TOWARD A MATHEMATICAL THEORY OF COUNTERTERRORISM

The Proteus Monograph Series

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The Proteus Management Group

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Toward a Mathematical Theory of Counterterrorism

Building the Perfect Terrorist Cell

by

Jonathan David Farley
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About the Author

Professor Jonathan David Farley is in the department of mathematics at the California Institute of Technology. He has formerly been a Science Fellow at Stanford University's Center for International Security and Cooperation and a professor at the Massachusetts Institute of Technology. *Seed* Magazine named him one of "15 people who have shaped the global conversation about science in 2005."

He is the 2004 recipient of the Harvard Foundation's Distinguished Scientist of the Year Award, a medal presented on behalf of the president of Harvard University in recognition of "outstanding achievements and contributions in the field of mathematics." The City of Cambridge, Massachusetts (home to both Harvard University and MIT) officially declared March 19, 2004 to be "Dr. Jonathan David Farley Day." In 2001-2002, Dr. Farley was a Fulbright Distinguished Scholar to the United Kingdom. He was one of only four Americans to win this award that year. He obtained his doctorate in mathematics from Oxford University in 1995, after winning Oxford's highest mathematics awards, the Senior Mathematical Prize and Johnson University Prize, in 1994. Professor Farley graduated *summa cum laude* from Harvard University in 1991 **with the second-highest grade point** average in his graduating class (earning **29 A's and 3 A-'s**).

Professor Farley's field of interest is lattice theory. He recently solved a problem posed by MIT professor Richard Stanley that had remained unsolved since 1981, a problem in "transversal theory" (posed by Richard Rado) that had remained unsolved for 33 years, and a problem from the 1984 Banff Conference on Graphs and Order that had remained unsolved for 22 years. Some of Dr. Farley's previous mathematical accomplishments include the resolution of a conjecture posed by Richard Stanley in 1975, and the solution to some problems in lattice theory that had remained unsolved for 34 years.

Professor Farley's work applying mathematics to counterterrorism has been profiled in *The Chronicle of Higher Education*, in *The Times Higher Education Supplement* in Great Britain, in *The Economist* Magazine, in *USA Today* and Associated Press newspaper articles across the United States, on Fox News Television, and on Air America Radio. Dr. Farley has been an invited guest on BBC World News Television and National Public Radio.

He is Chief Scientist of Phoenix Mathematics, Inc., a company that develops mathematical solutions to homeland security-related problems.

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How to Read This Paper

This is a mathematics monograph about a non-mathematical subject. As a mathematics paper, we will include theorems and proofs. It is not expected that the non-mathematically-inclined reader will wade through these proofs, or even the more symbol-laden definitions or statements of theorems. We do hope, however, that no reader will be dissuaded by the more technical portions of this paper, that he instead will choose to pass over them if the waters appear too rough; and we hope that all readers will at least be able to appreciate our main conclusions, the definitions of *poset*, *cutset*, and *tree*, and our elucidation of the structure of the "perfect" tree. At the same time, we hope more mathematical audiences will forgive the simplicity and verbosity of our exposition. When we write statements like "let $b \geq 2$," we mean, "Let b be a positive integer greater than or equal to 2." If a set S has k elements, we write "$|S|=k$."

Today we lack metrics to know if we are winning or losing the global war on terror.

— U.S. Secretary of Defense Donald Rumsfeld[1]

0.   A Return to Bletchley Park

When U.S. Secretary of Defense Donald Rumsfeld was making the above statement, the RAND journal *Studies in Conflict and Terrorism* was already attempting to address his concern (Farley 2003). Wars are composed of battles, so presumably the war on terror is composed—at least in part—of battles against terrorist cells.[2] But how can one tell if those battles have been won?

One could ask for the annihilation of the opposing side, but surely that is too crude a measure; Paris's Troy may have been sacked, but not Petain's Paris.[3]

One could declare a battle won if the terrorist cell has not conducted an attack, but of course it is the *potential* for attack that is the chief concern. How can we measure that?

In the first part of this paper, we will review a mathematical model for answering questions like this. This model has many shortcomings, but also perhaps some uses; details and suggestions for possible improvements will be found below.

But if one accepts the formalism of the model, with a few additional—and, we trust, reasonable—assumptions, one can ask, "What is the structure of the 'perfect' terrorist cell? the most robust terrorist cell? the cell that is least likely to be disrupted if a certain number of its members have been captured or killed?"[4] This becomes a precise mathematical question, which we address in the latter part of this paper. Finally we propose additional mathematics problems engendered by our research; we hope government investigators, academics, and students will pursue them.

1.   The Pentagon's New Math

In March 2006, *The New York Times* Magazine published an article entitled, "Can Network Theory Thwart Terrorists?"[5] When terrorist cells are depicted schematically, they are often shown as structures called *graphs*. These are not the graphs readers may have plotted in high school algebra, but collections of dots (called *nodes*), representing individuals, and lines (called *edges*) between nodes, representing any sort of relationship between the two corresponding individuals, such as a direct communications link. In Krebs 2002 [figure 1.1(i)], one finds graphs of the alleged September 11 hijackers,[6] and Rodríguez 2005 contains graphs of the alleged Madrid bombers. Of course a graph can represent any sort of social network, not just terrorists; for instance, your network of friends (Friendster 2006) or the leadership of the U.S. government [figure 1.1(ii)]. The graph of figure 1.2 represents a cell whose structure is given by telephone contacts. Of course, the physical location of a node in the real world or on the page or screen is less relevant than data about which node is connected to which.

The usual game is to identify what nodes to remove (corresponding to capturing or killing an enemy agent) in order to break up or disrupt a terrorist cell like that of figure 1.3. In figure 1.3(ii), certain nodes have been removed. The network is now *disconnected*: it has been broken up into several components that are not themselves joined by lines. The network has been disrupted.

Or has it? Consider the organization chart of a terrorist cell such as figure 1.4. In this picture, the top-to-bottom hierarchy is important: agents *A*, *B*, and *C* at the top of the picture are the leaders of the cell; agents *I*, *J*, and *K* at the bottom of the picture are the foot soldiers. Even if we remove agents *E*, *G*, and *K*—one quarter of the network [figure 1.4(ii)]—there is still a complete chain of command from *A* down to *I* (through *D* and *F*). Presumably terrorist cell leader *A* could transmit attack plans down to foot soldier *I*, so the cell remains a threat. Even though it has been disconnected, it has not been "disrupted."

The above example illustrates one reason why the graph-theoretic perspective is inadequate. (Others can be found in Farley 2006.) More

structure is needed: the actual hierarchy of the cell. A *partially ordered set*, or *poset*, is such a structure, and lattice theory is the branch of mathematics that deals with such structures. Below we will delve into posets in more detail, but first we give a brief overview.

A poset in our setting is just a fancy name for an organization chart. As with figure 1.4, the nodes at the top represent the people at the top of the organization; the nodes at the bottom represent the people at the bottom.[7] The difference between a poset and a graph is illustrated in figure 1.5. All a graph tells you is who is connected with whom [figure 1.5(i)]. It does not say whether the middle figure is the boss [figure 1.5(ii)] or the subordinate [figure 1.5(iii)]. [In mathematical parlance, the poset in figure 1.5(ii) is the *dual* of the poset in figure 1.5(iii)—one is the other, upside-down.]

Partially ordered sets are not wholly unfamiliar creatures. The set of all numbers [figure 1.6(i)] is a poset; equivalently, we could be looking at four military officers, a general, a colonel, a major, and a captain [figure 1.6(ii)]. (We call such simply ordered posets *chains*.) In fact, as with numbers, the notation "$a \leq b$" is used to indicate that a is either the same person as b ($a=b$), or else a is a subordinate of b ($a<b$), albeit not necessarily an immediate subordinate.

The advantage of this type of abstraction is that we do not need to be concerned (at least, at this level of analysis) with the precise mode by which two individuals communicate—by cell phone, email, Morse code, smoke signals, or ads in *The Washington Times*—we are only concerned with whether or not two individuals do or do not directly communicate.[8]

So the new model is that terrorist plans are formulated by the nodes at the top of the organization chart or poset (the leaders or *maximal* nodes); these plans are then transmitted down via the edges to the nodes at the bottom (the foot soldiers or *minimal* nodes), who presumably execute those plans (figure 1.7). The message, we assume, only needs to reach one foot soldier for damage to result. For example, suppose the poset represents a courier network. Only one messenger needs to succeed in parlaying the message; but the message must get through. We endeavor to block all routes from the maximal nodes (any one of

those nodes) to the minimal nodes (any one of them) by capturing or killing some subset of the agents. Note that the agents we remove need not be maximal *or* minimal (figure 1.8). Such a subset is called a *cutset*. The set of *k*-member cutsets in a poset *P* is denoted Cut(*P*,*k*).[9]

Imagine that the poset represents a system of pipes, down which water (information) may flow. Each node is a faucet, which can be on (letting water flow through) or off. If water only enters the system through the top, and all the faucets are initially on, we wish to find a collection of faucets to shut off so that no water drips off the bottom of the diagram. This cutset "cuts off" the flow of water, or, in the terrorism context, it cuts off the leaders from the foot soldiers.

In figure 1.9, we illustrate (inside the polygonal shape) a complete chain of command, or *maximal chain*; inside the oval we enclose the minimal nodes (these also form a cutset); the lighter circles represent another cutset; the darker circles a third cutset.

If *k* is a number, and *k* terrorists are killed or captured at random, the probability we have found a cutset (and hence disrupted the cell, according to our newer model) is the number of cutsets of size *k* divided by the total number of subsets of size *k*.[10]

A Mathematical Criterion for Determining Victory in the Shadow War?

In the war on terror, how can we tell if we have won a battle?

Gordon Woo, a Catastrophe Consultant for the company Risk Management Solutions, has suggested modeling terrorist cells as *graphs* or *networks*—that is, as collections of points or nodes connected by lines. The nodes represent individual terrorists, and a line is drawn between two nodes if the two individuals have a direct communications link. Figure 1.10 illustrates an organization with four members, Mel, Jean-Claude, Arnold, and Sylvester. Mel, Jean-Claude, and Arnold share a flat in Hamburg and communicate directly with each other, but Sylvester communicates only with Jean-Claude.

The task of law enforcement is to remove nodes from a graph representing a terrorist cell by capturing or killing members of that cell so that its organizational structure is disrupted. Woo suggests modeling this idea mathematically by asking the following question: How many nodes must you remove from the graph before it becomes disconnected (that is, before it separates into two or more pieces)? We might call this the *Connectedness Criterion*.[11]

Figure 1.11(i) illustrates a terrorist network Γ with seven members. If terrorist *A* is captured, the six remaining nodes are still connected and remain a cohesive whole [figure 1.11(ii)]. If, on the other hand, terrorists *E* and *G* are captured, then the graph breaks up into two parts that can no longer communicate directly with one another [figure 1.11(iii)].

There is a growing literature on modeling terrorist networks as graphs, an outgrowth of the existing literature concerning other types of criminal networks.[12] There is also literature on destabilizing networks, modeled as graphs, by seeing how connections do or do not dissipate when nodes are removed (Carley, Lee, and Krackhardt 2002).[13]

Our view is that modeling terrorist cells as graphs does not give us enough information to deal with the threat. Modeling terrorist cells as graphs ignores an important aspect of their structure, namely, their hierarchy, and the fact that they are composed of leaders and of followers. It is not enough simply to seek to disconnect terrorist networks. For while doing so may succeed in creating two clusters of terrorists incapable of communicating directly with each other, one of the clusters may yet contain a leader and enough followers to carry out a devastating attack.[14]

For example, consider the terrorist cell depicted in figure 1.12. If terrorists *B* and *E* were captured, the remaining cell would certainly be disconnected. Indeed, the cell would be broken into three components (figure 1.13). Nonetheless, there would still be a chain of command from the leader *A* down to two foot soldiers (*J* and *K*) capable of carrying out attacks.

The proper framework for our investigations is therefore that of *order theory*.[15] We do not merely want to break up terrorist networks into disconnected (non-communicating) parts. We want also to cut the leaders off from the followers. If we do that, then we can reasonably claim to have neutralized the network.

Why does this matter? It may not always be feasible to capture *every* member of a terrorist cell. It may not even be cost-effective to capture a majority of the members. The analysis we present below will enable intelligence agencies to estimate better the number of terrorist agents they must eliminate in order to cripple a cell. That way they may decide—based on *quantitative* information—how many millions of dollars they wish to devote towards targeting a particular cell, or whether they wish to spend their scarce resources in another theater of operations in the war on terror.

Refinements of our ideas should enable intelligence agencies to state, for example, that they are 85% certain that they have broken the terrorist cell they are investigating. Of course, our definition of what it means to have "broken" a terrorist cell is something one could debate, for even lone actors—from the Unabomber to the Shoe Bomber—can inflict serious damage. And we recognize the fact that, even if we are 85% sure that we have broken a terrorist cell, there is still a 15% chance that we have not—and hence there remains a chance that terrorists might commit another September 11. Nevertheless, law enforcement and intelligence agencies must allocate money and personnel, and our analysis would enable them to do so more rationally than at present.[16]

Breaking the Chains of Command

A common way to represent visually a group of people and the relationships between them is by means of a *graph* or *network*. We have seen several examples already. The individuals are represented by dots or nodes, and if two individuals are related in some fashion (for instance, if they are friends), then a line is drawn between the corresponding nodes. In the case of a terrorist cell, one might draw a line if the two individuals can communicate directly with one another.

A graph inadequately represents a terrorist cell, however, because it fails to capture the fact that, in any cell, there will most likely be a hierarchy—leaders and followers—with orders passed down from leaders to followers. Figure 1.5 makes this point clear.

All three graphs represent three people, Mary, James, and Robin. Mary can communicate to both James and Robin; Robin and James can each communicate only to Mary. What the graphs fail to capture is the fact that Mary might be the boss with two employees reporting to her, as in the middle picture, or Mary might be a secretary shared by two professors, as in the final picture. The last two pictures represent the same relationship-graph but different *ordered sets*.

Note that lines only connect individuals who are *directly* related to one another. It is unlikely that a general would communicate directly with a lieutenant. Of course a given general would normally have more than one colonel reporting to him, with neither colonel subordinate to the other, so in general we would not have a *total* order but a *partial* order.

Is it valid to use ordered sets to model terrorist cells? In his study of criminal networks, Klerks (Klerks 2002) argues that we should not assume that criminal networks are organized hierarchically simply because law enforcement agencies are so organized. If we were to follow Klerks, we would not be so quick to model criminal networks as ordered sets. But while Klerks' conclusions may be valid for ordinary criminal networks, it seems as if terrorist networks *are* in fact organized hierarchically, sometimes even along military lines.

Now suppose operations are being conducted against a specific terrorist cell. Short of capturing all of its members, how can we ascertain whether or not we have successfully disabled the cell? One criterion might be to say that a terrorist cell has been broken if it is no longer able to pass orders down from the leaders to the foot soldiers—the men and women who, presumably, would carry out the attacks. This is by no means the only possible criterion, but it enables us to make more reasonable quantitative estimates of the possibility that our operations have successfully disabled a terrorist cell.

