wave propagation theory. He generated sinusoidal waves with periods of 1, 2 and 3 seconds and ran them over bathymetry resembling a tilted cylinder. The plan view of the experimental layout is shown in Figure 3. Wave gauges located down the centerline of the tank measured the free surface elevation; these measurements were then processed to yield wave harmonic amplitudes. The experimental conditions ranged from deep ($\mu^2 \approx 2$) to shallow ($\mu^2 \approx 0.2$) water at the wave maker. We will concentrate on the 1 second period case. Comparisons to other cases in the experiment are detailed in the original papers.

For this $T = 1$ s case ($a_o = 0.0195$ m) $N = 2$ harmonics were used. This is in concert with the work detailed in the original studies; the inclusion of additional harmonics did not make a significant difference. For each case, all wave energy was placed in the first harmonic, with the higher harmonics initialized with zero energy. We compare the following models: the extended Boussinesq model of Kaihatu and Kirby (1998) with DSO parameters ($\alpha = -0.4111, \beta = -0.3188$); the extended Boussinesq model (expressed in terms of ϕ and η) of Kaihatu and Kirby (1994); and the nonlinear mild-slope equation model of Kaihatu and Kirby (1995) with the second-order correction of Kaihatu (2001). The exclusion of the second-order correction of Kaihatu (2001) did

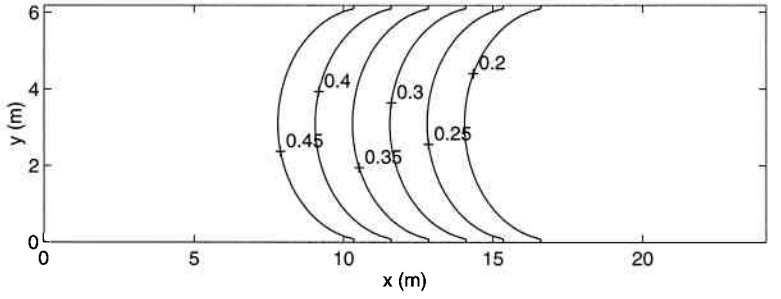

Figure 3. Plan view of experiment of Whalin (1971). Bathymetric contours labeled in meters.

not yield significant difference relative to inclusion. Additionally, comparisons between the DO ($\alpha = -0.3855, \beta = -0.3540$) and DSO parameters for this experiment are shown in the original paper of Kaihatu and Kirby (1998) and are not included here. Lastly, the model of Chen and Liu (1995) was not used in these comparisons; they used a second-order bound wave solution to force their model for the $T = 1$ s case, citing a large phase mismatch between the free and bound second harmonic otherwise. However, we are more interested in testing the model in general wave propagation scenarios, where *a priori* consideration of the free or bound wave nature of the forcing would not be desirable. Linear wave forcing of the model of Chen and Liu (1995) for this case reveals significant oscillation of the second harmonic over the domain.

Comparisons of the model to the $T = 1$ s, $a_o = 0.0195$ m case are shown in Figure 4. In this case the model of Kaihatu and Kirby (1995) with the second-order correction (Kaihatu, 2001) seems to work best for capturing evolution characteristics of both harmonics. The extended Boussinesq models of Kaihatu and Kirby (1994; 1998) appear to underpredict the energy transfer to the second harmonic in the focal region. The oscillations in the second harmonic as predicted by Kaihatu and Kirby (1994) are of the same order as those shown in Chen and Liu (1995), and are less severe than would be seen if the Chen and Liu (1995) model had been forced with a linear wave. This indicates that the presence of $O(\delta\mu^2)$ terms appear to have a stabilizing effect on the model, as commented by Tang and Ouelette (1997). We also note here that the model of Chen and Liu (1995) with bound wave forcing agrees with the second harmonic data for this case better than the model of Kaihatu and Kirby (1994) with linear wave forcing. Lastly, Tang and Ouelette (1997) show a better match to the $T = 1$ s case than seen here, possibly due to the inclusion of nonlinear diffraction terms.

6.2. Mase and Kirby (1992)

Mase and Kirby (1992) conducted a series of laboratory experiments in which irregular waves (Pierson-Moskowitz spectrum) were generated and allowed to shoal and break over a slope. The experimental set-up is shown in Figure 5. The case studied here had a peak period $T_p = 1$ s, leading to $kh \approx 2$ at the wave maker. Significant wave breaking occurred in this experiment beginning near the gauge at $h = 0.175$ m. Time series of free surface elevations were collected at 20 Hz and divided into 7 realizations of 2048 points each. Each realization was then put into a Fast Fourier Transform (FFT); the resulting energy density spectra were both Bartlett-averaged across all seven realizations and band-averaged across eight neighboring frequency bands. A gauge located 0.20 cm offshore of the toe of the slope provided the initial condition. Tests with $N = 300$ and $N = 500$ frequency components were run through the models for each realization, leading to a maximum frequency of 3 Hz and 5 Hz respectively.

