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14. ABSTRACT

The work is devoted to theoretical research leading to the fundamental understanding of the influence of thermal noise on the performance of nano-sized spin-torque microwave oscillators for military and space applications. In contrast with the existing theories, the developed theory is not be based on the assumption that the noise power is much smaller than the power generated by the oscillator. The theory takes into account strong nonlinearity of nano-scale spin-torque magnetic devices, i.e. dependence of device parameters (frequency, damping rate, etc.) on the amplitude of the generated microwave oscillations.

The developed theory lays foundation for the practical development of tunable nano-sized spin torque microwave oscillators.
Such devices may be desirable in the future nano-electronic circuits designed for military and space-oriented applications.
These devices could replace current semiconductor microwave generators, eliminating the sensitivity of such generators to ionizing radiation.

This research does not include the areas of intentional interference (jamming/RF weapons), noise in space environment (radiation hardening), or other countermeasure related issues.

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FINAL REPORT

on the ILIR project : "Influence of thermal noise on the performance of nano-sized spin-torque microwave oscillators for military and space applications"

Supported by the contract # W56HZV-08-P-L605 from the TACOM-Warren, AMSTA-AQ-ASRB
Name of Contractor: Oakland University (OU), Department of Physics, PI : Professor A.N. Slavin

Scope and objectives of the effort

The contractor is required to perform theoretical research leading to the fundamental understanding of the influence of thermal noise on the performance of nano-sized spin-torque microwave oscillators for military and space applications. In contrast with the existing theories, the developed theory should not be based on the assumption that the noise power is much smaller than the power generated by the oscillator. The theory should, also, take into account strong nonlinearity of nano-scale spin-torque magnetic devices, i.e. dependence of device parameters (frequency, damping rate, etc.) on the amplitude of the generated microwave oscillations.

If the research results are favorable, future effort toward actual application may lead to tunable nano-sized spin torque microwave oscillators. Such devices may be desirable in the future nano-electronic circuits designed for military and space-oriented applications. These devices could replace current semiconductor microwave transistors, eliminating the sensitivity of such semiconductors to ionizing radiation.

This research will not include the areas of intentional interference (jamming/RF weapons), noise in space environment (radiation hardening), or other countermeasure related issues.

Work Period 2 : 11/01/08- 12/31/08:

Theoretical study of the stochastic magnetization dynamics excited by spin-polarized current in magnetic nano-structures: calculation of the average values of generated power, generation frequency, and the generation linewidth of a nonlinear spin-torque oscillator. Analysis of the lineshape distortions near the auto-oscillation threshold. Comparison of the obtained theoretical results to the results of laboratory experiments.

The report covering the above described research topics and entitled: **"Stochastic magnetization dynamics excited by spin-polarized current in magnetic nano-structures"** is attached.

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Stochastic magnetization dynamics excited by spin-polarized current in magnetic nano-structures

I. INTRODUCTION

Thermal fluctuations and other noise sources play an important role in the dynamics of conventional auto-oscillators. Thermal fluctuations (or noise) determine one of the most important, from the practical point of view, parameters of the auto-oscillator – generation linewidth $\Delta\omega$. The influence of thermal noise is even more important for the nano-scale spin-torque oscillators. Due to the extremely small sizes of STO, the characteristic operational energy of these oscillators can be comparable with the thermal energy $k_{\rm B}T$ (where $k_{\rm B}$ is the Boltzmann constant and T is the thermodynamic temperature). As a result, not only the generation linewidth, but all the other characteristics of spin-torque oscillators can be dependent on the temperature. In particular, thermal fluctuations lead to the blurring of the generation threshold in STO and to a finite level of power fluctuations in the above-threshold operational regime, in which the influence of thermal fluctuations is, usually, negligible for conventional macro-sized oscillators.

In this section we describe how the influence of noise can be accounted for in the nonlinear oscillator model (Sec. II) and consider several important noise-related issues, namely, the influence of noise on the generation power of an oscillator (Sec. III) and the broadening of the oscillator linewidth under the action of thermal fluctuations (Sec. IV).

II. STOCHASTIC NONLINEAR OSCILLATOR MODEL

The stochastic dynamics of an auto-oscillator can be described in the framework of the nonlinear oscillator model (I.5) with external force $f_{\rm n}(t)$ representing the action of thermal fluctuations:

$$\frac{dc}{dt} + i\omega(|c|^2)c + \Gamma_+(|c|^2)c - \Gamma_-(|c|^2)c = f_{\rm n}(t)\ . \quad (1)$$

For any particular oscillator scheme (1) can be derived from the stochastic "material" equations. For instance, to obtain (1) for the Van der Pol oscillator (see Sec. I.IV-A) one has to add to the electrical scheme shown in Fig. I.2a an additional stochastic voltage source that describes thermal Johnson-Nyquist noise connected with the resistor R. For a spin-torque oscillator stochastic equation (1) can be derived from the Landau-Lifshits-Gilbert-Slonczewski equation (I.1) with an additional random thermal magnetic field.

For an auto-oscillator of an arbitrary nature, stochastic force $f_{\rm n}(t)$ in (1) can be considered as a phenomenological term describing action of thermal fluctuations. In such a case, the statistical properties of $f_{\rm n}(t)$ should be determined from the condition of a proper thermodynamical behavior of the oscillator in the state of thermal equilibrium (i.e., when the negative damping is vanishing $\Gamma_-(|c|^2) = 0$). This is achieved by selecting $f_{\rm n}(t)$ to be a white Gaussian noise with the zero mean and second-order correlator given by [1]

$$\langle f_{\rm n}(t) f_{\rm n}(t')\rangle = 0\,, \qquad \langle f_{\rm n}(t) f_{\rm n}^*(t')\rangle = 2D_{\rm n}\delta(t-t')\ . \quad (2)$$

Here $D_{\rm n}$ is the effective "diffusion coefficient" that characterizes the noise amplitude. To describe correctly the stochastic dynamics of a *nonlinear* oscillator with arbitrary dependences of the frequency $\omega(p)$ and natural damping $\Gamma_+(p)$ on the oscillation power p, one has to assume that the diffusion coefficient $D_{\rm n}$ also depends on p as [1]

$$D_{\rm n}(p) = \Gamma_+(p)\eta(p) = \Gamma_+(p)\frac{k_{\rm B}T}{\lambda\omega(p)} \quad (3)$$

where $\eta(p) = k_{\rm B}T/\lambda\omega(p)$ is the effective noise power in the nonlinear regime. As it will be shown below, this form of the diffusion coefficient provides correct equilibrium properties (see (7)) for the *arbitrary* power dependences $\omega(p)$ and $\Gamma_+(p)$.

In (3) factor λ is a scale factor that relates the oscillator energy $\mathcal{E}(p)$ to the dimensionless oscillator power $p = |c|^2$:

$$\mathcal{E}(p) = \lambda\int_0^p \omega(p')dp'\ . \quad (4)$$

Clearly, λ depends on the normalization of the oscillator amplitude c. For our choice of normalization of the amplitude c of the spin-torque oscillator (see (I.15)) λ has the form

$$\lambda = V_{\rm eff}M_0/\gamma$$

where $V_{\rm eff}$ is the effective volume of the magnetic material of the "free" layer involved in the auto-oscillation, M_0 is the saturation magnetization of the "free" layer, and γ is the gyromagnetic ratio.

For many problems, instead of solving the nonlinear stochastic Langevin equation (1) for the complex oscillation amplitude $c(t)$, it is convenient to consider the dynamics of the probability density function (PDF) $\mathcal{P}(t,p,\phi)$ that gives the probability of the auto-oscillator to have the power p and phase ϕ at the time t (given some probability distribution at the initial moment of time). While the dynamics of the complex amplitude $c(t)$ is described by the *stochastic* and *nonlinear* equation (1), the evolution of the PDF $\mathcal{P}$ is governed by the *deterministic* and *linear* Fokker-Planck equation that can be obtained from (1) using a standard formalism [2]

$$\frac{\partial\mathcal{P}}{\partial t} - \frac{\partial}{\partial p}\left[2p\left(\Gamma_+ - \Gamma_-\right)\mathcal{P}\right] - \omega\frac{\partial\mathcal{P}}{\partial\phi} = \frac{\partial}{\partial p}\left(2pD_{\rm n}\frac{\partial\mathcal{P}}{\partial p}\right) + \frac{\partial}{\partial\phi}\left(\frac{D_{\rm n}}{2p}\frac{\partial\mathcal{P}}{\partial\phi}\right)\ . \quad (5)$$

Terms in the left-hand-side part of (5) originate from the deterministic terms in (1), whereas the terms in the right-hand-side part of (5) describe the stochastic action of the thermal noise $f_{\rm n}(t)$. One can see, that the noise-related terms in (5) have the form of a usual diffusion operator (in the power-phase coordinates) with the power-dependent diffusion coefficient $D_{\rm n}$. Thus, thermal noise leads to the diffusion of the oscillator in the phase space.

