

# Coherent atom optics with optical potentials: a summary of new phenomena with Bose-Einstein Condensates at the University of Arizona

Final Project Report for ARO award DAAD19-03-1-0368

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## Abstract

The Bose-Einstein condensation (BEC) laboratory at the University of Arizona has built a program of research that focuses on two areas of BEC physics. First, we developed techniques necessary to experimentally investigate dynamic quasi-condensed states. With our system, we have instituted new directions of experimental BEC physics: studying spontaneous formation of topological defects during the BEC phase transition, and studying transitions to superfluid turbulent states. As phase transitions and turbulence occur throughout natural phenomena, this work is further relevant to a wide range of subjects in physics. Our first published results have been enabled using the support of ARO award DAAD19-03-1-0368; our research in these areas is also ongoing. Our second main efforts have revolved around refining capabilities for studying and manipulating vortices and ultra-cold gases in disc-shaped and annular potentials using optical potentials, with a particular eye towards quantum engineering of vortex and persistent current states. These efforts involve often separate areas of investigation: manipulating condensates with laser light, two-dimensional physics, fluid dynamics and turbulence, phase-transition dynamics, and vortices and persistent currents. Our efforts are collaborative, with theoretical work at multiple universities and experimental work in Arizona progressing in tandem.

## Project report

Among the research goals of the Bose-Einstein condensation lab at the University of Arizona is the development of advanced atom-optical techniques for optically manipulating Bose-Einstein condensates (BECs). Our goals have been oriented around the advancement of the following technical topics: (1) manipulation of BECs by using optical potentials in combination with magnetic field potentials; (2) a broad understanding of phenomena associated with merging and combining BECs, as might be sought with atom-beam combiners on atom chips; and (3) a thorough understanding of not only the process of Bose-Einstein condensation in two and three dimensions, as well as other various potential-well shapes, but also the loss of coherence and the transition to superfluid turbulent states. These last issues will become increasingly more important to understand as experiments achieve finer levels of control over BECs. These goals are motivated by a technological need to further advance the field of atom optics via the exploration of new experimental atom-optical techniques and matter-wave mode control, as well as inherent interest in exploration of previously unknown or unobserved physical phenomena accessible with coherent matter wave systems. In pursuing our goals, experimental work accomplished with this ARO award has included studying the BEC phase transition dynamics, studying topological defects that appear in the merging of condensates, building experimental capabilities to manipulate as well as help us better understand BECs, and building strong relationships with theoretical collaborators. Our accomplishments in these areas are summarized below.

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# 1 Bose-Einstein condensation at the University of Arizona.

[D.R. Scherer, *Ph.D Thesis, University of Arizona, 2007.*]

With partial support from ARO award DAAD19-03-1-0368, we completed construction of a robust BEC apparatus. We achieved our first  $^{87}\text{Rb}$  BECs near the end of 2005. Our apparatus consists of a differentially pumped two-cell vacuum system. A gas of Rb atoms, provided by SAES dispensers, fills a glass cell where laser cooling and magneto-optic trapping occur. In our experiments, a few billion  $^{87}\text{Rb}$  atoms are first laser-cooled, and atoms in the  $|F = 1, m_F = -1\rangle$  ground state are trapped in a spherical quadrupole magnetic field. A series of overlapping magnetic-field coils spans the length of the vacuum chamber, enabling the magnetic transfer of trapped atoms into a second glass cell [1]. A time-averaged orbiting potential (TOP) trap [2] is established, and radio-frequency-induced evaporative cooling proceeds as the bias magnetic field of the TOP trap is simultaneously reduced to 5 G. This method produces BECs of over  $10^6$  atoms; however, we pause cooling prior to condensation in order to utilize a final stage of evaporative cooling that takes place in a weak magnetic trap, one of the unique features of our experiments. Before condensation, the magnetic fields are adjusted such that the trap frequencies adiabatically relax to  $\sim 8$  Hz in the radial (horizontal) plane, and  $\sim 15$  Hz along the axial ( $z$ ) direction. BECs of up to  $\sim 10^6$  atoms are then created *directly in the weak trap* with a final 6 seconds (nominal) of cooling. Details of our experimental procedures are given in the PhD thesis of D.R. Scherer [3], available at our research webpage: <http://www.optics.arizona.edu/anderson/>.

