AFRL-RD-PS
TP-2009-1012
AFRL-RD-PS
TP-2009-1012

THE AGGREGATE BEHAVIOR OF BRANCH POINTS I- THE CREATION AND EVOLUTION OF BRANCH POINTS: POSTPRINT

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19 June 2009

Technical Paper

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KIRTLAND AIR FORCE BASE, NM 87117-5776

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED** *(From - To)*
19-06-2009	Technical Paper	January 1, 2006- June 19, 2009

4. TITLE AND SUBTITLE
The Aggregate Behavior of Branch Points I- The Creation and Evolution of Branch Points: Postprint

5a. CONTRACT NUMBER
In-House

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER
61102F

6. AUTHOR(S)
Darryl J. Sanchez, Denis W. Oesch

5d. PROJECT NUMBER
2301

5e. TASK NUMBER
SJ

5f. WORK UNIT NUMBER
DA

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Air Force Research Laboratory
3550 Aberdeen Ave SE
Kirtland AFB NM 87117-5776

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Air Force Research Laboratory
3550 Aberdeen Ave SE
Kirtland AFB NM 87117-5776

10. SPONSOR/MONITOR'S ACRONYM(S)
AFRL/RDSA

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
AFRL-RD-PS-TP-2009- 1012

12. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release

13. SUPPLEMENTARY NOTES
Accepted for publication in the Optical Society of America; SPIE Annual Conference; San Diego; 6 August 2009. 377ABW-2009-0885, 10 Jul 2009. "Government Purpose Rights"

14. ABSTRACT
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15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
					Darryl Sanchez
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**			**19b. TELEPHONE NUMBER** *(include area code)*
Unclassified	Unclassified	Unclassified	SAR	18	505-846-7209

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. 239.18

The Aggregate Behavior of Branch Points - The Creation and Evolution of Branch Points

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Abstract: It has long been known that branch points cause degradation in adaptive optic performance. Here, we begin a study on the aggregate nature of branch points, specifically beginning the process to relate branch points measured in the pupil to the upstream turbulence that created them. As such, we study not only the wave as measured in the telescope's pupil, but also the wave in the intervening region between the turbulence layer and the pupil with this paper's focus on the intervening region. We show that for optical waves propagating in atmospheric turbulence upstream of the pupil, branch points are created infinitesimally close together in pairs of opposite polarity. Branch points are shown to be enduring features of the propagating wave and their branch cuts are shown to evolve smoothly in time. It is postulated that atmospherically created branch point pairs separate as they propagate, and that they carry both the velocity of, and distance to, the turbulence layer that created them. Subsequent papers will demonstrate this to be true.

OCIS codes: 010.1300, 010.1330, 010.1285, 010.7350, 010.1080

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1. Introduction

As zeros of analytic functions, branch points are a well known and much studied in complex analysis [1]. The character zero in amplitude is accompanied by a 2π circulation in the phase and a choice of the principal domain which establishes the branch cut. It has been found that under the right conditions electromagnetic fields contain branch points, a monochromatic wave originating as a point source then propagating through atmospheric turbulence being one such example. It has been found that branch points pose a difficult problem for adaptive optics [2].

While the necessary and sufficient conditions for branch point formation are not known, generally branch points begin to form when the $(5/6)^{th}$ distance (propagation) moment of the atmospheric structure function exceeds 0.1 [3]. Hence, the distance to the onset of branch point formation varies depending on the atmospheric strength. The propagation is important; without it, atmospheric branch points would not form. Here, we are interested in how the propagation affects the creation and evolution of branch points.

Because we are interested in astronomical imaging, we restrict ourselves to the paraxial propagation region with the axis defined by the optic axis of the astronomical telescope. The main mirror of the telescope defines a plane orthogonal to the direction of propagation; we call this ‘the pupil plane’. A wave is said to be ‘in the pupil’ if it is evaluated at this plane. The pupil plane is the area of interest for the astronomical community since this is the part of the wave that forms the image. While ultimately we are solely interested in the wave in the pupil, the intervening region between the turbulence and the pupil is the main concern here. Also, as is well known, the propagtion the propagation is described by the Fresnel integral.

By the nature of the telescope in astronomical imaging, the optic axis is aligned with the direction of propagation. As such, we define $\hat{z}$ to be the direction of propagation. The pupil becomes a two dimensional $x-y$ slice of the three dimensional propagating wave. Later in this study, we will consider a sequence of these transverse planes along the z axis as the wave propagates from the turbulent region to the pupil.

Once in the pupil, the wave is conditioned using adaptive optics (AO). A conventional AO sytem only correct the phase. It does so by subdividing the telescope’s pupil into a hexagonal array of subapertures. Within each subaperture, a phase gradient measurement is made via a Shack-Hartmann wavefront sensor, and a deformable mirror actuator is placed at the corner of each subaperture. The gradient measurements are stitched together using a least square reconstructor. It has been shown that the least square reconstructor does not measure branch points; however, other reconstructors do [4] and we assume that one of these is used.

Previous studies in conventional adaptive optics solely considered branch points in the pupil plane, a single transverse slice of the propagating wave. Such a restriction comes about naturally from the nature of the wavefront sensor measurement; by its construction a wavefront sensor measures in a single transverse slice, always conjugate to the deformable mirror and sometimes conjugate to the pupil. In this frame, the branch points are assumed to be independent, although after the frame is reconstructed, branch cuts are placed between branch points typically using a minimum distance metric [5]. Here, we consider propagation prior the reaching the pupil and enumerate on how this formation impacts the pupil plane wavefront sensor measurement.

The ultimate goal of this study is to extend the state-of-the-art of adaptive optics such that it can compensate for atmospheric turbulence even when branch points are present. In this paper we demonstrate a narrow but crucial result: atmospherically created branch points must appear in pairs of opposite polarity and at the instant they appear, they have infinitesimal separation. In the later papers in the series, we will expand on this results with the eventual demonstration that a distribution of branch points measured in the pupil plane of a telescope can be used to estimate the three dimensional turbulence that created that distribution.

Although this initial demonstration is done under the assumptions of Kolmogorov turbulence and Taylor's frozen flow hypothesis, these restrictions are not severe. Assuming Kolmogorov turbulence is a weak assumption since almost all atmospheric turbulence is described by it [6]. Assuming frozen flow merely places an upper bound on sampling rate since any continuous function will appear linear when sampled fast enough. Later papers will relax this assumption and allow for evolving turbulence. In passing, it is important to note that we are not interested in lasers or cavity generated waves. Also, since we assume that the wave originates as a point source, speckle is also not an issue.

To this end, after a short background, we present in Section 3 the basis of the problem. In Section 4, we demonstrate a means to mathematically capture the notion of causality in the propagation apropos to creation of branch points. Following which, Section 5 demonstrates that upstream of the pupil, atmospherically created branch points are created in pairs infinitesimally close together and having opposite polarity. The latter part of Section 5 discusses both propagation of creation pairs and creation of large numbers of branch points in the pupil. Sections 6 and 7 discuss how branch cuts evolve in distance and how choice of the branch cut affects the principal domain. Next, in Section 8, we speculate on the nature of the proof to include that the branch points must have both the velocity and distance to the turbulence layer encoded in their statistics. Finally Section 9 wraps up the work.

