Australian Government
Department of Defence
Defence Science and
Technology Organisation

Fractal Effects in Lanchester Models of Combat

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DSTO-TR-2331

ABSTRACT

Lanchester's Equations are one of the most misunderstood and misused models of combat, yet they remain in widespread use as the combat mechanism behind many simulation systems. Previous work by the author examined the impact of a fractal distribution of forces on Lanchester's theory of combat. The present work extends that examination to conceptual issues regarding interpretation of Lanchester's Equations and to additional parameters beyond those examined previously.

RELEASE LIMITATION

Approved for public release

Published by

Joint Operations Division
DSTO Defence Science and Technology Organisation
Fairbairn Business Park Department of Defence
Canberra ACT 2600 Australia

Telephone: (02) 6265 9111
Fax: (02) 6265 2741

AR-014-597
August 2009

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Executive Summary

Lanchester's Equations are one of the most misunderstood and misused models of combat, yet they remain in widespread use as the combat mechanism behind many simulation systems. Previous work by the author examined the impact of a fractal distribution of forces on Lanchester's theory of combat. The present work extends that examination to conceptual issues regarding interpretation of Lanchester's Equations and to additional parameters beyond those examined previously.

A fractal model for the distribution of each side's forces in space is shown to produce general *force on force* attrition expressions which describe the ability of each side to apply its strength effectively, due to battlefield congestion and finite engagement ranges. This is reflected in the logarithmic dependence of the Combat Intensity, Relative Effectiveness and Defender's Advantage parameter on the initial force ratio.

It confirms the functional form for the equation of state which had previously been derived for a few special cases. This is consistent with previous derivations of that expression, which had also linked the functional form to limiting a force's ability to apply its strength, and agrees well with historical data analysis.

It concludes that spatial effects in combat are necessary to obtain this equation of state instead of Lanchester's original equation of state, which would suggest that the major shortfall of Lanchester's combat model is the lack of a mechanism describing the effects of movement and spatial limitations on interaction between forces.

An important consequence of the model proposed as a *force on force* attrition model incorporating battlefield congestion, is that all the previously derived quantities from the original Lanchester model still apply if the simple attrition coefficients are replaced by the force level attrition coefficients that are developed. This work appears to be the first report that the force level Combat Intensity and Relative Effectiveness, both of which are key parameters in the results of Lanchester's combat model, have a logarithmic dependence on force ratio. This has yet to be investigated using available historical data.

The major shortcoming of the present work is the lack of processes to determine the attrition rate coefficients and fractal dimensions. However, it should be realised that these limitations apply not only to Lanchester models of combat but to all models of combat. In spite of all these issues, the model developed here is useful as a combat metamodel to illustrate relationships between select combat parameters and facilitate comparison between model and historical data, as indeed are all "*Lanchester like*" models.

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Acronyms, Abbreviations and Symbols

a	Model Outcome
a, a′	Attacker's Attrition Coefficient
a_i	Independent model variables
a_i	Kill rate of force *i*
$a_x(i,j)$	Kill rate of X force of type *j* by Y force of type *i*
A_i	Co-ordinate transformation matrix
$A(x,y)$	Infinitesimal generator matrix between initial state *x* and final state *y*
$A(t)$	Strength of arbitrary force at time t
b_i	Dependent model variables
b_j	Kill rate of force *j*
$b_y(i,j)$	Kill rate of Y force of type *j* by X force of type *i*
b, b′	Defender's Attrition Coefficient
C..H	Arbitrary Exponents
dx/dt	Rate of change of X force strength with time
dy/dt	Rate of change of Y force strength with time
D_i	Fractal Dimension of i
$E=N^2$	Markov state space of integer pairs
$f(\ )\ g()$	Arbitrary functions
f_a	Side's termination fraction
$I(x,y)$	Unitary matrix
I, I_0	Combat Intensity
$\log_d$	Logarithm to the base *d*
ln	Natural logarithm
MANA	Map Aware Non-uniform Automata
p_i	Single shot probability of kill of force *i*
$P_t(x,y)$	Probability that initial state *x* will transition to final state *y* at time *t*
R, R_0	Relative Effectiveness
T	Combat Duration
V	Defenders Advantage Parameter
$X=(X_t)$	Regular continuous time Markov process
$x(t)$	Strength of X force at time t
$x_j(t)$	Strength of X force of type *j* at time t
x_0	Initial strength of X force
$y(t)$	Strength of Y force at time t
$y_j(t)$	Strength of Y force of type *j* at time t
y_0	Initial strength of Y force
α	Helmbold Equation gradient
β	Helmbold Equation Offset
Δt	Discrete time interval
Σ	Summation over the specified parameters
$\Phi(\ )$	Arbitrary dimensionless function
Π_j	Similarity Parameters

1. Introduction

Lanchester's Equations are one of the most misunderstood and misused models of combat, yet they remain in widespread use as the combat mechanism behind many simulation systems. Part of the problem stems from the example used by Lanchester to introduce his model [1] and the failure of many subsequent researchers to see further than his two coupled equations. This has allowed many conflicting definitions as to what comprises a Lanchester model of combat to develop, with the result that consideration of whether a particular model is a Lanchester model or not appears more a question of what particular definition was used. Nevertheless, many workers go to great length to justify claims that their work does not follow Lanchester [2] [3].

Previous work by the author [4] [5] has examined some of these themes and demonstrated how a better understanding of those aspects of Lanchester's theory of combat in addition to the equations for which he is known, can lead to closer correspondence between theory and historical data.

The present work will provide report on further examination of the conceptual issues regarding the relationship between Lanchester's Equations and models of combat. It will also extend the fractal concepts that proved so useful in the previous work to quantities beyond the equation of state.

2. Lanchester's Work

During the First World War F. W. Lanchester described one of the simplest, and most enduring, mathematical attrition models of force-on-force combat [1]. He proposed two systems of equations, depending on whether the fighting was "collective" or not. Collective combat between two sides of strength *x(t)* and *y(t)* being described by the equations:

$$\begin{aligned} \frac{dx}{dt} &= -ay(t), \quad x(0) = x_0 \\ \frac{dy}{dt} &= -bx(t), \quad y(0) = y_0 \end{aligned} \tag{1}$$

Which result in the equation of state (for the case where *a* and *b* are constant):

$$\frac{\left(x_0^2 - x^2(t)\right)}{\left(y_0^2 - y^2(t)\right)} = \frac{a}{b} = R \tag{2}$$

The quadratic form of which results in the system of equations known as the Lanchester Square Law. This restriction to constant coefficients is not as great a constraint as it appears. Kimball [6] has shown that the time scale is a rather arbitrary parameter in Lanchester's Equations and can be transformed to allow a new time scale to be defined in which the rate coefficients in equations (1) are constant.

Individual combat on the other hand is described by the equations:

$$\begin{aligned} \frac{dx}{dt} &= -a_x y(t)x(t), \quad x(0) = x_0 \\ \frac{dy}{dt} &= -b_y x(t)y(t), \quad y(0) = y_0 \end{aligned} \tag{3}$$

With its equation of state:

$$\frac{(x_0 - x(t))}{(y_0 - y(t))} = \frac{a_x}{b_y} \tag{4}$$

The linear form of which results in the system of equations known as the Lanchester Linear Law.