Equation (1) or (5) can be applied for the analysis of a number of noise-related problems of oscillator dynamics. It is clear, that one can easily add to this stochastic scheme additional action of deterministic external signals [3] or mutual coupling between oscillators. It is, also, possible to generalize (1) to include effects of other sources of noise, e.g., in the case of spin-torque oscillators, flicker noise related to the bias current or mag-noise related to the fluctuations in the orientation of the "fixed" layer magnetization.

III. Power Generated by an Oscillator in the Presence of Noise

In this section we consider the problem of determination of the power, generated by an auto-oscillator in the presence of thermal fluctuations. In the framework of the nonlinear auto-oscillator model one can find a complete analytical solution of this problem, valid both below the generation threshold (when, like in a passive oscillator, the oscillations exist only due to the thermal noise) and in the above-threshold generation regime (when the thermal noise leads to the fluctuations of the generated power). The obtained general results, applied to the case of an STO, suggest a precise method for the determination of the threshold current $I_{\rm th}$ and quantitatively describe experimental data in the near-threshold region, when the influence of the thermal fluctuations can not be ignored.

A. Theoretical Results

To find the average power, generated by an oscillator in the presence of thermal noise, it is sufficient to find the stationary solution $\mathcal{P}_0$ of the Fokker-Planck equation (5). It should be noted, that, since (5) does not depend on the oscillator phase ϕ explicitly (which is connected with the phase-invariance of the Langevin equation (1) or, in more general terms, with an autonomous character of the auto-oscillator's dynamics), the stationary PDF $\mathcal{P}_0$ is independent of the phase ϕ. Thus, in the stationary state any value of the oscillator phase has equal probability, and stationary PDF $\mathcal{P}_0$ is a function of only the oscillator power p: $\mathcal{P}_0 = \mathcal{P}_0(p)$. In this case (5) becomes an ordinary second-order differential equation

$$\frac{d}{dp}\left[2p\left(\Gamma_+ - \Gamma_-\right)\mathcal{P}_0 + 2pD_{\rm n}\frac{d\mathcal{P}_0}{dp}\right] = 0$$

and a physically-consistent solution of this equation has the form [1]

$$\mathcal{P}_0(p) = \mathcal{N}_0 \exp\left[-\frac{\lambda}{k_{\rm B}T}\int_0^p \omega(p')\left(1 - \frac{\Gamma_-(p')}{\Gamma_+(p')}\right)dp'\right] \ . \tag{6}$$

Here $\mathcal{N}_0$ is the normalization constant determined from the normalization condition

$$\int_0^\infty \mathcal{P}_0(p)dp = 1 \ .$$

One can see from (6), that at the thermal equilibrium ($\Gamma_-(p) = 0$) stationary PDF $\mathcal{P}_0(p)$ takes the form

$$\mathcal{P}_{\rm eq}(p) = \mathcal{N}_0 \exp\left[-\frac{\lambda}{k_{\rm B}T}\int_0^p \omega(p')dp'\right] = \mathcal{N}_0 \exp\left[-\frac{\mathcal{E}(p)}{k_{\rm B}T}\right] \tag{7}$$

which coincides with the standard Boltzmann distribution. We would like to stress, that our approach gives a correct equilibrium distribution (7) and, hence, satisfies the fluctuation-dissipation theorem [4] for an *arbitrary* dependence of the oscillator's frequency $\omega(p)$ and damping $\Gamma_+(p)$ on the power p, even if these functions significantly change on the thermal energy scale $k_{\rm B}T$.

Equation (6) allows one to determine any stationary characteristic of a nonlinear auto-oscillator in the presence of noise. For instance, the mean oscillation power $\overline{p}$ is found to be

$$\overline{p} = \int_0^\infty p\,\mathcal{P}_0(p)dp \ .$$

The power fluctuations Δp^2 can be calculated as

$$\Delta p^2 = \overline{p^2} - (\overline{p})^2$$

where

$$\overline{p^2} = \int_0^\infty p^2\,\mathcal{P}_0(p)dp \ .$$

When the temperature T is small, the argument of the exponential function in (6) is large and (6) can be simplified in two important cases. First, in the below-threshold regime (when $\Gamma_-(p) < \Gamma_+(p)$ for all p) the distribution $\mathcal{P}_0(p)$ has a maximum at $p = 0$. Expanding the integrand in (6) in a Taylor series near $p = 0$ and keeping only the first non-zero term, one obtains an approximate expression for the PDF $\mathcal{P}_0(p)$ in the below-threshold regime:

$$\mathcal{P}_<(p) \approx \frac{1-\zeta}{\eta(0)}\exp\left[-\frac{1-\zeta}{\eta(0)}\,p\right]$$

where the supercriticality is defined as $\zeta = \Gamma_-(0)/\Gamma_+(0)$, which coincides with the previously used notation (I.26) for the case of a spin-torque oscillator. The average power $\overline{p}$ and power fluctuations Δp^2 in this regime are given approximately by the expressions:

$$\overline{p} = \frac{\eta(0)}{1-\zeta} = \left(\frac{k_{\rm B}T}{\lambda\omega(0)}\right)\frac{\Gamma_+(0)}{\Gamma_+(0) - \Gamma_-(0)} \tag{8a}$$

$$\Delta p^2 = (\overline{p})^2 = \left(\frac{\eta(0)}{1-\zeta}\right)^2 \ . \tag{8b}$$

The below-threshold PDF $\mathcal{P}_<(p)$ has approximately the same Boltzmann-like form $\mathcal{P}_<(p) \sim \exp(-p/\overline{p})$ as the equilibrium distribution $P_{\rm eq}(p)$ (note, that for small temperature T one can approximate $\mathcal{E}(p)/k_{\rm B}T$ as $\lambda\omega(0)p/k_{\rm B}T = p/\eta(0)$ in (7)), but with the increased average power level $\overline{p}$. One may interpret this effect as an effective increase of the temperature $T \to T/(1-\zeta)$ (see, e.g., [5], [6]). We would like to note, however, that this interpretation is not completely correct, as the negative damping $\Gamma_-(p)$ in this regime does not only increase the power $\overline{p}$, but, also, reduces the linewidth of oscillations (see Sec. IV), which can not be described by any changes of the effective temperature.

In the above-threshold generation regime stationary PDF $\mathcal{P}_0(p)$ has a maximum at the stationary power $p = p_0$ (see (I.25)), for which $\Gamma_+(p_0) = \Gamma_-(p_0)$, and the integrand in (6) is zero. By expanding the integrand in (6) in a Taylor series near $p = p_0$, one can derive an approximate expression for the PDF $\mathcal{P}_0(p)$ in the above-threshold regime

$$\mathcal{P}_>(p) \approx \frac{1}{\sqrt{2\pi}\Delta p} \exp\left[-\frac{(p-p_0)^2}{2\Delta p^2}\right] .$$

Here

$$\Delta p^2 = \left(\frac{\Gamma_+(p_0)}{\Gamma_\mathrm{p}}\right) p_0 \eta(p_0) \approx \left(\frac{\Gamma_+(p_0)}{\Gamma_\mathrm{p}}\right) \frac{k_\mathrm{B}T}{\mathcal{E}(p_0)} p_0^2 \qquad (9)$$

is the level of power fluctuations of the oscillator in the above-threshold regime (Γ_p is defined by (I.28b)). The mean oscillation power $\overline{p}$ in this regime is approximately equal to the zero-temperature value p_0 (I.25).

One can see, that the probability distribution function $\mathcal{P}_>(p)$ in the above-threshold regime has a functional form that is substantially different from the form of the below-threshold PDF $\mathcal{P}_<(p)$. It is also interesting to note, that the power fluctuations in this regime increase with temperature as $\Delta p \sim \sqrt{T}$, while below the generation threshold $\Delta p \sim T$. Since for typical auto-oscillators $\Gamma_\mathrm{p} \sim \Gamma_+(p_0)$, the relative level of power fluctuations $\Delta p/p_0 \sim \sqrt{\eta(p_0)/p_0}$ decreases with p_0 (i.e., with the increase of the supercriticality) and, for sufficiently large p_0, power fluctuations become negligible.