The leaders are represented by the topmost nodes in the diagram of the ordered set, and the foot soldiers are represented by the bottommost nodes. (In order theory, these are called *maximal* and *minimal* nodes, respectively.) A chain of command linking a leader with a foot soldier is called a *maximal chain* in the ordered set.

In figure 1.4(i), the four agents *C*, *E*, *F*, and *J* form a maximal chain with the ordering from highest rank to lowest rank being *C>E>F>J*. We could also more simply write *CEFJ*.

Below we list all of the maximal chains:

ADFI	ADFJ	ADGI	ADGK
AEFI	AEFJ	AEHJ	
BEFI	BEFJ	BEHJ	
CEFI	CEFJ	CEHJ	
CK			

Each of these chains represents a chain of command through which terrorist leaders *A*, *B*, and *C* could pass instructions down to terrorist foot soldiers *I*, *J*, and *K*. In order to prevent such orders from being passed down and carried out, each of these 14 chains must be broken by means of the removal (death or capture) of at least one agent from each chain. A collection of nodes that intersects every maximal chain is called a *cutset*. (See El-Zahar and Zaguia 1986.)

In figure 1.4(i), the collection *DEK* forms a cutset, since every one of the maximal chains above contains one of *D*, *E*, and *K*. Another cutset would be *ABC*. The collection *DGHK* would *not* be a cutset since it misses the maximal chain *CEFJ*.

Quantifying the Effectiveness of an Operation against a Terrorist Cell

In what way can law enforcement *quantify* how effective it has been in disrupting a particular terrorist cell? As we have stated, one way to make this precise is to say that a terrorist cell has been disrupted *not* when all of its members have been captured or killed (which might be too costly in terms of money, agents, and agents' time), but when all

chains of command have been broken. That is, the collection of nodes in the network corresponding to the terrorists who have been killed or captured should be a cutset.

This enables us to *calculate*—not merely guess—the probability that a terrorist cell has been disrupted. Let Γ be a terrorist cell with n members (n=19 in the case of the alleged September 11 hijackers). Denote by $\Pr(\Gamma,k)$ the probability that Γ has been disrupted once k members have been captured or killed, where k is some whole number. Let $\mathrm{Cut}(\Gamma,k)$ be the number of cutsets in the ordered set Γ with k members. Then

the probability terrorist cell Γ has been broken

after k members have been captured = $\Pr(\Gamma,k)$

$$= \frac{\mathrm{Cut}(\Gamma,k)}{\dbinom{n}{k}}$$

where $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$ and $r!=r(r-1)(r-2)\bullet\ldots3\bullet2\bullet1$ for a positive

whole number r.

For example, consider the terrorist cell T with n=15 members in figure 1.14. What is the probability $\Pr(T,4)$ that T will be broken if k=4 members are captured or killed? We must first find the number of cutsets $\mathrm{Cut}(T,4)$ with 4 members. To do this, let us count the number of *minimal* cutsets with 4 or *fewer* members. These are the cutsets that cease to be cutsets if we ignore any one of the members.

minimal cutsets with 1 member	*A*	
minimal cutsets with 2 members	*BC*	
minimal cutsets with 3 members	*BFG*	*CDE*
minimal cutsets with 4 members	*BFNO*	*BGLM*
	CDJK	*CEHI*
	DEFG	

A simple calculation shows that the number of 4-member cutsets $\mathrm{Cut}(T,4)$ is

$$\binom{14}{3}+\binom{12}{2}+10+10+5=455=\binom{15}{3}$$

so

$$\Pr(T,4)=\frac{455}{\binom{15}{4}}=\frac{455}{1365}=\frac{1}{3}.$$

This means that our chances are 1 out of 3 that we will have broken this terrorist cell once we have captured 4 of its members. (We are assuming that we are as likely to capture the leader A as a foot soldier such as J.) Perhaps we will get lucky and the first four people we capture will be A, B, C, and D, in which case the cell T will have been broken. But we might also be unlucky and only capture D, H, I, and J, leaving several chains of command—like $A>C>G>O$—intact and capable of committing terrorist attacks.

The fact that $\Pr(T,4) = \frac{1}{3}$ means that we are twice as likely to be unlucky as lucky.

Note that if we were to use the Connectedness Criterion, which involves asking for the probability that T will become disconnected if we remove 4 members, then we would get a probability of

$$1-\frac{\binom{8}{4}+4\cdot 6}{\binom{15}{4}}=\frac{1271}{1365}>0.93.$$

In other words, we would feel 93% "safe" when in fact we would only be 33% "safe."

Warning Shots: Shortcomings of, and Possible Improvements to, the "Break the Chains" Model

There are many ways our model could be improved. First, we do not consider the situation where there are several terrorists in a particular cell who have the same rank. (For instance, suppose two or more terrorists share the same apartment in Hamburg. All of them are in direct communication with one another, but none of them outranks the others.) This can be handled by considering *preordered* or *quasiordered* sets.[17]

Second, we could consider the fact that terrorist operations take time. For instance, suppose agent B of figure 1.15 is captured on Wednesday, thus breaking that terrorist cell. Perhaps agent A passed down plans to B on Monday, and on Tuesday B passed those attack plans down to C. Then there might still be an attack even though the cell was broken.

Counterterrorist operations also take time. For instance, after B's capture, the cell will be broken. But if too much time passes, the cell may reorganize [figure 1.15(iii)]. For a general ordered set, the collection of cutsets will change after a reorganization. But, assuming the changes are local (that is, if they only involve a node and its neighboring nodes), this situation, too, can be handled with only a slightly more detailed analysis.[18]

Instead of assuming that the terrorists are removed simultaneously and that terrorist plans are transmitted instantaneously, queuing theory (see for instance Wein and Liu 2005) might give us a better model, or perhaps we could do experiments like McGough 2005.

Third, we assumed in our model that all the terrorists had an equal chance of getting captured. In reality, it may be the case that foot soldiers have the highest chance of being captured, since they are less well protected. Or it may be the case that leaders and foot soldiers are more likely to be captured than middle-level captains, since law enforcement might place a greater emphasis on capturing prominent

leaders than middle-level ones. Our overall analysis, however, remains the same even if we vary the probability distribution.

And although we assume that every terrorist has the same chance of being captured, whereas in reality some would be hard to catch and some not, from the terrorists' point of view, even if the United States were to target, say, the leaders or the foot soldiers more, obviously the terrorists cannot predict who will be arrested next (or else they would be able to evade capture). The safest way for them to model the situation (if they wanted to design the perfect cell) would be in fact to assume a "uniform probability distribution." Another idea that stems from this is an interesting metric of the importance of an individual X in a cell: given that we have captured a cutset, what is the probability we have captured X?[19]

Our model has more shortcomings. For instance, we do not necessarily know the structure of the particular terrorist cell under investigation. A priori, it could be any ordered set. A naive way around this might be to try to calculate $\Pr(\Gamma, k)$ for every possible ordered set Γ. This option is not feasible, however, as there are 4,483,130,665,195,087 possible ordered sets to which a 16-member cell (for instance) might correspond (Brinkmann and McKay 2002).

In fact, the situation is not as bleak as that. The order structure of a terrorist cell Γ is an empirical question. Presumably intelligence sources can tell us who the leaders are, who the captains are, and who the foot soldiers are. There are also tools available for piecing together the structure of a terrorist network (Dombroski and Carley 2002).

It is likely that terrorist cells are organized as *trees* (e.g., figures 2.2, 2.3, and 2.9), a common type of organizational structure. Trees have exactly one maximal element—corresponding to the fact that a terrorist cell, like a military unit, probably has just one leader—and no portion of a tree resembles the "V" structure of figure 2.6. This corresponds to the fact that it is unlikely that a terrorist would have direct contact with more than one superior: if he were captured, he could give away information about several other conspirators more valuable than himself. Another argument is that only in a tree can you

avoid the situation of non-complementary commands being issued to an agent from two different leaders.[20]

Of course, there are still 235,381 possible "tree" structures to which a 16-member cell might correspond (Sloane). We can eliminate most trees, however, as it is unlikely that a terrorist cell would have more than 20 or 30 members, and the hierarchy would probably have no more than 5 levels. This, along with empirical data concerning the cell under investigation, greatly reduces the number of possible order structures the cell might have.

Our model has more serious problems, however, namely its two assumptions. First, we assume that terrorist attacks occur when orders are passed down from leaders to foot soldiers. It may be instead that some terrorists act on their own (for instance, the so-called "Shoe Bomber"). This does not invalidate our model, though, as in these cases the terrorists are not properly part of a larger cell at all, but instead effectively form their own one-man cell.

Second, critics might charge that being 90% sure that a cell has been broken may be dangerously misleading. Even a 10% chance that another September 11 might occur gives the public little comfort. Nonetheless, decisions do have to be made about how to allocate scarce resources in the war against terror, and even when terrorist attacks do succeed, intelligence agencies will want quantitative data at their disposal to defend themselves from the ensuing public criticism (concerning why they did not devote the resources necessary to foil the attack). Our model enables law enforcement to plan its operations in less of an ad hoc fashion than they might be able to do otherwise.

One issue that deserves attention is the problem of redundancy. We know that it is good to safeguard a system by adding redundant components (although caution is urged by Perrow 1984). In most of our examples, if you remove a terrorist, no other terrorist has precisely the same connections as the removed individual. Where is the redundancy in our model? We have already said that redundancy could be incorporated by using quasiordered instead of partially ordered sets, but in fact redundancy already exists: the mere existence of multiple

chains of command by which terrorist plans could be transmitted represents redundancy.[21]

Laura Donohue has objected to this sort of model by asking how one distinguishes between the very different types of terrorist organizations, from the KKK and Al Qaeda to the IRA.[22] It is true that there is much that "pure" models fail to take into account. (We do not, for instance, get into how the cell members communicate or cryptological matters.)[23] It would be very interesting to see if data could be found relating to this issue. But our response is that we are not trying to precisely describe reality (although we believe our assumptions are not unduly unrealistic). That task is for experts in fields like cultural anthropology, psychology, criminology, and intelligence. We are merely trying to provide decision-making tools to people in law enforcement and intelligence, something better than guessing.

2.   Of Terrorists and Trees

In this section we will begin to try to determine the structure of the perfect terrorist cell.[24] There are two arguments in favor of trying to do this. The first is that, where we do have gaps in our knowledge of the structure of a cell, it is best to fill them in with the best possible structure—it is better to assume that terrorists are smart rather than stupid. (They may also have discovered the ideal structure through evolutionary processes.) Perhaps this will help us see into the shadow when drawing the map of terrorist networks. The second relates to the possibility that, by studying our work, terrorists could learn the most effective ways to organize. That, however, could be to our benefit because, if they do, it will decrease the possible number of organizational schemes from, say, 4 quadrillion (the number of ways a 16-member cell could organize) to something far more manageable and far easier to analyze. As with a jigsaw puzzle, the range of possibilities that law enforcement would need to consider would be reduced so that, when they capture someone, they might be able to deduce where he could fit in the organization.

Given all the assumptions of the previous section, our task translates into the following mathematical question:

Question. What partially ordered set with n elements (members) has the fewest cutsets of size k?

The answer is trivial: an *antichain* (figure 2.1), a poset with n members such that no two are connected by lines. This is so because each member is both a leader and a foot soldier, and so constitutes a maximal chain all by himself. Hence, to break all the chains of command, you have to capture all n members. Therefore, there are no cutsets of size k if $k<n$, and (exactly) one if $k=n$. (Every poset of size n has exactly one cutset of size n, namely the entire poset.)

As long as the poset has at least two members, however, the antichain is disconnected; there are at least two parts that are for all intents and purposes independent. It seems as if a terrorist cell ought to be a *connected* poset.

A simple way to avoid having a disconnected poset is to have the (almost) "flat" structure of figure 2.2 – one leader with n-1 (say, 18) followers: Unless you capture the leader, you must capture *all* the followers. But this may be unrealistic as well: it may be hard for one person to manage a large number of immediate subordinates.[25] Moreover, it is reasonable to suppose that the more direct links you have with other terrorists (the greater the *degree* of the node, in graph theory parlance), the greater the likelihood of betrayal: any one of the 18 could be captured and give you away. (For similar considerations see Gunther and Hartnell 1978.) Moreover, it may be the case that terrorists would not even want to take the small—but appreciable—risk that the whole cell could unravel with the capture of a single person. So another reasonable assumption is that there should be a (small) bound b on the number of immediate subordinates each person can have. (For instance, b might be 3 or 4.)

Definition. Let $b \geq 1$. A poset is *b-ary* if no member has more than b immediate subordinates. If $b=2$, we call such a poset *binary*.

Example. The poset in figure 2.8 is binary. Note we are not saying that every element must have exactly b immediate subordinates. The poset in figure 2.3 is ternary (3-ary) but not binary.

Of course the poset in figure 2.4 is binary, but seems unrealistic because half the cell members are "leaders." So perhaps there should be a (small) bound on the number of leaders as well. We will not necessarily assume that a cell has a single leader, which brings us a surprise: Which of the posets in figure 2.5 is better? (The reader may wish to pause to consider this before continuing.)

Let us count the cutsets. It helps to have the following notation.

Notation. Let Cutsets (a_1,a_2,a_3,a_4,a_5) denote the number of cutsets of size 1, 2, 3, 4, 5, respectively.

To calculate cutsets, it often helps to know the *minimal* cutsets. These are subsets that are just barely cutsets: no smaller subset of a minimal cutset is a cutset.[26]

Example. The poset in figure 2.5(ii) has 3 minimal cutsets: 2 of size 2 (*ab* and *ae*) and 1 of size 3 (*cde*). The cutset *cde* is minimal because none of *cd*, *ce*, or *de* is a cutset. (For example, the maximal chain *ae* misses *cd*, showing that *cd* is not a cutset.)

We will use the vector notation just established for minimal cutsets as well. (For us a "vector" is just a sequence of numbers.) So for the poset in figure 2.5(i), we write Min Cutsets (1,0,0,1,0) to indicate that there is 1 minimal cutset of size 1 (the leader), and 1 of size 4 (capture all 4 of the followers).

The surprise is that we have intuitively surmised that the flat structure of figure 2.5(i) ought to be the best, but the structure of figure 2.5(ii) has fewer cutsets of all sizes: both have the same number of 3-, 4-, or 5-member cutsets, but poset P has more 2-element cutsets than Q (4 versus 2), and Q lacks the vulnerability we noted with P: poset Q has *no* 1-member cutsets, compared to P's 1. This means that, while there is a chance of disrupting P by taking out one individual, there is no chance of this happening with Q.

Both P and Q have the same number of members and the same number of maximal chains (in this case, just the number of edges); but there seems to be no intuitive way to see that the structure Q is better. Intuition alone cannot be our guide; we must prove theorems.