2. Background

There has been effort in the adaptive optics community both to measure branch points and to unwrap the phase in the presence of branch points [2, 4, 7–12]. Unwrapping the phase is of fundamental importance because AO wavefront sensors eith measure gradient (e.g. Shack-Hartmann) or modulo 2π phase (e.g. Self-referenced Interferometer) the deformable mirrors which correct the atmospheric disturbance are continuous face sheet devices and can only correct continuous phase. Even with this unwrapping, Fried and Vaughn [2] demonstrated that there is ambiguity in assigning a continuous phase function with a scalar field due to the presence of branch points. These works make the implicit assumption that branch points are independent entities, i.e. they appear singely and can be treated as such.

In his seminal paper, Fried [7] demonstrated that the gradient of the phase is composed of the gradient of a smooth function (the scalar potential) and the curl of a discontinuous function (the vector potential) shown to be the Hertz potential. There has been work following Fried's foundational work [4, 8, 9, 11] and one [10] which demonstrated that the direction of the phase map (gradient) plays a role in the setting of branch points. For most of AO history, circulation in the phase, formerly called 'slope discrepancy', was caused only by noise. This is because in near-zenith imaging, the phase is irrotational, that is, there are no branch points. Brennan [13] demonstrated that slope space has as a basis the slope discrepancy subspace and the least squares subspace. In the regime in which we work, slope discrepancy consists of both noise and branch points.

3. Problem Formulation

Assume there is a monochromatic point source beyond the atmosphere such that $z < 0$. Propagate it as an expanding spherical wave until it reaches the atmosphere at z_{atm}, i.e. $0 \leq z_{\text{atm}} \leq z_{\text{pupil}}$ where it encounters the atmosphere. The finite extent of the atmospheric acts as an apodizing element giving the wave finite support, Ω_{atm}. Immediately prior to z_{atm}, the wave has constant amplitude and a spherical phase, $\phi_{sph}(z_{\text{atm}})$. At the atmospheric layer, as the beam propagates through the atmosphere's varying index of refraction, $n(\mathbf{r}_{atm})$, the phase of the beam takes phase $k\mathbf{r}_{atm}$ with λ the wavelength of the light, and $\mathbf{r}_{atm}$ the coordinates transverse to the direction of propagation. The phase in the plane z_{atm} is then

$$\phi_{z_{\text{atm}}} = \frac{2\pi}{\lambda} n(\mathbf{r}_{atm}) + \phi_{shp}$$

yielding for th wave at z_{atm}

$$f(\mathbf{r}_{atm}, z_{\text{atm}}) = A(\mathbf{r}_{atm}, z_{\text{atm}}) e^{i\phi_{z_{\text{atm}}}} \tag{1}$$

with $A(\mathbf{r}_{atm}, z_{\text{atm}})$ constant. The wave propagates from z_{atm} until it reaches the pupil at z_{pupil} where it is measured.

The wave for any plane $z > z_{\text{atm}}$ is given by the Fresnel propagator,

$$f(\mathbf{r}, z) = \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z}|\mathbf{r}|^2} \int_{\Omega_{atm}} f(\mathbf{r}_{atm}, z_{\text{atm}}) e^{i\frac{k}{2z}|\mathbf{r}_{atm}|^2} e^{i\frac{k}{z}\mathbf{r}_{atm}\cdot\mathbf{r}} d\mathbf{r}_{atm}, \tag{2}$$

where λ the wavelength of the light, $k = \frac{2\pi}{\lambda}$ the wavenumber, Ω_{atm} the aperture through which the wave passes at plane z_{atm}, z the coordinate location along the direction of propagation, $\mathbf{r} = (x, y)$ are the coordinates transverse to the direction of propagation, $\mathbf{r}_{atm}$ the transverse coordinates at the atmospheric interaction, $f(\mathbf{r}, 0)$ the field at the turbulence layer, and $f(\mathbf{r}, z)$ the field a distance z downrange from the turbulence layer. As is well known from the Plancherel Theorem and the Riemann-Lebesgue Lemma, under these conditions at each z, $f(\mathbf{r}, z) \in C^\infty(\mathbb{R}^2)$ almost everywhere [14], i.e. f is an element of the set of continuously differentiable functions on $\mathbb{R}^2$ except on sets of measure zero.

As the wave continues to propagate, it reaches the pupil at $z = z_{\text{pupil}}$ and is measured by an adaptive optics wavefront sensor on a telescope. This measurement is as discussed in [7] and the associated difficulties in estimating $\phi_{z_{\text{pupil}}}$ from these measurements as discussing in [2]. At z_{pupil}, the telescope both apertures the wave, applying the finite support $\Omega_{z_{\text{pupil}}}$, and also removes the spherical component, $\phi_{sph}(z_{\text{pupil}})$, of the wave. The wave can then be written

$$f(\mathbf{r}, z_{\text{pupil}}) = \Omega_{z_{\text{pupil}}} A(\mathbf{r}, z_{\text{pupil}}) e^{i\phi_{z_{\text{pupil}}}} \tag{3}$$

with $A(\mathbf{r}, z_{\text{pupil}})$ non-constant, $\phi_{z_{\text{pupil}}}$ the phase in the pupil after removal of the spherical focus term by the telescope, and

$$\Omega_{z_{\text{pupil}}} = \begin{cases} 1 & \mathbf{r} \leq D \\ 0 & \mathbf{r} > D \end{cases}$$

where D is the diameter of the telescope. Our interest here is in determining how $\phi_{z_{\text{pupil}}}$ came to be, and this is determined by the intervening propagation.

So following Fried [7,11], in any transverse plane along z with $z_{\text{atm}} \leq z \leq z_{\text{pupil}}$, let $w(x, y) := f(\mathbf{r}, z)$ be the scalar function representing an optical field in plane z. A scalar function can be assumed because, although it makes calculation easier, it does not affect the results. Let $u(x, y)$ and $v(x, y)$ be the real and imaginary parts of $w(x.y)$, respectively, with

$$w(x, y) = u(x, y) + iv(x, y). \tag{4}$$

The amplitude, $A(x,y)$, and phase, $\phi(x,y)$, of this field are given by

$$A(x,y) = \sqrt{u^2(x,y)+v^2(x,y)}, \tag{5}$$

$$\phi(x,y) = arg(w(x,y)) + 2\pi\kappa, \tag{6}$$

where on physical grounds ϕ is a continuous function [11] and this implies $\kappa = \kappa(x,y)$ with κ consisting entirely of two-dimensional step functions of height 2π. Since $w(x,y) := f(\mathbf{r},z) \in C^\infty(\mathbb{R}^2)$ almost everywhere, $w(x,y)$ is continuously differentiable over regions that do not include the discontinuities inherently defined by κ. In this way, we can write

$$w(x,y) = A(x,y)e^{i\phi(x,y)} \in C^2(\mathbb{R}^2 \backslash B) \tag{7}$$

with the notation $C^2(\mathbb{R}^2\backslash B)$ denoting the set of continuously differentiable functions on $\mathbb{R}^2$ except on the set of points given by set B which is the set of coordinates along which the branch cuts reside. Both $A(x,y)$ and $\phi(x,y)$ are continuously differentiable on $C^2(\mathbb{R}^2\backslash B)$.