Neglecting the dependence of the oscillator frequency $\omega(p)$ on the power p ($\eta(p) = \eta = \mathrm{const}$), which has only a minor influence on the form of the PDF, and using (I.20), one can derive from (6) an explicit general expression for the stationary probability distribution function of a spin-torque oscillator valid in the *whole range of bias currents*:

$$\mathcal{P}_0(p) = \sqrt{\frac{2\zeta}{\pi\eta}} \frac{\exp\left[-(1-\zeta+\zeta p)^2/2\zeta\eta\right]}{1+\mathrm{erf}\left[(\zeta-1)/\sqrt{2\zeta\eta}\right]} \qquad (10a)$$

for $Q = 0$ and

$$\mathcal{P}_0(p) = \frac{Q\exp\left[-(\zeta+Q)(1+Qp)/Q^2\eta\right]}{(1+Qp)^\beta E_\beta((\zeta+Q)/Q^2\eta)} \qquad (10b)$$

for $Q \neq 0$. Here $\beta = -(1+Q)\zeta/Q^2\eta$, $\mathrm{erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-t^2}dt$ is the error function, and $E_n(x) = \int_1^\infty e^{-xt}/t^n dt$ is the exponential integral function. Fig. 1 shows the profile of the probability distribution function (10) in the below-threshold ($\zeta = 0$), threshold ($\zeta = 1$), and above-threshold ($\zeta = 2$) regimes.

Using the analytical expression (10) for the probability distribution function, it is possible to obtain an analytical expression for the mean oscillation power $\overline{p}$ of a spin-torque oscillator:

$$\overline{p} = \sqrt{\frac{2\eta}{\pi\zeta}} \frac{\exp\left[-(\zeta-1)^2/2\zeta\eta\right]}{1+\mathrm{erf}\left[(\zeta-1)/\sqrt{2\zeta\eta}\right]} + \frac{\zeta-1}{\zeta} \qquad (11a)$$

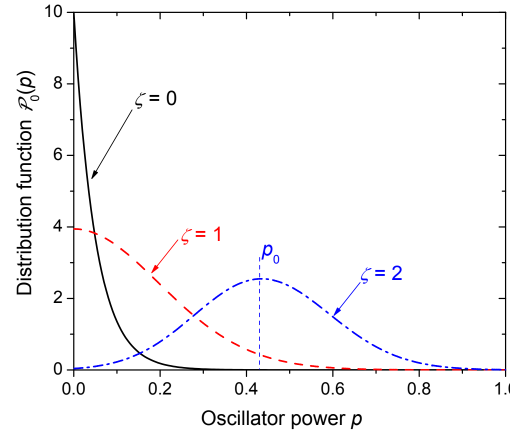

Fig. 1. Stationary probability distribution function $\mathcal{P}_0(p)$ (10) for several values of the supercriticality parameter ζ (from [1]). Nonlinear damping coefficient $Q = 0.3$, noise level $\eta = 0.05$. Data for $\zeta = 0$ (thermal equilibrium) have been divided by 2.

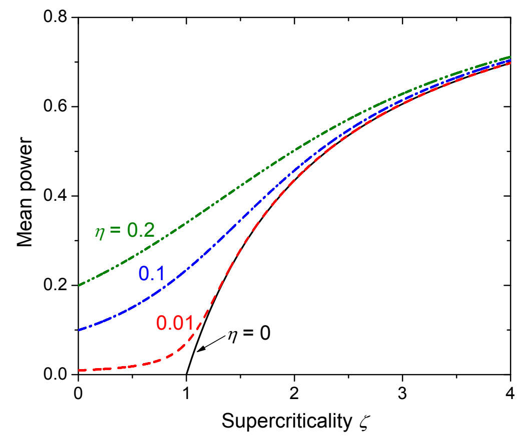

Fig. 2. Dependence of the mean oscillator power $\overline{p}$ (11) on the supercriticality ζ for several noise levels η (from [1]). Nonlinear damping coefficient $Q = 0.3$. The lower curve for $\eta = 0$ coincides with the deterministic value (I.25) of the oscillator power p_0.

for $Q = 0$ and

$$\overline{p} = \frac{Q\eta}{Q+\zeta}\left[1 + \frac{\exp(-(\zeta+Q)/Q^2\eta)}{E_\beta((\zeta+Q)/Q^2\eta)}\right] + \frac{\zeta-1}{\zeta+Q} \qquad (11b)$$

for $Q \neq 0$. Fig. 2 shows the dependence (11) of the mean power $\overline{p}$ on the supercriticality ζ for several values of the noise power η.

B. Comparison With Experiments

To illustrate the application of the above described theory to a real experimental situation we compare below the theoretical expressions for the auto-oscillator power in the presence of noise with the recent measurements of the generated power in a GMR spin-valve auto-oscillator based on a metallic nano-pillar [7]. The experimental data from [7] are shown by black dots in the main frame of Fig. 3. To analyze these experimental data it is important, first of all, to determine the threshold of

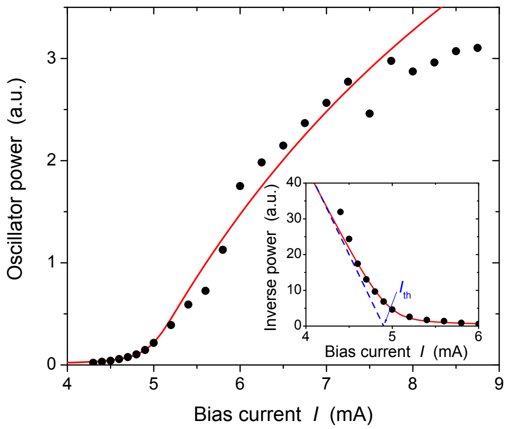

Fig. 3. Main panel: Dependence of the mean power $\overline{p}$ on the bias current I. Dots – experiment [7] for $T = 225$ K, solid line – theoretical dependence (11) for $I_{\rm th} = 4.9$ mA, $\eta = 4.2 \cdot 10^{-4}$, $Q = 0.3$ (from [1]). Inset shows the same data for *inverse* power $1/\overline{p}$ in near-threshold range of currents. Dashed line corresponds to the approximate expression (12) valid for small currents. Intersection of this line with x axis gives the value of the threshold current $I_{\rm th}$.

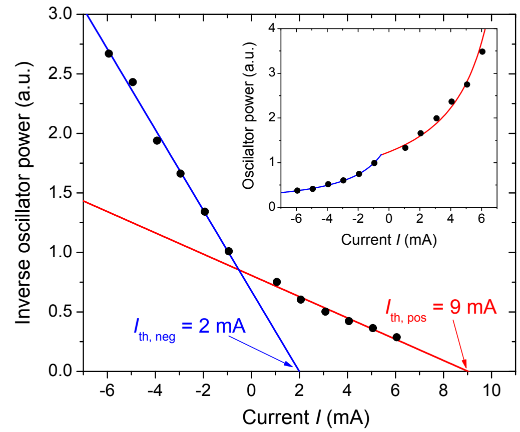

Fig. 4. Main panel: Dependence of the inverse power $1/\overline{p}$ on the bias current I. Dots – experiment [8] for a magnetic tunnel junction (the direction of the bias current was changed to coincide with our sign agreement), solid lines – theoretical dependences (12) demonstrating different values of the threshold current, determined from the data for positive ($I_{\rm th,\,pos}$) and negative ($I_{\rm th,\,neg}$) current directions. Inset shows the same set of data for the direct power $\overline{p}$.

microwave generation, i.e. the magnitude of the bias current at which the spin-torque oscillator, instead of passively filtering its eigen-frequency from the thermal noise, starts to actively self-generate. We believe that this can be done by analyzing the experimental data for very small values of the bias current. In the inset of Fig. 3 we show by dots the experimental dependence of the *inverse* mean power $1/\overline{p}$ on the bias current I. These dots can be fitted to the theoretical expression (8a) which gives a linear dependence of the inverse power on the bias current for small values of current

$$\frac{1}{\overline{p}} \propto (I_{\rm th} - I) \ . \qquad (12)$$

This linear dependence, taking place in the below-threshold regime $I < I_{\rm th}$ and shown by a dashed line in the inset of Fig. 3, crosses the horizontal axis (axis of the bias current) at the point $I = I_{\rm th}$ and, therefore, allows one to determine the $I_{\rm th}$ with a reasonably high precision.

Thus, the precise measurement of the oscillator output power as a function of the bias current *for low values of current* in combination with the theoretical formula (8a) provides a simple method for the precise determination of the threshold current $I_{\rm th}$ of microwave generation in a spin-torque oscillator in situations when the influence of thermal fluctuations is strong, and the determination of $I_{\rm th}$ by other means is difficult. In particular, from the inset of Fig. 3 one can immediately determine the value of the threshold current of $I_{\rm th} = 4.9$ mA, which is substantially larger than the current $I_* \approx 4.2$ mA, at which the thermally-induced oscillations become observable in the experiment [7].

