Australian Government
Department of Defence
Defence Science and
Technology Organisation

State Space Model for Autopilot Design of Aerospace Vehicles

Farhan A. Faruqi

Weapons Systems Division
Defence Science and Technology Organisation

DSTO-TR-1990

ABSTRACT

This report is a follow on to the report given in DSTO-TN-0449 and considers the derivation of the mathematical model for aerospace vehicles and missile autopilots in state space form. The basic equations defining the airframe dynamics are non-linear, however, since the non-linearities are "structured" (in the sense that the states are of quadratic form) a novel approach of expressing this non-linear dynamics in state space form is given. This should provide a useful way to implement the equations in a computer simulation program and possibly for future application of non-linear analysis and synthesis techniques, particularly for autopilot design of aerospace vehicles executing high g-manoeuvres.

This report also considers a locally linearised state space model that lends itself to better known linear techniques of the modern control theory. A coupled multi-input multi-output (MIMO) model is derived suitable for both the application of the modern control techniques as well as the classical time-domain and frequency domain techniques. The models developed are useful for further research on precision optimum guidance and control. It is hoped that the model will provide more accurate presentations of missile autopilot dynamics and will be used for adaptive and integrated guidance & control of agile missiles.

Published by

DSTO Defence Science and Technology Organisation
PO Box 1500
Edinburgh, SA 5111 Australia

Telephone: (08) 8259 7245
Fax: (02) 8259 6964

AR-013-894
March 2007

APPROVED FOR PUBLIC RELEASE

State Space Model for Autopilot Design of Aerospace Vehicles

Executive Summary

Requirements for next generation guided weapons and other aerospace vehicles, particularly with respect to their capability to engage high speed, highly agile targets and achieve precision end-game trajectory, have prompted a revision of the way in which the guidance and autopilot design is undertaken. This report considers the derivation of the mathematical models for aerospace vehicles and missile autopilots in state space form. The basic equations defining the airframe dynamics are non-linear, however, since the non-linearities are "structured" (in the sense that the states are of quadratic form) a novel approach of expressing this non-linear dynamics in state space form is given. This should provide a useful way to implement the equations in a computer simulation program and possibly for future application of non-linear analysis and synthesis techniques.

This report which is a follow on report to DSTO-TN-0449, also considers a locally linearised state space model that lends itself to better known linear techniques of the modern control theory. A coupled multi-input multi-output (MIMO) model is derived suitable for both the application of the modern control techniques as well as the classical time-domain and frequency domain techniques. The models developed are useful for further research on precision optimum guidance and control. It is hoped that the model will provide more accurate presentations of aerospace vehicles autopilot dynamics and will be used for adaptive and integrated guidance & control of agile missiles and other aerospace vehicles that do not necessarily have symmetric cruciform airframes.

Authors

Dr. Farhan A. Faruqi
Weapons Systems Division

Farhan A. Faruqi received B.Sc.(Hons) in Mechanical Engineering from the University of Surrey (UK), 1968; M.Sc. in Automatic Control from the University of Manchester Institute of Science and Technology (UK), 1970 and Ph.D from the Imperial College, London University (UK), 1973. He has over 20 years experience in the Aerospace and Defence Industry in UK, Europe and the USA. Prior to joining DSTO in January 1999 he was an Associate Professor at QUT (Australia) 1993-98. His research interests include: Missile Navigation, Guidance and Control, Target Tracking and Precision Pointing Systems, Strategic Defence Systems, Signal Processing, and Optoelectronics.

Contents

1. Introduction

Requirements for next generation guided weapons, particularly with respect to their capability to engage high speed, highly agile targets and achieve precision end-game trajectory, have prompted a revision of the way in which the guidance and autopilot design are undertaken. Integrating the guidance and control function is a synthesis approach that is being pursued as it allows the optimisation of the overall system performance. This approach requires a more complete representation of the airframe dynamics and the guidance system. The use of state space model allows the application of modern control techniques such as the optimal adaptive control and parameter estimation techniques [10] to be utilised. In this report we derive the autopilot model that will serve as a basis for an adaptive autopilot design and allow further extension of this to integrated guidance and control system design.

Over the years a number of authors [1-3, 6-9] have considered modelling, analysis and design of autopilots for atmospheric flight vehicles including guided missiles. In the majority of the published work on autopilot analysis and design, locally linearised versions of the model with decoupled airframe dynamics have been considered. This latter simplification arises out of the assumption that the airframe and its mass distribution are symmetrical about the body axes, and that the yaw, pitch and roll motion about the equilibrium state remain "small". As a result, many of the autopilot analysis and design techniques, considered in open literature, use classical control approach, such as: single input/single output transfer-functions characterisation of the system dynamics, Bode, Nyquist, root-locus and transient -response analysis and synthesis techniques [5, 7]. These techniques are valid for a limited set of flight regimes and their extension to cover a wider set of flight regimes and airframe configurations requires autopilot gain and compensation network switching.