We compare two models to the experimental data: the nonlinear mild-slope equation model of Kaihatu and Kirby (1995), with corrections by Kaihatu (2001); and the extended Boussinesq frequency

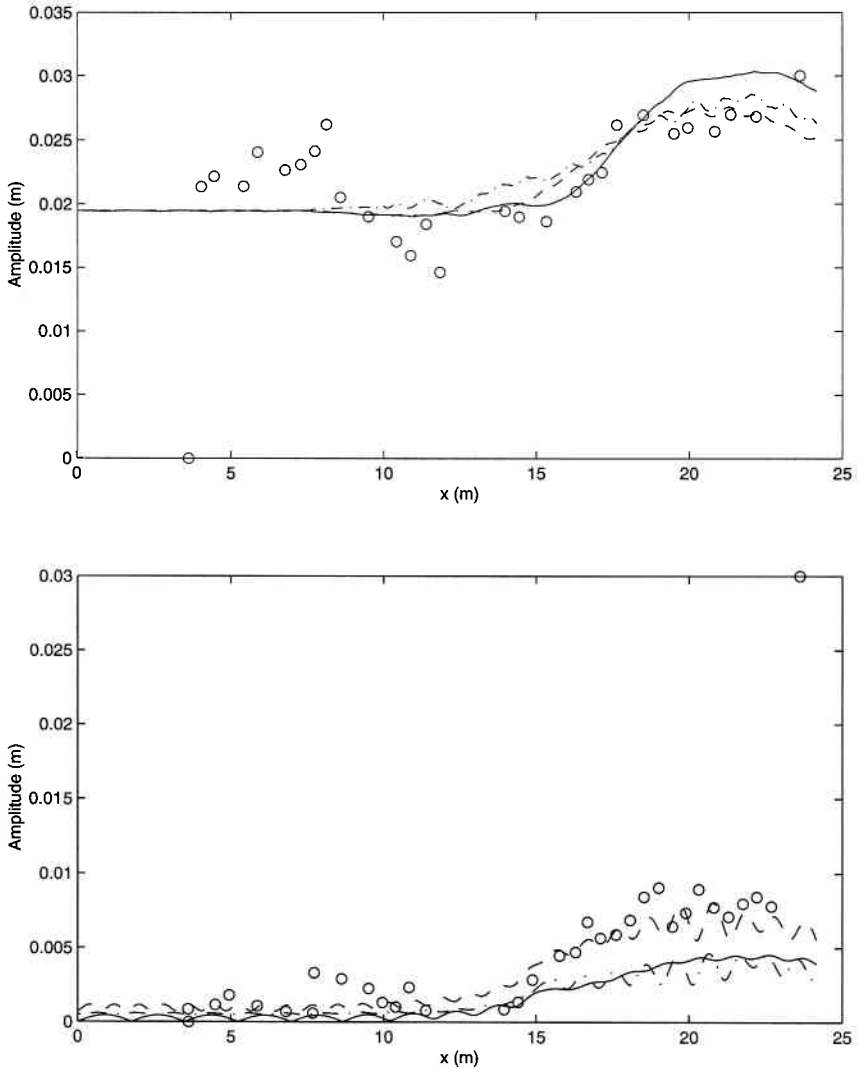

Figure 4. Comparison of nonlinear frequency domain models data of Whalin (1971). Solid line: model of Kaihatu and Kirby (1998) with DSO parameters. Dashed line: model of Kaihatu and Kirby (1995) with corrections from Kaihatu (2001). Dash-dot line: model of Kaihatu and Kirby (1994). Top: first harmonic amplitudes. Bottom: second harmonic amplitudes.

domain model of Kaihatu and Kirby (1998) with the DSO parameters. Both models were equipped with the dissipation mechanism of Thornton and Guza (1983) and the frequency-distribution methodology detailed by Mase and Kirby (1992). Both $\tilde{F} = 0.5$ and $\tilde{F} = 0$ were used, the latter corresponding to a full frequency-squared weighting of dissipation.

Comparisons of spectra at various locations are shown in Figure 6, with $N = 300$ and $\tilde{F} = 0$. It is apparent that, while both models tend to compare well to the data at the frequency peak and

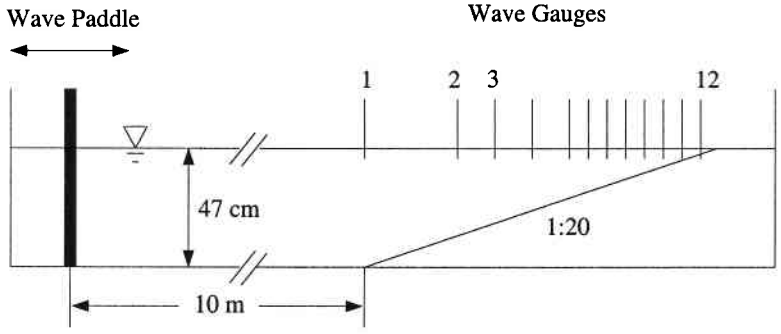

Figure 5. Layout of experiment of Mase and Kirby (1992).