Using thus determined value of the threshold current $I_{\rm th} = 4.9$ mA to calculate the supercriticality parameter $\zeta = I/I_{\rm th}$ in the theoretical expression (11), it is possible to describe theoretically the power $\overline{p}$, generated by an STO, in a wide range of bias currents (see main panel in Fig. 3). We would like to stress, that the correct determination of the threshold current $I_{\rm th}$ (and, therefore, the parameter ζ) is essential for the comparison of (11) with experiment, since the dependence of the oscillator power $\overline{p}$ on the other parameters of the model (like noise power η and nonlinear damping parameter Q) in the above-threshold region ($\zeta > 1$) is relatively weak.

Fig. 4 demonstrates another example of the threshold current determination using the linear dependence (12) in the below-threshold region. In this case the above described method of the threshold current determination is applied to a spin-torque device based on a magnetic tunnel junction (MTJ). Dots in the main panel of Fig. 4 show the experimental data for the inverse power $1/\overline{p}$ measured in the below-threshold regime in an MTJ nano-pillar (see [8]), in which the spacer between the "free" and "fixed" magnetic layers (see Fig. I.1b) was made of a non-magnetic *insulator* $\rm Al_2O_3$. Due to a much larger resistance of the MTJ nano-pillar (compared to fully-metallic GMR spin valve nano-pillars, where the spacer is made of a non-magnetic *metal*), the authors of [8] were able to measure the power of the thermally-induced oscillations of the device even for the negative values of the bias currents $I < 0$, when the spin-transfer torque creates effective *positive* damping and, according to (12), *reduces* the measured oscillation power $\overline{p}$.

One can see from Fig. 4 that the experimental dependences of the inverse output power on the bias current for positive $I > 0$ and negative $I < 0$ current directions are both linear (12), but have substantially different slopes. Respectively, values of the threshold current $I_{\rm th}$, estimated using (12) from the experimental data for positive ($I_{\rm th,\,pos} = 9$ mA) and negative ($I_{\rm th,\,neg} = 2$ mA) currents, are significantly different. Note, that both $I_{\rm th,\,pos}$ and $I_{\rm th,\,neg}$ correspond *to the same magnetization configuration*, but were calculated *for different current polarities*. Since the threshold current (I.23) is inversely proportional to the spin-polarization efficiency ε, this difference in $I_{\rm th}$ suggests that the spin-polarization efficiency ε for MTJ depends not only on the mutual orientation of the magnetizations of the "free" and "fixed" layers, but, also, on the direction of the bias current. In the case shown in Fig. 4 the ratio of spin-polarization efficiencies for positive and negative currents is $\varepsilon_{\rm pos}/\varepsilon_{\rm neg} = 2/9 \approx 0.22$.

Thus, the measurements of the current dependence of the noise-induced output power in MTJ-based STO can provide direct information about the spin-polarization efficiency, and can be used for verification of theories of spin-dependent tunneling in magnetic nano-structures.

We would like to mention, that the threshold current, estimated in [8] using the measurements of the linewidth and the data for the *direct* (rather than *inverse*) power $\overline{p}$, is $I_{\rm th} = 20 \pm 5$ mA, which is far above our estimates. We attribute this difference to the fact that other sources of noise can give a substantial contribution to the linewidth of a highly-resistive MTJ-based device, and, therefore, the estimation of the threshold current based on the linewidth data could be unreliable. On the other hand, the dependence of the direct power $\overline{p}$ on the bias current I (see inset in Fig. 4) is nonlinear, and it is much more difficult to make an estimate of the threshold current based on such a nonlinear dependence. Thus, the difference in the slopes for currents of different polarity, that is clearly seen in Fig. 4, which demonstrates the current dependence of the inverse power, is hardly noticeable in the inset of Fig. 4, where the current dependence of the direct power is presented.

IV. Generation Linewidth of an Auto-Oscillator

Thermal fluctuations do not only blur the stationary distribution $\mathcal{P}_0(p)$ of the oscillation power, but, also, alter the dynamical behavior of an auto-oscillator. In the presence of noise the auto-oscillation amplitude $c(t)$ is not anymore a harmonic function of time $c(t) \sim e^{-i\omega_{\rm g}t}$, corresponding to a simple monochromatic oscillation, but is, instead, a rather complicated stochastic function, and its spectrum

$$c_\Omega = \int_{-\infty}^{+\infty} c(t) e^{i\Omega t} dt$$

has, strictly speaking, non-zero spectral components, corresponding to every frequency Ω.

The *power spectrum* of the auto-oscillation $\mathcal{S}(\Omega)$, that can be defined as

$$\langle c_\Omega c^*_{\Omega'} \rangle = 2\pi\delta(\Omega - \Omega')\mathcal{S}(\Omega)$$

is a convenient quantity that gives a detailed description of the auto-oscillation spectrum in the presence of noise. Here the angular brackets $\langle \ldots \rangle$ stay for the averaging over the statistics of thermal fluctuations. For a coherent monochromatic oscillation with the frequency $\omega_{\rm g}$ the power spectrum in singular $\mathcal{S}(\Omega) \sim \delta(\Omega - \omega_{\rm g})$. In the presence of noise $\mathcal{S}(\Omega)$ has a finite width (called sometimes *generation linewidth*) $\Delta\omega$, which provides a simple way for a single-parameter characterization of a noisy auto-oscillation spectrum. The generation linewidth $\Delta\omega$ is one of the most important parameters of an auto-oscillator, especially with regards to its practical applications.

Alternatively, the coherence of oscillations can be characterized by an *autocorrelation function*

$$\mathcal{K}(\tau) = \langle c(t + \tau) c^*(t) \rangle \ .$$

The descriptions of noisy oscillations by a power spectrum $\mathcal{S}(\Omega)$ or an autocorrelation function $\mathcal{K}(\tau)$ are fully equivalent, since these two functions are connected by the Fourier transform

$$\mathcal{S}(\Omega) = \int \mathcal{K}(\tau) e^{i\Omega\tau} d\tau \ .$$

As it will be shown below, in many cases the power spectrum of an auto-oscillation can be approximately represented in a simple Lorentzian form

$$\mathcal{S}(\Omega) \approx \frac{2\Delta\omega\overline{p}}{(\Omega - \overline{\omega})^2 + \Delta\omega^2} \tag{13a}$$

where $\overline{p}$ is the average oscillation power, and $\overline{\omega}$ is the mean oscillation frequency. Note, that in (13a) and in the following text $\Delta\omega$ is defined as *half*-linewidth at a half-power level. The autocorrelation function, corresponding to (13a), has a simple exponential form

$$\mathcal{K}(\tau) = \overline{p} \exp(-i\overline{\omega}\tau - \Delta\omega|\tau|) \ . \tag{13b}$$

It is important to stress, that in many cases the experimental data for the generation linewidth are fitted to the simple Lorentzian function (13), even if the actual lineshape $\mathcal{S}(\Omega)$ (or autocorrelation function $\mathcal{K}(\tau)$) has more complicated form. Thus, usually, the generation linewidth $\Delta\omega$ is understood in a sense of an experimentally determined quantity obtained as a best fitting parameter in the Lorentzian expression (13).

In the next section we derive expressions for generation linewidth $\Delta\omega$ of an auto-oscillator is several important particular cases, using the stochastic Langevin equation (1).

A. Below-Threshold Regime

In the below-threshold regime ($\Gamma_-(p) < \Gamma_+(p)$), when the characteristic oscillation power $p \sim \eta$ is small, one can neglect all the nonlinearities in (1). In such a case equation (1) becomes an equation describing a usual linear oscillator in the presence of noise:

$$\frac{dc}{dt} + i\omega(0)c + [\Gamma_+(0) - \Gamma_-(0)]c = f_{\rm n}(t) \ .$$

One can easily find a solution of this linear equation in the form

$$c(t) = \int_{-\infty}^{t} f_{\rm n}(t') \exp\{-[i\omega(0) + \Gamma_+(0) - \Gamma_-(0)](t - t')\} dt'$$

Using the statistical properties of the thermal fluctuations (2), it is easy to show that the autocorrelation function $\mathcal{K}(\tau)$ in this linear case has a simple exponential form (13b) with the average power $\overline{p}$ given by (8a), mean frequency defined by $\overline{\omega} = \omega(0)$, and the full linewidth given by

$$2\Delta\omega = 2\Gamma_+(0) - 2\Gamma_-(0) = 2\Gamma_+(0)(1 - \zeta) \ . \tag{14}$$

It is clear, that the power spectrum $\mathcal{S}(\Omega)$ of the oscillation in this linear case has exactly the Lorentzian shape (13a).

For a spin-torque oscillator the expressions for the positive and negative damping are given by (I.20), and the linewidth

$2\Delta\omega$ of such an oscillator decreases linearly with the bias current I as

$$2\Delta\omega = 2\Gamma_{\rm G} - 2\sigma I = 2\Gamma_{\rm G}\left(1 - \frac{I}{I_{\rm th}}\right) . \quad (15)$$

The measurements of the current dependence of the linewidth $\Delta\omega$ in the below-threshold regime provide an alternative method of the experimental determination of the threshold current $I_{\rm th}$ for current-induced excitations. As it was discussed above, this method may not be very accurate for highly-resistive STO based on magnetic tunnel junctions (MTJ).