## 2 Vortex Formation by Merging of Multiple Trapped Bose-Einstein Condensates.

[D.R. Scherer, C.N. Weiler, T.W. Neely, and B.P. Anderson, *Phys. Rev. Lett.* 98, 110402 (2007).]

In this work [4], the basic BEC creation procedure of Sec. 1 was modified by the addition of a blue-detuned laser beam propagating along the tightly-confining ( $z$ ) axis of the TOP trap. The beam was shaped using a transmission mask [Fig. 1(b)] that was imaged onto the center of the trap. The sum of the magnetic and optical potentials created a triple-well trap, with three trap minima in the radial trapping plane arranged symmetrically about the  $z$  axis, as shown in Fig. 1(a). The laser beam was added to the harmonic TOP trap just prior to the final 6-s stage of evaporation in order to directly create three independent, isolated BECs out of a single thermal cloud [Fig. 1(d)]. This experiment demonstrated (i) the formation of vortices in a single BEC created from the controlled merging of the three initial BECs, and (ii) vortex formation during the growth of a single BEC in a potential well with the optical barriers present, but too weak to maintain isolated condensates throughout evaporative cooling. In both cases, a three-BEC merging process was involved, although in the latter case merging naturally occurred during BEC growth.

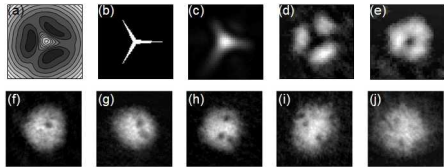


Figure 1: **Vortex formation by BEC merging.** (a) Potential energy contours of the 3-well trap in a horizontal slice through the trap center. (b) Transmission mask used to create the optical barrier, imaged in (c). (d) *In situ* phase-contrast image of three isolated BECs created in the 3-well trap. (e) *In situ* phase-contrast image of a single BEC that formed from the merging of three initial seed BECs. (f)-(j) Absorption images of expanded BECs, each created by the merging of three BECs. Vortex cores are visible as the “holes” in these column density images.

In these experiments, atomic fluid flow was established when BECs of indeterminate relative phases merged together; occasionally, merging would establish flow in an azimuthal direction, and hence vor-

ticity about the  $z$  axis would be generated. Single-vortex creation due to merging is expected for 25% of the BECs created, based on the initial indeterminacy of relative quantum phases [5, 6, 7]. Experimental results were consistent with statistical expectations. For fast merging, increased numbers of vortices were seen, but these rapidly decayed, a process attributed to vortex-antivortex annihilation.

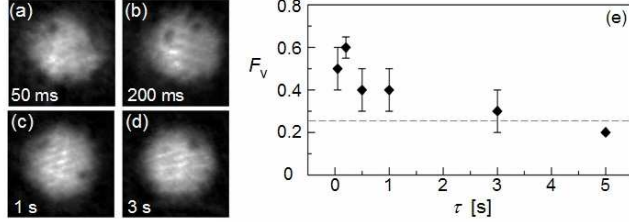


Figure 2: (a)-(d) Sample absorption images of BECs created by merging 3 BECs over the time scales indicated. Vortices are visible in these expansion images. (e)  $F_v$ , the fraction of images showing at least one vortex, plotted against barrier ramp-down time (merging time)  $\tau$ . For slow merging (large  $\tau$ ),  $F_v$  is consistent with a 25% probability of observing a vortex.

### 3 Spontaneous vortices in the formation of Bose-Einstein condensates.

[C.N. Weiler, T.W. Neely, D.R. Scherer, A.S. Bradley, M.J. Davis, and B.P. Anderson, *Nature* **455**, 948 (2008).]