We have elsewhere looked at all 63 five-member posets to calculate how many cutsets and minimal cutsets they have of each size. It is not clear how one could compare a poset that has Cutsets (1,4,8,5,1) with a poset that has Cutsets (0,5,9,5,1). The former has the aforementioned vulnerability—it can be disrupted by capturing a single person (namely, the leader)—whereas the latter poset cannot be so disrupted; but the latter poset has more cutsets of size 2 or 3 (with exactly the same number of cutsets of size 4 or 5 as the first poset). Which is better? A solution for now is to only compare posets whose smallest cutsets have the same size x—for instance, 2 in the case of second poset. Terrorist groups can then choose among structures for which there is *no* chance of disruption if fewer than x people are captured.[27]

The smallest x can be is 1; this is the case if the cell has a single leader (which seems to be the most reasonable case[28]). The smallest non-trivial value of b is 2—binary posets. And the simplest structures to consider are trees. So we will next examine binary trees.

Binary Trees

Examples of trees were given earlier. The simplest characterization is the following: a tree is a connected poset whose diagram does not contain the "V" shape of figure 2.6. Equivalently, a tree is a poset with a single leader (called the *root*) such that no member has more than one immediate superior. Thus the poset in figure 2.8 is a tree, but the poset in figure 2.7 is not.[29] We also saw a tree in figure 2.3. The technical definition is as follows.

Definition. A connected poset is a *tree* if, for any member p, the set of superiors of p forms a chain.

When most people think of "binary trees" (for instance, in computer science), they often think of trees like those in figure 1.14. All nodes except for the minimal ones (called *leaves* when we are talking about trees) have exactly $b=2$ immediate subordinates. The diagrams are pleasingly symmetric and, as the name *binary* suggests, the number of members is always a power of 2 minus 1. We cannot expect terrorist cells to oblige us and organize themselves only into cells of size 3, 7,

or 15, however; so we formulate the following (slightly informal) definition.

Definition. A binary tree is *complete* if every node has 2 immediate subordinates as far down as you are able to go; at most one node has 1 immediate subordinate; and the rest have none.

The trees of figure 1.14 and figure 3.1(iii) are complete, as is the binary tree of figure 2.8, but not the binary tree of figure 2.9(ii).

Intuition suggests that the best binary tree will be a complete one: these ought to have the most maximal chains, and the more maximal chains you have, the more people you might need to capture in order to break all the chains of command.[30] Hence, there ought to be fewer cutsets—or so intuition suggests.

Let us test our intuition. There are exactly six 5-member binary trees (figure 2.9). The best 5-member binary tree is indeed the complete one [figure 2.9(i)].

At first it seems reasonable to believe that if small "pruning" operations were done on trees, turning them into complete binary trees step by step, then the cutset vector would improve. Indeed, as one goes from figure 2.9(vi) to (v) to (iv) to (ii) to (i), the number of cutsets of each size grows smaller. (Looking at 2-member cutsets, for instance, the number shrinks from 10 to 10 to 9 to 7 to 5.)

Another strategy is just to directly compare an arbitrary binary tree with the complete binary tree having the same number of members, to see if the latter has the fewest cutsets.[31] It behooves us therefore to know how many cutsets complete binary trees have, which we do in Appendix 0, where we start by examining the balanced ones, that is, the ones whose size is a power of 2 minus 1.

3.   THE MAIN THEOREM

We expect that readers who are not mathematically inclined will only skim this section. The main result is that the special pure fishbone posets of Corollaries A0.1 and A0.2 have the fewest k-member cutsets among all n-member binary trees. Indeed, the analogue of this result

was extended by Campos, Chvátal, Devroye, and Taslakian to all *b*-ary trees. Our exposition is taken from Campos, Chvátal, Devroye, and Taslakian 2007.

Definition 3.1. Let $n \geq 1$ and let $b \geq 2$. The n-*element special pure fishbone* b-*ary poset* is like the fishbone posets of Corollaries A0.1 and A0.2, only the nodes with immediate subordinates instead all have *b* immediate subordinates, with the possible exception of the lowest such node, which is given as many immediate subordinates as possible so that the whole poset has *n* members.

Definition 3.2. Given trees T and U with n members, we will write $T \blacktriangleright U$ to mean that $\text{Cut}(T,k) \geq \text{Cut}(U,k)$ for all $k=1, 2, \ldots, n$ and that $\text{Cut}(T,k) > \text{Cut}(U,k)$ for at least one of these values of k.

Theorem 3.1 (Campos, Chvátal, Devroye, and Taslakian 2007)**.** Let $n \geq 1$ and let $b \geq 2$. Let T be a *b*-ary tree with *n* members. Let F be the special pure fishbone *b*-ary poset with *n* members.

Then either $T \blacktriangleright F$ or else $T = F$.

Definition 3.3. If x is a member of a tree T other than the leader, denote its unique immediate superior by *sup(*T,x*)*.

Lemma 3.1. Let T be a tree. Let x and y be members of T such that x is not the leader and y is a superior of $\sup(T,x)$ [that is, $y > \sup(T,x)$]. Define a new tree U with the same set of members as T, only for every member z of T that is not the leader,

$$\sup(U,z) = \begin{cases} \sup(T,z) & \text{if } z \neq x \\ y & \text{if } z = x. \end{cases}$$

Then $T \blacktriangleright U$.

Example 3.1. See figures 3.1(i) and 3.1(ii).

Proof of Lemma 3.1. If z is a foot soldier of T, then z is a foot soldier of U, and every node on the maximal chain of command in U from the leader to z lies on the chain of command from the leader to z in T. It follows that every cutset of U is a cutset of T, so that $\text{Cut}(T,k) \geq \text{Cut}(U,k)$ for all k.

Now consider the cutset C that consists of sup(T,x) and all foot soldiers of T that are not subordinates of sup(T,x). This is a cutset of T but not of U. [In Example 3.1 and figure 3.1(i), C is the set $\{v,w\}$. This is clearly not a cutset of figure 3.1(ii): $\{u,x,y\}$ is a maximal chain of U that does not intersect C.] ∎

Lemma 3.2. Let T be a tree. Let x and y be members of T such that x is neither the leader nor a foot soldier, and y is a foot soldier that is a subordinate of a subordinate of sup(x), but y is not a subordinate of x. [That is, there is a member w of T such that $w \neq x$ and $y < w < \text{sup}(x)$.] Define a new tree U with the same set of members as T, only for every member z of T that is not the leader,

$$\text{sup}(U, z) = \begin{cases} \text{sup}(T, z) & \text{if } \text{sup}(T, z) \neq x \\ y & \text{if } \text{sup}(T, z) = x. \end{cases}$$

Then $T \blacktriangleright U$.

Example 3.2. See figures 3.1(iii) and 3.1(iv).

Proof of Lemma 3.2. Given any set S of members of U, define

$$f(S) = \begin{cases} S & \text{if } S \text{ contains } y \text{ or a superior of } y \text{ in } U \\ S - \{x\} \cup \{y\} & \text{otherwise.} \end{cases}$$

[In Example 3.2, $f(\{t,u,w\})=\{t,u,w\}$, but $f(\{u,w\})=\{u,w,y\}$ and $f(\{u,w,x\})=\{u,w,y\}$.] If S is a cutset of U, then $f(S)$ is a cutset of T. [In our example, $\{u,v,w,x\}$ is a cutset of U, and $f(\{u,v,w,x\})=\{u,v,w,y\}$ is a cutset of T.] Also, S and $f(S)$ have the same number of members. If R and S are distinct subsets of U but $f(R)=f(S)$, then one of R and S contains neither x nor a superior of x in U; hence it is not a cutset of U. Thus Cut(T,k)≥Cut(U,k) for all k.

Now consider the cutset S of T consisting of sup(T,y) and all foot soldiers of T that are not subordinates of sup(T,y): there is no cutset R of U such that $f(R)=S$. [In our example, $S=\{t,v,w\}$.] ∎

Proof of Theorem 3.1. Assume there is no *b*-ary tree *U* with *n* members such that *T* ► *U*. Lemma 3.2 implies that two distinct nodes that are not foot soldiers cannot have the same immediate superior. By Lemma 3.1, every node that is not a foot soldier, except possibly the lowest one, has exactly *b* immediate subordinates. Hence *T*=*F*. ∎

The "Best" Cell Structures, Independent of Cutset Size

Elsewhere we have calculated the number of cutsets and minimal cutsets of all sizes for all posets with at most 5 members. The labeling scheme indicates the number of superior-subordinate pairs ("non-trivial incomparability relations") in a poset. For instance, the 5-member poset 6g (figure 3.2) has 6 superior-subordinate pairs: *ab*, *ac*, *bc*, *dc*, *ae*, and *de*. (Note that this is different from the number of edges in the graph.) Starting from the *n*-member antichain, you can obtain all *n*-member posets by adding one superior-subordinate pair at a time [up to $\binom{n}{2}$, for the *n* member chain]. For instance, figure 3.3 shows that one can add a superior-subordinate pair to the 4-member poset 3c in three different ways. The extra pair is shown in bold in each of the three augmented posets. (The bold line between diagrams indicates that the number of cutsets is *decreasing* when you would expect it to increase. See below.) In general, we merely list the posets you get by removing a superior-subordinate pair or by adding one.

As an aside, note that in almost every case, the number of cutsets increases when a superior-subordinate pair is added.

We can write down all of the cases where the number of cutsets decreases when a superior-subordinate pair is added. We have done it for 4- and 5-member posets.

Problem. Let *P* and *Q* be *n*-member posets such that *Q* is obtained from *P* by the addition of one superior-subordinate pair. Let $(p_1,\ldots,p_n)$ and $(q_1,\ldots,q_n)$ be the cutset vectors of *P* and *Q* respectively. Can one characterize, in terms of the structures of *P* and *Q*, the situations where it

is *not* the case that, for $1 \le i \le n$, $p_i \le q_i$? Can one characterize the situations where, for $1 \le i \le n$, $p_i \ge q_i$?

Using the information at our disposal, we can determine the best 1-, 2-, 3-, 4-, 5-, and 6-member posets in various categories—that is, the posets that, in their respective categories, have fewer cutsets than any other regardless of cutset size.

n=1
There is only one 1-member poset.

n=2
There is only one connected 2-member poset, the chain (figure 3.4).

n=3
The best connected posets are 2a and 2b (figure 3.5). The best connected poset with a single leader is 2b. (It happens to be a tree.)

n=4
The best connected 4-member poset is 4b (figure 3.6). The best binary poset with a single leader is 4c. (It happens to be a tree—indeed, one of the special pure fishbone posets of Corollary A0.2.)

n=5
The best connected 5-member posets are 6b and 6k (figure 3.7). The best connected 5-member binary poset is 6b. The best binary 5-member connected poset with at most two leaders is 4g. The best binary 5-member poset with at most two leaders is 4h. The best binary 5-member poset with a single leader is 6l. It happens to be a tree—indeed, one of the special pure fishbone posets of Corollary A0.1.

n=6
Using Lemma A0.1, we see that the best 6-member binary poset with a single leader is obtained from figure 3.7(iv) by adding a leader. The resulting poset (figure 3.8) happens to be a tree—indeed, one of the special pure fishbone posets of Corollary A0.2.

Problem. Does the trend we see for $1 \leq n \leq 6$ continue for $n \geq 7$, namely, that the best binary poset with a single leader is one of the special pure fishbone posets of Corollary A0.1 or A0.2?

Reality Check

The mathematics says that the special pure fishbone posets are optimal. But are they realistic? For instance, they have many short maximal chains (figure 3.9). Would a field operative be an immediate subordinate of a cell leader? Chin Peng, leader of the Malayan Communist Party during the so-called "Emergency," describes a situation that suggests the answer is yes—that, indeed, like our fishbone posets, there may just be one such field operative. He reports that message "couriers never learned of the exact location of the camps nor met guerrilla leaders," his key courier being "the one exception to these rules." (Chin 2003, pp. 336, 388) Senior Al Qaeda theorist Abu Mus'ab al-Suri has cautioned that previous jihads failed because foot soldiers did *not* have a personal connection to the leaders (Brachman and McCants 2006, p. 315). So a "James Bond"—an operative who reports directly to the head of his organization—might indeed be realistic. Brams, Mutlu, and Ramirez 2006 draw "influence posets" for figures connected to the September 11 plot; see their paper for a discussion of the short maximal chains (see also the diagrams in Krebs 2002 and Krebs 2006).

Fishbone posets also have long maximal chains (figure 3.10). Earlier it was suggested that terrorist cells might only have four or five levels at most. But such a restriction to four levels, for a 15-member binary tree cell, would force it to be a complete binary tree, without any consideration of cutsets, so our entire analysis would be irrelevant. Yet, it seems as if securing the chains of command *should* matter to the designer of a terrorist cell.

Bernard Brooks (personal communication 2006) uses the following heuristic to justify the structure of fishbone posets: the head of an organization may have a "right-hand man" who has no interest in having his own organization; he may also have a protégé, who would have a version of the same hierarchy beneath him.

Problems for Future Research

Problem 1. Calculate the (minimal) cutset vectors for small posets (at least size 7 or 8).[32] Is there a "best" poset? Or are there several "best" posets? Can something be seen concerning their structure?

Problem 2. Do our results extend to all posets with a single leader?

We found that, up to $n=6$ at least, the best binary poset with a single leader was a *tree*. Does this result hold for $n \geq 7$?

One way to attack this problem might be to consider *spanning trees of graphs*. (See the matroid theory text, Oxley 1993.) For every connected graph with n nodes, you can find a collection of edges that form a graph-theoretic tree with n nodes, a *spanning tree*. There may be more than one spanning tree [figures 3.11(ii) and (iii)].

As Bernd Schröder has pointed out (personal communication 2006), all posets with a single leader have a spanning tree that is also a tree when considered as a *poset*; perhaps one can show that this tree (or one of those trees) has fewer cutsets than the original poset.[33]

This would mean that, to find the perfect terrorist cell with a single leader, we would only have to consider trees. This would be exciting since it would be an *a priori* justification (in the sense of Kant) for organizing cells as trees (as opposed to the heuristic justifications given earlier).

4.   Conclusions

> *No proposition Euclid wrote,*
> *No formulae the text-books know,*
> *Will turn the bullet from your coat,*
> *Or ward the tulwar's downward blow*
> *Strike hard who cares—shoot straight who can—*
> *The odds are on the cheaper man.*
>
> — Rudyard Kipling, "Arithmetic on the Frontier"

Models are not perfect. Spulak and Turnley write, "Portraying terrorist groups as social networks…presents an untrue impression that the analysis has the accuracy and predictive power of physical theories." (Spulak and Turnley 2005, p. 15) We agree with them, and with Sagan 2004 and McGough 2005, who have suggested that human systems cannot be subjected to an engineering systems analysis without at least some reflection.