With the advent of fast processors it is now possible to take a more integrated approach to autopilot design. Modern optimal control techniques allow the designer to consider autopilots with high-order dynamics (large number of states) with multiple inputs/outputs and to synthesise controllers such that the error between the demanded and the achieved output is minimised. Moreover, with real-time mechanisation any changes in the airframe aerodynamics can be identified (parameter estimation) and compensated for by adaptively varying the optimum control gain matrix. This approach should lead to missile systems that are able to execute high g-manoeuvres (required by modern guided weapons), adaptively adjust control parameters (to cater for widely varying flight profiles) as well as account for non-symmetric airframe and mass distributions. Typically, for a missile autopilot, the input is the demanded control surface deflection and outputs are the achieved airframe (lateral) accelerations and body rates measured about the body axes. Other input/output variables (such as: the flight path angle and angle rate or the body angles) can be derived directly from lateral accelerations and body rates.

This report considers the derivation of the mathematical model for a missile autopilot in state space form. The basic equations defining the airframe dynamics are non-linear, however, since the nonlinearities are "structured" (in the sense that the states are of quadratic form) a novel approach of expressing this non-linear dynamics in state space form is given. This should provide a useful way to implement the equations in a computer simulation program and possibly for future application of non-linear analysis and synthesis techniques. Detailed consideration of the quadratic/bilinear type of dynamic systems is given in [4].

This report which is a follow on report from the previous report [1,2], also considers a locally linearised state space model that lends itself to better known linear techniques of the modern control theory. A coupled multi-input multi-output (MIMO) model is derived suitable for both the application of the modern control techniques as well as the classical time-domain and frequency domain techniques. For sake of clarity, Figure 2.1 is a symmetric cruciform missile, however the models developed are valid for non axis-symmetric aerospace vehicles. Tables A-1.1 to A-1.3 contain the various aerodynamic derivatives and coefficients.

2. State Space Aerodynamics Model

The airframe, actuation and sensor measurement equations have been derived in detail in Appendix A in this section we give the main results that will be used for matrix based computation of the state space model.

2.1 Nonlinear Airframe Model

Conventions and notations for vehicle body axes systems as well as the forces, moments and other quantities are shown in Figure 2.1 and defined in Table 2.1.

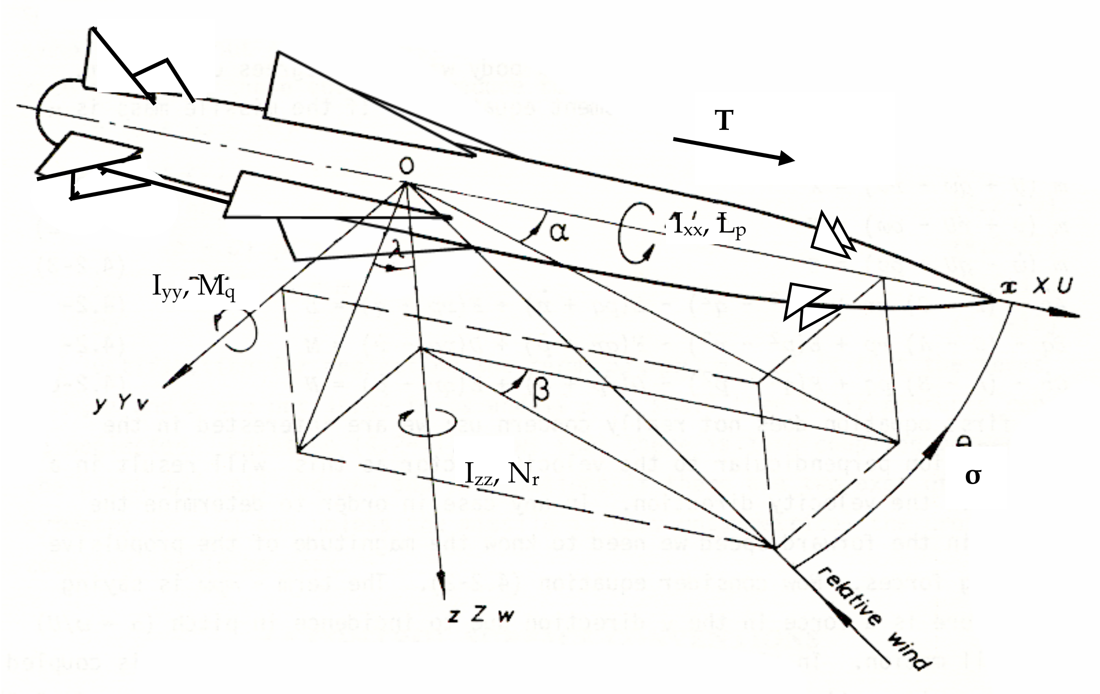

Figure 2.1 Motion variable notations

The variables shown in Figure 2.1 are defined as:
m - mass of a vehicle.

α - incidence in the pitch plane.
β - incidence in the yaw plane.
λ - incidence plane angle.
σ - total incidence, such that: $\tan\alpha = \tan\sigma\cos\lambda$, and $\tan\beta = \tan\sigma\sin\lambda$.
T – thrust.