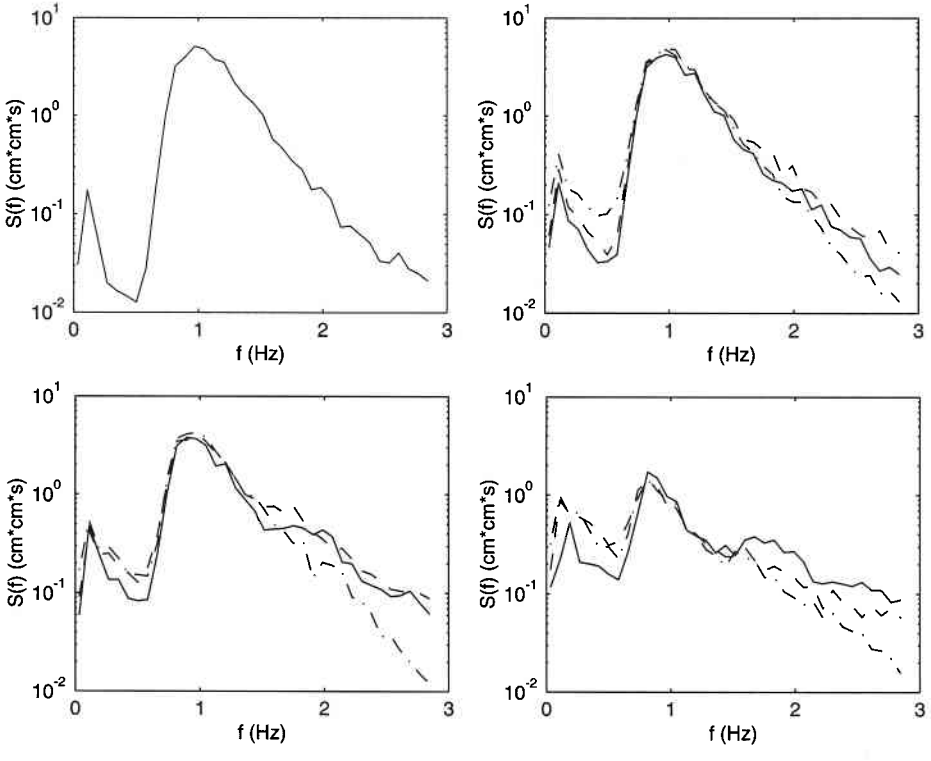

Figure 6. Comparisons of spectra from models to data from Mase and Kirby (1992): $N = 300$ frequencies, $F = 0$. Solid line: data. Dashed line: model of Kaihatu and Kirby (1995) with correction of Kaihatu (2001). Dash-dot line: model of Kaihatu and Kirby (1998) with DSO parameters. Top left: $h = 0.47$ m. Top right: $h = 0.2$ m. Bottom left: $h = 0.125$ m. Bottom right: $h = 0.05$ m.

lower, the extended Boussinesq model of Kaihatu and Kirby (1998) significantly underpredicts the high frequency evolution ($f > 1.75$ Hz). The case of $N = 500$ components reveal similar trends. No significant differences occur between the $\tilde{F} = 0$ and $\tilde{F} = 0.5$ cases.

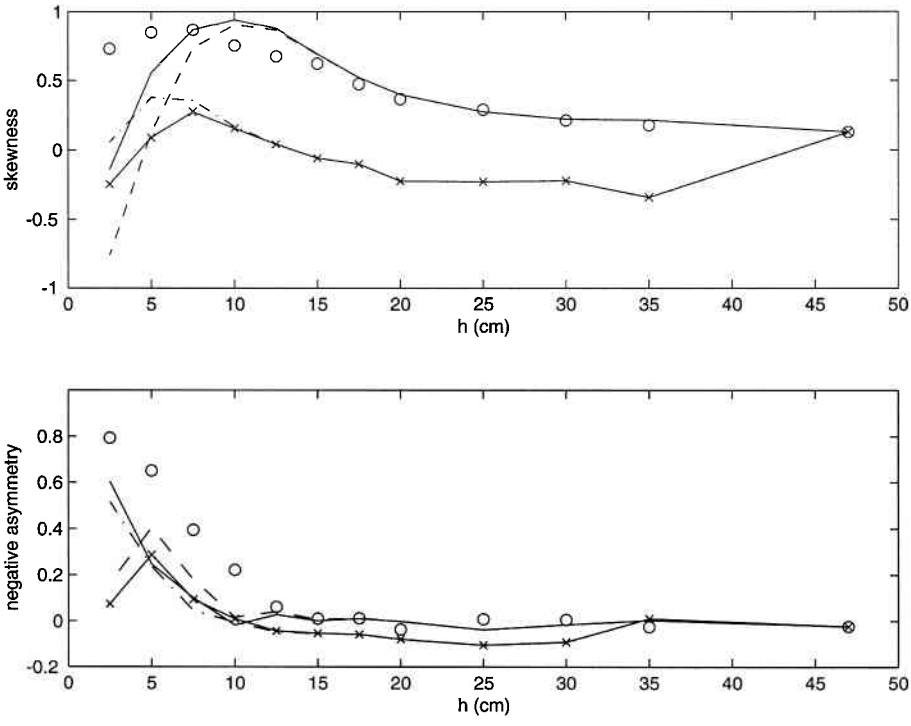

Figure 7. Comparisons of skewness and asymmetry from models to data from Mase and Kirby (1992): $N = 500$ frequencies. Open circles: data. Solid and dashed lines: model of Kaihatu and Kirby (1995) with correction of Kaihatu (2001) with $F = 0.5$ and $F = 0$, respectively. Dash-dot and dash-x lines: model of Kaihatu and Kirby (1998) with DSO parameters and $F = 0.5$ and $F = 0$, respectively. Top: skewness. Bottom: Negative asymmetry.