Over the intervening years there have been many attempts to validate the use of Lanchester's Equations to describe combat outcomes through analysis of historical data [7]. These efforts have largely been unsuccessful. Part of the reason for this lack of success is the failure of subsequent users of Lanchester's theory to understand its inherent assumptions, constraints and limitations. Typically, most users assume that the fighting will be entirely collective and that both sides will attrite each other evenly, with strengths asymptotically approaching zero. Yet random chance is a factor in warfare whose effect has been modelled by treating attrition as a stochastic process [8].

Equally, the assumption of "collective" combat is unlikely to apply throughout the entire battle and hence actual attrition results from a combination of collective and individual combats. This possibility has long been recognised and produced many attempts to generalise Lanchester's system of equations to better represent actual combat results. This will also be covered in greater detail below, as will discussion of some of the assumptions that underlay Lanchester's Equations.

These issues were recognised by Lanchester himself, and covered in his writings, through his attempt to validate his system of equations by analysis of the battle of Trafalgar. That analysis has recently been revisited [9] and is worth reviewing here as it illustrates a number of the problems encountered in appropriate use of Lanchester's Equations.

2.1 Application

While *real* battles are non-uniform in space and time, neither parameter appears in Lanchester's Equations. This results from an implicit assumption that all the forces of each side are able to engage the other (or in other words, the ability to engage is *not target limited*) in the derivation of the square law.

However, from a study of the evolution of the Battle of Trafalgar, Lanchester realised that a battle is more properly viewed as a series of concurrent and consecutive sub-battles separated by space and time. Lanchester proposed that each of these sub-battles be described using his square law and that the overall casualties of the battle are obtained by summation of those losses. Each of those sub-battles can also be deconstructed further into a set of smaller sub-battles, and the outcome of the battle can be shown to be strongly dependent on the ability of commanders to fragment the battle to their advantage.

In short, the successful application of Lanchester's equations to describe historical battle outcomes first requires an understanding of the structure of that battle to enable a hierarchical decomposition of the battle into ever smaller sub-battles until the assumptions underpinning Lanchester's equations are met by those sub-battles. It could be said that this decomposition results from the application of Command and Control on the battle's evolution. At any rate, the outcomes of those sub-battles can then be determined and the results propagated as the starting conditions for the following group of sub-battles. This process is repeated until the end of the battle. This is the process that Lanchester envisaged, but is not how most practitioners use his theory. Most practitioners have sought modifications or additional equations to replace Lanchester's essential process.

2.2 Interpretation

In the derivation of the square and linear laws, Lanchester interpreted their functional difference as resulting from changes in technology arising from the industrial revolution. His square law was considered to describe "modern" or "collective" combat while the linear described "ancient" or "individual" combat.

This is not the interpretation taken by many subsequent practitioners who subscribe to the view that the linear law describes attrition resulting from *indirect* or *area* fire, while the square law describes attrition resulting from *direct* or *point* fire.

However, both the linear and square laws are consistent with both the point and area fire interpretations, which can be found in Karr [8], and are thus seen to represent *sufficient* but not *necessary* conditions for the derivation of the loss rate terms in equations (1) and (3). This point is so frequently misunderstood that it is worth repeating the description here.

1. **Square Law**
    - *Point Fire Interpretation:* Targets are sufficiently numerous or the ability to locate them is sufficiently good that each attacker locates targets at a constant rate.
    - *Area Fire interpretation:* Each attacker engages all targets in a certain area per unit time while, targets are dispersed over a region maintaining a constant density so that a reduction in the number of targets reduces the area occupied.

2. **Linear Law**
    - *Point Fire Interpretation:* Targets are sufficiently few or sufficiently difficult to locate and attack so that each attacker locates targets at a rate proportional to the number of targets present.
    - *Area Fire interpretation:* Each attacker engages all targets in a certain area per unit time, while targets are dispersed over a region maintaining a constant area occupied so that a reduction in the number of targets reduces the target density.

2.3 Heterogeneous Models

The Lanchester equations above use a single force parameter to describe each side's strength. This was a reasonable description of the forces of Lanchester's day. It no longer holds true today, in which each side contains a number of different and effective force elements.

Heterogeneous Lanchester equations extend equations (1) and (3) to systems of equations where each side's single strength is replaced by a series of strengths describing the numbers of each type of combatant. In this case, the heterogeneous square law equations can be written as:

$$\begin{aligned} \frac{dx_i}{dt} &= -\sum_{j=1}^{N} a_x(i,j)\, y_j(t) \\ \frac{dx_j}{dt} &= -\sum_{i=1}^{M} b_y(j,i)\, y_i(t) \end{aligned} \tag{5}$$

In general, a state equation no longer exists. Similar equations can be developed for the Linear Law. This represents a considerable increase in complexity of the description of the attrition process and a consequent reduction in the model's utility. Alternatively, the heterogeneous system of equations can be replaced with an homogenous system of equations in which the overall force strength is measured using some form of *force scoring*. Force scoring is a means of aggregating a side's combat strength including both the number of weapons of a given type and an assessment of the relative contribution that each weapon type makes to that side's combat power. Such approaches have been used with varying degrees of success for many years and a number of different force scoring methods have been developed [10] [11].

Only homogenous systems of Lanchester Equations will be considered in the present work.

3. Models and Metamodels

The US Department of Defense defines the following [12]:

Model A physical, mathematical, or otherwise logical representation of a system, entity, phenomenon, or process.

Simulation A method for implementing a model over time.

Metamodel A model of a model. Metamodels are abstractions of the model or simulation being developed that use functional decomposition to show relationships, paths of data and algorithms, ordering, and interactions between model components and subcomponents.

A model can be considered as a representation of an actual situation that may be used to better understand that situation. Complex phenomena often require complex models if the model's behaviour is to reproduce that of the real world. The modelling of combat attrition is one such situation [3]. However, while such models produce reasonable agreement with real world results, they are less useful in understanding the functional dependence of the modelled quantity on the input parameters. In such cases it is useful to develop a (simpler) model of that model which, although providing lower fidelity results, is better at explaining the causes of those results. Such models are the *metamodels*.

Clearly, if a model can be described through simple functional relationships, then it will also be its own metamodel.

Metamodels are often developed using an *ad-hoc* unstructured approach such as dimensional analysis. In recent years a systematic process for the development of such metamodels has emerged that provides a degree of rigour to the undertaking [13].

A review of this "*intermediate asymptotic*" technique was provided by the author in the previous work on this subject [4]. In its simplest form, it is a means by which a functional relationship of the form:

$$a = f\left(a_1,\ldots a_k, b_1,\ldots b_m\right) \tag{6}$$

can be described by the approximation:

$$a = a_1^p \ldots a_k^r \Phi\left(\frac{b_1}{a_1^{p_1} \ldots a_k^{r_1}}, \ldots \frac{b_m}{a_1^{p_m} \ldots a_k^{r_m}}\right) \tag{7}$$

and facilitates approximation of Φ under certain conditions.