Table 2.1: Motion variables

Vehicle Body Axes System	Roll axis	Pitch axis	Yaw axis
Angular rates	p	q	r
Component of vehicle velocity along each axis	u	v	w
Component of aerodynamic forces acting on vehicle along each axis	X	Y	Z
Moments acting on vehicle about each axis	L	M	N
Moments of inertia about each axis	I_{xx}	I_{yy}	I_{zz}
Products of each inertia	I_{yz}	I_{zx}	I_{xy}
Longitudinal and lateral accelerations	a_x	a_y	a_z
Euler angles	ϕ	θ	ψ
Gravity along each axis	g_x	g_y	g_z
Vehicle thrust along the body axis	T		

Tail Control Configuration:

We shall use the following notation: ξ - aileron deflection; η - elevator deflection; ς - rudder deflection. Figure 2.2 defines the control surface convention. Here the control surfaces are numbered as shown and the deflections $(\delta_1, \delta_2, \delta_3, \delta_4)$ are taken to be positive if clockwise, looking outwards along the individual hinge axis. Thus, Aileron deflection: $\xi = \frac{1}{4}(\delta_1 + \delta_2 + \delta_3 + \delta_4)$, if all four control surfaces are active; or $\xi = \frac{1}{2}(\delta_1 + \delta_3)$, or $\xi = \frac{1}{2}(\delta_2 + \delta_4)$ if only two surfaces are active. Positive control defection (ξ) causes negative roll. Elevator deflection: $\eta = \frac{1}{2}(\delta_1 - \delta_3)$. Positive control deflection (η) causes negative pitch. Rudder deflection: $\zeta = \frac{1}{2}(\delta_2 - \delta_4)$. Positive control deflection (ζ) causes negative yaw.

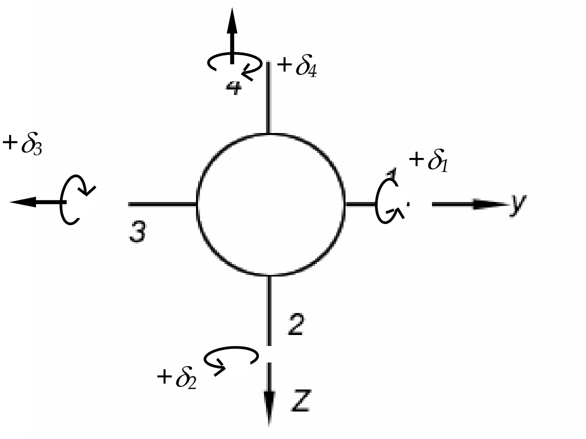

Figure 2.2 Control surfaces seen from the rear of a missile

Canard Control Configuration:

For a canard configuration, same convention will be used for control surface deflections; however, it is noted that the force and moment coefficients will have opposite signs. Canard control is generally not used for roll control.

Euler Equations of Motion

The six equations of motion for a body with six degrees of freedom may be written as [1-3]:

$$m(\dot{u} + wq - vr) = X + T + g_x m \tag{2-1}$$
$$m(\dot{v} + ur - wp) = Y + g_y m \tag{2-2}$$
$$m(\dot{w} - uq + vp) = Z + g_x m \tag{2-3}$$
$$I_{xx}\dot{p} - (I_{yy} - I_{zz})qr + I_{yz}(r^2 - q^2) - I_{zx}(pq + \dot{r}) + I_{xy}(rp - \dot{q}) = L \tag{2-4}$$
$$I_{yy}\dot{q} - (I_{zz} - I_{xx})rp + I_{zx}(p^2 - r^2) - I_{xy}(qr + \dot{p}) + I_{yz}(pq - \dot{r}) = M \tag{2-5}$$
$$I_{zz}\dot{r} - (I_{xx} - I_{yy})pq + I_{xy}(q^2 - p^2) - I_{yz}(rp + \dot{q}) + I_{zx}(qr - \dot{p}) = N \tag{2-6}$$

Here $(\dot{\cdot}) = \frac{d}{dt}$ - is the derivative operator.
Based on the Euler equations above, the nonlinear (quadratic) airframe state space model is given by (see equations A-1.1 to A-1.5):

$$\frac{d}{dt}\underline{x}_3^{[1]} = [F_0]\,\underline{x}_3^{[2]} + [G_0]\,\underline{u}_3^{[1]} + \underline{g}^{[1]} \tag{2-7}$$

Where:

$$\underline{x}_3^{[1]} = \begin{bmatrix} \underline{x}_1^{[1]} \\ -- \\ \underline{x}_2^{[1]} \end{bmatrix} = \begin{bmatrix} u & v & w & | & p & q & r \end{bmatrix}^T$$: is a 6x1 linear-state vector.

$$\underline{x}_3^{[2]} = \begin{bmatrix} \underline{x}_1^{[2]} \\ -- \\ \underline{x}_2^{[2]} \end{bmatrix} = \begin{bmatrix} uq & ur & vp & vr & wp & wq & | & p^2 & pq & pr & q^2 & qr & r^2 \end{bmatrix}^T$$: is a 12x1 quadratic-state vector.

$$\underline{u}_3^{[1]} = \begin{bmatrix} \underline{u}_1^{[1]} \\ -- \\ \underline{u}_2^{[1]} \end{bmatrix} = \begin{bmatrix} \tilde{X} + \tilde{T} & \tilde{Y} & \tilde{Z} & | & L & M & N \end{bmatrix}^T$$: is 6x1 a vector function of control inputs, forces and moments.

$$\underline{g}^{[1]} = \begin{bmatrix} \underline{g} \\ -- \\ \underline{0}_{3\times 1} \end{bmatrix} = \begin{bmatrix} g_x & g_y & g_z & 0 & 0 & 0 \end{bmatrix}^T$$: is the 6x1 gravity (or disturbance) vector.