In his essay "Lewis F. Richardson's Mathematical Theory of War," Anatol Rapoport observes, "A mathematical model is more characteristically a point of departure rather than a point of arrival in the construction of a theory," adding, "the builder of a mathematical social science must virtually pull assumptions out of his hat." (Rapoport 1957, pp. 258, 281) Models can always be made more sophisticated, but the modeler should nonetheless endeavor to avoid the mistake of Lewis Carroll's mapmaker. There are also other aspects besides reliability that one might wish to optimize for, such as susceptibility to betrayals. (See Gunther and Hartnell 1978.) With our assumptions, we have found that the n-member terrorist cell organized as a b-ary tree is least likely to be disrupted when it is structured like the special pure b-ary fishbone poset (of Corollary A0.1 or A0.2, when b equals 2). It would now be interesting to see if examples could be found of terrorist or insurgent networks organizing in this way. It would also be interesting to see how much better the "perfect" or most efficient cell structure is than other common cell structures. Our ideas could be pushed further, to handle more general categories of cell structures, not just (binary) trees. And of course, if our assumptions are accepted, the same ideas could be applied to designing the "perfect" organizational structure for counterinsurgency teams. Wein suggests that perhaps government forces could do a game-theoretic analysis: given that the terrorists form the "perfect" cell, what are the optimal ways of combating it.[34] Paté-Cornell adds that perhaps there should be multiple layers of links, representing different types of networks (communications, financial, etc.), and some sort of scale to indicate the intensity of the link (whatever that is).[35]

Terrorism is not an academic subject. When academics suggest new tools for combating terrorism, we should be skeptical. This is especially

true when it comes to an abstruse field such as mathematics and, in particular, order theory.

Yet it remains true that, in the war on terror, decisions have to be made—quantitative decisions, concerning the allocation of resources, money, and manpower. Our methods should help law enforcement and intelligence agencies make these decisions—or at least give them credible arguments with which to defend their decisions before the public and congressional oversight committees.

Our tools help answer the question, "Have we disabled a terrorist cell, or is it still capable of carrying out attacks?" While we cannot answer such a question with certainty, our methods help us determine the probability that we have disrupted a particular terrorist cell.

We treat the cell Γ as an *ordered set*, a network with a built-in hierarchy (leaders, foot soldiers, and so on). We say that a terrorist cell has been rendered incapable of carrying out attacks if we have *broken the chains of command*, that is, disrupted every possible line of communication between leaders and foot soldiers. We do this by capturing or killing terrorists who collectively form a *cutset* of the ordered set. By counting all of the cutsets with k members in Γ, we can compute the probability $\Pr(\Gamma, k)$ of disrupting Γ by capturing or killing k of its members.

In conclusion, while the model can be improved, we do not believe our assumptions do too much violence to the real world. We hope others can use our approach to help reduce violence in the real world.

Figures
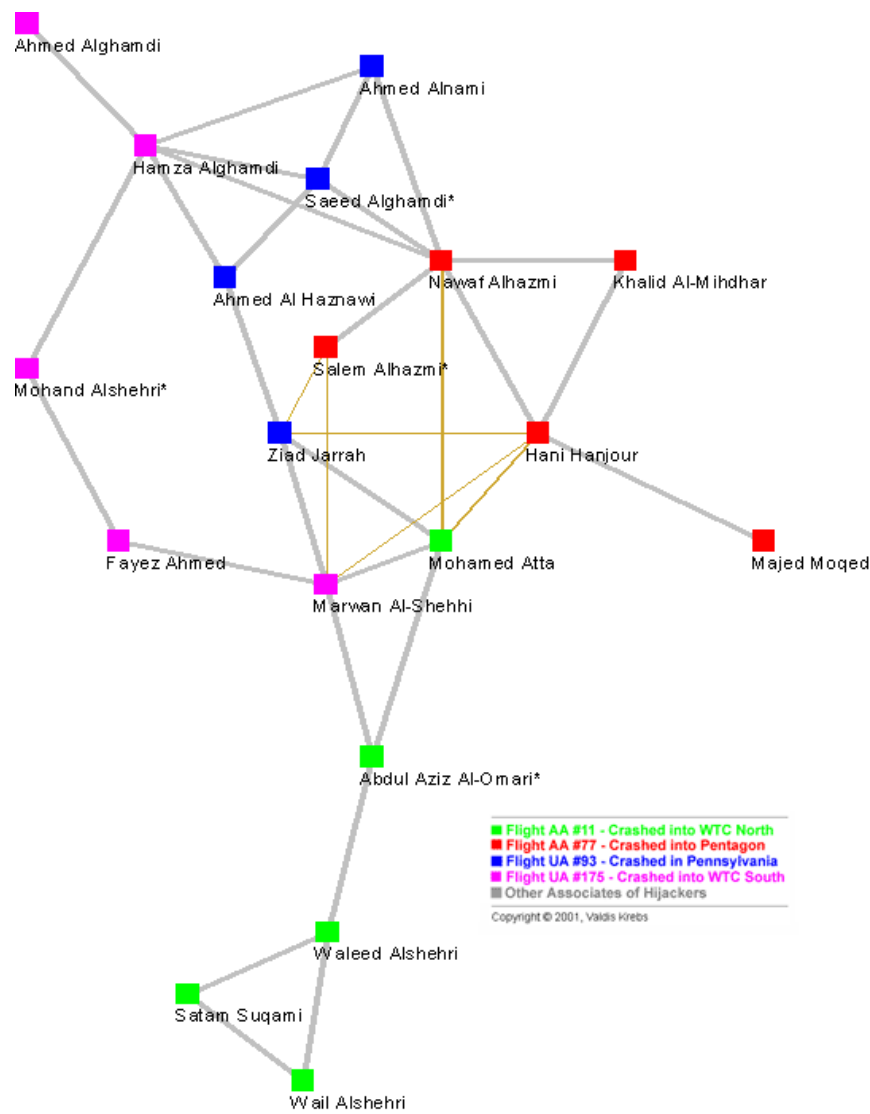
Figure 1.1(i). A graph representing the alleged September 11 hijackers (Krebs 2002; used by permission).
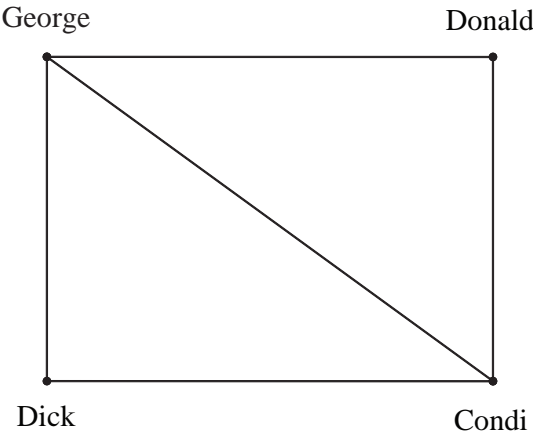
Figure 1.1(ii). A graph representing the U.S. government's executive branch.

Figure 1.2. Detection of terrorist network from communications traffic.
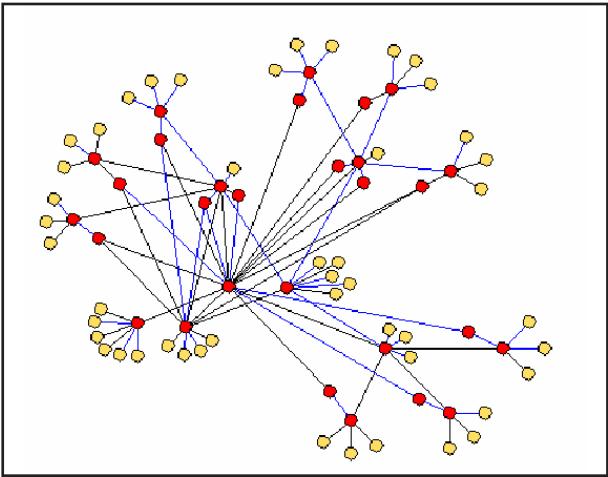
Figure 1.3(i). Network breaking.
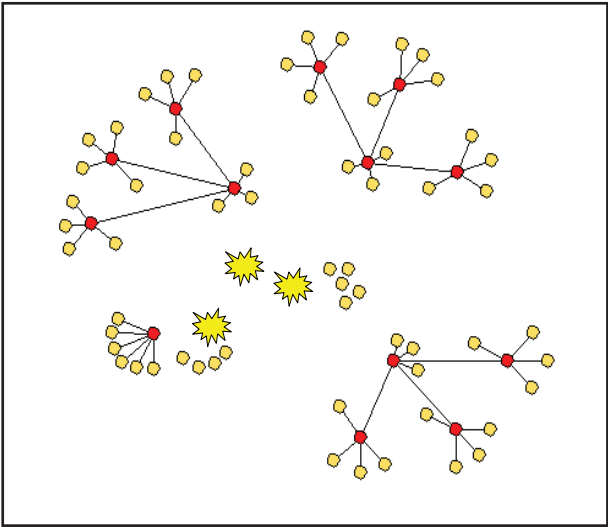
Figure 1.3(ii). Network breaking.
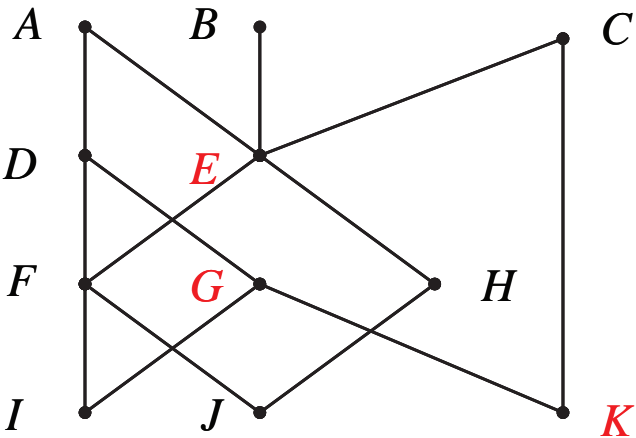
Figure 1.4(i). Attacking a terrorist network.
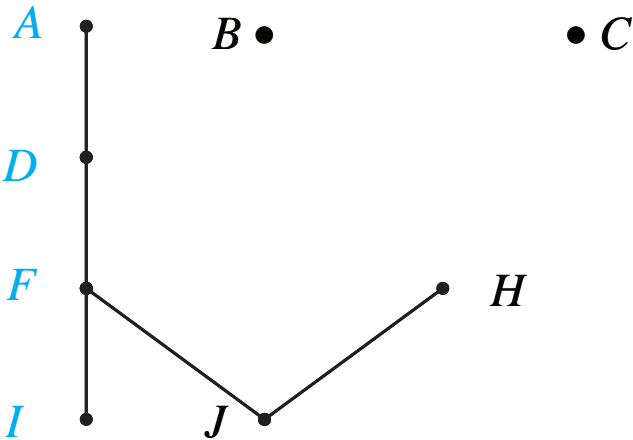
Figure 1.4(ii). Mission accomplished?

James   Mary   Robin

Figure 1.5(i). A graph.

Mary

James   Robin
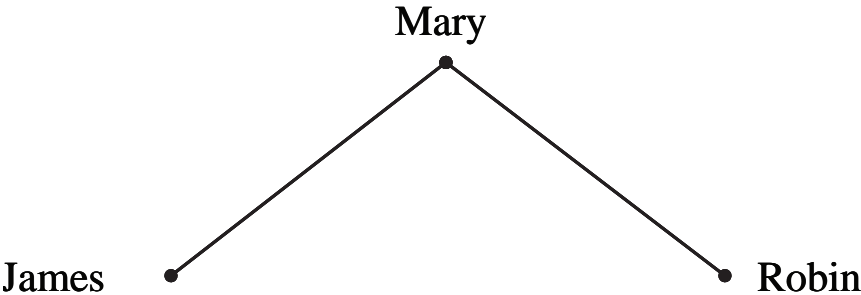
Figure 1.5(ii). A poset with the same graph.