In general, the combat attrition of engaged forces be described through a series of coupled equations of the form:

$$\begin{aligned} \frac{dx_i}{dt} &= f_i\left(x_1,\ldots x_k, y_1,\ldots y_n, a_1,\ldots a_m\right) \\ \frac{dy_j}{dt} &= g_j\left(x_1,\ldots x_k, y_1,\ldots y_n, a_1,\ldots a_m\right) \end{aligned} \tag{8}$$

Applying the intermediate asymptotic approximation leads to equations with the form:

$$\begin{aligned} \frac{dx}{dt} &= -a_B^C\, y(t)^D\, x(t)^E, \quad x(0) = x_0 \\ \frac{dy}{dt} &= -a_R^F\, x(t)^G\, y(t)^H, \quad y(0) = y_0 \end{aligned} \tag{9}$$

of which the established Lanchester's Equations are no more than special cases where the assumptions and constraints that apply (linear, square or mixed cases) are used to determine values for the exponents. More generally, it is possible to regard Lanchester models of combat as those metamodels which arise from the application of the intermediate asymptotic approximation to the general attrition expressions of equation (8) and not just the well known square law formulation with constant coefficients.

This is the definition that will be used in the present work.

4. Classic Lanchester Theory

Following Taylor [14], Lanchester models of combat are generally considered to be capable of providing insight into a limited number of questions:

- How do the force strengths change in time?
- What are the conditions required for success? (Who will win?)
- How many will survive?
- How long will the combat last?
- How do the initial conditions affect the outcome?

For brevity, discussion will be limited to the "square law" case with constant coefficients.

4.1 Square Law Constant Coefficient Model

Considering each Lanchester sub-battle (section 2.1) separately, the state equation (equation 2) and solutions to the differential equations (equation 1) represent possible starting points:

$$\frac{\left(x_0^2 - x^2(t)\right)}{\left(y_0^2 - y^2(t)\right)} = \frac{a}{b} = R_0 \qquad (10)$$

where R_0 is the relative fire effectiveness.

$$\begin{aligned} x(t) &= x_0 \cosh I_0 t - y_0 R \sinh I_0 t, \quad x(0) = x_0 \\ y(t) &= y_0 \cosh I_0 t - \frac{x_0}{R} \sinh I_0 t, \quad y(0) = y_0 \end{aligned} \qquad (11)$$

where $I_0 = \sqrt{ab}$ and is known as the combat Intensity as it controls the rate at which the time solutions above evolve.

Lanchester's original work on air combat assumed that each sub-battle would continue until one side was destroyed. Much subsequent work has been concerned with developing more general battle termination conditions. Extensions to the classic Lanchester theory will be covered in the following two sections. For the time being, each sub-battle will be assumed to continue until one side is destroyed.

There are three possible outcomes for battles of duration T:

- X wins with $x(T) = x_f > 0$ and $y(T) = y_f = 0$,
- Y wins with $y_f > 0$ and $x_f = 0$, (12)
- neither side wins with $x_f = y_f = 0$.

It can be easily shown [14] that which of these outcomes prevails is determined by comparing the initial force ratio to the relative fire effectiveness.

- X wins if $x_0/y_0 > \sqrt{R_0}$,
- Y wins if $x_0/y_0 < \sqrt{R_0}$, (13)
- neither side wins if $x_0/y_0 = \sqrt{R_0}$.

For further brevity, let side Y be the winner. Again following Taylor [14]:

$$T = \frac{1}{2I}\ln\left(\frac{\sqrt{R_0} + x_0/y_0}{\sqrt{R_0} - x_0/y_0}\right) = \frac{1}{I}\tanh^{-1}\left(\frac{x_0}{\sqrt{R_0}\,y_0}\right) \tag{14}$$

and

$$\frac{y_f}{y_0} = \sqrt{1 - \frac{1}{R_0}\left(\frac{x_0}{y_0}\right)^2} \tag{15}$$

with the general result that

$$\frac{y}{y_0} = \sqrt{1 - \frac{1}{R_0}\left\{\left(\frac{x_0}{y_0}\right)^2 - \left(\frac{x}{y_0}\right)^2\right\}} \tag{16}$$

Equations 10 to 16 form the standard or classic Lanchester results and enable the five questions to be answered. A similar set of results can be determined for the Lanchester "linear law" case with constant coefficients [15].

4.2 Shortcomings

Lanchester's original models have a number of shortcomings. Limiting consideration to the ten most important shortcomings identified by Taylor [15] gives:

1. Constant attrition rate coefficients
2. No force movement in space
3. Homogenous forces
4. Battle termination not modelled
5. Deterministic and not stochastic
6. Not verified against historical data
7. Cannot predict attrition rate coefficients
8. Tactical decision processes not considered
9. Battlefield Intelligence not considered
10. Command Control and Communications not considered

However, it can easily be demonstrated that a number of these shortcomings arise from the failure to consider Lanchester's fragmentation process (section 2.1) as an integral part of the model. Many of them can be, at least in part, addressed through application of that process.

In particular, the decisions to initiate combat, commit forces and allocate effort (Tactical decisions) require knowledge of the opponents location (Battlefield Intelligence) and the ability to arrange your own forces to carry out those decisions (Command Control and Communications). All must be considered, in both space and time, in order to enable a hierarchical decomposition of the battle into smaller sub-battles to which Lanchester's Equations are then applied.

Furthermore, each sub-battle can be defined with different attrition rate coefficients and with movement of forces between sub-battles separated in space. Such decomposition can also handle factors such as reinforcements or withdrawals. It just that all these factors need to be incorporated manually through decisions rather than through a series of auxiliary equations.

The interaction of forces made up from many different types of participant has already been discussed (section 2.3) and solutions using either some form of aggregate force scoring or heterogeneous systems of equations are widely used.

Of the list of ten shortcomings, the only ones that are not addressed through the process component of a Lanchester model are:

- Battle termination not modelled
- Deterministic and not stochastic
- Not verified against historical data
- Cannot predict attrition rate coefficients

These do represent potential serious problems, although it should be noted that they apply to many combat models and not just those based on Lanchester's Equations. This point has also been made by Taylor [15], who goes on to note that in spite of all its shortcomings, Lanchester's simple differential equation model of attrition is widely used even if only as a metamodel for a much more sophisticated combat simulation [3].

5. Extending Lanchester's Equations

The first reported development of Lanchester's Equations was by Morse and Kimball [6] in the 1940s, which considered the effects of reinforcements. Taylor [15] has a comprehensive review of subsequent development of the Lanchester theory of combat to 1983, with a list of 20 extensions that have been researched over the years.

An examination of that list shows that the majority of those investigations have sought to resolve one or more of three basic shortcomings in the Lanchester model.

- No spatial distribution of forces. The effect of the distribution, movement and localisation of each side's forces is not represented.
- No temporal distribution of forces other than through combat attrition. The effect of reinforcements and variation in combat effectiveness of each side's forces, as well as the non-simultaneous (stochastic) nature of attrition, is not represented.

- No rules for battle termination, other than annihilation of one side. The causes and conditions required for a battle to end are not represented.

This section will consider the work to resolve each of these shortcomings in turn. This discussion is not intended to be a comprehensive review of the extensions to Lanchester's theory. For this the interested reader is referred to Taylor [15]. Given references are not always the first, nor the most important, but are representative of those that are more readily available. How well the theory agrees with historical data will also be considered.