$$[F_0] = \begin{bmatrix} [C_0] & | & [0_{3\times 6}] \\ --- & | & ------ \\ [0_{3\times 6}] & | & [A_0]^{-1}[B_0] \end{bmatrix}$$: is a 6x12 state-coefficient matrix.

$$[G_0] = \begin{bmatrix} [I_{3\times 3}] & | & [0_{3\times 3}] \\ --- & | & --- \\ [0_{3\times 3}] & | & [A_0]^{-1} \end{bmatrix}$$: is a 6x6 input-coefficient matrix.

$$[A_0] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{zx} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{yz} & I_{zz} \end{bmatrix}$$: is a 3x3 matrix.

$$[B_0] = \begin{bmatrix} 0 & I_{zx} & -I_{xy} & I_{yz} & (I_{yy} - I_{zz}) & -I_{yz} \\ -I_{zx} & -I_{yz} & (I_{zz} - I_{xx}) & 0 & I_{xy} & I_{zx} \\ I_{xy} & (I_{xx} - I_{yy}) & I_{yz} & -I_{xy} & -I_{zx} & 0 \end{bmatrix}$$: is a 3x6 matrix.

$$[C_0] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$: is a 3x6 matrix.

Subscripts under **[*I*]** and **[*0*]** matrices denote the matrix dimensions. Generally, not all state variables in the state equation are accessible or measurable. The vehicle angular rate components (roll rate p, pitch rate q, and yaw rate r) and the acceleration components (a_x, a_y, a_z) are commonly available and can be measured using the IMU.

2.2 Body Acceleration Model

Equations for the vehicle actual body accelerations and body rates are derived in Appendix A equations (A-1.6) to (A-1.11). These equations allow position offsets from c.g for observing the body accelerations. The acceleration components at point O (where O is at a distance of d_x, d_y and d_z from the central point of gravity, c.g., along x-, y- and z-axis, respectively), may be written as:

$$\begin{aligned} a_x &= \dot{u} + qw - rv - d_x(q^2 + r^2) + d_y(pq - \dot{r}) + d_z(pr + \dot{q}) \\ &= \tilde{X} + \tilde{T} + g_x - d_x(q^2 + r^2) + d_y(pq - \dot{r}) + d_z(pr + \dot{q}) \end{aligned} \tag{2-8}$$

$$\begin{aligned} a_y &= \dot{v} + ru - pw + d_x(pq + \dot{r}) - d_y(p^2 + r^2) + d_z(qr - \dot{p}) \\ &= \tilde{Y} + g_y + d_x(pq + \dot{r}) - d_y(p^2 + r^2) + d_z(qr - \dot{p}) \end{aligned} \tag{2-8}$$

$$\begin{aligned} a_z &= \dot{w} + pv - qu + d_x(pr - \dot{q}) + d_y(qr + \dot{p}) - d_z(p^2 + q^2) \\ &= \tilde{Z} + g_z + d_x(pr - \dot{q}) + d_y(qr + \dot{p}) - d_z(p^2 + q^2) \end{aligned} \tag{2-9}$$

After some matrix manipulations, the body acceleration model may be written as (see equations (A-1.6) to (A-1.11):

$$\underline{y}_3^{[1]} = [J_0]\underline{x}_3^{[1]} + [K_0]\underline{x}_3^{[2]} + [L_0]\underline{u}_3^{[1]} + [M_0]\underline{g}^{[1]} \tag{2-10}$$

Where:

$\underline{y}_1^{[1]} = [a_x \quad a_y \quad a_z]^T$: is a 3x1 body acceleration vector.

$\underline{y}_2^{[1]} = \underline{x}_2^{[1]} = [p \quad q \quad r]^T$: is a 3x1 body rate vector

$\underline{y}_3^{[1]} = [\underline{y}_1^{[1]} \;\; | \;\; \underline{y}_2^{[1]}]^T$: is a 6x1 output vector.

$$[D_0] = \begin{bmatrix} 1 & 0 & 0 & 0 & d_z & -d_y \\ 0 & 1 & 0 & -d_z & 0 & d_x \\ 0 & 0 & 1 & d_y & -d_x & 0 \end{bmatrix}$$: is a 3x6 accelerometer position 'offset' matrix.

$$[D_1] = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & d_y & d_z & -d_x & 0 & -d_x \\ 0 & 1 & 0 & 0 & -1 & 0 & -d_y & d_x & 0 & 0 & d_z & -d_y \\ -1 & 0 & 1 & 0 & 0 & 0 & -d_z & 0 & d_x & -d_z & d_y & 0 \end{bmatrix}$$: is a 3x12 accelerometer position 'offset' matrix.

$$[J_0] = \begin{bmatrix} [0_{3\times 6}] \\ ---------- \\ [0_{3\times 3} \;|\; I_{3\times 3}] \end{bmatrix}$$: is a 6x6 linear-state output coefficient matrix.

$$[K_0] = \begin{bmatrix} [D_0 F_0 + D_1] \\ ------ \\ [0_{3\times 12}] \end{bmatrix}$$: is a 6x12 quadratic-state output coefficient matrix.

$$[L_0] = \begin{bmatrix} [D_0 G_0] \\ ------ \\ [0_{3\times 6}] \end{bmatrix}$$: is a 6x6 coefficient matrix.

$$[M_0] = \begin{bmatrix} [D_0] \\ --- \\ [0_{3\times 6}] \end{bmatrix}$$: is a 6x6 output coefficient matrix.