James   Robin

Mary
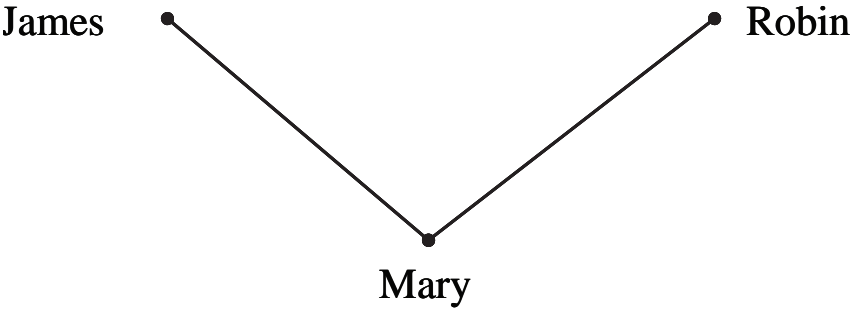
Figure 1.5(iii). A different poset with the same graph.
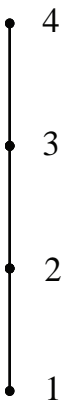
Figure 1.6(i). A chain of numbers.   **Figure 1.6(ii). A chain of people.**
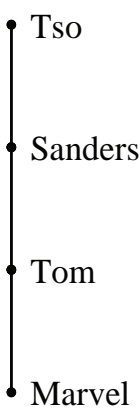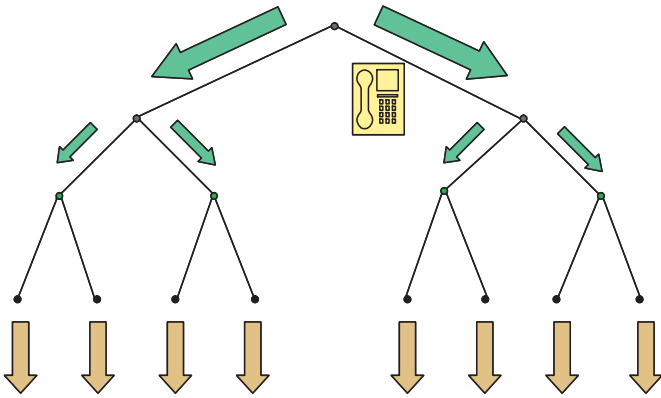
Figure 1.7. Simulating the chains of command.
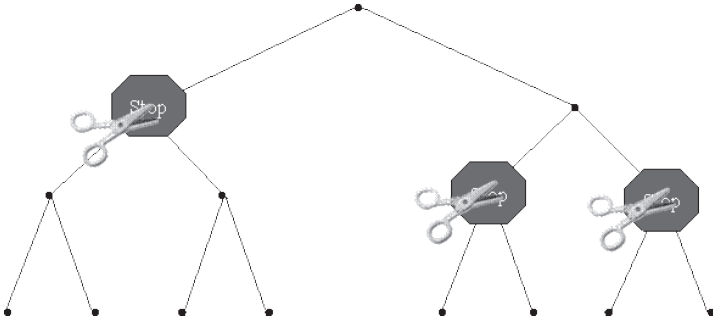
Figure 1.8. Simulating the removal of people from the cell.
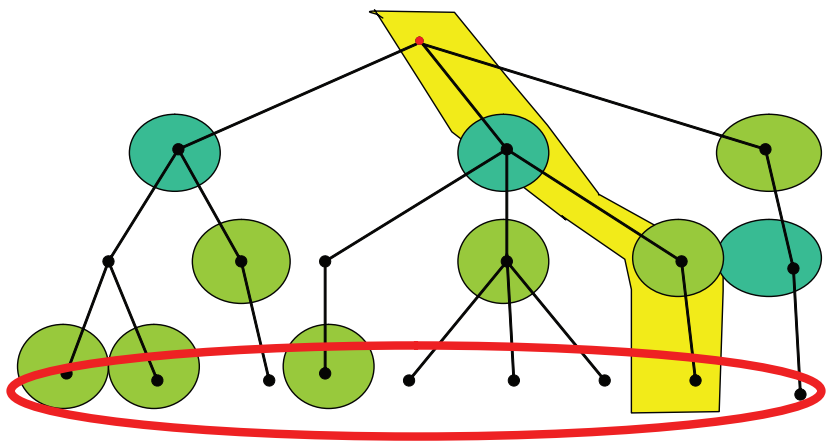
Figure 1.9. Modeling terrorist cells as partially ordered sets.
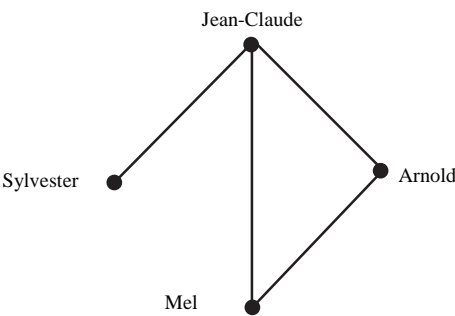
Figure 1.10. A graph illustrating an organization.
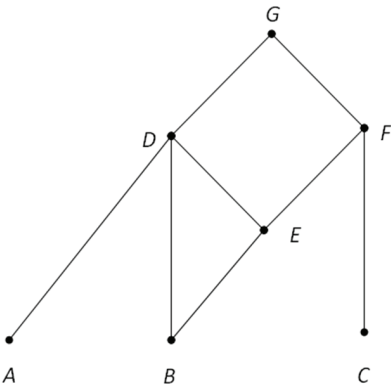
Figure 1.11(i). A graph illustrating a terrorist network Γ.
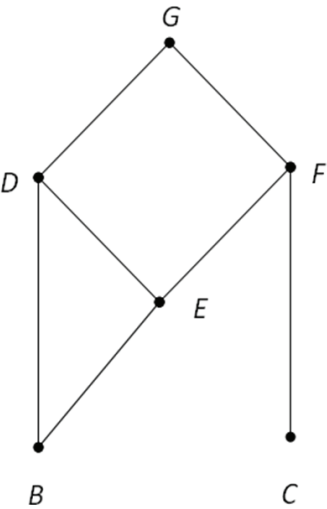
Figure 1.11(ii). The graph Γ after agent A is captured.
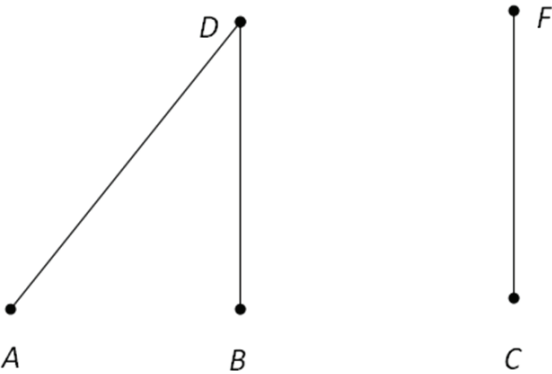
Figure 1.11(iii). The graph Γ of (i) is disconnected after the capture of terrorist agents E and G.
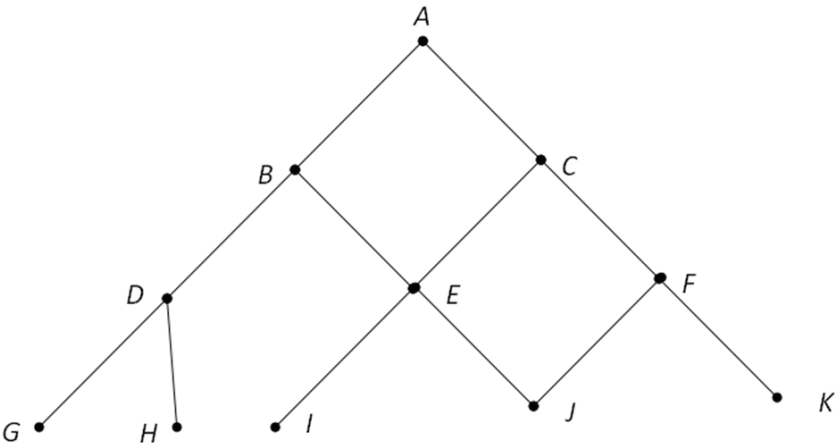
Figure 1.12. A graph Γ of a terrorist cell.
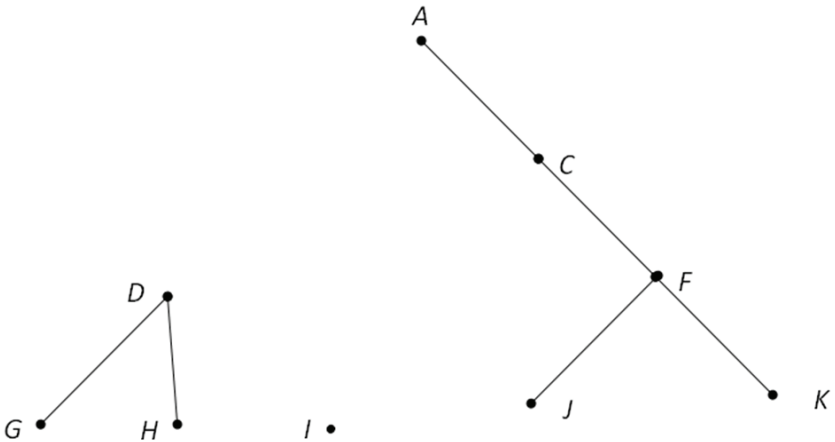
Figure 1.13. The terrorist cell Γ after the capture of two agents.
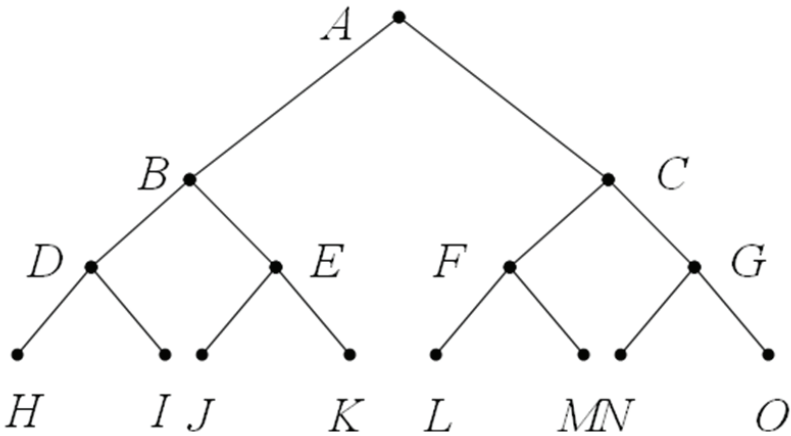
Figure 1.14. A "binary tree" T.

Figure 1.15(i). A terrorist cell Γ before agent B is captured.

Figure 1.15(ii). The terrorist cell Γ after agent B is captured.

Figure 1.15(iii). The terrorist cell Γ after a reorganization.

Figure 2.1. An antichain.
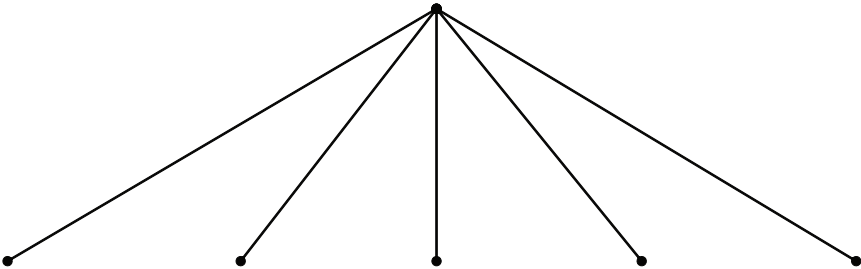
Figure 2.2. Is the flat structure ideal?
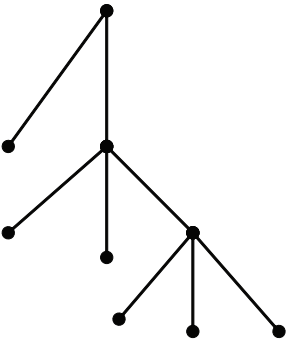
Figure 2.3. A ternary tree that is not binary.

Figure 2.4. Would a terrorist cell have this many "leaders"?

Poset *P*

Min Cutsets (1,0,0,1,0)
Cutsets (1,4,6,5,1)
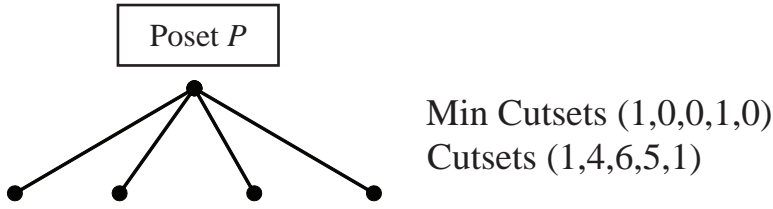
Figure 2.5(i). Which poset is better?

Poset *Q*

a   *b*

Min Cutsets (0,2,1,0,0) (ab,ae;cde)
Cutsets (0,2,6,5,1)

c   d   e
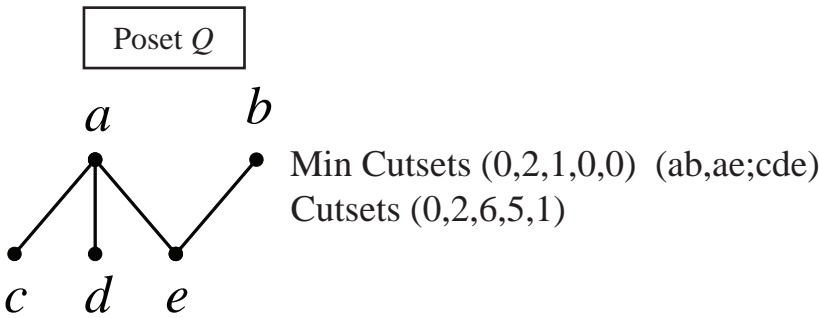
Figure 2.5(ii). Which poset is better?
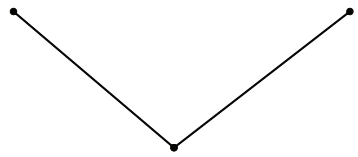
Figure 2.6.   A forbidden "covering subposet" for a tree.
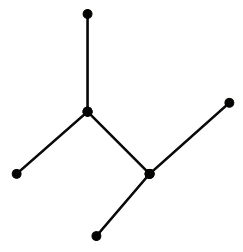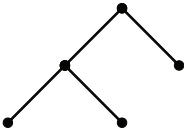

Figure 2.7.   A poset that is not a tree.
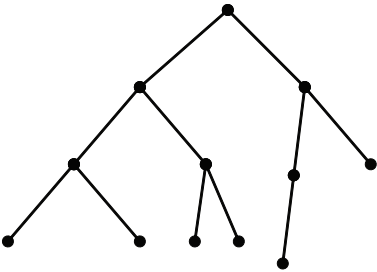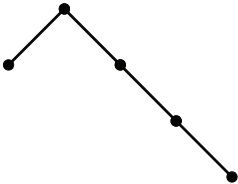
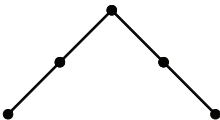Figure 2.8. A complete binary tree.

Cutsets (1,5,9,5,1)

Figure 2.9(i). A 5-member binary tree.
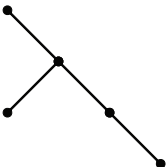

Cutsets (1,7,9,5,1)

Figure 2.9(ii). A 5-member binary tree.

Cutsets (1,8,10,5,1)

Figure 2.9(iii). A 5-member binary tree.

Cutsets (2,9,10,5,1)

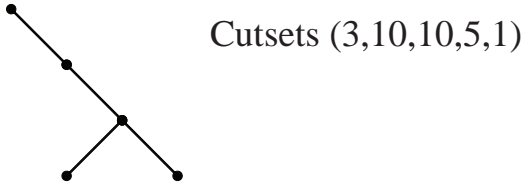Figure 2.9(iv). A 5-member binary tree.

Cutsets (3,10,10,5,1)

Figure 2.9(v). A 5-member binary tree.
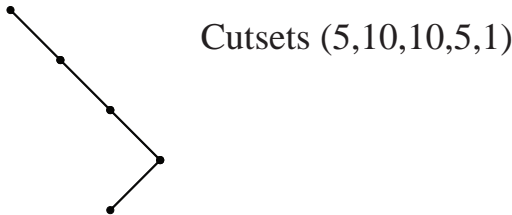
Cutsets (5,10,10,5,1)

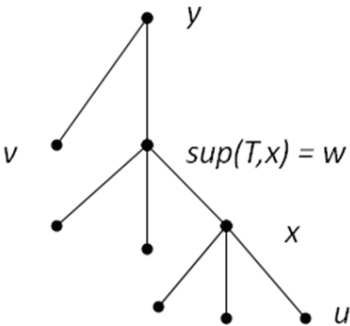**Figure 2.9(vi). A 5-member binary tree.**

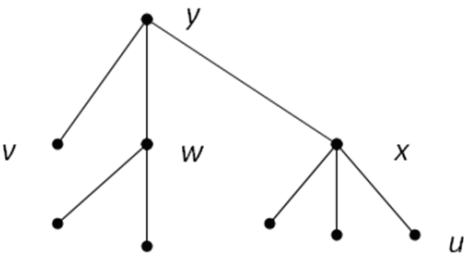**Figure 3.1(i). A tree T illustrating Lemma 3.1.**

Figure 3.1(ii). The tree U illustrating Lemma 3.1.
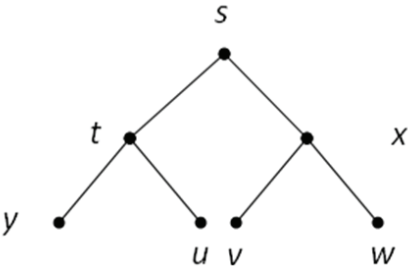
Figure 3.1(iii). The tree T illustrating Lemma 3.2.
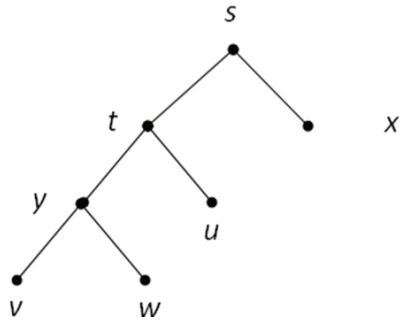
Figure 3.1(iv). The tree U illustrating Lemma 3.2.
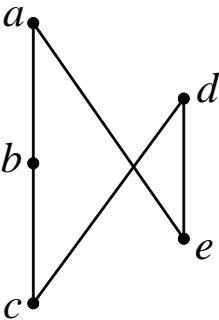
Figure 3.2. The 5-member poset 6g.
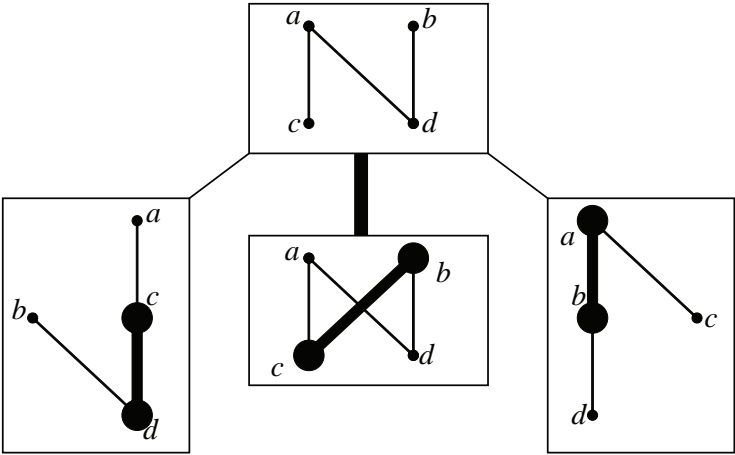
Figure 3.3. Adding superior-subordinate pairs to a poset.