It was shown by Fried [7] that phase found in adaptive optics caused by atmospheric turbulence is given by is a smooth component and a discontinuous component due to branch points and can be written

$$\phi_{pupil} = \phi_{lms} + \phi_{hid} \tag{8}$$

where ϕ_{pupil} is the phase of the wave in the pupil, ϕ_{lms} is the smooth component, and ϕ_{hid} is the discontinuous component due to branch points. The branch point component is given by

$$\phi_{\text{hid}}(\mathbf{r}) = Im\left\{ log\left[\frac{\Pi_{k=1}^{K}(x-x_k)+i(y-y_k)}{\Pi_{l=1}^{K'}(x-x'_l)+i(y-y'_l)}\right]\right\} \tag{9}$$

with (x_k,y_k) the coordinates of branch points with positive polarity, (x'_l,y'_l) the coordinates of branch points with negative polarity, K the number of branch points with positive polarity, and K' the number of branch points with negative polarity.

From the theory of analytic functions, a branch point at (x_p,y_p) is equivalent to both a zero in the amplitude, $w(x_p,y_p) = 0$, and a circulation in phase. The circulation in phase is given by

$$\int_{C'} d\vec{l'} \cdot \vec{\nabla}_\phi(x',y') = \pm 2\pi \tag{10}$$

for C' a closed curve encircling the one branch point at (x_p,y_p). and $\vec{\nabla}_\phi$ the gradient operator acting on ϕ. There are many ways of writing Equation 10, but the form chosen here mimics how the branch points are measured by an AO system's wavefront sensor. Also, for discussional clarity, we call $sign\left(\int_{C'} d\vec{l'} \cdot \vec{\nabla}_\phi(x',y') = \pm 2\pi\right)$ the polarity of the branch point. Also, it is well known that the zeros of an analytic function are isolated and so the set of all branch points in $w(x,y)$ are also isolated. It could be said, in a one sentence description, that this paper is about how zeros of the analytic wave are created and evolve.

As is well known [1], mappings with branch points are multi-valued and in order to create a single valued function, a principal domain must be chosen. Choosing a principal domain establishes the location of the branch cut, and hence the set B. Along the branch cut, a 2π discontinuity exists. Typically in complex analysis, the branch cut (and hence B) can be chosen arbitrarily, and when calculating residues, it is chosen to minimize the difficulty in integration. Here, we will show that this cannot be done, i.e. that the location of the branch cut (and hence the principal domain) is determined by the physical system; this will be clarified in Section 7.

To establish notation, for branch point p, let $B_p = \{(x,y)_c\}_{c\in \text{branch cut of P}}$ be the set of coordinate values along which the branch cut lies. Define $B(x,y;z)$ to mean "at each propagation distance z, $B(x,y;z) = \{(x,y)_c\}_{c\in\text{all branch cuts}}$ is the set of coordinates of all branch cuts in the plane transverse to z". For a wave propagating in z, no apriori restrictions are placed on varyations in B although it will be demonstrated later that B is greatly constrained by causality.

Finally, in order to study the regions where branch points form, we use uniform convergence of a sequence of functions. Uniform convergence is covered in standard textbooks, for instance Rudin [15]. Heuristically, uniform convergence occurs when a sequence of functions, f_n, converges to a function simultaneously at all coordinate values. Precisely, given a sequence of functions $\{f_n\}_{n=1,2,3,...}$ on a set $\mathbb{R}^2$. $\{f_n\}_{n=1,2,3,...}$ converges uniformly on $\mathbb{R}^2$ if for every $\varepsilon > 0$, $\exists N$ s.t. $n > N \Rightarrow |f_n(x) - f(x)| \leq \varepsilon$, $\forall x \in \mathbb{R}^2$. Here for ease of discussion:

A wave is said to propagate uniformly if and only if $\{f_n\} \to f$ uniformly.

Uniform propagation is equivalent to enforcing causality, i.e. to preventing a wave from changing instantaneously across all space when propagating from cross sectional plane z to cross sectional plane $z+\varepsilon$ where ε is an arbitrarily small distance. Interestingly, we will show that after interaction with a turbulence layer, propagation is uniform, and this by itself is sufficient to make statements about how branch points appear in the pupil.

4. Fresnel Propagation is Uniform

Say there is a monochromatic wave propagating along $\hat{z}$ and say the telescope pupil exists at plane z_{pupil}. We study the behavior of waves as they move forward in z. So, create a sequence in z, labeled z_n, such that $\{z_n\}$ is monotonically increasing and $z_n \to z_{\text{pupil}}$ as $n \to \infty$. Let $w(x,y) = f(x,y,z_{\text{pupil}})$ be the two dimensional scalar function representing a cross section of the traveling wave in the pupil. Then create a set of two dimensional functions of transverse planes approaching plane z_{pupil} in steps of $\frac{1}{n}$ by using $f(x,y,z_n)$ with

$$z_n := z_{\text{pupil}} - \frac{1}{n} \quad n \in \mathbb{Z}, \tag{11}$$

and to preclude the un-physical occurrence of propagating back in time, assume without loss of generality that $z > 1$. Note in passing that the choice of $\{z_n\}$ is arbitrary as long as z_n is monotonically increasing with n and $z_n \to z_{\text{pupil}}$. Then, let

$$f_n := f(x,y,z_n) \tag{12}$$

and create a set $\{f_n\}$.

First, assume that the first branch points form after the pupil, i.e. for $z \leq z_{\text{pupil}}$ there are no branch points. (This restriction will be lifted in a subsequent Section 5.) Hence, $\phi_{hid} = 0$ and

$$\phi_{hid} = 0 \quad \Rightarrow \quad \begin{cases} \phi_{AO} = \phi_{lms} \\ \\ B(x,y;z < z_{\text{pupil}}) = \varnothing \end{cases} \tag{13}$$

with $\varnothing$ denoting the null set.