Differences between the $\tilde{F} = 0$ and $\tilde{F} = 0.5$ cases begin to appear when comparing skewness and asymmetry. Figure 7 shows a comparison of both models, each using $\tilde{F} = 0.5$ and $\tilde{F} = 0$, to the skewness and asymmetry values from the data for the case of $N = 500$. (It was shown by Kaihatu, 2001 that good comparisons to higher-order moments required $N = 500$ frequency components.) Both skewness and asymmetry are better predicted by the model of Kaihatu and Kirby (1995), with corrections by Kaihatu (2001), than with the extended Boussinesq model of Kaihatu and Kirby (1998). Additionally, $\tilde{F} = 0.5$ results in a slightly better skewness prediction and a slightly worse asymmetry prediction than $\tilde{F} = 0$, except for the gauge nearest to shore. At that location the modeled asymmetry falls off dramatically if $\tilde{F} = 0$ in either shoaling model. The reason for this sudden dropoff is not clear. We note here that this behavior at the last gauge is not an indictment of the $\tilde{F} = 0$ value; Kennedy et al. (2000) showed excellent agreement between the time domain Boussinesq model of Wei et al. (1995) and the Mase and Kirby (1992) data. An eddy viscosity mechanism was used for dissipation in the time domain model, the formulation of which is equivalent to $\tilde{F} = 0$ in the frequency domain. Rather, the problem in the frequency domain may lie in the use of the bulk energy dissipation mechanism used here; it is quite likely that a frequency domain formulation of the local eddy viscosity mechanism in the Kennedy et al. (2000) model would address the problems seen at the shallowest gauge.

6.3. Discussion

Based on the data-model comparisons shown, it appears that frequency domain formulations of the extended Boussinesq model of Nwogu (1993) may be inferior to those based on the nonlinear mild-slope equation (Kaihatu and Kirby, 1995; Kaihatu, 2001). One possible reason for the relatively poor performance of the frequency domain model of Kaihatu and Kirby (1998) may lie in the order of truncation of the original equations of Nwogu (1993). These equations were truncated to retain only $O(\delta, \mu^2)$ terms. As seen in the (ϕ, η) model of Kaihatu and Kirby (1994), it is possible to retain $O(\delta\mu^2)$ terms and still treat the equations using the frequency domain transformation; perhaps the inclusion of these terms in the frequency domain model of Kaihatu and Kirby (1998) would improve the results (though, in the author's experience, the derivation process is exceedingly complicated and ambiguous).

Interestingly, however, Wei and Kirby (1995) show very good agreement between their numerical treatment of the model of Nwogu (1993) and the Mase and Kirby (1992) data. As mentioned early in the chapter, there is often some discrepancy between the performance of the original time domain model and its corresponding frequency domain formulation. Additionally, Kofoed-Hansen and Rasmussen (1998) showed that the model of Madsen and Sørensen (1993) (a frequency domain shoaling model based on the extended Boussinesq model of Madsen et al., 1991) demonstrates favorable agreement with the Mase and Kirby (1992) data set, despite the order of truncation being the same as that in Kaihatu and Kirby (1998). This is likely due to differences in the underlying time domain equations and in the derivation procedure of the corresponding frequency domain models.

7. STOCHASTIC MODELS

While frequency domain models offer great utility in simulating shallow water processes, this is done at some computational expense, particularly when they are initialized with smoothed spectra (as would be the case for field studies, for example). Initial phases are required for these models, so multiple temporal realizations are run with random initial phases and the results averaged until acceptably smooth, which can require considerable computation time. This has motivated research into stochastic models of triad interactions. In this section, we briefly describe a few approaches, referring interested readers to the papers cited herein and the overview of Agnon and Sheremet (2000) for further details.

Abreau et al. (1992) developed a statistical model for triad wave evolution, suitable as a source function in a spectral balance model such as WAM (Komen et al., 1994) or SWAN (Booij et al., 1999). The model was developed using the non-dispersive asymptote, with the concomitant assumption that resonant interactions are only possible among collinear shallow water waves. This inherently disallows vector-sum interactions among spectral components in the wave field, negating a significant portion of the potential nonlinear behavior.