5.1 Spatial Effects

Two main approaches have been used to explore spatial effects in attrition modelling.

The dynamics which distribute forces about the battlefield themselves can be modelled directly, usually by extending Lanchester's Equations as standard form partial differential equations [16], which in the case of one dimensional movement can be written as:

$$\begin{aligned} \frac{\partial B}{\partial t} &= D_b \frac{\partial^2 B}{\partial x^2} + V_b \frac{\partial B}{\partial x} - B\int \beta(x,y)R(y,t)dy \\ \frac{\partial R}{\partial t} &= D_r \frac{\partial^2 R}{\partial x^2} + V_r \frac{\partial R}{\partial x} - R\int \gamma(x,y)B(y,t)dy \end{aligned} \tag{17}$$

The diffusion term describes the loss of spatial cohesion as a result of its motion and combat, while the advection term results from the collective movement of that sides forces as a whole. The attrition rate coefficient also depends on the distance between opposed forces and overall attrition rate is the convolution of that rate coefficient with the distribution of firing units. The solutions to these equations are more complex than for simple Lanchester Equations, and require more detailed boundary conditions. As a result, they describe specific types of battle (such as a set piece infantry attack) rather than general combat situations [17].

The effect of spatial distribution can also be modelled empirically by localising the interaction between opposing force elements or adding auxiliary equations to describe the movement of a notional front line [18]. The difficulty here is that there is no generally accepted theory of the relationship between force movement and force ratio or casualties.

There are only a few empirical studies into the relationship between rates of advance and battlefield characteristics, the most influential of which has been reviewed by the US General Accounting Office [19]. It notes that the data was originally correlated with the strength of the enemy's resistance, with only a minor mention of the force strength. However, it goes on to note that the presentation of the data has changed and is now primarily correlated with force ratio and only against enemy posture as a secondary consideration. Analytic approximations to that data have been used in models that try to link force movement with combat attrition [20]. Independent studies [21] have found that advance rate is more closely correlated with periods of inactivity during a battle rather than with any combat metric.

More frequently, such studies use simple analytic approximations for spatial effects [22], which appear to be chosen primarily for their suitability to analytic solution. These studies are also limited to specific types of battle rather than general combat situations.

Alternatively, spatial effects can be introduced through modifications to the attrition equations themselves. Asymmetric attrition equations (mixed law) are sometimes used to represent the differing ability of each side to use the battlefield to its advantage [15]. Helmbold [23] has also suggested modifying the attrition equations by functions of the force ratio to describe the effect of "diminishing returns" in the ability of a large force to apply its force effectively when engaging a smaller force.

5.2 Temporal Effects

The inclusion of temporal effects such as reinforcements, replacements and non-combat losses were among the first extensions to Lanchester's Equations that were examined [6] and lead to modified Lanchester Equations of the form:

$$\begin{aligned} \frac{dx}{dt} + ay(t) + cx(t) = H(t), \quad x(0) = x_0 \\ \frac{dy}{dt} + bx(t) + dy(t) = G(t), \quad y(0) = y_0 \end{aligned} \tag{18}$$

where the additional terms represent continuous non-combat force losses and discrete changes to the forces at specified times. However, apart from permitting closer agreement between theory and historical results they do not further an understanding of combat.

Of greater importance is the recognition that combat is a stochastic process. Stochastic analogues to Lanchester's Equations have been extensively researched [6] [24]. A continuous time, discrete space, Markov process can lead to stochastic attrition models analogous to the deterministic Lanchester equations in a rigorous manner. The treatment summarised here follows that of Karr [8] with the following assumptions:

1. All combatants on each side are identical (homogenous).
2. Times between detections by a surviving red combatant are independent and identically exponentially distributed with mean r_1^{-1}, regardless of the (non-zero) number of surviving blue combatants.
3. When a red combatant detects a blue combatant an instantaneous attack occurs, in which the blue combatant is destroyed with probability p_1 and survives with probability $1 - p_1$. Total loss of contact immediately takes place.
4. Blue combatants satisfy assumptions 2 and 3 with parameters r_2 and p_2 respectively.
5. Conditioned on survival, detection and attack processes of all combatants are mutually independent (in the probabilistic sense).

If X is a regular continuous time Markov process with countable space $E = N^2$ the set of integer pairs (i,j) with both (i,j) ≥ 0, then:

$$\begin{aligned} X &= (X_t)_{t\geq 0} \\ P_t(x,y) &= P_x\{X_t = y\} \\ P_{t+s}(x,y) &= \sum_{z\varepsilon E} P_t(x,z)P_s(z,y) \end{aligned} \tag{19}$$

where P_x is the probability that the initial state (t = 0) is x. The infinitesimal generator matrix A is then defined by:

$$\begin{gathered} A(x,y)=\lim_{h\to 0}\frac{P_h(x,y)-I(x,y)}{h} \\ P\{X_{t+h}=y|X_t=x\}=A(x,y)h+o(h) \\ P_t'=P_tA \end{gathered} \tag{20}$$

A Markov process (B_t, R_t) with state space E can be regarded as a combat attrition model between a homogenous R side and a homogenous B side provided that the paths t $\rightarrow$ (B_t, R_t) are non-increasing. Here both B_t and R_t are random variables for each t. Equation (20) is then analogous to the Lanchester attrition equations given the derivation of an appropriate generator matrix A with loss rates $k_i = r_i p_i \ldots i = 1,2$:

$$\begin{gathered} A((i,j),(i-1,j))=k_1 j \\ A((i,j),(i,j-1))=k_2 i \\ A((i,j),(l,m))=0\ldots\forall \text{ other pairs of } (i,j),(l,m) \\ A((i,j),(i,j))=-(k_1 j+k_2 i) \end{gathered} \tag{21}$$

This is recognisable as analogous to Lanchester's Square Law (equations 1). The expectation values of the forces' strengths do not in general satisfy the deterministic Lanchester Equations in the square law case. However, given the additional assumption that the loss rates are small, an equation of state for the continuous time Markov process can eventually be found from:

$$\frac{d}{dt}E\left[k_2\left(B_t^2+B_t\right)-k_1\left(R_t^2+R_t\right)\right]=0 \tag{22}$$

5.3 Battle Termination Conditions

In Lanchester's model, a battle ends when one side is eliminated. This is rarely a realistic outcome. A more realistic model of how and why a battle ends is required if the results given in equations 13 to 15 (victory, duration and casualties) are to be improved upon.

Helmbold [25] undertook one of the earliest and most influential studies of a theory of battle termination. He constructed a detailed theory, from first principles, that could be applied both deterministically and stochastically. The critical assumption of his battle termination theory is that a unit will cease to be an effective fighting force when its strength has dropped below a specified force level. That force level will depend on the unit's type, size and mission. For brevity, again consider the case where side Y wins at time T. Three conditions must be met:

- $x(T) = x_f = x_{BP} = f_{BP}^{x} x_0$,
- $y(T) = y_f > y_{BP} = f_{BP}^{y} y_0$, (23)
- $x(t) > x_f$ and $y(t) > y_f$ for $0 < t < T$.