[C.N. Weiler, *Ph.D Thesis, University of Arizona, 2008.*]

Our 2007 Physical Review Letters article [4] described how vortices may be created in the merging of condensates. Subsequent efforts in our laboratory, completed early in 2008, showed that vortices may spontaneously form during the formation of a single 3D condensate. The physics involved in this process may be related to the Kibble-Zurek (KZ) mechanism, a topic broadly applicable to continuous phase transitions in a wide variety of physical systems [8, 9, 10]. The KZ mechanism provides a prescription for estimating a correlation length  $\xi$  and hence the density of defects, proportional to  $1/\xi^2$ , that may spontaneously form in the phase transition. For a continuous phase transition that proceeds quasi-statically,  $\xi$  diverges at the critical point and therefore no defects are expected. However, in the KZ mechanism the phase transition occurs over a finite time, and the system falls out of equilibrium when the thermalization (or relaxation) rate drops below a quench rate  $1/\tau_Q$ . At this point  $\xi$  is frozen in and essentially remains constant through the critical point. A principle result is that faster quenches lead to an earlier freeze-in time, and hence smaller values of  $\xi$  and higher defect densities. For our experimental parameters,  $\xi \sim 0.6\mu\text{m}$ , about 6 times smaller than the radial oscillator length, and  $\tau_Q \sim 5\text{s}$ .

The KZ mechanism is appealing due to its potential for characterizing a wide variety of phase transitions, irrespective of the microscopic processes involved. An alternative model with specific

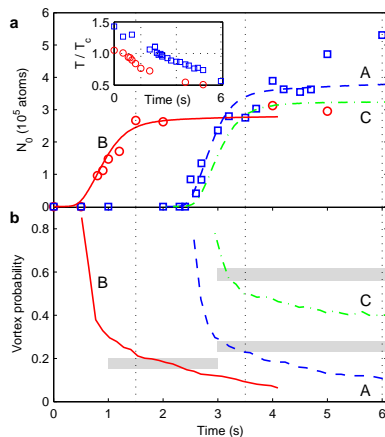
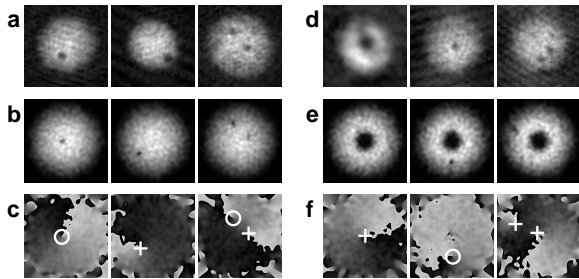


Figure 3: **Condensate formation and vortices.** (a) Condensate number  $N_0$  versus time. Blue squares (red circles) indicate experimental data for two quenches, and lines indicate corresponding numerical simulations. The green dot-dashed line is the numerical result for a toroidal trap. Vertical dotted lines indicate the observation times for which experimental statistics are generated. Inset: experimentally measured temperatures. (b) The probability of finding at least one vortex passing through the  $z = 0$  plane plotted for all three simulated quenches. Gray regions indicate the experimental measurement range for each data set.

application to condensation dynamics in a homogeneous Bose gas describes the transition from a weak-turbulent (kinetic) stage to strong-turbulent (coherent) state [11, 12, 13, 14, 15]. In this scenario, as energy is removed from the system the low-energy atomic field modes become macroscopically occupied. Destructive interference between these modes leads to nodes in the field, which appear as lines of zero atomic density. Subsequently, a *quasi-condensate* having local coherence but no long-range coherence grows around the lines of zero density, which then evolve into well-structured vortex cores spread throughout the quasi-condensate as coherence builds. Such a state has been referred to as a superfluid turbulent state. Eventually the turbulent superfluid relaxes into equilibrium and a true condensate with global phase coherence is achieved.