Note: equation (2-10) represents the actual accelerations and body rates outputs; these have to be measured using a body fixed IMU (accelerometers and gyros).

2.3 Accelerometer Dynamics Model

The dynamic model for the accelerometers is derived in Appendix A, equations (A-1.12) to (A-1.17). A second order linear dynamics is assumed for the accelerometer model.

$$\frac{d}{dt}\underline{x}_4^{[1]} = [W_0]\;\underline{x}_4^{[1]} + [W_2]\underline{x}_3^{[2]} + [W_3]\underline{u}_3^{[1]} + [W_4]\underline{g}^{[1]} \tag{2-11}$$

Where:

$$\underline{x}_4^{[1]} = \begin{bmatrix} a_{xo} & \dot{a}_{xo} & a_{yo} & \dot{a}_{yo} & a_{zo} & \dot{a}_{zo} \end{bmatrix}^T$$: is a 6x1 accelerometer state vector.

$$[W_0] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_x^2 & -2\zeta_x\omega_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_y^2 & -2\zeta_y\omega_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_z^2 & -2\zeta_z\omega_z \end{bmatrix}$$: is a 6x6 coefficient matrix containing accelerometer parameters.

$$[W_1] = \begin{bmatrix} 0 & 0 & 0 \\ \omega_x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_y^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_z^2 \end{bmatrix}$$: is a 6x3 coefficient matrix containing accelerometer parameters.

$[W_2] = [W_1 D_0 F_0 + W_1 D_1]$: is a 6x12 coefficient matrix.

$[W_3] = [W_1 D_0 G_0]$: is a 6x6 coefficient matrix.

$[W_4] = [W_1 D_0]$: is a 6x6 coefficient matrix.

The accelerometer measurement model is given by (see equation (A-1.18)):

$$\underline{z}_4^{[1]} = [J_1]\,\underline{x}_4^{[1]} + \underline{v}_{a_b}^{[1]} + \underline{v}_{a_d}^{[1]} + \underline{v}_{a_s}^{[1]} + \underline{v}_{a_n}^{[1]} \tag{2-12}$$

Where:

$\underline{z}_4^{[1]} = [a_{x_m} \quad a_{y_m} \quad a_{z_m}]^T$: is a 3x1 accelerometer measurement vector.

$\underline{v}_{a_b}^{[1]} = [v_{x_b} \quad v_{y_b} \quad v_{z_b}]^T$: is a 3x1 accelerometer bias error vector.

$\underline{v}_{a_d}^{[1]} = [v_{x_d} \quad v_{y_d} \quad v_{z_d}]^T$: is a 3x1 accelerometer drift error vector.

$\underline{v}_{a_s}^{[1]} = [v_{x_s} \quad v_{y_s} \quad v_{z_s}]^T$: is a 3x1 accelerometer scale factor error vector.

$\underline{v}_{a_n}^{[1]} = [v_{x_n} \quad v_{y_n} \quad v_{z_n}]^T$: is a 3x1 accelerometer noise error vector.

$$[J_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$: is a 3x6 matrix.

2.4 Gyro Dynamics Model

The dynamic model for the gyros is derived in Appendix A, equations (A-1.19) to (A-1.21). A second order linear dynamics is assumed for the gyro model.

$$\frac{d}{dt}\underline{x}_5^{[1]} = [W_5]\underline{x}_5^{[1]} + [W_7]\ \underline{x}_3^{[1]} \qquad (2\text{-}13)$$

Where:

$$\underline{x}_5^{[1]} = \begin{bmatrix} p_o & \dot{p}_o & q_o & \dot{q}_o & r_o & \dot{r}_o \end{bmatrix}^T$$: is a 6x1 gyro state vector.

$$[W_5] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_p^2 & -2\zeta_p\omega_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_q^2 & -2\zeta_q\omega_q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_r^2 & -2\zeta_r\omega_r \end{bmatrix}$$: is a 6x6 coefficient matrix containing gyro parameters.

$$[W_6] = \begin{bmatrix} 0 & 0 & 0 \\ \omega_p^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_q^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_r^2 \end{bmatrix}$$: is a 6x3 coefficient matrix containing gyro parameters.

$[W_7] = [0_{6\times 3} \quad | \quad W_6]$: is a 6x6 coefficient matrix.

The gyro measurement model is given by (see equation (A-1.22)):

$$\underline{z}_5^{[1]} = [J_2]\,\underline{x}_5^{[1]} + \underline{v}_{gb}^{[1]} + \underline{v}_{gd}^{[1]} + \underline{v}_{gs}^{[1]} + \underline{v}_{gn}^{[1]} \qquad (2\text{-}14)$$

Where:

$\underline{z}_4^{[1]} = \left[p_m \quad q_m \quad r_m \right]^T$: is a 3x1 gyro measurement vector.

$\underline{v}_{gb}^{[1]} = \left[v_{pb} \quad v_{qb} \quad v_{r_b} \right]^T$: is a 3x1 gyro bias error vector.

$\underline{v}_{gd}^{[1]} = \left[v_{pd} \quad v_{qd} \quad v_{r_d} \right]^T$: is a 3x1 gyro drift error vector.