It must be shown that the sequence $\{f_n\}$ is such that $f_n \to f(\mathbf{r},z)$ uniformly. Using results from functional analysis, this is easy, if opaque. Following [14, Ch. IX], given $\mathscr{L}(\mathbb{R}^n)$ the Schwartz space of C^∞ functions of rapid decrease, $\mathscr{L}(\mathbb{R}^n)$ is a Fréchet space and hence is complete, and further the Fourier transform is a linear bicontinuous bijection from $\mathscr{L}(\mathbb{R}^n)$ onto $\mathscr{L}(\mathbb{R}^n)$. Since $cos() \in C^\infty(\mathbb{R}^n)$ and $sin() \in C^\infty(\mathbb{R}^n)$ with $||\cdot||_\infty < 1$, given a function, $g \in \mathscr{L}(\mathbb{R}^n)$, multiplication by either the sine or cosine also yields a function in $\mathscr{L}(\mathbb{R}^n)$. Hence, the Fresnel transform given by Equation 2 is also a linear bicontinuous bijection from $\mathscr{L}(\mathbb{R}^2)$

onto $\mathscr{L}(\mathbb{R}^2)$. So, given a function, $f \in \mathscr{L}(\mathbb{R}^2)$, there exists a sequence $\{f_n\}$ with $f_n \in \mathscr{L}(\mathbb{R}^2)$ such that $f_n \to f$. Since the Fresnel transform is linear, choosing f_n as in Equation 12 with z_n as in Equation 11 yields the desired result, $f \to f_n$ uniformly [14]. A more lucid demonstration, which only uses properties of the Fresnel integral, is given in Appendix A.

This shows that when no branch points are present, $f_n \to f(\mathbf{r}, z)$ uniformly on $\mathbb{R}^2 \backslash \varnothing$. Moreover, since analytic functions are unique and given Equation 7, this implies that both the amplitude and phase propagate uniformly, i.e. create sets $\{A_n\}$ and $\{\phi_n\}$ with $A_n = A(x,y,z_n)$ and $\phi_n = \phi(x,y,z_n)$ and $z_n = z_{\text{pupil}} - \frac{1}{n}$ as before. Then given that $f_n \to f(\mathbf{r},z)$ uniformly , $A_n(\mathbf{r}) \to A(\mathbf{r})$ uniformly and $\phi_n(\mathbf{r}) \to \phi(\mathbf{r})$ uniformly.

This is the crux of the matter; the rest that is presented here follows easily from this result. Interestingly, this by itself is sufficient to make statements about how branch points appear in the pupil; how this is so is shown in the subsequent sections.

5. Paired Branch Points and Separation

To prove that branch points are only created infinitesimally close together in pairs of opposite polarity, first, in Section 5.2, it is shown that pairs of branch points of opposite polarity can be created without violating causality but only when created infinitesimally close together, while in Section 5.3 demonstrates this is false for odd numbers of branch points.

5.1. Arbitrary Non-trivial Branch Point Phase

We will demonstrate that paired branch points can occur, but only when created infinitesimally close together.

So, suppose we have a wave propagating in the $\hat{z}$ direction such that, prior to the plane z_{pupil}, the wave does not contain branch points. Suppose further that in plane z_{pupil} an arbitrary distribution of branch points appear, i.e. for $z < z_{\text{pupil}}$, $\phi_{hid} = 0$ and for $z \geq z_{\text{pupil}}$, $\phi_{hid} \neq 0$ for which

$$\phi_{hid}(z < z_{\text{pupil}}) \neq 0 \quad \Rightarrow \quad \begin{cases} \phi_{z_{\text{pupil}}} = \phi_{lms} \\ \\ B(x,y;z < z_{\text{pupil}}) \neq \{\} \end{cases} \tag{14}$$

and

$$\phi_{hid}(z \geq z_{\text{pupil}}) \neq 0 \quad \Rightarrow \quad \begin{cases} \phi_{z_{\text{pupil}}} = \phi_{lms} + \phi_{hid} \\ \\ B(x,y;z < z_{\text{pupil}}) \neq \{\} \end{cases} \tag{15}$$

Let $f(x,y,z)$ denote the wave prior to plane z and create the sequence $\{f_n\}$ with $f_n = f(x,y,z_n)$ as before by using the sequence $z_n = z_{\text{pupil}} - \frac{1}{n}$. Let $f(x,y,z_{\text{pupil}})$ denote the wave in the pupil. Let $w(x,y) = f(x,y,z_{\text{pupil}})$ be the two dimensional scalar function representing a cross section of the traveling wave in the pupil, and as was shown by Fried,

$$w(x,y) = A(x,y)e^{\imath(\phi_{lms}(x,y)+\phi_{hid}(x,y))} \tag{16}$$

For the general case of an arbitrary distribution of branch points in plane z_{pupil}, ϕ^{hid} is given by

$$\phi_{\text{hid}}(\mathbf{r}) = Im\left\{log\left[\frac{\Pi_{k=1}^{K}(x-x_k)+i(y-y_k)}{\Pi_{l=1}^{K'}(x-x'_l)+i(y-y'_l)}\right]\right\}$$

and this is obviously non-trivial in the general case. But since $z < z_{\text{pupil}}$, for each ϕ_n, $\phi^n_{hid}(z) = 0$. It is immediately obvious that in the general case that each of the ϕ_n only contain least square phase. So $\phi_n \nrightarrow \phi_{z_{\text{pupil}}}$ since $\phi_{hid} \neq 0$. Therefore, $f_n \nrightarrow f(x,y,z_{\text{pupil}})$. So, an arbitrary distribution of branch points cannot be created in the propagation from plane z to plane $z+\varepsilon$ with ε arbitrarily small.

5.2. Atmospherically Created Branch Points Can Appear in Pairs

But consider a very specific case: suppose that in plane z_{pupil} two branch points with opposite polarity are created a distance d apart. Suppose without loss of generality that the two branch points are located at $(0,0)$ and $(0,d)$. Let $\Phi(x,y)$ be the phase of a single branch point. Then the phase of a branch point pair is

$$\Phi(x,y+d) - \Phi(x,y) = \text{atan}\frac{y+d}{x} - \text{atan}\frac{y}{x}$$

which for the special case of $d < x$ and $d < y \implies \frac{y(y+d)}{x^2} > 0$, gives

$$\Phi(x,y+d) - \Phi(x,y) = \text{atan}\frac{dx}{x^2+y^2+yd}. \tag{17}$$

Then it is trivial to show that in this regime

$$\begin{aligned}\lim_{d\to 0}(\Phi(x,y+d) - \Phi(x,y)) &= \lim_{d\to 0}\frac{dx}{x^2+y^2+yd} - \frac{1}{3}\left(\frac{dx}{x^2+y^2+yd}\right)^3 + \cdots \\ &= \lim_{d\to 0} d\left\{\frac{x}{x^2+y^2+yd} - \frac{d^2}{3}\left(\frac{x}{x^2+y^2+yd}\right)^3 + \cdots\right\} \\ &\to 0 \quad \text{as } d \to 0.\end{aligned} \tag{18}$$

So pairs of branch points can be created at z_{pupil} such that $\phi_{hid} \approx 0$, but Equation 18 omits the region when $x < d$ or $y < d$, i.e. this is not sufficient to demonstrate uniform propagation but it is evocative.