Figure 3.4. The only (hence, the best) connected 2-member cell.
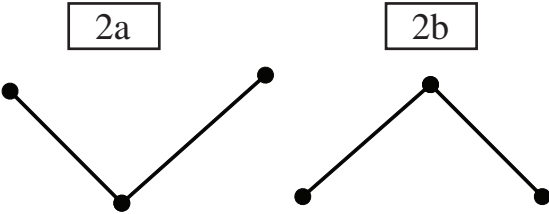
Figure 3.5(i). The best connected 3-member posets.
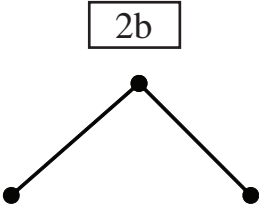
Figure 3.5(ii). The best connected 3-member poset with a single leader.
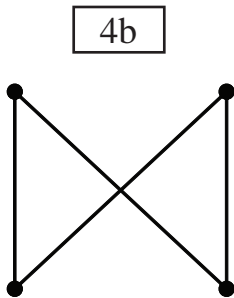
Figure 3.6(i). The best connected 4-member poset.
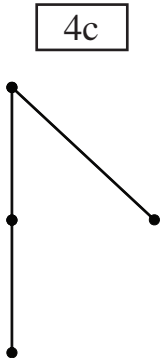
Figure 3.6(ii). The best 4-member binary poset with a single leader.
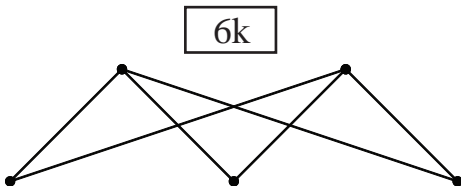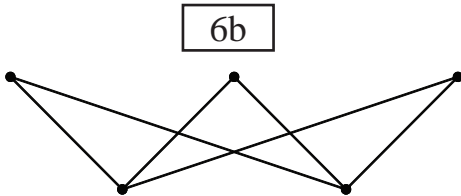
Figure 3.7(i). The best 5-member connected posets.

6b
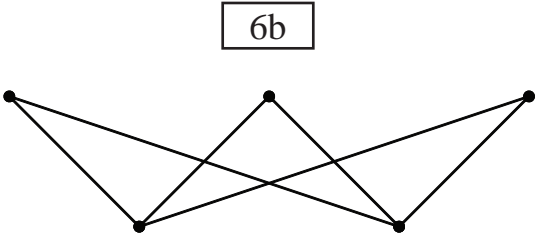
Figure 3.7(ii). The best 5-member binary connected poset.

4g
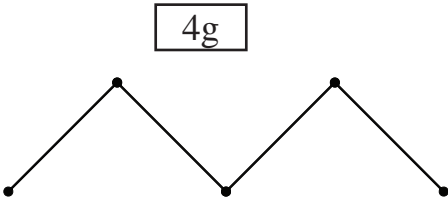
Figure 3.7(iii). The best binary 5-member connected poset with
at most 2 leaders.

4h
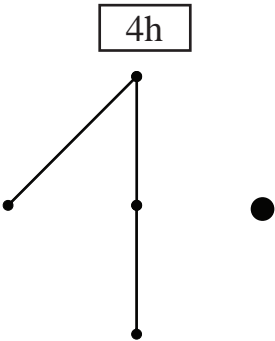
Figure 3.7(iv). The best binary 5-member poset with at most 2 leaders.

6l
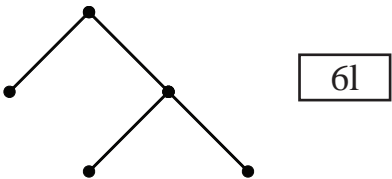
Figure 3.7(v). The best binary 5-member poset with a single leader.

Figure 3.8. The best binary 6-member poset with a single leader.

Figure 3.9. Are such short maximal chains realistic?

Figure 3.10. Are such long maximal chains realistic?
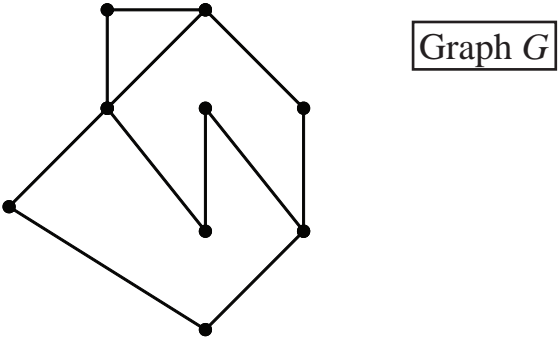
Figure 3.11(i). A connected graph *G*.
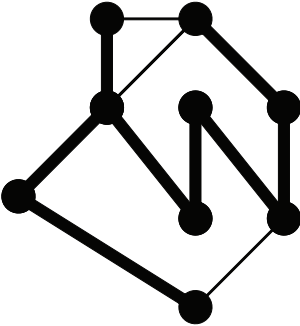
Figure 3.11(ii). A spanning tree of *G*.
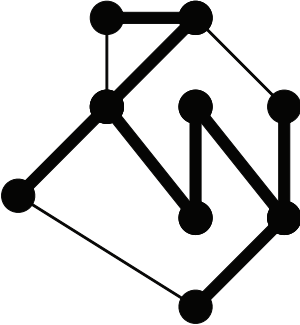
Figure 3.11(iii). A spanning tree of *G*.

4b
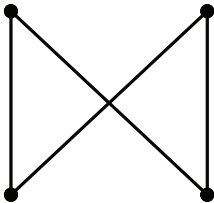
Cutsets (0,2,4,1)

Figure 3.12(i). A four-crown tower
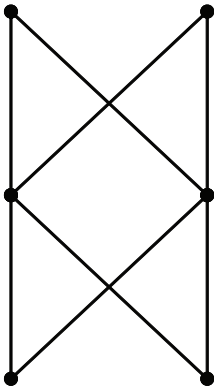
Cutsets (0,3,12,15,6,1)

Figure 3.12(ii). A four-crown tower
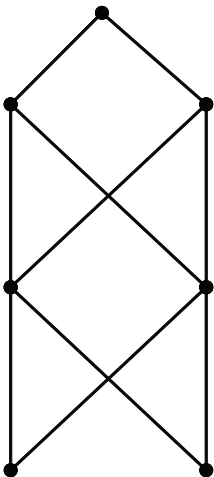
Cutsets (1,9,27,35,21,7,1)

Figure 3.12(iii). A binary poset with a single leader.

Appendix 0.

Counting Cutsets in Complete Binary Trees[36]

Let us consider the complete binary trees with $n=2^m-1$ members. When $m=3$, we have $n=7$ (figure A0.1). The minimal cutsets are as follows:

Minimal 1-member cutset:	*a*
Minimal 2-member cutset:	*bc*
Minimal 3-member cutsets:	*bfg* *cde*
Minimal 4-member cutset:	*defg*

The cutset and minimal cutset vectors are shown in figure A0.1.

Let us see how to calculate the number of *m*-member cutsets. (Recall that *m*=3.) Remember that every cutset must contain one of the five minimal ones. We will try to list all the cutsets that contain *a*; then all the cutsets that contain *bc* but not *a*; then all the cutsets that contain *bfg* but not *a* or *bc*; then all the cutsets that contain *cde* but not *a*, *bc*, or *bfg*; then all the cutsets that contain *defg* but not *a*, *bc*, *bfg*, or *cde*. Note that any subset containing a cutset is also a cutset.

The 3-member cutsets containing a. Any subset containing *a* is a cutset, so we can take *a* and any 2 of the remaining 6 members of the tree; so that are 6C2=15 such cutsets.

The remaining 3-member cutsets containing bc. We already have *bc*, so we can take 1 of the remaining 4 members of the tree (4 not 5 since we have already counted the cutsets containing the 5th element *a*). So there are 4C1=4 such cutsets.

The remaining 3-member cutsets containing bfg. The cutset *bfg* already has 3 members, so there is just this 1 cutset.

The remaining 3-member cutsets containing cde. Again, there is just 1.

The remaining 3-member cutsets containing defg. There are no 3-member cutsets containing a 4-member set.

Thus the total number of 3-member cutsets is 15+4+1+1+0=21. Looking at row 7 of Al-Karaji's triangle (Table A1.1), this happens to equal $nC2$. Indeed, looking at the cutset vector in figure A0.1, the first m entries happen to equal the first m entries in row n of Al-Karaji's triangle. Additionally, we found earlier that there were $\binom{15}{3}$ 4-member cutsets of a complete binary tree with $15=2^4-1$ members. Unless this is a total coincidence, it suggests the following result:

Proposition A0.1. Let $m\geq1$. Let T be the complete binary tree with $n=2^m-1$ members. Then for $1\leq k\leq m$, the number of k-member cutsets is $\binom{n}{k-1}$.

Before we prove this, we record the following (the proofs are clear):

Lemma A0.1. Let P be an n-member poset with a single leader r. Then the number of k-member cutsets containing r is $\binom{n-1}{k-1}$.■

Lemma A0.2. Let T be a tree such that the root r has exactly two immediate subordinates. Then if we ignore the root, we are left with two smaller trees T_1 and T_2 that have no edges between them. (We denote this situation by writing T_1+T_2.) The cutsets of T not containing r are just the cutsets of T_1+T_2. Every k-member cutset of T_1+T_2 consists of a k_1-member cutset of T_1 and a k_2-member cutset of T_2, where $1\leq k_1,k_2\leq k-1$ and $k_1+k_2=k$. (Thus there are no 0- or 1-member cutsets of T_1+T_2.) Moreover, if $1\leq k_1,k_2\leq k-1$ and $k_1+k_2=k$, when we combine a k_1-member cutset of T_1 with a k_2-member cutset of T_2, we obtain a k-member cutset of T_1+T_2.■

Example. Let T be the binary tree of figure A0.2(i).

1-member cutsets of T_1:   *a*   *b*
2-member cutset of T_1:   *ab*

1-member cutset of T_2: c
2-member cutsets of T_2: cd ce de

Thus all the 3-member cutsets of T not containing r are:

  acd ace ade bcd bce bde
  abc

Proof of Proposition A0.1. If $m=1$ then $n=2^1-1=1$ and k can only take the value 1 (since $1 \le k \le m$), so T has only 1 member, and sure enough there is just one 1-member cutset, and 1 equals $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} n \\ k-1 \end{pmatrix}$.

If $m \ge 2$, then $n \ge 3$, so the root r of T has exactly two immediate subordinates. (Remember T is a *complete* binary tree.) Let T_1 and T_2 be the (identically-structured) subtrees obtained if we remove r from T. Note that T_1 and T_2 are also complete binary trees of size $n_1 = n_2 = \frac{n-1}{2} = 2^{m-1}-1$. If $k=1$, then clearly there is only one 1-member cutset of T, namely, r itself, so

$$|\mathrm{Cut}(T,1)| = 1 = \begin{pmatrix} n \\ 0 \end{pmatrix}$$

as we sought to show. So finally suppose $2 \le k \le m$. Let us first count the cutsets containing r, then the cutsets not containing r. According to Lemma A0.1, there are $\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$ cutsets containing r. According to Lemma A0.2, the number of cutsets not containing r is

$$(*) \qquad \sum_{\substack{k_1+k_2=k \\ 1 \le k_1, k_2 \le k-1}} |\mathrm{Cut}(T_1, k_1)| \, |\mathrm{Cut}(T_2, k_2)|$$

By induction we can assume that the statement of the proposition is true for trees strictly smaller than T (i.e., T_1 and T_2), so we can assert that

$$| \text{Cut}(T_1, k_1) | = \binom{n_1}{k_1 - 1} = \binom{\frac{n-1}{2}}{k_1 - 1} = \binom{2^{m-1} - 1}{k_1 - 1}$$

and similarly

$$| \text{Cut}(T_2, k_2) | = \binom{\frac{n-1}{2}}{k_2 - 1}$$

Thus the number of cutsets not containing r is, from (*),

$$\sum_{k_1=1}^{k-1} \binom{\frac{n-1}{2}}{k_1 - 1} \binom{\frac{n-1}{2}}{(k - k_1) - 1} = \sum_{k_1=1}^{k-1} \binom{\frac{n-1}{2}}{k_1 - 1} \binom{\frac{n-1}{2}}{k - 2 - (k_1 - 1)}$$

or

$$\sum_{a=0}^{k-2} \binom{\frac{n-1}{2}}{a} \binom{\frac{n-1}{2}}{(k-2) - a}$$

Clearly this is just the number of ways to pull k-2 objects from a pool of n-1 objects, i.e., $\binom{n-1}{k-2}$. (Simply artificially split the n-1 objects into two groups, each with $\frac{n-1}{2}$ objects; choose a objects from the first group and $(k$-2$)$-a objects from the other.)

Thus the total number of k-member cutsets of T is

$$\binom{n-1}{k-1} + \binom{n-1}{k-2}$$

By properties of Al-Karaji's triangle (see Appendix 1), this equals $\binom{n}{k-1}$,

as we sought to show. ∎

The following problem will only be of interest to those familiar with ideas from enumerative combinatorics (Stanley 1997):

Problem.[37] What is the "bijective" proof of Proposition A0.1? That is, what is a natural one-to-one correspondence between k-member cutsets of T and $(k\text{-}1)$-member subsets of T (for $1 \le k \le m$)?

Note that Proposition A0.1 only tells us the number of k-member cutsets of an n-member complete binary tree when $n=2^m\text{-}1$ and $1 \le k \le m$. We do not know $|\mathrm{Cut}(T,k)|$ when n is not of this special form (a power of 2 minus 1) or, even if n has this form, when k is bigger than m; but we can make some progress on the latter question. Let us consider counting k-member cutsets when $k=m+1$.