Helmbold's theory provides a procedure for determining the breakpoint fractions f in this equation. For simplicity, the remainder of this treatment will only consider the deterministic (fixed breakpoint) application of this theory to the Lanchester square law case. Following equation 13, side Y will be the winner if:

$$\frac{x_0}{y_0} < \sqrt{R_0 \frac{1-(f_{BP}^{y})^2}{1-(f_{BP}^{x})^2}}\,, \tag{24}$$

Following equation 14, the battle will last a time T:

$$T = \begin{cases} \dfrac{-1}{I}\ln\left(1 - f_{BP}^{x}\right) & \dfrac{x_0}{y_0} = \sqrt{R_0} \\ \dfrac{1}{I_0}\ln\left\{\dfrac{-\left(x_0/y_0\right)f_{BP}^{x} + \sqrt{R_0 - \left(x_0/y_0\right)^2\left(1-(f_{BP}^{x})^2\right)}}{R_0 - \left(x_0/y_0\right)}\right\} & \dfrac{x_0}{y_0} \neq \sqrt{R_0} \end{cases} \tag{25}$$

Following equation 15, the survivors will number:

$$\begin{aligned} \frac{y_f}{y_0} &= \sqrt{1 - \frac{1}{R_0}\left(\frac{x_0}{y_0}\right)^2\left(1-(f_{BP}^{s})^2\right)} \\ \frac{x_f}{x_0} &= f_{BP}^{x} \end{aligned} \tag{26}$$

Unfortunately, as Helmbold himself demonstrated [25], the results of this simple deterministic model and its more sophisticated stochastic application, are not in good agreement with the available historical data. This has prompted research into more sophisticated models of battle termination. Nevertheless, this simple model is still the most widely used model of battle termination in conjunction with Lanchester's Equations.

Hawkins [26] has conducted the most important work on battle termination conditions since Helmbold. His resulting model includes factors other than casualty fractions as triggers for battle termination. His model is well suited to implementation in computer based simulations, but is less amenable for inclusion with analytic equations. Jaiswal [27] has examined the implications of the other major termination trigger that uses attrition, but notes that a full analytic examination of Hawkins' model is not possible without explicit modelling of spatial effects.

There is one further application of equation 23 developed by Helmbold [28]. The equation of state (equation 2) can be rewritten to define:

$$\mu^2 = \frac{1-\left(x/x_0\right)^2}{1-\left(y/y_0\right)^2} = R_0\left(\frac{y_0}{x_0}\right)^2 \tag{27}$$

Helmbold has demonstrated that μ is a measure of the relative advantage of Y over X. In particular, the Advantage Parameter V is defined as $V = \ln\mu$ shows a correlation with the side that wins the battle. For a deterministic model it should indicate which side will win. For a stochastic model, the value of V is an indication of the probability that a given side will win. If $V < 0$ then X has the advantage while for $V > 0$ Y has the advantage.

Helmbold's advantage parameter can be derived through other means, such as the battle trace concept developed by Bitters [29].

5.4 Comparison with Historical Data

As is often the case with simple analytic models, Lanchester's equations may be too abstract to describe many real operational problems. Nevertheless, simple analytic models (or metamodels of more complex combat simulations) do provide insight into combat dynamics. For this reason there have been a number of attempts to validate Lanchester's model of combat attrition against historical combat data.

Rigorous comparison of theory with historical results requires time correlated combat strengths and casualties for both sides. Such data is rarely available. Engel [30] found good agreement using the Iwo Jima campaign provided that reinforcements were included. More recently, the Ardennes campaign [31] and the battle of Kursk [32] have enabled similar examinations to be made, with less satisfactory results. Johnson [33] has reviewed the history of such attempts with the conclusion that a Lanchester attrition process between the square and linear laws is operative.

Recognising the difficulty in obtaining satisfactory data, Helmbold [28] developed an alternative approach that only required knowledge of the duration, initial and final strengths. Instead of detailed knowledge about a single battle, this restricted knowledge was collected for several hundred battles. Such databases have inherent limitations and restrict what comparisons may be usefully made, as was recognised by Helmbold [28]. However, the determination of advantage parameter (equation 27), its comparison with actual outcomes and correlation with force ratio, provides a useful check of a major consequence of Lanchester's Equations. Using the data from this alternative approach, Helmbold [34] also empirically discovered a relationship between the force ratio and the ratio of the difference squared between initial and final strengths:

$$\begin{gathered}\ln\left(\frac{x_0^2 - x_f^2}{y_0^2 - y_f^2}\right) = \alpha \ln\left(\frac{x_0}{y_0}\right) + \beta \\ \ln(HelmboldRatio) = \alpha \ln(ForceRatio) + \beta\end{gathered} \tag{28}$$

which is in contrast to that given by Lanchester's Equations (equation 2).

This equation was later shown to be a natural consequence of a particular type of mixed law Lanchester attrition process [35]. As previously mentioned, historical battles will likely involve data with widely varying attrition coefficients. Hartley [36] has considered the effect this will have on the expected relationship between initial and final strengths, and concluded that a sample of real data, with different attrition coefficients, should be clustered about a best fit relationship having the same form as the applicable attrition process. In other words, equation 28 should describe the trend observed from an ensemble of real data.

6. Fractal Lanchester Equations

One of the many complex combat models for which the Lanchester Equations have been developed as a metamodel is the cellular automata simulation MANA [37] [38]. Previous work by the author [4] [5] has shown that a fractal spatial distribution of forces metamodel produces a discrete time Markov process attrition expression, for which an equation of state very similar to equation 28 can be obtained. The present work considers the implications of that spatial metamodel to other parameters of interest in a Lanchester combat metamodel.

6.1 Equation of State

Application of the intermediate asymptotic approximation [38] produces the following attrition equations as the metamodel for MANA:

$$\begin{aligned} \frac{\Delta x}{\Delta t} &= -\frac{(a\Delta t)^{D_y/2}}{\Delta t} y \quad x(0) = x_0 \\ \frac{\Delta y}{\Delta t} &= -\frac{(b\Delta t)^{D_x/2}}{\Delta t} x \quad y(0) = y_0 \end{aligned} \tag{29}$$

let $a = R_0 b$, then:

$$\frac{x\Delta x}{y\Delta y} = \begin{Bmatrix} (a\Delta t)^{(D_y - D_x)/2} R_0^{D_x/2} \\ (b\Delta t)^{(D_y - D_x)/2} R_0^{D_y/2} \end{Bmatrix} = R_0^{(D_y + D_x)/2} (I_0 \Delta t)^{(D_y - D_x)/2} \tag{30}$$

Where the expressions in brackets are equivalent forms and result from collecting terms according to the fractal dimension of side *X* or side *Y*, while the rightmost expression is their geometric mean.

Previous work [38] has shown that the fractal dimension which describes the clustering of units on each side depends on the size of the forces involved and how closely the forces are examined. The metamodel proposed in [38] relates the force's spatial resolution to the time step of this Markov attrition process, with the result that:

$$(D_y - D_x)/2 = \frac{1}{2}\log_{\Delta t}\left(\frac{x_0}{y_0}\right) \tag{31}$$

Using Appendix A it is easy to demonstrate that:

$$(I_0 \Delta t)^{(D_y - D_x)/2} = \left(\frac{x_0}{y_0}\right)^{\frac{1}{2}\log_{\Delta t}(I_0 \Delta t)} \tag{32}$$

Let the force strengths $x >> 1$ so that $x(x+1) \approx x^2$, and approximate the discrete Markov process by a continuous time process. Equation 28 can be shown to result from substituting equation 32 into equation 30, setting $\alpha = ½\log_{\Delta t}(I_0 \Delta t)$ and $\beta = ¼(D_x + D_y)\ln R_0$ and summing the equation between its initial and final states.