Recently, the linear dependence (15) of the linewidth $\Delta\omega$ on the bias current I has been observed both in fully metallic GMR spin valves [9] and in MTJ based on the TMR effect [8].

B. Above-Threshold Regime: "Linear" Oscillator

Far above the generation threshold the power fluctuations in an auto-oscillator are negligible compared to the mean generated power. In this regime the complex amplitude $c(t)$ becomes a pure phase-modulated process $c(t) \approx \sqrt{p_0}e^{i\phi(t)}$. Consequently, the generation linewidth $\Delta\omega$ and the overall shape of the auto-oscillation power spectrum $\mathcal{S}(\Omega)$ is determined only by the phase fluctuations. This property is general for auto-oscillators of any nature, and is connected with different types of stability of the oscillator dynamics with respect to the power and phase fluctuations (see Sec. I.V-A2).

Thus, to find a generation linewidth of an auto-oscillator in the above-threshold generation regime, it is sufficient to determine the statistical properties of the auto-oscillator phase $\phi(t)$. Here we will consider this problem for a "linear" oscillator, i.e. for an auto-oscillator with frequency independent of the oscillation power $\omega(p) = \omega_{\rm g} = \omega_0 = {\rm const}$. As it was noted before, the majority of conventional oscillators belong to this class, whereas spin-torque oscillators, in general, are characterized by a strong dependence of the oscillation frequency $\omega(p)$ on the power p (this case will be considered in the following section).

Using (1), one can derive a stochastic equation for the phase $\phi = \arg(c)$ of an auto-oscillator

$$\frac{d\phi}{dt} + \omega_{\rm g} = \frac{1}{\sqrt{p}}{\rm Im}[\widetilde{f}_{\rm n}(t)] \quad (16)$$

where $\widetilde{f}_{\rm n}(t) = f_{\rm n}(t)e^{-i\phi(t)}$. Note, that the statistical properties of the function $\widetilde{f}_{\rm n}(t)$ are the same as for the function $f_{\rm n}(t)$, i.e. $\widetilde{f}_{\rm n}(t)$ is a white Gaussian stochastic process with the second-order correlator given by (2).

Far above the generation threshold one can substitute the stationary power p_0 for p in the right-hand-side part of (16), thus reducing this equation to a closed-form equation for the phase ϕ, which has the following solution for the phase as a function of time:

$$\phi(t) = \phi(0) - \omega_{\rm g}t + \frac{1}{\sqrt{p_0}}\int_0^t {\rm Im}[\widetilde{f}_{\rm n}(t')]dt' ,$$

where it was assumed that the phase at the initial moment of time $t = 0$ has fixed value $\phi(0)$. This solution describes the "Brownian motion" of the phase $\phi(t)$ under the action of thermal fluctuations. It is clear, that $\phi(t)$ is a Gaussian process with the mean value

$$\langle\phi(t)\rangle = \phi(0) - \omega_{\rm g}t \quad (17a)$$

and the variance

$$\Delta\phi^2(t) = \langle\phi^2(t)\rangle - [\langle\phi(t)\rangle]^2 = \frac{\Gamma_+(p_0)\eta(p_0)}{p_0}|t| . \quad (17b)$$

The phase variance $\Delta\phi^2(t)$ increases linearly with the time interval $|t|$, exactly as the square of a particle displacement for a usual Brownian motion.

The autocorrelation function $\mathcal{K}(t)$, corresponding to the Gaussian random phase $\phi(t)$, can be written as

$$\mathcal{K}(t) = p_0\left\langle\exp\{i[\phi(t)-\phi(0)]\}\right\rangle = p_0 e^{i\langle\phi(t)-\phi(0)\rangle}\exp[-\Delta\phi(t)^2/2] .$$

For a "linear" auto-oscillator ($\Delta\phi^2(t) \sim |t|$, see (17b)) the auto-correlation fuction $\mathcal{K}(t)$ has the Lorentzian form (13b) with the average frequency $\overline{\omega} = \omega_{\rm g}$ and full linewidth

$$2\Delta\omega_0 = \frac{\Delta\phi^2(t)}{|t|} = \Gamma_+(p_0)\frac{\eta(p_0)}{p_0} .$$

The ratio of the noise power $\eta(p_0)$ to the stationary auto-oscillation power p_0 in the above derived linewidth expression can be rewritten as a ratio of corresponding energies $k_{\rm B}T/\mathcal{E}(p_0)$, yielding a physically transparent expression for the Lorentzian generation linewidth of a "linear" auto-oscillator:

$$2\Delta\omega_0 = \Gamma_+(p_0)\frac{k_{\rm B}T}{\mathcal{E}(p_0)} . \quad (18)$$

To the best of our knowledge, the generation linewidth in a general form (18) was first written in [10].

The damping rate $\Gamma_+(p_0)$ in (18) determines the overall scale of the possible linewidth variations, and the ratio $k_{\rm B}T/\mathcal{E}(p_0)$ describes the linewidth reduction due to a smaller influence of noise (having the constant energy $k_{\rm B}T$) for above-threshold oscillations, the energy $\mathcal{E}(p_0)$ of which increases with the oscillation power p_0. Note, also, that if we formally use (18) for the evaluation of the linewidth in the state of thermodynamic equilibrium ($\mathcal{E}(p_0) \to k_{\rm B}T$, $\Gamma_+(p_0) \to \Gamma_+(0)$), we obtain the linewidth value that is 2 times smaller than the value given by (14) in the same limit ($\Gamma_-(0) \to 0$). This difference is explained by the fact that *only phase fluctuations* lead to the linewidth broadening of an auto-oscillator in the above-threshold generation regime, whereas, for a passive oscillator, both phase and amplitude fluctuations play the same role.

The expressions for the generation linewidth of many different types of conventional ("linear") auto-oscillators have been derived previously (see, e.g., a classical book [11]). It is important to note, that all these expressions can be rewritten in the general form (18). In particular, the full generation linewidth of an electrical auto-oscillator (similar to the one considered in Sec. I.IV-A) is given by the classical formula [11]:

$$2\Delta\omega_0 = \frac{k_{\rm B}TR\,\omega_0^2}{U_0^2}$$

where R is the resistance of the circuit, $\omega_0 = 1/\sqrt{LC}$ is the resonance frequency of the electrical circuit having inductance L and capacitance C, and U_0 is the voltage amplitude of self-sustained electrical oscillations on the capacitor C. This expression can be exactly rewritten in the form (18), taking into account that, for an electrical oscillator, the damping rate is $\Gamma_+ = R/(2L)$, and the oscillation energy is $\mathcal{E} = CU_0^2/2$.

The general expression (18) for the oscillator linewidth, sometimes, can be used even for the description of quantum systems. For example, the quantum limit for the linewidth of a single-mode laser (Schawlow-Townes limit) leads to the following expression [12], [13]

$$2\Delta\omega_0 = \frac{\hbar\omega_0}{2}\frac{\Gamma_0^2}{P_{\rm out}} \ .$$

Here ω_0 is the generation frequency, Γ_0 is the loss rate, which is assumed to be caused only by the energy radiation out of the laser cavity (losses in the active medium were neglected), and $P_{\rm out}$ is the output power of the laser. This expression, also, can be rewritten in the form (18), if one uses the energy of quantum fluctuations $\hbar\omega_0/2$ instead of the thermal noise energy $k_{\rm B}T$, and takes into account, that the output power $P_{\rm out}$ of the laser is connected with the oscillation energy $\mathcal{E}$ by the expression $P_{\rm out} = \Gamma_0\mathcal{E}$.

An attempt to calculate the generation linewidth of a spin-torque oscillator has been undertaken in [14], and resulted in an expression (see equation (28) in [14]), that, also, can be rewritten in the form (18). However, the dependence of the generated frequency of the generation power was ignored in [14], and, therefore, the linewidth expression obtained in [14] is correct only qualitatively. Thus, it is necessary to consider the case of a substantially "nonlinear" auto-oscillator (i.e. an oscillator with a large nonlinear frequency shift) to quantitatively describe the generation linewidth of an STO.

C. Above-Threshold Regime: "Nonlinear" Oscillator

The expression for the auto-oscillator generation linewidth (18), which was verified for many physical realizations of auto-oscillators, underestimates the generation linewidth of an STO by one–two orders of magnitude. The reason for such a large difference is that in the derivation of (18) the oscillation frequency was assumed to be independent of the oscillation power $\omega(p) = \text{const}$. For spin-torque oscillators, as it was already emphasized, the influence of the nonlinear frequency shift is very important. To find the correct expression for the generation linewidth of an STO, as well as for any other "nonlinear" oscillator, one has to explicitly take into account the power dependence of the frequency $\omega(p)$ in the theory.