In collaboration with theorists Dr. Matthew Davis (Univ. of Queensland) and Dr. Ashton Bradley (Univ. of Otago), we experimentally and numerically characterized spontaneous vortex formation rates. In the experiment, we collected images of condensates formed under various growth conditions, and determined the fraction of images that clearly showed at least one vortex core. Such fractions were nominally in the 0.2 range (i.e., 20% of the time a vortex would form). On the numerical front, Davis and Bradley used the stochastic Gross-Pitaevskii Equation (SGPE) [16, 17, 18] formalism to simulate the process of cooling a gas through the BEC phase transition. This model captures the physics of BECs in a finite-temperature environment near the critical point. In this approach, numerical runs are meant to give results similar to single experimental runs, while an average of numerous simulation runs approximates expectation values and quantum kinetic theory. The 3D simulations showed the formation of spontaneous vortex formation and the further dynamics of the vortices, including merging, annihilation, and damping. (We will continue working together with Dr. Davis and Dr. Bradley on continuing projects in our laboratory.)

The results of both the experimental and numerical approaches were in remarkably good quantitative agreement for condensate growth in 3D harmonic traps and 3D toroidal traps (Fig. 3). Example images are shown in Fig. 4. Our results are also consistent with both the KZ scenario for topological defect trapping during continuous phase transitions, as well as the superfluid turbulence models. The results of our studies were published in *Nature* [19].



**Figure 4: Vortices in harmonic and toroidal traps.** (a) Expansion images of BECs created in a harmonic trap showing spontaneously formed vortices. (b,c) Sample simulation results showing in-trap integrated column densities along the  $z$  symmetry axis (in (b)) and associated phase profiles in the  $z = 0$  plane (in (c)). Vortices are indicated by crosses and circles at  $\pm 2\pi$  phase windings. (d) Left image: experimental image of a BEC in a toroidal trap. Remaining images: vortices in BECs after expansion from a toroidal trap. (e,f) Simulations of BEC growth in a toroidal trap show vortices (as in (b),(c)) and persistent currents.

## 4 The merging of experimental capabilities. *[ongoing work.]*

Subsequent to our work on the dynamics of the BEC phase transition in a 3D trap, we set out to accomplish similar work in a 2D system. We have obtained new data for this project, and are working with our theoretical collaborators to bring this project to completion. In this project, we have accomplished the following technical tasks:

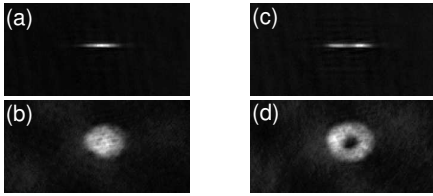
(1) We perfected capabilities for confining thermal atoms in a trap formed from a 3D magnetic trap and a sheet of laser light, creating a pancake or disc-shaped atom trap (henceforth our "2D trap").

(2) We efficiently evaporatively cool atoms through the condensation threshold in this trap, obtaining BECs with up to 2 million atoms;

(3) We have implemented expansion imaging of condensates created in our 2D trap such that vortices are visible.

(4) We have implemented new tools for the laser manipulation of condensates, one of the primary goals of our ARO-sponsored research.

We emphasize that our trap is particularly unique because it is both cylindrically symmetric and is formed from a single sheet of light rather than an optical lattice. To our knowledge, only one other group (NIST Gaithersburg) is currently using similar a atom trapping configuration. Our recent success with expansion imaging of atoms from the 2D trap is also particularly unique for our 2D system. A BEC in our 2D trap is shown from the side and from the  $z$  axis in figure 5(a) and (b).



**Figure 5: BEC in the 2D trap.** (a) Side view of BEC in 2D trap. (b) Top view of BEC in 2D trap. (c) Side view of BEC in washer trap. (d) Top view of BEC in washer trap. The radius of the BEC in the horizontal ( $z=0$ ) plane is about 50 microns. The trap aspect ratio is at least 20:1.

We have further extended our capabilities by combining our 2D trap with a laser beam propagating along the  $z$  axis to make a toroidal potential. This potential creates BECs in the shape of a washer, as shown in figure 5(c) and (d). This potential is currently permitting us to study persistent currents in a flat trap geometry. Not only is such a geometry relevant and promising for applications, but it is new to the BEC research field.