To remedy this, consider that the mapping atan() is smooth except along the branch cut; smoothness implies that the derivative exists and is finite. So again consider the two branch points. As is typically done in AO, the two branch points will be connect with a branch cut. The phase of the first branch point is $\text{atan}(\frac{y}{x})$ with principal domain $[0,2\pi]$; this places the branch cut along the points $\{(x,y\} = ([0,\infty],0)$. The phase of the second branch point is $\text{atan}(\frac{y+d}{x})$ with principal domain $[-\pi,\pi]$; this places the branch cut along the points $\{(x,y\} = ([-\infty,d],0)$. So with $\text{atan}(\frac{y}{x}) - \text{atan}(\frac{y+d}{x})$, the branch cuts cancel outside $\{(x,y\} = ([0,d],0)$, yielding the line $B(x,y;z_{\text{pupil}}) = \{[0,d],0\}$.

Since atan() is smooth, $\left(\text{atan}(\frac{y}{x})\right)'$ exists except along the branch cuts. So, as is given by elementary calculus,

$$\text{atan}(\frac{y+d}{x}) - \text{atan}(\frac{y}{x}) = d\ \text{atan}\left(\frac{y}{x}\right)' + O(d^2) \tag{19}$$

which in-turn implies

$$\left|\text{atan}(\frac{y+d}{x}) - \text{atan}(\frac{y}{x})\right| < d\,C + O(d^2) \tag{20}$$

where $C = \left|lim\ sup\ \left(arctac(\frac{y}{x})\right)'\right|$ and $x,y \in \mathbb{R}/B$, and $O(\cdot)$ is "on the order of" Hence,

$$\text{given } \varepsilon,\ \exists D \le \frac{\varepsilon}{4C} \text{ s.t.} \quad \forall d < D, \left|\text{atan}(\frac{y+d}{x}) - \text{atan}(\frac{y}{x})\right| < \frac{\varepsilon}{4}\ \forall x,y \in \mathbb{R}^2/B.$$

There may be question about choice of the principal domain. Recall, for some argument $(x,y) \in \mathbb{R}^2$, $\text{atan}(\frac{y}{x}) = \theta + n2\pi$. In $mod_{2\pi}$ algebra, $\theta + n2\pi = \theta$, so choice of the principal domain does not change θ merely the location of the 2π discontinuity and the preceeding argument goes through unchanged. The restriction to separation in the vertical axis is unimportant, and all orientations can be easily obtained via rotation of coordinates.

So, for a pair of branch points of opposite polarity, when their separation is small enough,

$$\left|f_n - f(x,y,z_{\text{pupil}})\right| \tag{21}$$
$$= \left|f(\mathbf{r},z_n) - f(\mathbf{r},z_{\text{pupil}})\right| \tag{22}$$
$$= \left|f(\mathbf{r},z_n) - A(\mathbf{r},z_{\text{pupil}})e^{\imath\Phi_{lms}(\mathbf{r},z_{\text{pupil}})+\Phi_{hid}(\mathbf{r},z_{\text{pupil}})}\right| \tag{23}$$
$$= \left|f(\mathbf{r},z_n) - A(\mathbf{r},z_{\text{pupil}})e^{\imath\Phi_{lms}(\mathbf{r},z_{\text{pupil}})} + A(\mathbf{r},z_{\text{pupil}})e^{\imath\Phi_{lms}(\mathbf{r},z_{\text{pupil}})} - A(\mathbf{r},z_{\text{pupil}})e^{\imath\Phi_{lms}(\mathbf{r},z_{\text{pupil}})+\Phi_{hid}(\mathbf{r},z_{\text{pupil}})}\right| \tag{24}$$
$$< \left|f(\mathbf{r},z_n) - A(\mathbf{r},z_{\text{pupil}})e^{\imath\Phi_{lms}(\mathbf{r},z_{\text{pupil}})}\right| + \left|A(\mathbf{r},z_{\text{pupil}})e^{\imath\Phi_{lms}(\mathbf{r},z_{\text{pupil}})} - A(\mathbf{r},z_{\text{pupil}})e^{\imath\Phi_{lms}(\mathbf{r},z_{\text{pupil}})+\Phi_{hid}(\mathbf{r},z_{\text{pupil}})}\right| \tag{25}$$
$$< \frac{\varepsilon}{4} + \left|A(\mathbf{r},z_{\text{pupil}})\right| \left|1 - e^{i\Phi_{hid}(\mathbf{r},z_{\text{pupil}})}\right| \tag{26}$$

Then examaning the last term

$$\left|A(\mathbf{r},z_{\text{pupil}})\right| \left|1 - e^{i\Phi_{hid}(\mathbf{r},z_{\text{pupil}})}\right| = \left|A(\mathbf{r},z_{\text{pupil}})\right| \left|1 - \cos(\Phi_{hid}(\mathbf{r},z_{\text{pupil}})) - i\sin(\Phi_{hid}(\mathbf{r},z_{\text{pupil}}))\right| \tag{27}$$
$$< \left|A(\mathbf{r},z_{\text{pupil}})\right| \left|1 - \cos(\Phi_{hid}(\mathbf{r},z_{\text{pupil}}))\right| + \left|i0 + i\sin(\Phi_{hid}(\mathbf{r},z_{\text{pupil}}))\right| \tag{28}$$

then let $d < D$ where D is given as above and $D << 1$

$$< \left|A(\mathbf{r},z_{\text{pupil}})\right| \left|1 - 1 - dC\right| + \left|0 + (dC)^2\right| \tag{29}$$
$$= \left|A(\mathbf{r},z_{\text{pupil}})\right| \left|dC\right| + \left|(dC)^2\right| \tag{30}$$
$$< \frac{\varepsilon}{4} + \frac{\varepsilon^2}{16} < \varepsilon \qquad \forall d < D \text{ and } \forall x,y \in \mathbb{R}^2/B \tag{31}$$

where $D < \left[\left|A(\mathbf{r},z_{\text{pupil}})\right| C\right]^{-1} \in \mathbb{R}^2/B$. [Recall, that the zeros of $w(x,y)$ are isolated and they exist at the endpoints of B]. Then

$$\left|f_n - f(x,y,z_{\text{pupil}})\right| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \left(\frac{\varepsilon}{4}\right)^2 < \varepsilon \tag{32}$$

So,

$$\text{given } \varepsilon, \exists D \text{ and } N \text{ s.t. } \forall n > N \text{ and } d < D, \left|f_n - f(x,y,z_{\text{pupil}})\right| < \varepsilon \; \forall x,y \in \mathbb{R}/B$$

that is

$$f_n \to f(x,y,z_{\text{pupil}}) \text{ uniformly}$$

but only in the very special case when two branch points of opposite polarity are separated by d, arbitrarily small.

That is, if a wave propagates through turbulence and a pair of branch points of opposite polarity spaced infinitesimally close together does not violate causality.