This is an extension of previously published work [5], which had only considered special cases of equation 28.

6.2 Fractal Distribution Interpretation

The metamodel used here [38] assumes that each side's force is fractally distributed about the battlefield. The distance over which a unit can interact with units from the other side limits the spatial resolution for the fractal analysis. The force movement model links this with the time resolution in the Markov process. Hence the fractal dimensions that appear in the attrition expressions (equation 29) depend on the force sizes and the kill rates of individual units.

While equation 29 appears to depend on six parameters $\{x_0, y_0, a, b, D_x, D_y\}$, two of those parameters are determined by the other four. The choice of which two parameters to express in terms of the other four is arbitrary, being made according to what relationships between the remaining parameters are to be explored. The increased complexity in the dependence of casualty rates on the four independent parameters results from inclusion of the relationship between all six parameters, which arises from the fractal distribution of the forces in space [38]. The previous section approximated the discrete Markov process by a continuous time process, which can be expressed as:

$$\begin{aligned} a' &= \lim_{\Delta t \to 0} \frac{(a\Delta t)^{D_y/2}}{\Delta t} \\ b' &= \lim_{\Delta t \to 0} \frac{(b\Delta t)^{D_x/2}}{\Delta t} \end{aligned} \tag{33}$$

Where the attrition coefficients *a'* and *b'* represent kill rates for the forces as a whole, and are determined from the individual unit kill rates (*a* and *b*) moderated by battlefield congestion. Substituting equation 34 into equation 29, and taking the limit as $\Delta t \to 0$, produces Lanchester's original square law attrition equations (equation 1). The end result of the application of fractal spatial force distributions has been to include the spatial effects in the definition of new *force on force* attrition rate parameters:

$$\{x_0, y_0, a, b\} \Rightarrow \underbrace{\{x_0, y_0, a, b, D_x, D_y\}}_{\substack{\Downarrow \\ \{x_0, y_0, a', b'\}}} \tag{34}$$

An important result of this interpretation of the Fractal Attrition Equations (equation 29) as a *force on force* attrition expression incorporating battlefield congestion is that all the derived quantities from the original Lanchester model (sections 4 and 5 above) still apply if the simple attrition coefficients are replaced by the force level coefficients *a'* and *b'* defined in equation 33.

6.3 Attacker's Advantage

Previous work [35] on the origin of equation 28, considered a linear-logarithmic Lanchester attrition process, the general case of which can be written as:

$$\begin{aligned} \frac{dx}{dt} &= -f\left(\frac{x}{y}\right) y(t), \quad x(0) = x_0 \\ \frac{dy}{dt} &= -g\left(\frac{x}{y}\right) x(t), \quad y(0) = y_0 \end{aligned} \tag{35}$$

where the functions *f* and *g* limit the ability of each side to apply its strength due to battlefield congestion (popularly known as the law of diminishing returns). This approach also involves the definition of new force level attrition rate parameters that depend on the force ratio, which is the same conclusion reached in the previous section for fractal spatial force distributions.

Equation 28 has been shown to be consistent with the patterns observed in historical data [34] as seen in Figure 1. The line shown is the best fit to the entire data set without consideration of the winning side's posture. This is quite an important distinction as, again following Taylor [14] for clarity, the losers were defined as side X and the winners as side Y. The colour of the data points in Figure 1 indicates the posture of the winning side (Y).

If similar lines of best fit were determined separately for each winning side's posture, the gradient of each (α value), at 95% significance, cannot be regarded as different. Figure 1 also shows that the average values for ln(HelmboldRatio) for defenders are higher than the corresponding values for attackers (larger β value). It should be noted that for the overall distribution $\beta \approx 0$. Finally, Figure 1 shows that the ln(HelmboldRatio) for attackers is pretty evenly distributed about the overall line of best fit, while the corresponding values for defenders are heavily clustered above that line.

Figure 1 indicates that for a winning defender $\beta > 0$. If $\beta = ¼(D_x + D_y)\ln R_0$, this means $R_0 > 1$. A larger value for R_0 or D_x and D_y will produce the larger value of β observed for winning defenders in Figure 1.

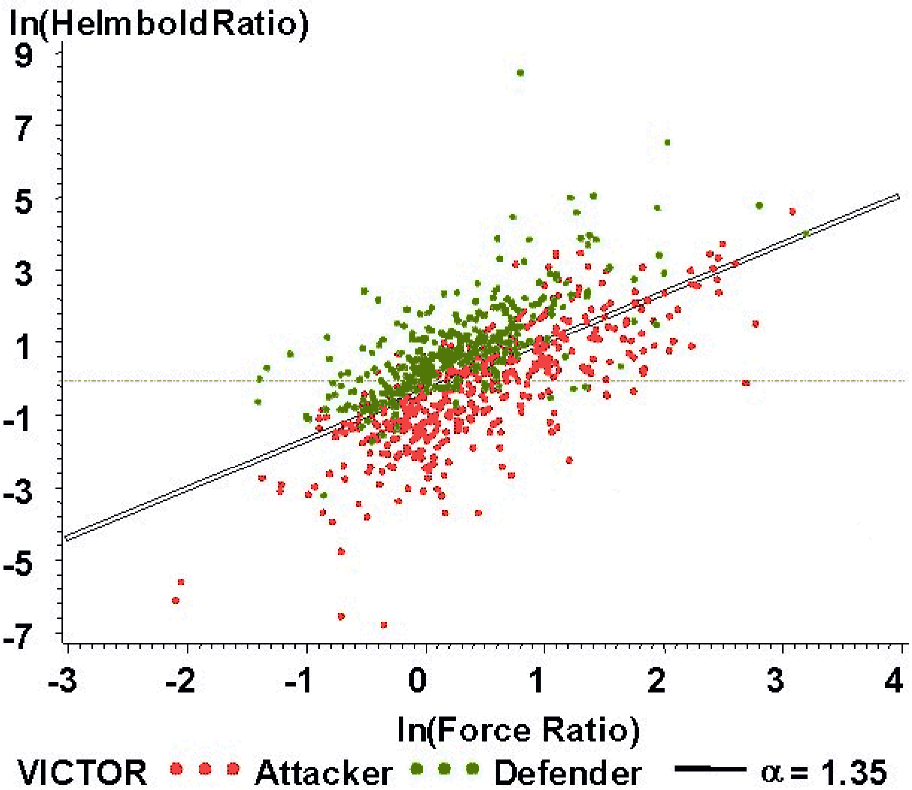

Figure 1: Helmbold's Relationship, from Hartley [7]

These conditions for successful defenders can be interpreted to mean:

- the kill rate of defenders is less than the kill rate of attackers;
- for the same force ratio, the kill rate of defenders is less when defenders win than when the attackers win;
- the attackers have an insufficient force concentration.