Table A0.1 lists the k-member cutsets in a complete binary tree with $n=2^m\text{-}1$ members for $m=1$, 2, and 3 (i.e., for $n=1$, 3, and 7). We compare this with the quantity $\dbinom{n}{k-1} = \dbinom{n}{m}$ occurring in the statement of Proposition A0.1 for $m=1$, 2, 3, and 4 (i.e., $n=1$, 3, 7, and 15). The entries in the last column can be obtained as follows: For $m=1$ ($n=1$), clearly there are no 2-member subsets in a 1-member set. For $m=2$ ($n=3$), there is only one 3-member cutset, the entire tree. For $m=3$ ($n=7$), we can count 4-member cutsets the way we counted 3-member cutsets in figure A0.1: We can take a and any 3 of the remaining 6 members; bc and any 2 of the remaining 4 members; bfg and any 1 of the remaining 2 members; cde and any 1 of the remaining 2 members; or $defg$ for a total of

$$\binom{6}{3}+\binom{4}{2}+\binom{2}{1}+\binom{2}{1}+\binom{1}{0} = 20+6+2+2+1 = 31$$

(It should now be clear that the number of 5-member cutsets is

$$\binom{6}{4}+\binom{4}{3}+\binom{2}{2}+\binom{2}{2} = 15+4+1+1 = 21$$

and the number of 6-member cutsets is

$$\binom{6}{5}+\binom{4}{4} = 6+1 = 7)$$

The reader now has enough information to guess—without having to count—what number should go into the last box of the last column

of Table A0.1: that is, how many 5-member cutsets are in a 15-member complete binary tree? (The reader may wish to pause here before going on.)[38]

To confirm our intuition that the complete binary trees have the fewest cutsets, let us compare the number of 3-member cutsets in the 7-member complete binary tree T of figure A0.1—which we calculated to be 21—with the number of 3-member cutsets in the binary tree F of figure A0.3(i). Let us calculate the minimal cutsets (Table A0.2). The last column is computed as follows. The first entry (1) is clear. The second entry is $\binom{6}{1} + 1 = 6 + 1 = 7.$ The third entry is

$$\binom{6}{2} + \binom{4}{1} + 1 = 15 + 4 + 1 = 20.$$

The fourth entry is

$$\binom{6}{3} + \binom{4}{2} + \binom{2}{1} + 1 = 20 + 6 + 2 + 1 = 29.$$

We note that

$$|\mathrm{Cut}(F,3)| = 20 < 21 = |\mathrm{Cut}(T,3)|,$$

so, contrary to what our intuition told us, complete binary trees are not optimal after all. So what trees, if any, are optimal?

Fishbone Posets

Let us make a new guess that binary trees of the type in figure A0.3 are optimal. Intuition has led us astray; now we must resort to logic and to proof. Is there some way we can describe such posets?

Firstly, they have a long spine, and then spokes or ribs protruding from the spine at various vertebrae. The following definition makes this precise. The parameter s is the number of nodes along the spine; t is the number of ribs; and $v_1, \ldots, v_t$ are the locations of the vertebrae where those ribs are joined.

Definition A0.1. Let $s \geq 1$. Let $0 \leq t \leq s-1$. Let
$$1 \leq v_1 < \cdots < v_t < s.$$
Let $F := F(s,t;v_1,\ldots,v_t)$ be the poset with $s+t$ members $\{p_1,\ldots,p_s,q_1,\ldots,q_t\}$ such that
$$p_1 > \cdots > p_s$$
$$p_{v1} > q_1$$
$$p_{v2} > q_2$$
$$.$$
$$.$$
$$.$$
$$p_{vt} > q_t$$

with no other non-trivial comparabilities. We call F a *pure fishbone poset of type* $(s,t;v_1,\ldots,v_t;s+t)$.

Example. The binary tree of figure A0.3(i) is a pure fishbone poset of type $(4,3;1,2,3;7)$. The binary tree G of figure A0.3(ii) is a pure fishbone poset of type $(5,2;1,3;7)$. Let us count the minimal cutsets of G (Table A0.3). As before, the first entry of the last column (1) is clear. The second entry is
$$\binom{6}{1} + 1 + 1 = 6 + 1 + 1 = 8.$$
The third entry is
$$\binom{6}{2} + \binom{4}{1} + \binom{3}{1} + 1 + 1 = 15 + 4 + 3 + 1 + 1 = 24$$
The fourth entry is
$$\binom{6}{3} + \binom{4}{2} + \binom{3}{2} + \binom{1}{1} = 20 + 6 + 3 + 1 = 30$$

The point of introducing pure fishbone posets is that all binary trees have a "frame" that is a pure fishbone poset. This fact can be used to get a lower bound for the number of cutsets (Proposition A0.3). For example, the complete binary tree with 15 members in figure 1.14 has as a "frame" the pure fishbone poset of figure A0.4 (ii); see figure A0.4(i). To precisely define what we mean by "frame," we resort to the following. (Note that instead of saying a poset X has a frame that is a pure fishbone poset, we simply say that X *is* a fishbone poset.)

Definition A0.2. Let F be the poset of Definition A0.1. A poset X with n members is called a *fishbone poset of type* $(s,t;v_1,\ldots,v_r;n)$ if

    (a) F is a subposet of X

    (b)   Every member of X besides $p_1,\ldots,p_{s-1}$ is subordinate or equal to one of p_s, q_1, …, q_t.

We also say X is a fishbone poset of *type* $(s;n)$.

Note that the spine of the tree with $n=15$ members in figure A0.4(i) has $s=4$ nodes and that $4=3+1=\lfloor \log_2 15 \rfloor+1$.

Proposition A0.2 Let $n\geq 1$. Let $s:=\lfloor \log_2 n \rfloor+1$. A binary tree with n members is a fishbone poset of type $(s;n)$.

Proof. A binary tree with n members has a maximal chain whose top s members are

$$p_1 > p_2 > \cdots > p_s$$

where p_1 is the root of the tree. Let t be the size of the set

$$\{1\leq i\leq s\text{-}1 \mid p_i \text{ has an immediate subordinate besides } p_{i+1}\};$$

Let $v_1<\cdots<v_t$ be the numbers in this set and let q_1, …, q_t be the respective immediate subordinates. The pure fishbone poset $F(s,t;v_1,\ldots,v_t)$ of Definition A0.1 meets the conditions of Definition A0.2 with respect to the original tree X.∎

Proposition A0.3. Let k, n, $s\geq 1$. Let $0\leq t\leq s\text{-}1$. Let $v_0:=0$ and $v_{t+1}:=s$; and let $1\leq v_1<\cdots<v_t<s$. Let X be a fishbone poset of type $(s,t;v_1,\ldots,v_r;n)$.

Then $|\mathrm{Cut}(X,k)|$ is at least equal to

$$\binom{n-1}{k-1}+\binom{n-2}{k-1}+\cdots+\binom{n-v_1}{k-1}$$

$$+\binom{n-v_1-2}{k-2}+\binom{n-v_1-3}{k-2}+\cdots+\binom{n-v_2-1}{k-2}$$

$$+\binom{n-v_2-3}{k-3}+\binom{n-v_2-4}{k-3}+\cdots+\binom{n-v_3-2}{k-3}$$

$$+\binom{n-v_3-4}{k-4}+\binom{n-v_3-5}{k-4}+\cdots+\binom{n-v_4-3}{k-4}$$

$$+\cdots$$

$$+\binom{n-v_t-(t+1)}{k-(t+1)}+\binom{n-v_t-(t+1)-1}{k-(t+1)}+\cdots+\binom{n-s-t}{k-(t+1)}$$

with equality if X is pure (i.e., $n=s+t$).

There are s terms in the above sum. If $1 \leq l \leq s$, then the l-th term is

$$\binom{n-I-l}{k-(I+1)}$$

where I is the largest number i such that $0 \leq i \leq t$ and $v_i < l$.

Proof. Each term corresponds to a minimal cutset
$$\{p_l\} \cup \{q_i | 1 \leq i \leq t \text{ and } v_i < l\} \qquad (1 \leq l \leq s);$$
it is clear that the l-th term is

$$\binom{n-I-l}{k-(I+1)} . \blacksquare$$

Among the pure fishbone posets, two special ones will turn out to be of interest, depending on whether n is odd (Corollary A0.1) or even (Corollary A0.2).

Corollary A0.1. Let k, $r \geq 1$. Let $n := 2r-1$. Let F be a pure fishbone poset of type $(r,r-1;1,2,\ldots,r-1;n)$.

Then $|\mathrm{Cut}(F,k)|$ equals

$$\binom{n-1}{k-1} + \binom{n-3}{k-2} + \binom{n-5}{k-3} + \cdots + \binom{n-(2r-3)}{k-(r-1)} + \binom{n-(2r-1)}{k-r} . \blacksquare$$

Example. Consider the pure fishbone poset of figure A0.3(i). Here $n=7$, so $r=4$. Thus $|\mathrm{Cut}(F,3)|$ equals

$$\binom{6}{2} + \binom{4}{1} + \binom{2}{0} + \binom{0}{-1} = 15 + 4 + 1 + 0 = 20$$

and $|\mathrm{Cut}(F,4)|$ equals

$$\binom{6}{3} + \binom{4}{2} + \binom{2}{1} + \binom{0}{0} = 20 + 6 + 2 + 1 = 29$$

as we computed in Table A0.2.

Corollary A0.2. Let $k \geq 1$. Let $r \geq 2$. Let $n := 2r-2$. Let F be a pure fishbone poset of type $(r,r-2;1,2,\ldots,r-2;n)$.

Then $|\text{Cut}(F,k)|$ equals

$$\binom{n-1}{k-1}+\binom{n-3}{k-2}+\binom{n-5}{k-3}+\cdots+\binom{n-(2r-3)}{k-(r-1)}+\binom{n-(2r-2)}{k-(r-1)}. \blacksquare$$

Example A0.1. Consider the pure fishbone poset G of figure A0.5. Here $n=6$ so $r=4$. Then $|\text{Cut}(G,1)|$ equals

$$\binom{5}{0}+\binom{3}{-1}+\binom{1}{-2}+\binom{0}{-2}=1+0+0+0=1;$$

$|\text{Cut}(G,2)|$ equals

$$\binom{5}{1}+\binom{3}{0}+\binom{1}{-1}+\binom{0}{-1}=5+1+0+0=6;$$

$|\text{Cut}(G,3)|$ equals

$$\binom{5}{2}+\binom{3}{1}+\binom{1}{0}+\binom{0}{0}=10+3+1+1=15;$$

$|\text{Cut}(G,4)|$ equals

$$\binom{5}{3}+\binom{3}{2}+\binom{1}{1}+\binom{0}{1}=10+3+1+0=14;$$

$|\text{Cut}(G,5)|$ equals

$$\binom{5}{4}+\binom{3}{3}+\binom{1}{2}+\binom{0}{2}=5+1+0+0=6;$$

$|\text{Cut}(G,6)|$ equals

$$\binom{5}{5}+\binom{3}{4}+\binom{1}{3}+\binom{0}{3}=1+0+0+0=1.$$

Figures

Min Cutsets (1,1,2,1,0,0,0)
Cutsets (1,7,21,31,21,7,1)
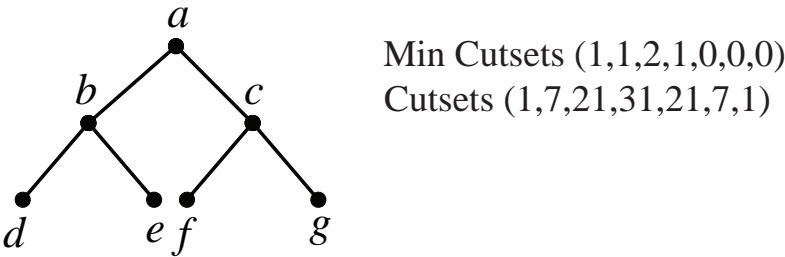
Figure A0.1. A complete binary tree.

Tree T
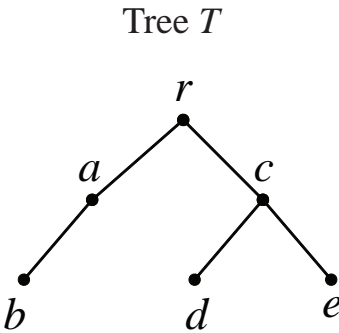
Figure A0.2(i). A complete binary tree.

Subtree T_1 Subtree T_2
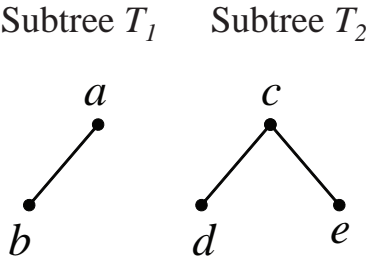
Figure A0.2(ii). The "forest" T_1+T_2.

$a=p_1$
$b=p_2$
$c=q_1$
$d=p_3$
$e=q_2$

$s=4$ $t=3$
$v_1=1$ $v_2=2$ $v_3=3$

$f=p_4$ $g=q_3$
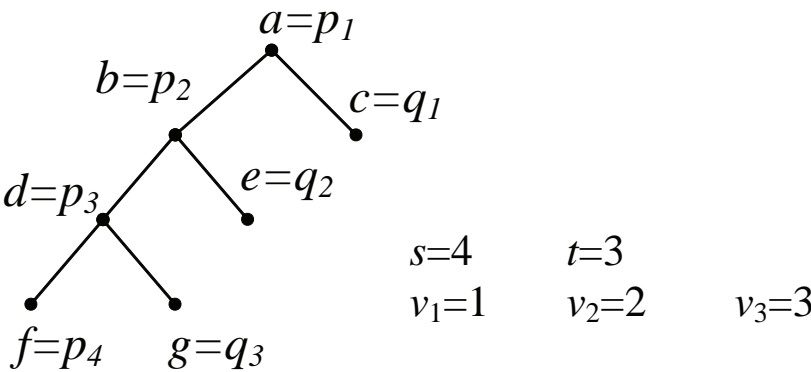
Figure A0.3(i). A pure fishbone poset of type (4,3;1,2,3;7).

Poset G
$a=p_1$
$b=p_2$
$c=q_1$
$d=p_3$
$e=p_4$
$f=q_2$

$s=5$ $t=2$
$v_1=1$ $v_2=3$

$g=p_5$
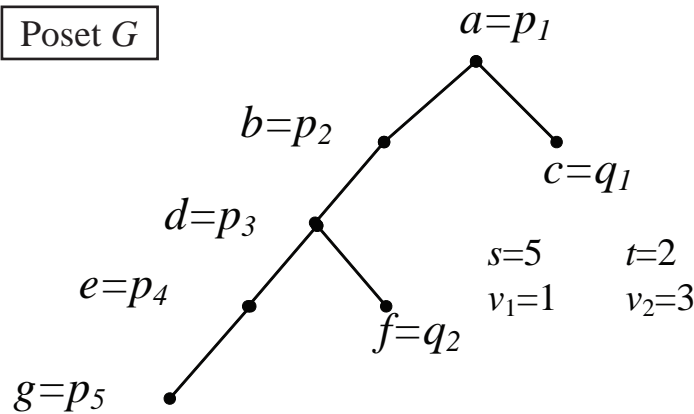
Figure A0.3(ii). A pure fishbone poset of type (5,2;1,3;7).
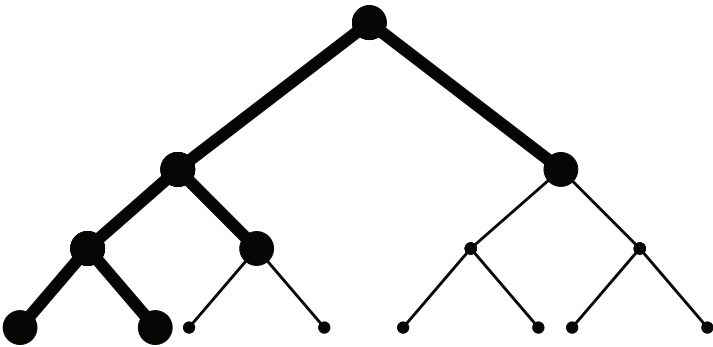
Figure A0.4(i). Every binary tree is a fishbone poset.
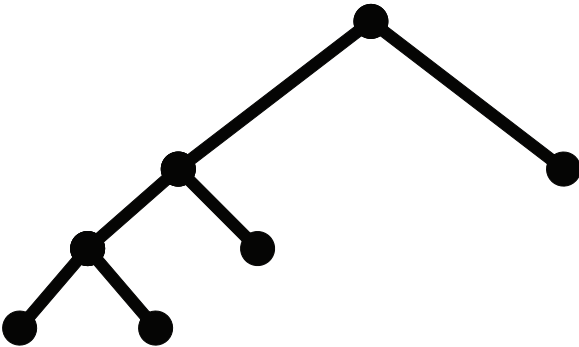
Figure A0.4(ii). The pure fishbone poset "frame" of the binary tree in figure A0.4(i).
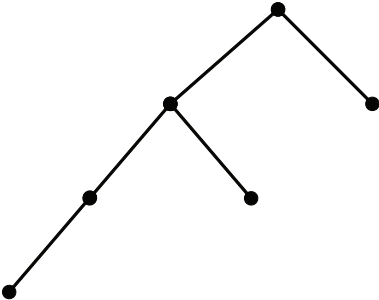
Figure A0.5. A pure fishbone poset of type (4,2;1,2;6).