5.3. *Atmospherically Created Branch Points Cannot Appear Singly*

Next, consider the case of a wave that is branch point free prior to the pupil but contains a single branch point in the pupil. Specifically, suppose we have a wave propagating in the $\hat{z}$ direction such that, prior to the plane z_{pupil}, the wave does not contain branch points, and in plane z_{pupil} a single branch point appears. Suppose further, without loss of generality, that the branch point is created at $x = y = 0$. Then for $z < z_{\text{pupil}}$

$$\phi_{hid} = 0 \quad \Rightarrow \quad \begin{cases} \phi_{AO} = \phi_{lms} \\ B(x,y;z < z_{\text{pupil}}) \neq \varnothing \end{cases} \tag{33}$$

and for $z \geq z_{\text{pupil}}$

$$\phi_{hid} = \text{atan}\left(\frac{y}{x}\right) \quad \Rightarrow \quad \begin{cases} \phi_{AO} = \phi_{lms} + \text{atan}\left(\frac{y}{x}\right) \\ B(x,y;z < z_{\text{pupil}}) \neq \varnothing \end{cases} \tag{34}$$

Then, as before, let

$$f_n := f(\mathbf{r}, z - \frac{1}{n}) = A_n(\mathbf{r}) e^{\imath \Phi_n(\mathbf{r})}$$

with $\Phi_n(\mathbf{r}) \in \phi_{lms}(\mathbf{r})$. As before,

$$\phi_n \nrightarrow \phi_{lms}(x,y) + \text{atan}\left(\frac{y}{x}\right)$$

which implies

$$f_n \nrightarrow f(x,y,z_{\text{pupil}}).$$

So a single branch point cannot occur without violating causality.

This is easily extended to show that any odd number of branch points cannot occur.

5.4. *Arbitrary numbers of Branch Points*

Given a wave with a creation pair, suppose the wave propagates further and additional branch points begin to form. Following the same type of proof as in Sections 4 and 5, it can be shown that the second onset of branch point formation also occurs in pairs of opposite polarity with infinitesimal separation. In this way, the pupil can be fully populated with branch points.

5.5. *Propagation with a Creation Pair*

For completeness, it must be shown that once created, a creation pair can propagate uniformly.

The proof is identical to that in Section 4, with the exception that $B(x,y;z)$ is no longer the null set, i.e. it contains the branch cuts associated with the creation pairs.

5.6. *Section Summary*

Given a wave that is branch-point-free prior to plane z. A branch point pair can be created at $z+\varepsilon$ where ε is arbitrarily small as long as their separation is infinitely small and the branch points are of opposite polarity.

We call this a creation pair.

6. Evolution of the Branch Cuts

Suppose a single creation pair has formed and there is additional propagation with no further branch point formation. Following the same type of proof as in Section 4, it can be shown that $f_n \to f$ uniformly requires that $B_n \to B$ uniformly where $B_n \in \{B(x,y;z_n)\}$ with $z_n := \frac{1}{n}$ as before. Note, although B is changing with increasing z, this B is not a wave and hence is not propagating; we say here for discussional purposes that B is evolving.

So, not surprisingly then, the branch cuts evolve uniformly as the wave moves in z. Uniform evolution of B means that B is a smooth function of z which implies that branch points are a persistent feature of the propagating wave. This stands in direct contradiction to standard practice in adaptive optics where typically branch cuts are assigned via a frame-to-frame minimum distance metric [5]. In short, we have shown that the branch cuts evolve smoothly in z and hence in time since they travel with the wave.

Also, see Appendix B for a stronger statement on the condition on B.

7. Creation Pairs, Branch Cuts, and Principal Domains

Each branch point has a branch cut. Since we have shown that branch points appear in pairs of opposite polarity, we would expect that the branch cut lies along the branch point pair. This is in keeping with the observation in [2] that branch cuts lie along minima in intensity.

The principal domain of each branch point is then chosen so that the branch cut lies along the direction to the other branch point. Therefore, since choosing the line along which the branch cut resides is equivalent to a choice of principal domain, (1) each branch point in the creation pair has a principal domain different than the other by π, and (2) each creation pair, depending on its orientation, has a different base interval for its principal domain.

8. Discussion

We have proven that when created by atmospheric turbulence, branch points must be created in pairs of opposite 'polarity' with infinitesimal separation. Yet in real systems, they are observed with physically measurable separation.

Based on this evidence, three interesting implications presents themselves. First, the creation pairs will have the velocity of the turbulence layer that created them. An experimental demonstration confirming this observation will be presented in the second paper in this series.

Second, since we have shown that branch points must be created in pairs infinitesimally close together, yet they are observed to have finite separations in wavefront sensor data, they must drift apart as the wave propagates. The forth paper in this series will prove this to be true and also determine the magnitude of this drift component.

Thirdly, it appears that the separation of the paired branch points yields (stochastically) the distance to the turbulence layer which caused their creation. That is, given a turbulence layer which generates an ensemble of pairs of branch points some distance later, the distance to the turbulence layer can be obtained from the mean separation between the paired branch points. Again an experimental demonstration will be presented in the fourth paper of this series.

9. Summary

Here we have begun a study on the aggregate behavior of branch points and have shown a narrow but crucial result: atmospherically created branch points are created infinitesimally close together in pairs of opposite polarity and drift apart as they propagate. This proves that atmospherically created plane branch points measured in the pupil of a telescope contain information about the turbulence layer that created them. Furthermore, we have shown that branch cuts are also a persistant feature of the wave and that they evolve uniformly much as the wave does while propagating.

In short, we have taken the first step in demonstrating that branch points have a rich aggregate behavior previously unmentioned in the literature.

Acknowledgements

We would like to thank the Air Force Office of Scientific Research for their support in funding this research.

A. Appendix - Enumertion of the Derivation demonstrating Uniform Propagation

It must be shown that the sequence $\{f_n\}$ is such that $f_n \to f(\mathbf{r},z)$ uniformly. To do so it must first be shown that $f(\mathbf{r},z)$, outside some radius, R, is arbitrarily close to zero, i.e. given $\varepsilon > 0, \exists R > 0$ s.t. $\forall |\mathbf{r}_0| > R, |f(\mathbf{r}_0,z)| < \varepsilon$. So consider

$$f(\mathbf{r},z) = \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z}|\mathbf{r}|^2} \int_\Omega f(\mathbf{r}_0,0) e^{i\frac{k}{2z}|\mathbf{r}_0|^2} e^{i\frac{k}{z}\mathbf{r}_0\cdot\mathbf{r}} d\mathbf{r}_0, \tag{35}$$

and let

$$I(\mathbf{r},z) := \int_\Omega f(\mathbf{r}_0,0) e^{i\frac{k}{2z}|\mathbf{r}_0|^2} e^{i\frac{k}{z}\mathbf{r}_0\cdot\mathbf{r}} d\mathbf{r}_0.$$

It suffices to show that $\forall |\mathbf{r}| > R, |I(\mathbf{r},z)| < \varepsilon$.