Eldeberky and Battjes (1995) developed a parameterized triad energy exchange mechanism which depends on the evolution of the biphase, a higher-order statistical quantity of the wave train which is zero at low nonlinearity (low Ursell number) and asymptotically approaches $-\pi/2$ as the Ursell number increases. The evolution of this quantity was determined from experimental measurements as a function of Ursell number. Additionally, nonlinear interaction was further limited to self-self interactions at the spectral peak. This allowed energy to move from the peak to its second harmonic, but did not allow for any feedback transfer. This parameterized model is a selectable option in the SWAN model (see Chapter 5).

Recently, much has been done on developing stochastic models from the phase-resolving dynamical equations (for example, Boussinesq equations). The evolution equation and its conjugate are each multiplied by their corresponding conjugates, then added together, resulting in an energy equation

with triple products of amplitudes in the nonlinear summations. These triple products are related to the bispectrum (a higher-order spectrum), for which an additional evolution equation is needed for system closure. The bispectral evolution equation is derived by applying the triple product definition of the bispectrum to the original evolution equation, resulting in an equation for bispectra with quadruple products of amplitudes in the nonlinear terms (the trispectrum). At this point a Gaussian closure assumption is made, which has the effect of reducing the trispectrum to products of the spectrum, thus creating a finite system of equations. Herbers and Burton (1997) developed a directional stochastic model from the Boussinesq equations of Peregrine (1967) using the same periodicity assumptions as Kirby (1990) and the procedure described above. Agnon and Sheremet (1997) worked from the nonlinear dispersive model of Agnon et al. (1993) to develop a stochastic model using the same closure hypothesis as Herbers and Burton (1997), but in the form of a single equation model for the spectral energy. Kofoed-Hansen and Rasmussen (1998) operated on the extended Boussinesq equations of Madsen and Sørensen (1993) and developed a corresponding stochastic model. They showed that good model comparisons were possible so long as the Ursell number $U_r < 1.5$, was in accord with the observations of Agnon and Sheremet (1997). Eldeberky and Madsen (1999) revisited Agnon and Sheremet (1997) and rederived a stochastic model, taking an additional second-order effect (noted earlier) into account.

One potential limitation has been the specification of the closure required to truncate the system to a finite number of solvable equations; the subset of trispectra used in the bispectral evolution equation represent the lowest order contributions. However, Holloway (1980), in the context of triad interactions among internal waves, hypothesized a different closure for the system. Rather than discarding a significant portion of the trispectra, Holloway (1980) suggested that the trispectrum is also proportional to the bispectrum; this, in addition to the products of energy terms, make up the contributions from the trispectrum. Becq-Girard et al. (1999), working from the extended Boussinesq model of Madsen and Sørensen (1993), developed a stochastic model with Holloway's (1980) closure hypothesis included. This inclusion essentially adds a linear term (multiplied by an empirical proportionality coefficient) to the bispectral evolution equation, with the effect of broadening the resonance condition, and adding higher-order contributions that may improve overall performance at moderate Ursell numbers.

8. CONCLUSIONS

Nonlinear frequency domain models have undergone rapid development, apace with corresponding advances in the time domain realm (particularly with respect to Boussinesq models). They have also increased in utility with the incorporation of enhanced frequency dispersion effects, improved shoaling and energy dissipation from wave breaking. We investigated several formulations for nonlinear frequency domain models; ensuing data-model comparisons demonstrate that nonlinear models based on the mild-slope equations appear to be more accurate than frequency domain transformations of the extended Boussinesq equations explored herein.

Initial phases of the irregular wave train were available to drive the models for data-model comparisons to the laboratory data. In most general field situations, however, smoothed spectra from pressure gauges, wave buoys or forecast model output would be the only source of data. Using smoothed spectra as an initial condition requires multiple runs of the model with random phases, a time consuming task. With the advent of the SWAN model, the consideration of triad interactions (even in the parameterized form used by the model) has become more widespread, and the need for a useful operational form of these interactions more apparent. This is particularly evident as more model systems linking wave, hydrodynamic and sediment modules are developed. The stochastic models described previously exhibit great potential for operational use; more development in this area is warranted.

ACKNOWLEDGMENTS

Support for the writing of this chapter was provided by the Office of Naval Research (ONR) through two projects: the National Ocean Partnership Program (NOPP) project "Development and Verification of a Nearshore Community Model for Physical Processes in the Coastal Ocean;" and the 6.2 Naval Research Laboratory (NRL) Core 6.2 project "Dynamically Constrained Nowcasting of Near Coastal Waves and Bathymetry." The author would also like to express his appreciation to Dr. James T. Kirby of the Center for Applied Coastal Research, University of Delaware, for his patient instruction in the fine art of nonlinear frequency domain models during the author's graduate study. This paper, NRL contribution NRL/BC/7320-02-0001, is approved for public release; distribution unlimited.