$\underline{v}_{gs}^{[1]} = \left[v_{p_s} \quad v_{q_s} \quad v_{r_s} \right]^T$: is a 3x1 gyro scale factor error vector.

$\underline{v}_{gn}^{[1]} = \left[v_{pn} \quad v_{qn} \quad v_{r_n} \right]^T$: is a 3x1 gyro noise error vector.

$[J_2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$: is a 3x6 matrix.

2.5 Actuation Servo Model

The dynamic model for the actuation system is derived in Appendix A, equations (A-1.23) to (A-1.28). A second order linear dynamics is assumed for the gyro model.

This equation is of the form:

$$\frac{d}{dt}\underline{x}_6^{[1]} = [V_0]\,\underline{x}_6^{[1]} + [V_1]\,\underline{u}_4^{[1]} \qquad (2\text{-}15)$$

Where:

$\underline{x}_6^{[1]} = \left[\xi_o \quad \dot{\xi}_o \quad \eta_o \quad \dot{\eta}_o \quad \zeta_o \quad \dot{\zeta}_o \right]^T$: is a 6x1 state vector.

$\underline{u}_4^{[1]} = \underline{\alpha}_i = \left[\xi_i \quad \eta_i \quad \zeta_i \right]^T$: is a 3x1 control (servo actuator) input vector.

$$[V_0]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_\xi^2 & -2\zeta_\xi\omega_\xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_\eta^2 & -2\zeta_\eta\omega_\eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_\zeta^2 & -2\zeta_\zeta\omega_\zeta \end{bmatrix}$$: is a 6x6 servo actuator coefficient matrix.

$$[V_1]=\begin{bmatrix} 0 & 0 & 0 \\ \omega_\xi^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_\eta^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_\zeta^2 \end{bmatrix}$$: is a 6x3 servo input coefficient matrix.

If the actuator system noise is included in the model then the actual output from the actuator servo may be written as:

$$\frac{d}{dt}\underline{x}_6^{[1]}=[V_0]\underline{x}_6^{[1]}+[V_1]\underline{u}_4^{[1]}+\underline{v}_{s_n}^{[1]} \tag{2-16}$$

We may also write for the actuator output:

$$\underline{y}_6^{[1]}=[\xi_o \quad \eta_o \quad \varsigma_o]^T=[V_2]\underline{x}_6^{[1]} \tag{2-17}$$

Where:

$\underline{v}_{s_n}^{[1]}=\left[v_{\xi_n} \quad v_{\dot{\xi}_n} \quad v_{\eta_n} \quad v_{\dot{\eta}_n} \quad v_{\zeta_n} \quad v_{\dot{\zeta}_n}\right]^T$: is a 6x1 actuator servo noise error vector.

$$[V_2]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$: is a 3x6 matrix.

Note that:

$$\begin{aligned}\underline{u}_3^{[1]}&=\left[\tilde{X}(..) \quad \tilde{Y}(..) \quad \tilde{Z}(..) \quad L(..) \quad M(..) \quad N(..)\right]^T \\ &=\underline{f}(u,v,w,p,q,r,\dot{u},\dot{v},\dot{w},\xi_o,\eta_o,\zeta_o)=\underline{f}\left(\underline{x}_3^{[1]},\underline{\dot{x}}_3^{[1]},[V_2]\underline{x}_6^{[1]}\right)\end{aligned} \tag{2-18}$$

2.6 Overall Nonlinear (Quadratic) Airframe Model including IMU

Equations (A-1.5), (A-1.17), (A-1.21) and (A-1.26) combine to give us an overall airframe, IMU and actuator dynamic model; this equation is of the form:

$$\frac{d}{dt}\underline{x}_7^{[1]} = [F_1]\underline{x}_7^{[1]} + [F_2]\underline{x}_3^{[2]} + [G_1]\underline{u}_3^{[1]}(..) + [G_2]\underline{u}_4^{[1]} + [H_1]\underline{g}^{[1]} + [H_2]\underline{v}_{s_n}^{[1]} \tag{2-19}$$

Where:

$\underline{x}_7^{[1]} = \left[\underline{x}_3^{[1]T} \;|\; \underline{x}_4^{[1]T} \;|\; \underline{x}_5^{[1]T} \;|\; \underline{x}_6^{[1]T} \right]^T$: is a 24 x1 state vector.

$[F_1] = \begin{bmatrix} [0_{6\times6}] & | & [0_{6\times6}] & | & [0_{6\times6}] & | & [0_{6\times6}] \\ --- & | & --- & | & --- & | & --- \\ [0_{6\times6}] & | & [W_0] & | & [0_{6\times6}] & | & [0_{6\times6}] \\ --- & | & --- & | & --- & | & --- \\ [W_7] & | & [0_{6\times6}] & | & [W_5] & | & [0_{6\times6}] \\ --- & | & --- & | & --- & | & --- \\ [0_{6\times6}] & | & [0_{6\times6}] & | & [0_{6\times6}] & | & [V_0] \end{bmatrix}$: is a 24x24 coefficient matrix.

$[F_2] = \left[[F_0]^T \;|\; [W_2]^T \;|\; [0_{12\times6}] \;|\; [0_{12\times6}] \right]^T$: is a 24x12 coefficient matrix.