All of which are consistent with conventional military wisdom regarding the conditions required for a successful attack.

Lastly, the observation from Figure 1that attacker wins are more evenly distributed about the overall mean than defender wins implies that the conditions determined above constitute necessary (the defender is unlikely to win when they are not met) but not sufficient conditions (attackers can still win even when the conditions are met) for a successful defence.

However, this last observation probably results from the advantage that holding the initiative confers. The attacker can choose whether to attack or not. Examples where an attacker did not think the outcome worth the cost and chose not to attack do not appear in the database. The "missing" data entries below the line of best fit in Figure 1 are the attacker's real advantage and comes from holding the initiative. Figure 1, therefore, also demonstrated the advantage in taking the initiative.

6.4 Intensity

From equation 11, the combat Intensity controls the rate at which the time solutions to Lanchester's Equations evolve. Using the *force on force attrition* equations (29), defines the Intensity:

$$I^2 = a'b' = a^{D_y/2} b^{D_x/2} \Delta t^{(D_x+D_y)/2-2} = \begin{cases} (ab)^{D_x/2} (a\Delta t)^{(D_y-D_x)/2} \Delta t^{D_x-2} \\ (ab)^{D_y/2} (b\Delta t)^{-(D_y-D_x)/2} \Delta t^{D_y-2} \end{cases} \tag{36}$$

where the rightmost expressions are equivalent forms that can be obtained by collecting terms according to the fractal dimension of side X or side Y. Taking their geometric mean gives:

$$I = I_0^{(D_x+D_y)/4} R_0^{(D_y-D_x)/8} \Delta t^{(D_x+D_y)/4-1} \tag{37}$$

Using equation 31 to evaluate the difference between the fractal dimensions and applying the technique from Appendix A gives:

$$I = I_0^{(D_x+D_y)/4} \left(\frac{x_0}{y_0} \right)^{1/8 \log_{\Delta t} R} \Delta t^{(D_x+D_y)/4-1} \tag{38}$$

The intensity of a combat should increase, albeit slowly, when the force ratio favours the more effective side. The author is not aware of any systematic studies of the relationship between Intensity and force ratio to date. Equation 38 also predicts an increase in Intensity with the fractal dimensions for both sides. At first sight this appears strange, as a larger fractal dimension results from greater force dispersion which usually lower the intensity of combat. On closer examination, the original metamodel [38] assumes a uniform level of activity for all of a side's units. Hence each unit has the same propensity to engage the enemy and the attrition rate will depend in large part on the number of such encounters. This will be increased by each side's improved spatial coverage, or larger fractal dimension. Hence it seems that the fractal dimension that should be used here describes more than the force's geometric distribution, it describes the distribution of activity throughout that side.

6.5 Relative Effectiveness

The force level relative effectiveness is given by the right hand side of equation 30. Using equation 31 to evaluate the difference between the fractal dimensions and applying the technique from Appendix A gives the right hand side below:

$$R' = \frac{a'}{b'} = R_0^{(D_y+D_x)/2} (I_0 \Delta t)^{(D_y-D_x)/2} = R_0^{(D_y+D_x)/2} \left(\frac{x_0}{y_0} \right)^{\frac{1}{2}\log_{\Delta t}(I_0 \Delta t)} \tag{39}$$

Similarly to the effect of dispersion/concentration for combat Intensity, relative effectiveness increases with dispersion (increasing fractal dimension) due to the increased number of unit encounters. The author is not aware of any systematic studies of the relationship between relative effectiveness and force ratio to date. This may be in part due to the standard Lanchester model prediction that relative effectiveness is a constant.

6.6 Advantage Parameter

The definition of the defender's Advantage Parameter V given in section 5.3 was:

$$V = \ln \mu = 2 \ln \left(\frac{1-(x/x_0)^2}{1-(y/y_0)^2} \right) \tag{40}$$

It can be easily shown that:

$$\ln \left(\frac{x_0^2 - x^2}{y_0^2 - y^2} \right) = 2 \ln \left(\frac{x_0}{y_0} \right) + \ln \left(\frac{1-(x/x_0)^2}{1-(y/y_0)^2} \right) \tag{41}$$

where the left hand side is readily obtained from equation 28 for the force level defender's Advantage:

$$V = 2\beta + 2(\alpha - 2) \ln \left(\frac{x_0}{y_0} \right) \tag{42}$$

A systematic study of the value of α [7], from which Figure 1 was sourced, has found its mean value as 1.35 with a maximum possible value of less than 2. The second term in equation 42 therefore reduces the defender's advantage when the attackers force ratio is increased. A logarithmic dependence on force ratio is also predicted by the standard Lanchester theory and has been observed in the historical data [28].

6.7 Termination Conditions

It was demonstrated in section 6.2 that defining force level attrition coefficients *a'* and *b'* incorporating battlefield congestion enables all the results of section 5 to be reused, by substituting them for the single unit attrition coefficients *a* and *b*.

As a result, equations 23 to 26 which describe the battle termination conditions including the probability that the defender won the battle, its duration and the number of predicted survivors for both sides are still applicable. All that is required is that force level values for the Combat Intensity *I* (equation 38) and relative effectiveness *R* (equation 39) must be substituted for the single unit values I_0 and R_0.

As no worthwhile simplification of those equations, or new relationships or dependencies, have been found to result from this action, those equations are omitted here.

7. Conclusions

The present work has shown that Lauren's fractal metamodel [38] of the distribution of forces in space results in general *force on force* attrition expressions (equation 29) which include the effect of that distribution. Most importantly, they describe the ability of each side to apply its strength effectively, due to battlefield congestion. This is reflected in the logarithmic dependence of the Combat Intensity (equation 37), Relative Effectiveness (equation 39) and Defender's Advantage parameter (equation 42) on the initial force ratio.

It has confirmed the functional form for the equation of state (equation 28) which had previously been derived for a few special cases [5]. This is consistent with previous derivations of that expression [35], which had also linked the functional form to limiting a forces ability to apply its strength. This relationship agrees very well with Helmbold's historical data analysis [34]. More importantly, the conclusion that spatial effects in combat are necessary to obtain this equation of state instead of Lanchester's original equation of state (equation 2), would suggest that the major shortfall of Lanchester's combat model is the lack of a mechanism describing the effects of movement and spatial limitations on interaction between forces. In contrast, while this shortcoming did make Taylor's list of issues with Lanchester's equations [14], it did not make the top ten.

An important consequence of the model proposed here as a *force on force* attrition model incorporating battlefield congestion, is that all the previously derived quantities from the original Lanchester model [14] still apply if the simple attrition coefficients are replaced by the force level coefficients *a'* and *b'* defined in equation 33.

This work appears to be the first report that the force level Combat Intensity and Relative Effectiveness, both of which are key parameters in the results of Lanchester's combat model, have a logarithmic dependence on force ratio. This has yet to be investigated using available historical data.

The major shortcoming of the present work is the lack of processes to determine the attrition rate coefficients (*a* and *b*) and fractal dimensions (D_x and D_y). However, it should be emphasised that these limitations, and indeed most of the limitations on Taylor's list, apply not only to Lanchester models of combat but to all models of combat. In spite of all these issues, the model developed here is useful as a combat metamodel to illustrate relationships between select combat parameters and facilitate comparison between model and historical data as indeed are all Lanchester models.