In the above-threshold generation regime the power fluctuations $\delta p(t) = p(t) - p_0$ are small even for a "nonlinear" oscillator $\delta p(t) \ll p_0$. However, to obtain a correct result for the generation linewidth one should not ignore them altogether, but, instead, should use this smallness to linearize (1) near the stationary generation power $p = p_0$ to derive an approximate system of equations for the power fluctuations $\delta p(t)$ and the phase $\phi(t)$ in a "nonlinear" auto-oscillator:

$$\frac{d\delta p}{dt} + 2\Gamma_{\rm p}\delta p = 2\sqrt{p_0}\,{\rm Re}[\widetilde{f}_{\rm n}(t)] \qquad (19a)$$

$$\frac{d\phi}{dt} + \omega_{\rm g} = \frac{1}{\sqrt{p_0}}{\rm Im}[\widetilde{f}_{\rm n}(t)] - N\delta p \qquad (19b)$$

where, as in the previous subsection, $\widetilde{f}_{\rm n}(t) = f_{\rm n}(t)e^{-i\phi(t)}$ is a stochastic process with the same properties as $f_{\rm n}(t)$. The left-hand side of (19a) is the same as (I.27) for power fluctuations δp of free auto-oscillations.

One can see from (19b), that the frequency nonlinearity N creates an addition noise term $-N\delta p$ in the the phase equation (19b). The physical mechanism behind this additional noise is clear: for a nonlinear auto-oscillator the power fluctuations $\delta p(t)$ lead to the frequency modulation $\omega(p(t)) = \omega(p_0 + \delta p(t)) \approx \omega_{\rm g} + N\delta p$. In some sense this additional noise term can be considered as an inhomogeneous broadening of the oscillator linewidth: oscillators with different (due to thermal fluctuations) powers p have different oscillation frequencies $\omega(p)$.

Since the stochastic system (19) is a *linear* system of equations, and the noise $\widetilde{f}_{\rm n}(t)$ is a Gaussian process, both $\delta p(t)$ and $\phi(t)$ are the Gaussian processes also. One can easily obtain a complete set of statistical characteristics of these processes. Namely, the mean value of the power fluctuations following from (19) is zero $\langle\delta p(t)\rangle = 0$, whereas its correlation function has the form

$$\langle\delta p(t)\delta p(t')\rangle = \frac{\Gamma_+(p_0)}{\Gamma_{\rm p}}\eta(p_0)p_0 e^{-2\Gamma_{\rm p}|t-t'|} \ . \qquad (20)$$

The mean value of the phase $\phi(t)$ (for a fixed value $\phi(0)$ of the phase at the initial moment of time $t = 0$) is given by

$$\langle\phi(t)\rangle = \phi(0) - \omega_{\rm g}t \qquad (21a)$$

and the variance of the phase fluctuations is expressed as

$$\Delta\phi^2(t) = 2\Delta\omega_0\left[(1+\nu^2)|t| - \nu^2\frac{1 - e^{-2\Gamma_{\rm p}|t|}}{2\Gamma_{\rm p}}\right] \qquad (21b)$$

where ν is the dimensionless nonlinear frequency shift coefficient (I.34).

In contrast to the case of a "linear" oscillator (17b), the variance $\Delta\phi^2(t)$ (21b) has a nonlinear dependence on the time interval $|t|$. This is caused by a finite correlation time (see (20)) of the additional "nonlinear" noise term $-N\delta p(t)$ in (19b). Only for the time intervals $|t|$, that are much larger than the correlation time $1/\Gamma_{\rm p}$ of the power fluctuations, the phase variance $\Delta\phi^2(t)$ starts to grow linearly with $|t|$. Consequently, the spectrum of a "nonlinear" auto-oscillator is, in general, non-Lorentzian. There are, however, two limiting cases when a rather complicated expression (21b) for the phase variance of a nonlinear auto-oscillator can be substantially simplified.

If the temperature T and, consequently, the generation linewidth $\Delta\omega$ is sufficiently small $\Delta\omega \ll \Gamma_{\rm p}$, one can neglect the exponential factor in the second term of the right-hand-side part of (21b) at the characteristic decoherence time scale of $|t| \sim 1/\Delta\omega$. Then, the following expression for the phase variance is obtained in the low-temperature limit:

$$\Delta\phi^2(t) \approx 2\Delta\omega_0(1+\nu^2)|t| - \frac{\nu^2}{2\Gamma_{\rm p}} \ .$$

In this limit the autocorrelation function $\mathcal{K}(t)$ becomes exponential (13b), and the power spectrum $\mathcal{S}(\Omega)$ has a Lorentzian shape (13a) with the full linewidth of generation given by the expression

$$2\Delta\omega = 2\Delta\omega_0(1+\nu^2) = (1+\nu^2)\Gamma_+(p_0)\frac{k_{\rm B}T}{\mathcal{E}(p_0)} \ . \qquad (22)$$

One can see, that the nonlinearity of the auto-oscillator frequency $\omega(p)$ leads to the increase of the generation linewidth by $(1+\nu^2)$ times. The expression for the generation linewidth of a "nonlinear" auto-oscillator (22) was derived on the basis of a nonlinear oscillator model and published for the first time in [10].

One can show, that the condition $\Delta\omega \ll \Gamma_{\rm p}$ of applicability of (22) can be rewritten as a condition for the temperature

$$k_{\rm B}T \ll \left(\frac{\Gamma_{\rm p}}{\Gamma_+(p_0)}\right)\frac{\mathcal{E}(p_0)}{1+\nu^2} \qquad (23a)$$

or, using (9), as a condition for the relative power of thermal fluctuations

$$\frac{\Delta p}{p_0} \ll \frac{1}{\sqrt{1+\nu^2}} \ . \qquad (23b)$$

Thus, (22) represents a low-temperature limit of the generation linewidth of a "nonlinear" auto-oscillator. The estimations for the typical parameters of a spin-torque oscillator (permalloy circular nano-pillar of the radius $R_{\rm c} = 50$ nm and thickness of the "free" layer $L = 5$ nm, generation frequency $\omega_{\rm g}/2\pi \simeq 30$ GHz) show that the expression (22) is quantitatively correct for the temperatures $T \leq 10 - 100$ K, depending on the supercriticality parameter ζ.

Another limiting case, which allows an easy analysis of (21b), is the opposite limiting case of relatively long (compared to the inverse linewidth $1/\Delta\omega$) correlation times of the power fluctuations $1/\Gamma_{\rm p}$, or the case of relatively large generation linewidths $\Delta\omega \gg \Gamma_{\rm p}$. In this case the exponential function in (21b) can be developed in a Taylor series to give:

$$\Delta\phi^2(t) \approx 2\Delta\omega_0(|t| + \nu^2\Gamma_{\rm p}t^2) \ .$$

For sufficiently large frequency nonlinearities $\nu \gg 1$ one can drop the first (linear in $|t|$) term in the brackets, and retain only the second term that is one quadratic in time. Then, the phase variance $\Delta\phi^2(t) \sim t^2$, and the power spectrum takes the form

$$\mathcal{S}(\Omega) \sim \exp\left[-\frac{(\omega(p_0)-\Omega)^2}{2\Delta\omega_*^2}\right]$$

with the characteristic linewidth

$$\Delta\omega_* = |\nu|\sqrt{\Gamma_+(p_0)\Gamma_{\rm p}}\sqrt{\frac{k_{\rm B}T}{\mathcal{E}(p_0)}} \ . \qquad (24)$$

In this limit the auto-oscillator linewidth is large $\Delta\omega_* \gg \Gamma_{\rm p}$ and proportional to the normalized nonlinear frequency shift coefficient ν in the first power. Also, in contrast with the low-temperature linewidth expression (22), giving the linear dependence on temperature, the expression (24) for the auto-oscillator linewidth is proportional to $\sqrt{T}$. It is interesting to note, that formula for the auto-oscillator linewidth, which is qualitatively similar to (24), was for the first time proposed in [15] for the case of a spin-torque auto-oscillator (see (2) in [15]). In the same paper the $\sqrt{T}$ temperature dependence of an STO linewidth was obtained as a result of direct numerical simulations.

The expression (24) for the generation linewidth is valid in the temperature range

$$\left(\frac{\Gamma_{\rm p}}{\Gamma_+(p_0)}\right)\frac{\mathcal{E}(p_0)}{\nu^2} \ll k_{\rm B}T \ll \left(\frac{\Gamma_{\rm p}}{\Gamma_+(p_0)}\right)\mathcal{E}(p_0) \qquad (25a)$$

or, in terms of power fluctuations,

$$\frac{1}{|\nu|} \ll \frac{\Delta p}{p_0} \ll 1 \ . \qquad (25b)$$

Thus, (24) is the asymptotic expression of the generation linewidth of a strongly-nonlinear ($|\nu| \gg 1$) auto-oscillator in the intermediate temperature interval. For larger temperatures or smaller oscillation powers p_0 (i.e., in the near-threshold region) expansion (19) is not valid anymore, and one has to analyze the full equations (1) or (5), which can be done only using numerical methods.