We have also built the ability to spatially and temporally control the positioning of two focused laser beams for addressing the atoms in the trap. We use two beams reflecting from two mirror mounts, with the axes of each mount controlled by piezo-electric transducers. We have the ability to create such time-averaged potentials for use in condensate manipulation, or we may more slowly modulate the beams in order to control the BEC atoms in other ways. Using these capabilities, we have initiated a study of spontaneous vortex formation in the flat trap BEC transition. The completion of this project is underway. Results so far indicate strong signatures of exciting new physics, with a proliferation of vortices obtained after a temperature quench, and reliable trapping of persistent currents in our washer trap.

## 5 Dynamics of Vortex Formation in Merging Bose-Einstein Condensate Fragments.

[R. Carretero-González, B.P. Anderson, P.G. Kevrekidis, D.J. Frantzeskakis, and C.N. Weiler, Phys. Rev. A 77, 033625 (2008).]

In order to better understand vortex formation by BEC merging (Section 2), we collaborated with theorists Panos Kevrekidis, Ricardo Carretero-González, and Dimitris Frantzeskakis to numerically simulate BEC merging [20]. Modeling was based on the trap geometry, barrier shapes, and atom numbers of our experiments. BEC dynamics were simulated with the 2D and 3D Gross-Pitaevskii equation (GPE) [21] for a range of relative phases and merging rates. The numerical results show that the main vortex formation mechanisms postulated in the experimental paper are reproduced in the numerics; namely, with initially indeterminate relative phases, single vortices are observed  $\sim 25\%$

of the time when three BECs merge.

## 6 Structure and stability of two-dimensional Bose-Einstein condensates under both harmonic and lattice confinement.

[K.J.H. Law, P.G. Kevrekidis, B.P. Anderson, R. Carretero-González, and D.J. Frantzeskakis, J. Phys. B: At. Mol. Opt. Phys. 41, 195303 (2008).]

This work numerically investigated BECs confined in two dimensions by both a cylindrically symmetric harmonic potential and an optical lattice with equal periodicity in two orthogonal directions [22]. The goal of the study was to examine the stability of the lowest-energy states. It was found that the ground state as well as some of the excited states may be stable or weakly unstable for both attractive and repulsive interatomic interactions. Higher-lying excited states were typically found to be increasingly more unstable. This was a numerical project, without an experimental component. However, it is briefly mentioned here to emphasize that we are in the initial stages of a long-term experiment-theory collaboration, and that these initial theoretical projects are a foundation for continued projects that will involve experimental observations as well as numerical.

## 7 Impacts of the accomplished research

**Student Research and Education.** The College of Optical Sciences provides students with research and academic experience and training in a wide range of Optics subjects, from lens design to laser cooling and Bose-Einstein condensation. The experiments accomplished with this ARO award have allowed the continued development of a broad research program that explores the relationships between atoms and light, a fundamental component of our more general emphasis on Optics. The PI has also taught graduate-level classes in optical physics, including a laboratory course on lasers and solid-state devices, and a required introductory class on quantum mechanics and quantum optics designed primarily for *optical engineers* rather than physicists. The PI currently teaches a quantum mechanics course for graduate students specializing in optical physics research, and a graduate course in Atom Optics for all optics students, regardless of specialty area. The education of students is enhanced in these courses by incorporating discussions of current research topics into the curriculum. Broad student educational experiences are also important for future technological advancement, as many of our graduates become technical leaders in Optics. As the fields within Optical Physics continue to take enormous strides, our students will be prepared in this increasingly important component of technology.

**Scientific Community.** Results from the completed projects have been disseminated through refereed journal articles, technical meetings, seminars, and colloquia. Multiple ongoing collaborations with theorists have been established, and are envisioned to continue to carry broad impact within the scientific community as such give-and-take experimental-theoretical collaborations are rare. Our previous results will likely become a foundational component in continued development of a comprehensive understanding of classical and quantum physics in 3D and 2D trapped gases. Our work has also started to have an even broader scientific impact, as the concepts involved are truly interdisciplinary: our experiments have involved concepts in AMO physics, condensed matter physics, and fluid dynamics. From the first years of BEC experimentation, research goals have drawn from many areas of physics, and broad impacts of research have extended well beyond atomic physics. Such interdisciplinary efforts are often at the heart of emerging new fields of physics, and we will continue to push our efforts along these fascinating frontiers.

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