8. References

1 *Aircraft in Warfare: The Dawn of the Fourth Arm*, F W Lanchester, Constable & Co., London, 1916.

2 *The Calculus of Conventional War*, J M Epstein, The Brookings Institution, Washington DC, 1985.

3 *Technology for the USN and USMC 2000-2035 Volume 9 Appendix I*, P K Davis, D B Blumenthal and D Gaver, National Academy of Science, Washington DC, 1997.

4 *Verification and Validation of the Fractal Attrition Equation*, N D Perry, DSTO-TR-1822, Australian Department of Defence, 2006.

5 *Art of war hidden in Kolmogorov's equations*, M K Lauren, G C McIntosh, N D Perry and J Moffat, Chaos **17**, 2007.

6 *Methods of Operations Research*, G Kimball and P Morse, MIT Press, 1951.

7 *Predicting Combat Effects*, D S Hartley III, INFORMS, Linthicum Md, 2001.

8 *Lanchester Attrition Processes and Theatre-Level Combat Models*, A F Karr, Mathematics of Conflict, pp 89-126, Elsevier (North-Holland), 1983.

9 *Combat Analysis-Lanchester and the Battle of Trafalgar*, B McCue, PHALANX, MORS, 1999.

10 *Numbers, Predictions and War*, T N Dupuy, MacDonald and Janes, London, 1979.

11 *Situational Force Scoring (SFS) in the Joint Integrated Contingency Model (JICM)*, B Wilson, J Rothenburg, RAND, 1999.

12 *DoD Modelling and Simulation Glossary*, DOD 5000.59-M, US Department of Defence, 1998.

13 *Scaling, Self-similarity and Intermediate Asymptotics*, G I Barenblatt, CUP, 1996.

14 *Lanchester Models of Warfare Volumes 1 and 2*, J G Taylor, Operations Research Society of America, 1983.

15 *Military Operations Research*, N K Jaiswal, Kluwer, 1997.

16 *Combat modelling with partial differential equations*, V Protopopescu, R T Santoro and J Dockery, European Journal of Operations Research, **38**, 178-183, 1989.

17 *Combat Modelling with Spatial Effects*, N K Jaiswal and B S Nagabhushana, Computers and Operations Research, **21**, 615-628, 1994.

18 *Lanchester Type Models of Warfare*, H K Weiss, Proc.First International Conference on Operations Research, ORSA, 1957.

19 *Models, Data and War*, US General Accounting Office, PAD-80-21, 1980.

20 *The Calculus of Conventional War*, J M Epstein, Brookings Institute, 1985.

21 *Rates of Advance in Historical Land Combat Operations*, R L Helmbold, US Army Concepts Analysis Agency, CAA-RP-90-1, 1990.

22 *Requirements for the theory of combat*, H K Weiss, Mathematics of Conflict, Elsevier, 1983.

23 *A Modification of Lanchester's Equation*, R Helmbold, Ops. Res. **12**, 857-859, 1965.

24 *Stochastic Attrition Models of Lanchester Type*, A F Karr, report no. P-1030, Institute for Defense Analyses, Arlington VA, 1974.

25 *Decision in Battle: Breakpoint Hypotheses and Engagement Termination Data*, R L Helmbold, RAND R-772-PR, 1971.

26 *Modelling the Breakpoint Phenomena*, C F Hawkins, Signal, July, 1989.

27 *Termination Decision Rules in Combat Attrition Models*, N K Jaiswal and B S Nagabhushana, Naval Combat Logistics 42, 419-433, 1995.

28 *Historical Data and Lanchester's Theory of Combat*, R L Helmbold, US Army Training and doctrine Command, AD480975, 1961.

29 Properties of the Battle Trace, D L Bitters, Military Operations Research **5**, 67, 2000.

30 *A verification of Lanchester's Law*, J H Engel, Operations Research **2**, 163-171, 1954.

31 *Lanchester Models of the Ardennes Campaign*, J Bracken, Warfare Modelling, 417-435, MORS, 1995.

32 *Quantification of the Battle of Kursk*, W J Bauman, USCAA ,1999.

33 *Lanchester's Square Law in Theory and Practice*, R L Johnson, US CGSC , AD A225 484-5, 1990.

34 *Air Battles and Land Battles- a common pattern?*, R L Helmbold, AD 718 975, DTIC, 1971.

35 *Historical Support for a Mixed law Lanchester attrition Model*, D S Hartley III and K L Kruse, Oak Ridge TN K/DSRD-113, 1989.

36 *Predicting Combat Effects*, D S Hartley III, INFORMS, 2001.

37 *Metamodels and Emergent Behaviour in Models of Conflict*, J Moffat and M Passman, Simulation Study Group Workshop, March, 2002.

38 *A Metamodel for describing the outcomes of the MANA Cellular Automaton Combat Model based on Lauren's attrition equation*, M K Lauren, DTA Report 205, NR 1409, ISSN 1175-6594, New Zealand Defence Force, 2005.

Appendix A: Manipulating Logarithmic Exponents

The following relationship occurs commonly throughout the present work.

$$z = x^{a \log y} \qquad \text{A1}$$

The manipulations to permit a change of variable are given here

$$\log z = \log x^{a \log y} \qquad \text{A2}$$

$$\log z = a \log y \log x = a \log x \log y \qquad \text{A3}$$

$$\log z = \log y^{a \log x} \qquad \text{A4}$$

and hence

$$z = y^{a \log x} \qquad \text{A5}$$

DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION DOCUMENT CONTROL DATA	1. PRIVACY MARKING/CAVEAT (OF DOCUMENT)

2. TITLE	3. SECURITY CLASSIFICATION (FOR UNCLASSIFIED REPORTS THAT ARE LIMITED RELEASE USE (L) NEXT TO DOCUMENT CLASSIFICATION)
Fractal Effects in Lanchester Models of Combat	Document (U) Title (U) Abstract (U)
4. AUTHOR(S)	5. CORPORATE AUTHOR
Nigel Perry	DSTO Defence Science and Technology Organisation Fairbairn Business Park Department of Defence Canberra ACT 2600 Australia

6a. DSTO NUMBER	6b. AR NUMBER	6c. TYPE OF REPORT	7. DOCUMENT DATE
DSTO-TR-2331	AR-014-597	Technical Report	August 2009

8. FILE NUMBER	9. TASK NUMBER	10. TASK SPONSOR	11. NO. OF PAGES	12. NO. OF REFERENCES
2009/1030745	07/034	LRR	23	38

13. URL on the World Wide Web	14. RELEASE AUTHORITY
http://www.dsto.defence.gov.au/corporate/reports/DSTO-TR-2331.pdf	Chief, Joint Operations Division

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Lanchester's equation, Combat simulation, Force on force attrition model

19. ABSTRACT
Lanchester's Equations are one of the most misunderstood and misused models of combat, yet they remain in widespread use as the combat mechanism behind many simulation systems. Previous work by the author examined the impact of a fractal distribution of forces on Lanchester's theory of combat. The present work extends that examination to conceptual issues regarding interpretation of Lanchester's Equations and to additional parameters beyond those examined previously.