$[G_1] = \left[[G_0]^T \;|\; [W_3]^T \;|\; [0_{6\times6}] \;|\; [0_{6\times6}] \right]^T$: is a 24x6 coefficient matrix.

$[G_2] = \left[[0_{3\times6}] \;|\; [0_{3\times6}] \;|\; [0_{3\times6}] \;|\; [V_1]^T \right]^T$: is a 24x3 coefficient matrix.

$[H_1] = \left[[I_{6\times6}] \;|\; [W_4]^T \;|\; [0_{6\times6}] \;|\; [0_{6\times6}] \right]^T$: is a 24x6 coefficient matrix.

$[H_2] = \left[[0_{6\times6}] \;|\; [0_{6\times6}] \;|\; [0_{6\times6}] \;|\; [I_{6\times6}] \right]^T$: is a 24x6 coefficient matrix.

A block diagram of the decomposed version (derived by considering the sub-matrices) of the overall model is given in Figure A-1.1. See appendix section 3.

2.7 The Measurement Model

Equation (2-12) and (2-14) may be combined to give the overall airframe and IMU (Gyros, Accelerometers) measurement model (see equation (A-1.30)):

$$\underline{z}_7^{[1]} = [J_6]\underline{x}_7^{[1]} + \underline{v}_b^{[1]} + \underline{v}_d^{[1]} + \underline{v}_s^{[1]} + \underline{v}_n^{[1]} \tag{2-20}$$

Where:

$\underline{z}_7^{[1]} = \left[\underline{z}_4^{[1]T} \quad | \quad \underline{z}_5^{[1]T} \right]^T = \left[a_{x_m} \quad a_{y_m} \quad a_{z_m} \quad p_m \quad q_m \quad r_m \right]^T$: is a 6x1 IMU (gyro + accelerometer) measurement vector.

$\underline{v}_b^{[1]} = \left[\underline{v}_{a_b}^{[1]T} \quad | \quad \underline{v}_{g_b}^{[1]T} \right]^T = \left[v_{x_b} \quad v_{y_b} \quad v_{z_b} \quad v_{p_b} \quad v_{q_b} \quad v_{r_b} \right]^T$: is a 6x1 IMU bias error vector.

$\underline{v}_d^{[1]} = \left[\underline{v}_{a_d}^{[1]T} \quad | \quad \underline{v}_{g_d}^{[1]T} \right]^T = \left[v_{x_d} \quad v_{y_d} \quad v_{z_d} \quad v_{p_d} \quad v_{q_d} \quad v_{r_d} \right]^T$: is a 6x1 IMU drift error vector.

$\underline{v}_s^{[1]} = \left[\underline{v}_{a_s}^{[1]T} \quad | \quad \underline{v}_{g_s}^{[1]T} \right]^T = \left[v_{x_s} \quad v_{y_s} \quad v_{z_s} \quad v_{p_s} \quad v_{q_s} \quad v_{r_s} \right]^T$: is a 6x1 IMU scale factor error vector.

$\underline{v}_n^{[1]} = \left[\underline{v}_{a_n}^{[1]T} \quad | \quad \underline{v}_{g_n}^{[1]T} \right]^T = \left[v_{x_n} \quad v_{y_n} \quad v_{z_n} \quad v_{p_n} \quad v_{q_n} \quad v_{r_n} \right]^T$: is a 6x1 IMU noise error vector.

$$[J_6] = \begin{bmatrix} 0_{3\times6} & | & J_1 & | & 0_{3\times6} & | & 0_{3\times6} \\ --- & | & --- & | & --- & | & --- \\ 0_{3\times6} & | & 0_{3\times6} & | & J_2 & | & 0_{3\times6} \end{bmatrix}$$: is a 6x24 matrix.

3. Linearised State Space Airframe Model

The linearised state space model is derived in Appendix A, section 2 (equations (A-2.1) to (A-2.9). It is assumed that $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$, L, M and N are functions of $u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w}\, \xi, \eta, \varsigma$ and first order linearization of these nominal values u_0, v_0, w_0, p_0, q_0, r_0, ξ_0, η_0 and ς_0, are considered. The linearised airframe model is given by:

The 'small' perturbation model for the quadratic dynamic model of equation (A-1.29) may be written as (see equation (A-2.5)):

$$\begin{aligned} \frac{d}{dt}\Delta\underline{x}_7^{[1]} &= [F_1]\Delta\underline{x}_7^{[1]} + [F_2 E_4 + G_1 E_0]\Delta\underline{x}_3^{[1]} + [G_1 E_2]\ \Delta\underline{x}_6^{[1]} + [G_1 E_3]\frac{d}{dt}\Delta\underline{x}_3^{[1]} \\ &+ [G_2]\Delta\underline{u}_4^{[1]} + [H_2]\Delta\underline{v}_{s_n}^{[1]} \end{aligned} \qquad (3\text{-}1)$$

This equation will be referred to as the decomposed form (see Appendix A section 3)