LIST OF SYMBOLS

α	—	dispersion optimization parameter from Nwogu (1993)
$\tilde{\alpha}$	—	dissipation coefficient from Mase and Kirby (1992)
β	—	shoaling optimization parameter from Kaihatu and Kirby (1998)
$\tilde{\beta}$	—	dissipation coefficient from Mase and Kirby (1992)
Γ^+, Γ^-	—	nonlinear coefficients from Eldeberky and Madsen (1999)
γ	—	shoaling gradient
$\tilde{\gamma}$	—	free parameter in calculation of H_{max} from Battjes and Janssen (1978)
$\overline{\gamma}$	—	dissipation free parameter from Thornton and Guza (1983)
δ	—	nonlinearity scaling parameter
ϵ_b	—	dissipation
η	—	free surface elevation
$\hat{\eta}$	—	time-periodic free surface elevation
Λ	—	$O(\delta\mu^2)$ term in extended Boussinesq model of Kaihatu and Kirby (1994)
μ	—	dispersion scaling parameter
π	—	3.1415927....
ρ	—	mass density
ϕ	—	velocity potential
$\hat{\phi}$	—	time-periodic velocity potential
ϕ_α	—	velocity potential at depth z_α
$\hat{\phi}_\alpha$	—	time-periodic velocity potential at depth z_α
Υ	—	$O(\delta\mu^2)$ term in extended Boussinesq model of Kaihatu and Kirby (1994)
ω	—	radian frequency
A	—	complex amplitude of free surface
a	—	complex amplitude of free surface in parabolic model
a_o	—	reference amplitude
$\tilde{a}$	—	complex amplitude of free surface with second-order correction of Kaihatu (2001)
b	—	complex amplitude of velocity potential
C	—	phase speed
C_g	—	group velocity
$c.c.$	—	complex conjugate
E	—	term in shoaling relation of Kaihatu and Kirby (1998)

$\overline{E}$ — energy
F — term in shoaling relation of Kaihatu and Kirby (1998)
$\tilde{F}$ — breaking parameter controlling frequency dependence of dissipation from Mase and Kirby (1992)
f — cyclic frequency
f_{peak} — peak frequency
$\tilde{f}$ — depth dependence from linear theory
$\overline{f}$ — average frequency
G — term in shoaling relation of Kaihatu and Kirby (1998)
G' — depth dependence term
g — gravitational acceleration
H_{max} — maximum wave height
H_{rms} — root-mean-square wave height
h — water depth
h_o — reference water depth
i — $\sqrt{-1}$
k — wave number
$\mathbf{k}$ — wave number vector
$\overline{k}$ — longshore-averaged wave number
k_x — cross-shore component of wave number vector
k_y — longshore component of wave number vector
N — total number of frequencies
n — frequency index
p — pressure
Q_b — percentage of breaking waves
R — nonlinear coefficient term from Kaihatu and Kirby (1998)
$\tilde{R}$, $\tilde{R}'$ — nonlinear coefficient from Chen and Liu (1994)
$\tilde{\tilde{R}}$, $\tilde{\tilde{R}}'$ — nonlinear coefficient from Kaihatu and Kirby (1994)
S — nonlinear coefficient term from Kaihatu and Kirby (1998)
$\tilde{S}$, $\tilde{S}'$ — nonlinear coefficient from Chen and Liu (1994)
$\tilde{\tilde{S}}$, $\tilde{\tilde{S}}'$ — nonlinear coefficient from Kaihatu and Kirby (1994)
T — wave period
T_p — peak period
t — time
u — cross-shore velocity
u_α — cross-shore velocity at z_α
$\mathbf{u}_\alpha$ — horizontal velocity vector at z_α
$\hat{\mathbf{u}}_\alpha$ — time-periodic horizontal velocity vector at z_α
$\overline{\mathbf{u}}$ — depth-averaged horizontal velocity vector
U_r — Ursell number (δ/μ^2)
v — longshore velocity
W — shoaling coefficient term
x — cross-shore coordinate
Y — nonlinear coefficient of Kaihatu and Kirby (1995)
Y' — nonlinear coefficient from Eldeberky and Madsen (1999)
Y'' — nonlinear coefficient from Kaihatu (2001)
y — longshore coordinate
Z — nonlinear coefficient from Kaihatu and Kirby (1995)

Z' — nonlinear coefficient from Eldeberky and Madsen (1999)
Z'' — nonlinear coefficient from Kaihatu (2001)
z — vertical coordinate
∇ — horizontal gradient operator

REFERENCES

Abreau, M., Larraza, A., and Thornton, E.B., 1992. Nonlinear transformation of directional wave spectra in shallow water, Journal of Geophysical Research, 97: 15,579–15,589.

Agnon, Y., Sheremet, A., Gonsalves, J., and Stiassnie, M., 1993. Nonlinear evolution of a unidirectional shoaling wave field. Coastal Engineering, 20: 29–58.

Agnon, Y., and Sheremet, A., 1997. Stochastic nonlinear shoaling of directional spectra. Journal of Fluid Mechanics, 345: 79–99.

Agnon, Y., and Sheremet, A., 2000. Stochastic evolution models for nonlinear gravity waves over uneven topography. In: Liu, P.L.-F. (Editor), Advances in Coastal and Ocean Engineering, Vol. 6. Singapore: World Scientific Publishing Co.

Battjes, J.A., and Groenendijk, H.W., 2000. Wave height distributions on shallow foreshores. Coastal Engineering, 40: 161–182.