First note that $\mathbf{r}_0$ is bounded since Ω is compact in $\mathbb{R}^2$. Then, since $\mathbf{r}$ is unbounded,

$$\exists R \text{ s.t. } \forall |\mathbf{r}| > R \text{ s.t. } f(\mathbf{r}_0,0) \text{ and } e^{i\frac{k}{2z}|\mathbf{r}_0|^2} \text{ vary slowly w.r.t } \frac{k}{z}|\mathbf{r}_0\cdot\mathbf{r}|. \tag{36}$$

Therefore to good approximation, both $f(\mathbf{r}_0,0)$ and $e^{i\frac{k}{2z}|\mathbf{r}_0|^2}$ can be taken out of the integral. Hence,

$$|I(\mathbf{r},z)| \approx \left| f(\mathbf{r}_0,0) e^{i\frac{k}{2z}|\mathbf{r}_0|^2} \int_\Omega e^{i\frac{k}{z}\mathbf{r}_0\cdot\mathbf{r}} d\mathbf{r}_0 \right| \tag{37}$$

$$\approx \left| f(\mathbf{r}_0,0) e^{i\frac{k}{2z}|\mathbf{r}_0|^2} \frac{z}{ik|\mathbf{r}|} \int_{\Omega(\theta(\rho))} e^{i\frac{k}{z}|\mathbf{r}|\rho\cos(\theta)} d\theta \right]_{\Omega(\rho)} \Bigg| \tag{38}$$

$$\lesssim \frac{1}{|\mathbf{r}|} \left| f(\mathbf{r}_0,0) \frac{z}{k} \int_{\Omega(\theta(\rho))} e^{i\frac{k}{z}|\mathbf{r}|\rho\cos(\theta)} d\theta \right]_{\Omega(\rho)} \Bigg| \tag{39}$$

$$\tag{40}$$

where Ω is integrated in cylindrical coordinates $(\rho,\theta) \implies \mathbf{r}_0\cdot\mathbf{r} = \rho|\mathbf{r}|cos(\theta)$, $\Omega(\rho)$ is the boundary of Ω in the ρ coordinate, $\Omega(\theta(\rho))$ is the remaining boundary of Ω over θ, and $]_{\Omega(\rho)}$ evaluates the expression along the ρ boundary. Then note in the last expression that since f has compact support, and since z and k are fixed, and since $\int_{\Omega(\theta(\rho))} e^{(\cdot)} < +\infty$, $|\cdot| < +\infty$ Hence, letting $C := |\cdot|$,

$$|I(\mathbf{r},z)| \lesssim \frac{C}{|\mathbf{r}|} \tag{41}$$

$$\implies |f(\mathbf{r},z)| \lesssim \frac{1}{|\mathbf{r}|}\frac{C}{\lambda z} \tag{42}$$

$$\implies \text{given } \varepsilon > 0, \exists R > 0 \text{ s.t. } \forall |\mathbf{r}| > R,\ |f(\mathbf{r},z)| < \frac{1}{4}\varepsilon. \tag{43}$$

Then, it suffices to show that given $\varepsilon > 0, \exists N > 0$ s.t. $\forall \mathbf{r} \in \mathbb{R}^2$ and $n > N$, $|f(\mathbf{r}, z-\frac{1}{n}) - f(\mathbf{r},z)| < \varepsilon$. So consider,

$$\begin{aligned}
|f_n - f(\mathbf{r},z)| &= \Big|\frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z}|\mathbf{r}|^2} I(\mathbf{r},z) - \frac{e^{ik(z-\frac{1}{n})}}{i\lambda(z-\frac{1}{n})} e^{i\frac{k}{2(z-\frac{1}{n})}|\mathbf{r}|^2} I(\mathbf{r},z)\Big| \\
&\leq \frac{1}{\lambda(z-\frac{1}{n})}\Big|I(\mathbf{r},z) - I(\mathbf{r},z-\frac{1}{n}) - \frac{1}{nz}e^{ik\frac{1}{n}} e^{-i\frac{k}{2}|\mathbf{r}|^2\frac{1}{n}\frac{1}{z(z-\frac{1}{n})}} I(\mathbf{r},z)\Big| \qquad (44)
\end{aligned}$$

and using the triangle inequality and since $0 < z-1 < z - \dfrac{1}{n}$,

$$\leq \frac{1}{\lambda(z-1)}\Big|I(\mathbf{r},z) - I(\mathbf{r},z-\frac{1}{n})\Big| + \frac{1}{n}\Big|\frac{I(\mathbf{r},z)}{z}\Big|$$

and since $\frac{I(\mathbf{r},z)}{z}$ is bounded, $\exists M$ s.t. $\forall n > M$,

$$\frac{1}{n}\frac{1}{\lambda(z-\frac{1}{n})}\Big|\frac{I(\mathbf{r},z)}{z}\Big| < \frac{1}{2}\varepsilon. \qquad (45)$$

Therefore,

$$\text{for } z > 1 \text{ and } \forall n > M,\ |f(\mathbf{r},z-\frac{1}{n}) - f(\mathbf{r},z)| < \frac{1}{\lambda(z-1)}\Big|I(\mathbf{r},z) - I(\mathbf{r},z-\frac{1}{n})\Big| + \frac{1}{2}\varepsilon \qquad (46)$$

Now consider two cases, $|\mathbf{r}| > R$ and $|\mathbf{r}| \leq R$ with R chosen as in Equation 36.

Case 1: $|\mathbf{r}| > R$

$$\begin{aligned}
|f(\mathbf{r},z-\frac{1}{n}) - f(\mathbf{r},z)| &< \frac{1}{\lambda(z-1)}\Big|I(\mathbf{r},z) - I(\mathbf{r},z-\frac{1}{n})\Big| + \frac{1}{2}\varepsilon \\
&< \frac{1}{\lambda(z-1)}|I(\mathbf{r},z)| + |I(\mathbf{r},z-\frac{1}{n})| + \frac{1}{2}\varepsilon \\
&< \frac{1}{|\mathbf{r}|}\frac{(C_1+C_2)}{\lambda(z-1)} + \frac{1}{2}\varepsilon \qquad (47)
\end{aligned}$$

where C_1 and C_2 are as given in Equation 42. Then letting $R := \frac{C_1+C_2}{\lambda(z-1)}$,

$$\implies \text{for } z > 1 \text{ and } \forall n > M,\ \text{given } \varepsilon > 0, \exists R \text{ s.t. } \forall |\mathbf{r}| > R, |f(\mathbf{r},z-\frac{1}{n}) - f(\mathbf{r},z)| < \varepsilon$$