It is not clear, *a priori*, which of the above derived approximate expressions for the "nonlinear" auto-oscillator linewidth (22) or (24) is a better approximation for the description of the experimentally measured linewidth magnitudes in strongly nonlinear spin-torque oscillators. On one hand, many of the experiments, where the STO linewidth was measured, were done at a room temperature $T \simeq 300$K, which is higher than the upper limit of applicability for the low-temperature expression (22). On the other hand, in many cases the experimentally observed linewidths of spin-torque oscillators are rather narrow $\Delta\omega < \Gamma_{\rm G}$, so that the conditions of applicability of the other asymptotic expression (24) are, also, violated. As it will be demonstrated below, the direct comparison of the currently available experimental data with the above developed theory, in general, demonstrates that the low-temperature expression (22) gives a much better account of the existing experimental data. However, additional measurements of the generation linewidth of STO performed in a wide range of temperatures are necessary to clarify this important question and to, possibly, see in the experiment the cross-over from the linear to a squate root dependence on temperature, when the temperature is increased.

D. Near-Threshold Region

The region near the generation threshold $\Gamma_+(0) \simeq \Gamma_-(0)$ (or, for a spin-torque oscillator, $I \simeq I_{\rm th}$) is the most difficult region for the theoretical analysis. In this region there are no obvious linearization schemes by which (1) can be simplified, and one needs to employ various numerical methods to evaluate the auto-oscillator linewidth near the generation threshold.

The distortions of the power spectrum $\mathcal{S}(\Omega)$ in the vicinity of the generation threshold have been studied in [16] using analytical and numerical methods based on a non-stationary Fokker-Planck equation (5) for the probability distribution function $\mathcal{P}(t,p,\phi)$. The results of this analysis show that, in the near-threshold region, the power spectrum $\mathcal{S}(\Omega)$ does not have a simple Lorentzian form (13a), but rather can be represented as a sum of partial Lorentzians

$$\mathcal{S}(\Omega) = \sum_j \frac{2\Delta\omega_j \overline{p}_j}{(\Omega - \overline{\omega}_j)^2 + \Delta\omega_j^2} \ .$$

For a "linear" auto-oscillator ($\omega(p) = \omega_0 = \text{const}$) the central frequencies $\overline{\omega}_j$ for all the partial Lorentzians are equal $\overline{\omega}_j = \omega_0$, and the overall power spectrum $\mathcal{S}(\Omega)$ remains symmetric with respect to the central frequency ω_0.

In contrast, for a "nonlinear" auto-oscillator ($\omega(p) \neq \text{const}$) all the frequencies $\overline{\omega}_j$ are distinct and, in general, distributed non-symmetrically around the zero-temperature frequency $\omega_{\rm g} = \omega(p_0)$. As a result, the power spectrum $\mathcal{S}(\Omega)$ becomes asymmetric in the near-threshold region. Such asymmetry of the power spectrum has been recently observed in experiments with spin-torque oscillators based on metallic GMR nano-pillar [7].

Another specific feature of the behavior of a nonlinear auto-oscillator linewidth in the near-threshold region is an apparent broadening of this linewidth. Like in the case of a usual inhomogeneous linewidth broadening, the sum of several partial Lorentzians with different central frequencies $\overline{\omega}_j$ results in the power spectrum $\mathcal{S}(\Omega)$ that has the overall width that is larger, than the partial linewidths $\Delta\omega_j$ of individual Lorentzians. This effect is illustrated by Fig. 5a, where the linewidth $\Delta\omega$, obtained from a single-Lorentzian fit to the power spectrum $\mathcal{S}(\Omega)$, is shown as a function of the bias current I for a a spin-torque oscillator. Good agreement is found between the numerically and analytically calculated linewidths far below and far above the generation threshold in all cases, as indicated by the dashed lines representing the linewidth values calculated using equations (15) and (22). In general, the overall linewidth $\Delta\omega$ decreases with the increasing current, except in the region near the threshold, where a local maximum of the linewidth can appear in the case of a strong frequency nonlinearity $|\nu| \gg 1$. Depending on the magnitude of the nonlinear frequency shift $|N|$, this maximum can be several times larger than the generation linewidth in both below- and above-threshold regimes [16]. The experimental measurements of the power spectra of current-induced microwave oscillations in STO, also, show the line narrowing followed by line broadening near the generation threshold with the increase of the bias current (see Fig. 3c in [7]).

Fig. 5b shows the variation of the generation frequency $\overline{\omega}$, obtained from the single-Lorentzian fit to the power spectrum $\mathcal{S}(\Omega)$, with the bias current I [16]. This variation is relatively slow below the generation threshold $\overline{\omega} \approx \omega_0$ and exhibits a fast quasi-linear change with current above the generation threshold due to the nonlinear frequency shift, in agreement with (I.30).

E. Comparison with Experiments

The above presented theory of the generation linewidth of nonlinear auto-oscillators successfully explains a number of qualitative features, experimentally observed in the linewidth measurements on spin-torque oscillators. First of all, the inverse dependence of the generation linewidth (22) on the

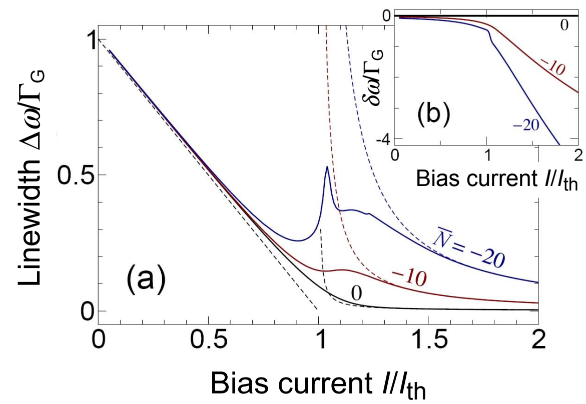

Fig. 5. Generation linewidth $\Delta\omega$ (panel (a)) and frequency shift $\delta\omega = \overline{\omega} - \omega_0$ (panel (b)) as a function of the bias current I obtained from single-Lorentzian fits to spectra computed for different values of the frequency nonlinearity $\overline{N} = N/\Gamma_{\rm G}$ (from [16]). Noise power $\eta = 10^{-3}$, nonlinear damping coefficient $Q = 2$. Dashed lines in (a) show the below-threshold (15) and above-threshold (22) linewidth asymptotics. Used values of the parameters are typical for in-plane magnetized spin-torque oscillators (note that the nonlinear frequency shift N is negative).

energy of oscillations $\mathcal{E}(p_0)$ explains the general linewidth narrowing with the increase in the bias current and oscillation amplitude observed in, e.g., Fig. 4 in [17]. Note, also, that the energy $\mathcal{E}(p_0)$ is proportional to the effective volume $V_{\rm eff}$ of the auto-oscillator, which is much larger for a spin-torque auto-oscillator, based on a magnetic nano-contact, than for a similar device based on a nano-pillar. Thus, the linewidth expression (22) provides a natural explanation for a well-known experimental fact (see [18]–[20]) that the auto-oscillation linewidths associated with devices based on magnetic nano-pillars are, in general, several times broader that those in the devices based on magnetic nano-contacts.

In Fig. 6 we show the comparison of the generation linewidth calculated using two asymptotic expressions (22) and (24) with the results of experimental measurements of the temperature dependence of the linewidth of a spin-torque oscillator performed on the nano-pillar devices #1 (Fig. 6a) and #2 (Fig. 6b) in [15] (see Fig. 2 in [15]). The geometrical parameters of the nano-pillar devices were taken from [15] and it was assumed that the excited magnetization oscillation mode is pinned at the pillar lateral boundaries (see [21] for details), the Gilbert damping parameter $\alpha_{\rm G} = 0.01$, the nonlinear damping parameter $Q = 3$, and the polarization efficiency $\varepsilon = 0.4$ were assumed to be the same for both devices [10]. As it is clear from Fig. 6, the simple analytical expression (22) obtained in the low-temperature limit gives a reasonably good estimate of the observed linewidths at different temperatures for *both* nano-pillar devices with *the same* parameters. At the same time, the second asymptotic expression (24) obtained in the relatively high-temperature limit significantly overestimates the generation linewidth. Assuming that the parameters of the two devices are slightly different (which is possible due to different nano-patterning and different thicknesses of the "free" magnetic layer), one can obtain much better quantitative agreement with the experiment [15] using the low-temperature expression (22). Although it is difficult to determine from

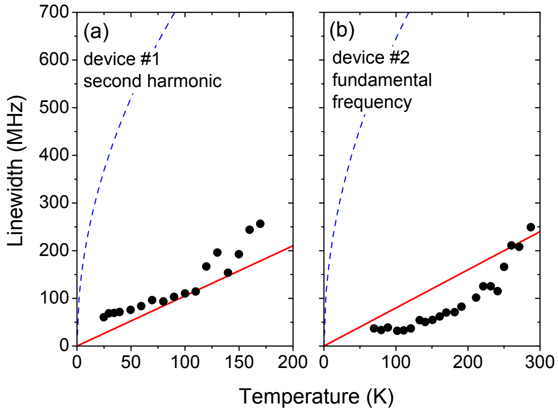

Fig. 6. Generation linewidth $2\Delta\omega$ of a spin-torque oscillator calculated from the approximate expressions (22) (solid lines) and (24) (dashed lines) in comparison with the temperature dependence of the linewidth in a nanopillar device #1 (a) and device #2 (b) measured in [15] (black dots). Panel (a) shows the linewidth measured at the *second* harmonic of the signal $2\Delta\omega_2 = 8\Delta\omega$.