Battjes, J.A., and Janssen, J.P.F.M., 1978. Energy loss and set-up due to breaking of random waves. Proceedings of the 16th International Conference on Coastal Engineering, ASCE, pp. 569–587.

Battjes, J.A., and Stive, M., 1985. Calibration and verification of a dissipation model for random breaking waves. Journal of Geophysical Research, 90: 9159–9167.

Becq-Girard, F., Forget, P., and Benoit, M., 1999. Non-linear propagation of unidirectional wave fields over varying topography. Coastal Engineering, 38: 91–113.

Berkhoff, J.C.W., 1972. Computation of combined refraction-diffraction. Proceedings of the 13th International Conference on Coastal Engineering, ASCE, pp. 471–490.

Booij, N., 1981. Gravity waves on water with non-uniform depth and current. Report 81–1. Delft, The Netherlands: Department of Civil Engineering, Delft University of Technology.

Booij, N., Ris, R.C., and Holthuijsen, L.H., 1999. A third-generation wave model for coastal regions. 1. Model description and validation. Journal of Geophysical Research, 104: 7649–7666.

Bryant, P.J., 1973. Periodic waves in shallow water. Journal of Fluid Mechanics, 59: 625–644.

Bryant, P.J., 1974. Stability of periodic waves in shallow water. Journal of Fluid Mechanics, 66: 81–96.

Chawla, A., Özkan-Haller, H.T., and Kirby, J.T., 1998. Spectral model for wave evolution over irregular bathymetry. Journal of Waterway, Port, Coastal and Ocean Engineering, 124: 189–198.

Chen, Y., and Liu, P.L.-F., 1995. Modified Boussinesq equations and associated parabolic models for water wave propagation. Journal of Fluid Mechanics, 288: 351–381.

Chen, Y., Guza, R.T., and Elgar, S., 1997. Modeling spectra of breaking surface waves in shallow water. Journal of Geophysical Research, 102: 25,035–25,046.

Dally. W.R., 1990. Random breaking waves: a closed-form solution for planar beaches. Coastal Engineering, 14: 233–263.

Eldeberky, Y., and Battjes, J.A., 1995. Parameterization of triad interaction in wave energy models. Proceedings of Coastal Dynamics '95, pp. 140–148.

Eldeberky, Y., and Battjes, J.A., 1996. Spectral modeling of wave breaking: Application to Boussinesq equations. Journal of Geophysical Research, 101: 1253–1264.

Eldeberky, Y., and Madsen, P.A., 1999. Deterministic and stochastic evolution equations for fully dispersive and weakly nonlinear waves. Coastal Engineering, 38: 1–24.

Freilich, M.H., and Guza, R.T., 1984. Nonlinear effects on shoaling surface gravity waves. Philosophical Transactions of the Royal Society of London, Series A, 311: 1–41.

Herbers, T.H.C., and Burton, M.C., 1997. Nonlinear shoaling of directionally spread waves on a beach. Journal of Geophysical Research, 102: 21,101–21,114.

Holloway, G., 1980. Oceanic internal waves are not weak waves. Journal of Physical Oceanography, 10: 906–914.

Kadomtsev, B.B., and Petviashvili, V.I., 1970. On the stability of solitary waves in weakly dispersing media. Soviet Physics - Doklady, 15: 539–541.

Kaihatu, J.M., 1994. Frequency domain models for nonlinear finite depth water wave propagation. Ph.D. thesis. Newark, DE: University of Delaware, 169p.

Kaihatu, J.M., 2001. Improvement of nonlinear parabolic dispersive wave model. Journal of Waterway, Port, Coastal and Ocean Engineering, 127: 113–121.

Kaihatu, J.M., and Kirby, J.T., 1994. Parabolic and angular spectrum modeling of a fully nonlinear extended Boussinesq equation. Proceedings of the International Symposium: Waves—Physical and Numerical Modeling, pp. 514–523.

Kaihatu, J.M., and Kirby, J.T., 1995. Nonlinear transformation of waves in finite water depth. Physics of Fluids, 7: 1903–1914.

Kaihatu, J.M., and Kirby, J.T., 1998. Two-dimensional parabolic modeling of extended Boussinesq equations. Journal of Waterway, Port, Coastal and Ocean Engineering, 124: 57–67.

Keller, J.B., 1988. Resonantly interacting water waves. Journal of Fluid Mechanics, 191: 529–534.

Kennedy, A.B., Chen, Q., Kirby, J.T., and Dalrymple, R.A., 2000. Boussinesq modeling of wave transformation, breaking, and runup. I: 1-D. Journal of Waterway, Port, Coastal and Ocean Engineering, 126: 39–47.

Kirby, J.T., 1986a. Higher-order approximations in the parabolic equation method for water waves. Journal of Geophysical Research, 91: 933–952.

Kirby, J.T., 1986b. Rational approximations in the parabolic equation method for water waves. Coastal Engineering, 10: 355–378.

Kirby, J.T., 1990. Modeling shoaling directional wave spectra in shallow water. Proceedings of the 22nd International Conference on Coastal Engineering, ASCE, pp. 109–122.