Tables

m	*n*	*n*C*m*	# (*m*+1)-member cutsets
1	1	1	0
2	3	3	1
3	7	35	31
4	15	1365	?

Table A0.1. Cutsets of complete binary trees.

k	Minimal *k*-member cutsets	|Cut (*F*,*k*)|
1	*a*	1
2	*bc*	7
3	*cde*	20
4	*cefg*	29

Table A0.2. Cutsets in a binary tree of figure A0.3.

k	Minimal k-member cutsets	|Cut (G,k)|
1	a	1
2	bc,cd	8
3	cef,cfg	24
4		30

Table A0.3. Cutsets in a binary tree of figure A0.3.

1
11
121
1331
14641
15101051
1615201561
172135352171

Table A1.1. Al-Karaji's triangle.

APPENDIX 1. AL-KARAJI'S TRIANGLE

Suppose you have four people, *A*, *B*, *C*, and *D*, and you wish to select two of them to form a doubles tennis team. (The other two must return home.) How many teams can be formed? The answer is 6:

$$AB \quad AC \quad AD \quad BC \quad BD \quad CD$$

This number is denoted $\binom{4}{2}$ or 4C2, read "4 choose 2." In general, the number of ways you can choose a set of *k* objects from among *n* objects is denoted $\binom{n}{k}$ or *n*C*k*. (If *k*<0 or *k*>*n*, then this number is 0 by convention.) An exact formula for *n*C*k* when 0≤*k*≤*n* is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where for any *r*≥1,

$$r! = r \cdot (r\text{-}1) \cdot (r\text{-}2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

and 0! is defined to be 1. For example,

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(2 \cdot 1)} = \frac{24}{2 \cdot 2} = \frac{24}{4} = 6$$

The number $\binom{n}{k}$ can also be found by looking at the *k*-th number from the left in the *n*-th row of *Al-Karaji's triangle* (erroneously referred to sometimes as *Pascal's triangle*). The first few rows are in Table A1.1. (The top row is row 0 and the leftmost number in each row is the 0-th.) The rule for constructing it is that the initial row is a single 1 (surrounded by 0's on either side, out to infinity). Given a row, we construct the row below it by taking two adjacent numbers, adding them, and placing the sum below and between the numbers we have added. This corresponds to the identity

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Another fact about Al-Karaji's triangle is that the numbers in each row increase up until the middle:

$$\binom{n}{0} \le \binom{n}{1} \le \binom{n}{2} \le \cdots \le \left(\left\lfloor \frac{n}{2} \right\rfloor\right) = \left(\left\lceil \frac{n}{2} \right\rceil\right)$$

where for any real number r, $\lfloor r \rfloor$ is the greatest integer less than or equal to r and $\lceil r \rceil$ is the least integer greater than or equal to r. For basic ideas see Grinstead and Snell.

ENDNOTES

1. From a leaked 2003 memo (*USA Today*, October 22, 2003).

2. Of course the war on terror is not simply about battling terrorist cells, or even networks of cells, as one reader of this manuscript has noted: a more important goal is to stop the propagation of terrorist cells (see Farley 2007), or even to eliminate the causes of terrorism itself.

3. It is not even an accurate measure. Algeria defeated France, even though the French lost 15,000 soldiers and the Algerians nearly one million (Encyclopedia Britannica Online 2006).

4. Note that we are not talking about battling terrorist cells that have already received their orders and are completely prepared to execute them; in such cases there may no longer be a need for any further communication. We are talking about terrorist cells that are in the process of hatching plots that still require communication within the cell in order for the plans to be carried out.

5. Keefe 2006. In this monograph, the words "network" and "cell" have the same meaning. We represent both with the mathematical concepts of *graph* or *poset* (technical terms with distinct meanings, to be defined later). In the terrorism literature, the term "network" is often taken to mean (although often it is not spelled out) a loose, non-hierarchically-organized confederation of individuals, usually wide-ranging, as in the term "the global Al Qaeda network," or the phrase, "It takes a network to fight a network" (Lewis 2006). When we speak of cells or networks, by contrast we *will* mean a hierarchical structure close to the ground, like the nineteen alleged September 11 hijackers. Of course, one might try to handle the situation of multiple cells that might be coupled in some way.

6. See also http://www.firstmonday.org/issues/issue7_4/krebs/.

7. Technically a poset is a set with a binary relation that is reflexive, transitive, and antisymmetric (Davey and Priestley 2002). In this paper, all posets are finite (a statement known as Trotter's Axiom).

8. Of course it goes without saying that law enforcement would be able to do much more if they did know the mode by which terrorists communicated and could intercept messages.

9. Alternatively, we could focus on the communications links between nodes and seek to sever (or redirect) these. It would be worthwhile to do the "link" analysis; the mathematics involved would be similar to what we do here. In fact, one ought to consider the result of removing both nodes and links together.

10. Again, this only means the cell would be unable to carry out new plans, plans that still required communication from above. The cell could still perhaps carry out plans that had already been disseminated.

11. One could also ask for the number of *edges* that need to be removed in order for

the graph to become disconnected, or the number of nodes and edges.

12. See, for example, Klerks 2002 and Krebs 2002.

13. A reader of a draft of this manuscript emphasizes that one could focus on edges. Removing edges would be tantamount to preventing terrorists from communicating—for instance, by blocking certain cell phones or crashing jihadist websites. Focusing on edges also enables one to track the nodes—the senders and receivers of messages—and possibly to "hijack" lines of communication so as to send confusing, false, or misleading information. We choose to focus on nodes instead of edges since the mathematics involved is virtually the same, and hence what we do here could be easily mimicked for edges; but our intuition is that, from a practical standpoint, there are so many lines of communication available to terrorists that it might be impossible to block all the ways one terrorist could communicate with another without having a fix on the terrorists themselves.

14. Indeed, according to the United States Defense Department's transcript of the video allegedly made by Osama bin Laden, bin Laden says, "Those who were trained to fly didn't know the others. One group of people did not know the other group." One might infer, therefore, that at that low level in the network, the clusters of nodes corresponding to the hijackers were already disconnected. But they posed a danger because it was still possible for orders to filter down from above: According to the tape, "[T]hey were trained and we did not reveal the operation to them until…just before they boarded the planes." (U.S. Department of Defense 2001) We make no claims about the authenticity of the video or the accuracy of the transcript.

15. Order theory does not preclude looking at edges; indeed, in a partially ordered set, the partial order relation—essentially, the set of edges—is at least as important as the set of nodes.

16. See Conclusions for how to incorporate an analysis of edges—and our estimate of their strength and nature—in order to construct a better estimate of the likelihood that a cell has been disrupted.

17. These are sets with a binary relation that is reflexive and transitive but not necessarily antisymmetric.

18. Of course, understanding how terrorist cells "heal"—that is, recruit new members, acquire new leaders when old ones are captured, with intermediate captains moving up in the hierarchy—requires looking at empirical data. For instance, how long does it take for a communications link to reform, or for nodes to be replaced?

19. The author would like to thank an attendee of one of his talks for suggesting an idea close to this.

20. Chin Peng believes Malayan communist Lee Meng was captured by the British because she was not "made to undertake only one specific task at a time. Either she should operate only as a courier under the direction of the Central

Committee, or, she should be restricted to guerrilla activities under the state committee." (Chin 2003, p. 348)

21. In the case of the alleged September 11 hijackers, for instance, it was not enough for the commands to get through to one person, since a team of four or five was needed to hijack an aircraft. In cases like this, a single minimal node (or a single equivalence class in a quasiorder) might correspond to four or five individuals. Also, as Earl Burress (personal communication 2006) has suggested, a terrorist cell might want to maximize the chance that paths to at least two minimal nodes remain unbroken. The case of the jumbo jet that crashed in New York City in November 2001 ("Feds Eye Engines in Air Crash," CNN.com, November 12, 2001) was regarded as an accident and not an Al Qaeda attack because it was a solitary incident; three planes striking their targets on September 11, 2001, however, could not be so dismissed. On the other hand, US Federal Aviation Administration counterterrorism expert Bogdan Dzakovic states, "If I were a terrorist mastermind plotting another big attack…and I could muster up another 20 guys, I'd scatter them around to different airports around the country. I would give each one of them three bombs and three different sets of luggage. Some of those bombs will make it onto flights." (Katovsky 2006) This suggests that perhaps our model—where only one maximal chain need be left unblocked—is not so far off the mark.

    We are perhaps being too neglectful of redundancy in *communications*—for instance, a website could be used as a backup in case there is a breakdown in cell phone communications—which could be represented by having multiple links between two nodes.

22. Personal communication 2005.

23. This, as a reader points out, might be viewed as the key, since how cell members communicate relates directly to how vulnerable they are to message interception and surveillance.

24. We will not (see White 2002) discuss the ideal size of a terrorist cell based on what functions the cell needs to perform (although, as one commentator has suggested, this could be incorporated into our model). We will try to determine the ideal structure of a terrorist cell *given* that it has a certain size. Also, the edges add another layer of complexity, as in reality there could be multiple edges, two-way edges, etc., which we do not consider here.

25. Academics will appreciate the difficulty of managing 18 graduate students simultaneously.

26. To calculate the number of cutsets, you often need to know more than just the *number* of minimal cutsets.

27. This leads us to consider how terrorist cells adapt once it is discovered that their structure is known and hence they cannot be as effective as they would desire. Ideas from reflexive control (Thomas 2004) or game theory should also be utilized to see how a terrorist cell balances its goals of minimizing the number of

cutsets and keeping its structure relatively unknown. Farley (2007) discusses the use of cellular automata and the mathematics of diffusion processes; in future work we hope to draw on mathematical models of tumor formation to see how we might describe the transformation of a terrorist cell as it is being attacked. We also hope to use tools from mathematical epidemiology in order to model the long-term spread of radical ideologies, viewed as a contagion infecting a population.

28. Guerrilla forces of the Malayan Communist Party, however, were jointly headed by two individuals (Chin 2003, p. 71).

29. The poset of figure 2.7 is a tree according to *graph* theory, however. A *graph* is a tree if between any two nodes there is exactly one way to get from one to the other. See the directions for further research at the end of the monograph.

30. We actually have not checked to see if complete binary trees have the most maximal chains, nor have we confirmed that there is a relationship between the number of cutsets and the number of maximal chains.

31. Note that there is no reason to believe—just a hope—that there is a "best" binary tree, or that there is a "best" tree for all sizes of cutsets. A priori, for trees with 15 members, there may be *several* trees that have the smallest possible number of 7-member cutsets, and another set of trees that have the smallest possible number of 8-member cutsets. (Whether or not this is true, we shall see later.)

32. John Stembridge's main contribution to the theory of ordered sets has been to develop Maple packages and a database of small posets. This could be used for Problem 1. (See University of Michigan website of John Stembridge.)

33. Every finite poset with a single leader has a spanning tree that is also a tree as a poset: Let the *dual rank* of a member x of a finite poset be the size of the largest maximal chain in the subposet of elements bigger than x. For each element of dual rank k erase all connections to immediate superiors except for one that connects to an immediate superior of dual rank k-1. The resulting structure is a tree, even with the same dual rank function.

   One might naively believe that a four-crown tower [figure 3.12(i) and (ii); see also Farley 1997-1998, figure 2.2] with an additional leader added on would be a binary poset with fewer cutsets than its spanning trees, since it has so many maximal chains. This is false, however. The four-crown tower with 4 members [figure 3.12(i)] has cutset vector (0,2,4,1). The four-crown tower with 6 members [figure 3.12(ii)] has cutset vector (0,3,12,15,6,1): The cutsets consist of the cutsets of the four-crown tower with 4 members, and either none or one of the new leaders; or both of the leaders along with *any* subset of the remaining four members. Lemma A0.1 gives us the cutsets of figure 3.12(iii). Yet this has at least as many cutsets of every cardinality than even the complete binary tree with 7 members (see figure A0.1), which has at least as many cutsets of every cardinality than the pure fishbone poset with 7 members, which is a spanning tree of figure 3.12(iii). We thank Bernd Schröder for suggesting this example.

34. Personal communication 2006.

35. Personal communication 2006.

36. Readers wishing to refresh themselves on Al-Karaji's triangle should consult Appendix 1.

37. We are not asserting that this is a difficult problem, merely that it would be interesting to obtain the answer.

38. The laborious calculation is as follows. (An easier method is below.) Consider the 15-member complete binary tree of figure 1.14. We list the minimal cutsets:

Minimal 1-member cutset:	A		
Minimal 2-member cutset:	BC		
Minimal 3-member cutsets:	BFG	CDE	
Minimal 4-member cutsets:	BFNO	BGLM	CDJK
		CEHI	DEFG
Minimal 5-member cutsets:	BLMNO	CHIJK	DEFNO
	DEGLM	DFGJK	EFGHI

The number of 5-member cutsets is (we have done this type of calculation several times now):

$$\binom{14}{4}+\binom{12}{3}+\binom{10}{2}+\binom{10}{2}+\binom{8}{1}+\binom{8}{1}+\binom{8}{1}+\binom{8}{1}+\binom{8}{1}+1+1+1+1+1+1$$

$$= 1001+220+45+45+8+8+8+8+8+1+1+1+1+1+1$$

$$= 1357.$$

Alternatively, look at the difference between the numbers in the last two columns of Table A0.1. Starting with the first row, we get 1, 2, and 4—or 2^0, 2^1, and 2^2—suggesting that the final difference ought to be $2^3=8$—that is, the final entry in the table ought to be 1365-8=1357. This also gives us a conjecture for the number of $(m+1)$-member cutsets in a complete binary tree with $n=2^m-1$ members. We do not even have a conjecture for the number of k-member cutsets where $k>m+1$.

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