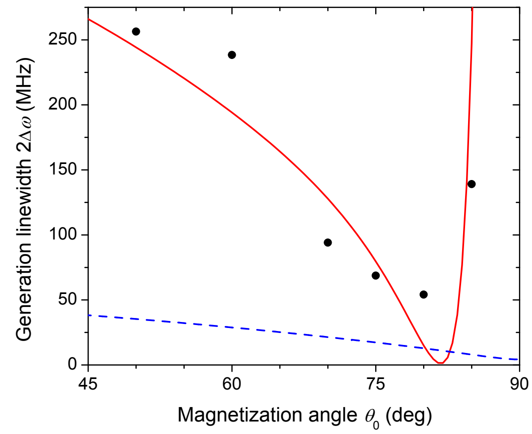

Fig. 7. Generation linewidth $2\Delta\omega$ of a spin-torque auto-oscillator calculated from (22) (solid line) in comparison with the angular dependence of the nano-contact STO linewidth measured at a room temperature in [17] (black dots) (from [10]). The dashed line represents the classical result for a "linear" auto-oscillator linewidth calculated from (18) and multiplied by 10 .

the experimental data presented in Fig. 6 what is the real temperature dependence of the generation linewidth, it is clear that the low-temperature asymptotic expression (22) gives a much better estimate of the generation linewidth magnitude.

In Fig. 7 we compared the theoretical (22) and experimental [17] (see Fig. 6 in [17]) dependences of the full generation linewidth $2\Delta\omega$ on the out-of-plane magnetization angle θ_0 of nano-contact spin-torque oscillator. All the parameters of the nano-contact device Fig. 7 were taken from [17], while the current $I = 9$ mA and magnetic field $H_0 = 9$ kOe correspond to the center of the experimentally studied region, and the nonlinearity parameter of positive damping was again chosen to be equal to $Q = 3$ [10]. It is clear from Fig. 7, that the linewidth dependence on the bias field orientation calculated using the low-temperature asymptotic expression (22) is in good quantitative agreement with the experimental results from [17]. In contrast, the classical result for a "linear" auto-oscillator (18) (see dashed line in Fig. 7, which shows classical result (18), multiplied by 10) predicts a much narrower lines and a monotonous decrease of the linewidth as a function of the out-of-plane magnetic field angle θ_0. The "nonlinear" result (22) gives a reasonable qualitative and quantitative description of the experimentally observed behavior of the auto-oscillator linewidth, and, in particular, it predicts the linewidth minimum around $\theta_0 \approx 80^\circ$, which is clearly seen in the experimental data. This linewidth minimum occurs at the "linear" magnetization angle $\theta_{0,\,\mathrm{lin}}$, for which the nonlinear frequency shift coefficient N vanishes (see Fig. I.6a), and, therefore, at this magnetization angle the spin-torque oscillator behaves as a classical "linear" oscillator with the generation linewidth determined by (18).

For an STO based on an anisotropic nano-pillar one may, also, expect a non-trivial dependence of the full generation linewidth $2\Delta\omega$ on the in-plane magnetization angle ϕ_0, since in this case the nonlinear frequency shift coefficient N significantly varies with ϕ_0 (see Fig. I.6b). Recent experiments [22] performed using a spin-torque oscillator based on a nano-pillar geometry where the "free" layer was magnetized by the bias magnetic field oriented in the "free" layer plane, indeed, demonstrated strong variation of the oscillation linewidth with the variation of the in-plane magnetization angle ϕ_0. This behavior of the generation linewidth finds a natural explanation in the framework of the asymptotic linewidth expression (22). Fig. 8 shows the comparison of the experimental data from [22] with the angular dependence of the linewidth $2\Delta\omega$ calculated from the asymptotic equation (22) using the general expression for the nonlinear frequency shift coefficient in an anysotropic magnetic film and taking all the other parameters of the spin-torque oscillators from [22]. It is clear from Fig. 8, that the linewidth corresponding to the magnetization along the hard in-plane axis ($\phi_0 = 90^\circ$) of the nano-pillar "free" layer is much smaller than the linewidth in the case of the magnetization along the easy axis ($\phi_0 = 0$). It is, also, clear, that the linewidth expression (22) gives a good qualitative and a reasonable quantitative description of the experimental data for both nano-oscillator devices used in the experiments [22].

We would, also, like to note that, similar to the case of an spin-torque oscillator with an out-of-plane magnetized "free" layer, in the case of an STO based on an anisotropic in-plane magnetized nano-pillar it is possible to choose the direction and magnitude of the bias magnetic field in such a way that the nonlinear frequency shift coefficient N vanishes (see Fig. I.6b). Under these conditions the linewidth of the anisotropic STO will have a minimum possible value, corresponding to the case of a "linear" auto-oscillator (18).

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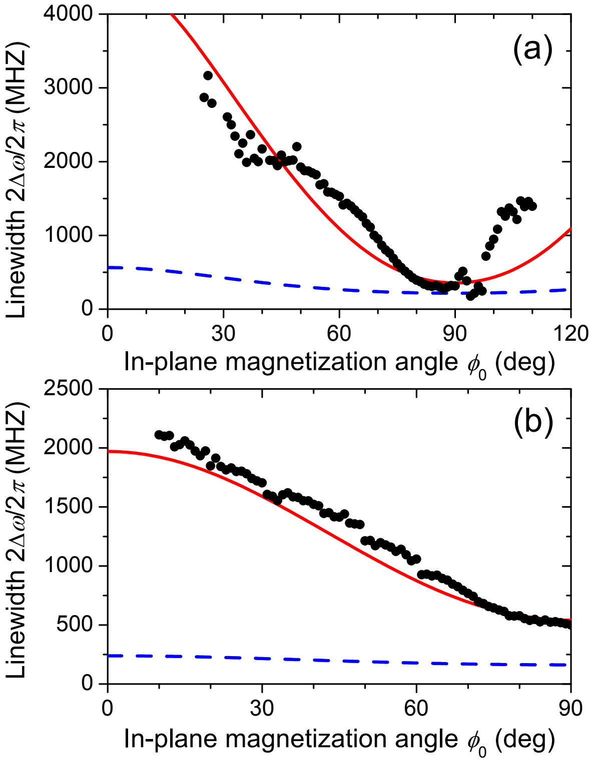

Fig. 8. Dependence of the generation linewidth $2\Delta\omega$ (22) on the in-plane magnetization angle ϕ_0 for two spin-torque oscillators based on anisotropic magnetic nano-pillars: (a) oscillator with a synthetic "fixed" layer Py-IrMn; (b) oscillator with a thick Py "fixed" layer (after [22]). Black dots – experimental data measured at room temperature from [22], solid lines – calculation using the expression (22) for a "nonlinear" auto-oscillator, dashed lines – calculation using the classical "linear" expression (18) multiplied by 10. All the calculations were done for the parameters of the experiment [22]: saturation magnetization $4\pi M_0 = 8$ kOe, bias magnetic field $H_0 = 1.08$ kOe (a) and $H_0 = 1.2$ kOe (b), in-plane anisotropy field $H_A = 0.2$ kOe (a) and $H_A = 0.1$ kOe (b), Gilbert damping constant $\alpha_G = 0.015$, nonlinear damping coefficient $Q = 3$, "free" layer thickness $L = 4$ nm, shape of the nano-pillar – elliptical with the sizes 150 nm x 50 nm (a) and 130 nm x 70 nm (b), spin-polarization efficiency $\varepsilon = 0.32$ (a) and $\varepsilon = 0.375$ (b), bias current $I = 5$ mA, temperature $T = 300$ K.

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