Kirby, J.T., and Kaihatu, J.M., 1996. Structure of frequency domain models for random wave breaking. Proceedings of the 25th International Conference on Coastal Engineering, ASCE, pp. 1144–1155.

Kofoed-Hansen, H., and Rasmussen, J.H., 1998. Modelling of nonlinear shoaling based on stochastic evolution equations. Coastal Engineering, 33: 203–232.

Komen, G.J., Cavaleri, L., Donelan, M.A., Hasselmann, K., Hasselmann, S., and Janssen, P.A.E.M., 1994. Dynamics and Modeling of Ocean Waves. Cambridge: Cambridge University Press. 531p.

Liu, P.L.-F., Yoon, S.B., and Kirby, J.T., 1985. Nonlinear refraction-diffraction of waves in shallow water. Journal of Fluid Mechanics, 153: 185–201.

Lozano, C.J., and Liu, P.L.-F., 1980. Refraction-diffraction model for linear surface water waves. Journal of Fluid Mechanics, 101: 705–720.

Madsen, P.A., Murray, R., and Sørensen, O.R., 1991. A new form of the Boussinesq equations with improved linear dispersion characteristics. Coastal Engineering, 15: 371–388.

Madsen, P.A., and Schäffer, H.A., 1998. Higher-order Boussinesq-type equations for surface gravity waves: Derivation and analysis. Philosophical Transactions of the Royal Society of London, Series A, 356: 3123–3184.

Madsen, P.A., and Schäffer, H.A., 1999. A review of Boussinesq-type equations for surface gravity waves. In: Liu, P.L.-F. (Editor), Advances in Coastal and Ocean Engineering, Vol. 5. Singapore: World Scientific Publishing Co.

Madsen, P.A., and Sørensen, O.R., 1992. A new form of the Boussinesq equations with improved linear dispersion characteristics. Part 2. A slowly-varying bathymetry. Coastal Engineering, 18: 183–204.

Madsen, P.A., and Sørensen, O.R., 1993. Bound waves and triad interactions in shallow water. Ocean Engineering, 20: 359–388.

Mase, H., and Kirby, J.T., 1992. Hybrid KdV frequency domain equation for random wave transformation. Proceedings of the 23rd International Conference on Coastal Engineering, ASCE, pp. 474–482.

McCowan, A.D., and Blackman, D.R., 1989. Proceedings of the 9th Australasian Conference on Coastal and Ocean Engineering, pp. 412–416.

Mei, C.C., 1983. The Dynamics of Ocean Surface Waves. Singapore: World Scientific Publishing Co. 740p.

Nwogu, O., 1993. Alternative form of Boussinesq equations for nearshore wave propagation. Journal of Waterway, Port, Coastal and Ocean Engineering, 119: 618–638.

Peregrine, D.H., 1967. Long waves on a beach. Journal of Fluid Mechanics, 27: 815–827.

Phillips, O.M., 1960. On the dynamics of unsteady gravity waves of finite amplitude. Part 1. The elementary interactions. Journal of Fluid Mechanics, 9: 193–217.

Radder, A.C., 1979. On the parabolic equation method for water-wave propagation. Journal of Fluid Mechanics, 95: 159–176.

Rygg, O.B., 1988. Nonlinear refraction-diffraction of surface waves in intermediate and shallow water. Coastal Engineering, 12: 191–211.

Schäffer, H.A., and Madsen, P.A., 1995. Further enhancements of Boussinesq-type equations. Coastal Engineering, 26: 1–14.

Smith, R., and Sprinks, T., 1975. Scattering of surface waves by a conical island. Journal of Fluid Mechanics, 72: 373–384.

Tang, Y., and Ouelette, Y., 1997. A new kind of nonlinear mild-slope equation for combined refraction-diffraction of multifrequency waves. Coastal Engineering, 31: 3–36.

Thornton, E.B., and Guza, R.G., 1983. Transformation of wave height distribution. Journal of Geophysical Research, 88: 5925–5938.

Ursell, F., 1953. The long wave paradox in the theory of gravity waves. Proceedings of the Cambridge Philosophical Society, 49: 685–694.

Wei, G., and Kirby, J.T., 1995. A time-domain numerical code for extended Boussinesq equations. Journal of Waterway, Port, Coastal and Ocean Engineering, 120: 251–261.

Wei, G., Kirby, J.T., Grilli, S.T., and Subramanya, R., 1995. A fully nonlinear Boussinesq model for surface waves. 1: Highly nonlinear, unsteady waves. Journal of Fluid Mechanics, 294: 71–92.

Whalin, R.W., 1971. The limit of application of linear wave refraction theory in a convergence zone. Report. H-71-3. Vicksburg, MS: U.S. Army Corps of Engineers Waterways Experiment Station.

Witting, J.M., 1984. A unified model for the evolution of nonlinear water waves. Journal of Computational Physics, 56: 203–236.

Wu, T.Y., 1981. Long waves in ocean and coastal waters. Journal of Engineering Mechanics, 107: 501–522.