$\square_{Case1}$

Case 2: $n > M$ **and** $|\mathbf{r}| \leq R$

$$\begin{aligned}
&|I(\mathbf{r},z) - I(\mathbf{r},z-\frac{1}{n})| \\
&\quad = \Big|\int_\Omega f(\mathbf{r}_0,0)e^{i\frac{k}{2z}|\mathbf{r}_0|^2}e^{i\frac{k}{z}\mathbf{r}_0\cdot\mathbf{r}}d\mathbf{r}_0 - \int_\Omega f(\mathbf{r}_0,0)e^{i\frac{k}{2(z-1/n)}|\mathbf{r}_0|^2}e^{i\frac{k}{(z-1/n)}\mathbf{r}_0\cdot\mathbf{r}}d\mathbf{r}_0\Big| \\
&\quad = \Big|\int_\Omega d\mathbf{r}_0 f(\mathbf{r}_0,0)e^{i\frac{k}{2(z-1/n)}|\mathbf{r}_0|^2}e^{i\frac{k}{(z-1/n)}\mathbf{r}_0\cdot\mathbf{r}}\left(e^{i\frac{k}{2}\frac{1}{n}\frac{1}{z(z+1/n)}|\mathbf{r}_0|^2}e^{ik\mathbf{r}_0\cdot\mathbf{r}\frac{1}{n}\frac{1}{z(z+1/n)}} - 1\right)\Big| \\
&\quad < \int_\Omega d\mathbf{r}_0\Big|f(\mathbf{r}_0,0)\Big|\Big|\left(e^{\frac{\alpha+\beta}{n}} - 1\right)\Big| \qquad (48)
\end{aligned}$$

where

$$\alpha := i\frac{k}{2}\frac{1}{z(z+1/n)}|\mathbf{r}_0|^2 \qquad \text{and} \qquad \beta := ik\mathbf{r}_0\cdot\mathbf{r}\frac{1}{z(z+1/n)}. \tag{49}$$

Then,

$$\begin{aligned}
&|I(\mathbf{r},z)-I(\mathbf{r},z-\frac{1}{n})| \\
&\quad = \int_\Omega d\mathbf{r}_0\Big|f(\mathbf{r}_0,0)\Big|\Big|\frac{1}{1!}\left(\frac{\alpha+\beta}{n}\right)^1+\frac{1}{2!}\left(\frac{\alpha+\beta}{n}\right)^2+\frac{1}{3!}\left(\frac{\alpha+\beta}{n}\right)^3+\cdots\Big| \\
&\quad \leq \int_\Omega d\mathbf{r}_0\Big|f(\mathbf{r}_0,0)\Big|\frac{1}{n}\left(\frac{1}{1!}|\alpha+\beta|^1+\frac{1}{2!}|\alpha+\beta|^2+\frac{1}{3!}|\alpha+\beta|^3+\cdots\right) \\
&\quad < \frac{1}{n}\int_\Omega d\mathbf{r}_0\Big|f(\mathbf{r}_0,0)\Big|\left(e^{|\alpha+\beta|}-1\right)
\end{aligned} \tag{50}$$

but $|\mathbf{r}|\leq R$ and $\mathbf{r}_0\in\Omega \implies \alpha,\beta<+\infty \implies \int_\Omega(\cdot)<+\infty$. So letting $C_3 := \int_\Omega(\cdot)<+\infty$,

$$\implies |I(\mathbf{r},z)-I(\mathbf{r},z-\frac{1}{n})| < \frac{1}{n}C_3 \tag{51}$$

$$\implies |f(\mathbf{r},z-\frac{1}{n})-f(\mathbf{r},z)| < \frac{1}{n}\,C_3\frac{1}{\lambda(z-1)}+\frac{1}{2}\varepsilon \tag{52}$$

and since $\exists N > 2\frac{1}{\varepsilon}\frac{C_3}{\lambda z} \implies \exists N>0,$ s.t. $\forall n>N, |f(\mathbf{r},z-\frac{1}{n})-f(\mathbf{r},z)| < \frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon$ then letting $P := max(N,M)$

$$\implies \text{given } \varepsilon>0, \exists P>0 \text{ s.t. } \forall|\mathbf{r}|<R \text{ and } n>P \implies |f(\mathbf{r},z-\frac{1}{n})-f(\mathbf{r},z)|<\varepsilon \tag{53}$$

$\square_{Case2}$

So, we have shown that $f_n \to f(\mathbf{r},z)$ uniformly on $\mathbb{R}^2\backslash B$ for both $|\mathbf{r}|>R$ and for $|\mathbf{r}|\leq R$. Therefore, free space Fresnel propagation is uniform. Moreover, since analytic functions are unique and given Equation 7, this implies that both the amplitude and phase propagate uniformly, i.e. create sets $\{A_n\}$ and $\{\phi_n\}$ with $A_n = A(x,y,z_n)$ and $\phi_n = \phi(x,y,z_n)$ and $z_n = z_{\text{pupil}} - \frac{1}{n}$ as before. Then given that $f_n \to f(\mathbf{r},z)$ uniformly , $A_n(\mathbf{r}) \to A(\mathbf{r})$ uniformly and $\Phi_n(\mathbf{r}) \to \Phi(\mathbf{r})$ uniformly.

B. Appendix - Constraint on the Evolution of B

The further stronger statement can be made: the newly created branch point pair induces both real and imaginary parts to the wave. Initially, the imaginary part is zero and the real part is one. As the wave propagates past z_{pupil}, the imaginary and real parts of the branch point phase uniformly evolve away from zero. This can be seen explicitly if d is taken away from zero like

$$d_m := \frac{1}{m} \text{ with } m\in\mathbb{Z} \text{ s.t. } \frac{1}{M}<D.$$

Then let $\{\Phi_{hid}^{(m)}(\mathbf{r}, z_{\text{pupil}})\}$ be a sequence created using the d_m and from Equations 28 and 29,

$$\begin{aligned}
\left|1 - e^{i\Phi_{hid}(\mathbf{r}, z_{\text{pupil}})}\right| &< \left|1 - \cos(\Phi_{hid}(\mathbf{r}, z_{\text{pupil}}))\right| + \left|i0 + i\sin(\Phi_{hid}(\mathbf{r}, z_{\text{pupil}}))\right| \\
&< |0 - dC| + \left|i0 - i(dC)^2\right| \\
&= \left|0 - \frac{C}{m}\right| + \left|i0 - i\left(\frac{C}{m}\right)^2\right| \\
&\quad \text{Since } C \text{ is a constant, } \exists M \text{ s.t. } \forall m > M, C/m < \varepsilon/2 \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon^2}{4} < \varepsilon \qquad \forall m > M \text{ and } \forall x, y \in \mathbb{R}^2/B
\end{aligned} \tag{54}$$

So,

$$\text{given } \varepsilon,\ \exists M \text{ such that } \forall m > M,\ \left|1 - \cos(\Phi_{hid}(\mathbf{r}, z_{\text{pupil}}))\right| < \varepsilon\ \forall x, y \in \mathbb{R}$$

and

$$\text{given } \varepsilon,\ \exists M \text{ such that } \forall m > M,\ \left|i0 - i\sin(\Phi_{hid}(\mathbf{r}, z_{\text{pupil}}))\right| < \varepsilon\ \forall x, y \in \mathbb{R}$$

as was to be shown.

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