Where:

$$\left[F_2E_4+G_1E_0\right]=\begin{bmatrix}\left[F_0E_4+G_0E_0\right]\\ ----------\\ \left[W_2E_4+W_3E_0\right]\\ ----------\\ \left[0_{6\times 6}\right]\\ ----------\\ \left[0_{6\times 6}\right]\end{bmatrix};\ \left[G_1E_2\right]=\begin{bmatrix}\left[G_0E_2\right]\\ ----\\ \left[W_3E_2\right]\\ ----\\ \left[0_{6\times 6}\right]\\ ----\\ \left[0_{6\times 6}\right]\end{bmatrix};\ \left[G_1E_3\right]=\begin{bmatrix}\left[G_0E_3\right]\\ ----\\ \left[W_3E_3\right]\\ ----\\ \left[0_{6\times 6}\right]\\ ----\\ \left[0_{6\times 6}\right]\end{bmatrix}$$

Equation (3-1) may be written in a compact form as:

$$\frac{d}{dt}\Delta\underline{x}_7^{[1]}=\left[F_5\right]\Delta\underline{x}_7^{[1]}+\left[G_3\right]\Delta\underline{u}_4^{[1]}+\left[H_3\right]\Delta\underline{v}_{s_n}^{[1]} \tag{3-2}$$

Where:

$$\left[E_0\right]=\begin{bmatrix}\tilde{X}_u & \tilde{X}_v & \tilde{X}_w & \tilde{X}_p & \tilde{X}_q & \tilde{X}_r\\ \tilde{Y}_u & \tilde{Y}_v & \tilde{Y}_w & \tilde{Y}_p & \tilde{Y}_q & \tilde{Y}_r\\ \tilde{Z}_u & \tilde{Z}_v & \tilde{Z}_w & \tilde{Z}_p & \tilde{Z}_q & \tilde{Z}_r\\ L_u & L_v & L_w & L_p & L_q & L_r\\ M_u & M_v & M_w & M_p & M_q & M_r\\ N_u & N_v & N_w & N_p & N_q & N_r\end{bmatrix}$$: is a 6x6 aero-derivative matrix.

$$\left[E_1\right]=\begin{bmatrix}\tilde{X}_\xi & \tilde{X}_\eta & \tilde{X}_\varsigma\\ \tilde{Y}_\xi & \tilde{Y}_\eta & \tilde{Y}_\varsigma\\ \tilde{Z}_\xi & \tilde{Z}_\eta & \tilde{Z}_\varsigma\\ L_\xi & L_\eta & L_\varsigma\\ M_\xi & M_\eta & M_\varsigma\\ N_\xi & N_\eta & N_\varsigma\end{bmatrix}$$: is a 6x3 control-derivative matrix.

$\left[E_2\right]=\left[E_1V_2\right]$: is a 6x6 matrix.

$$\left[E_3\right]=\begin{bmatrix}\tilde{X}_{\dot{u}} & \tilde{X}_{\dot{v}} & \tilde{X}_{\dot{w}} & 0 & 0 & 0\\ \tilde{Y}_{\dot{u}} & \tilde{Y}_{\dot{v}} & \tilde{Y}_{\dot{w}} & 0 & 0 & 0\\ \tilde{Z}_{\dot{u}} & \tilde{Z}_{\dot{v}} & \tilde{Z}_{\dot{w}} & 0 & 0 & 0\\ L_{\dot{u}} & L_{\dot{v}} & L_{\dot{w}} & 0 & 0 & 0\\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & 0 & 0 & 0\\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & 0 & 0 & 0\end{bmatrix}$$: is a 6x6 aero-derivative matrix.

$$[E_4] = \begin{bmatrix} q_0 & 0 & 0 & 0 & u_0 & 0 \\ r_0 & 0 & 0 & 0 & 0 & u_0 \\ 0 & p_0 & 0 & v_0 & 0 & 0 \\ 0 & r_0 & 0 & 0 & 0 & v_0 \\ 0 & 0 & p_0 & w_0 & 0 & 0 \\ 0 & 0 & q_0 & 0 & w_0 & 0 \\ 0 & 0 & 0 & 2p_0 & 0 & 0 \\ 0 & 0 & 0 & q_0 & p_0 & 0 \\ 0 & 0 & 0 & r_0 & 0 & p_0 \\ 0 & 0 & 0 & 0 & 2q_0 & 0 \\ 0 & 0 & 0 & 0 & r_0 & q_0 \\ 0 & 0 & 0 & 0 & 0 & 2r_0 \end{bmatrix}$$: is a 12x6 matrix of steady state values.

$[F_3] = [I_{24\times 24}] - [G_1 E_3 \;\;|\;\; 0_{24\times 6} \;\;|\;\; 0_{24\times 6} \;\;|\;\; 0_{24\times 6}]$: is a 24x24 coefficient matrix.

$[F_4] = [F_1] + [F_2 E_4 + G_1 E_0 \;\;|\;\; 0_{24\times 6} \;\;|\;\; 0_{24\times 6} \;\;|\;\; 0_{24\times 6}]$: is a 24x24 coefficient matrix.

$[F_5] = [F_3]^{-1}[F_4]$: is a 24x24 coefficient matrix.

$[G_3] = [F_3]^{-1}[G_2]$: is a 24x3 coefficient matrix.

$[H_3] = [F_3]^{-1}[H_2]$: is a 24x3 coefficient matrix.

3.1 Linearised Measurement Model

Small perturbation model of the measurement model (see equation (A-1.30) may be written as:

$$\underline{\Delta z}_7^{[1]} = [J_6]\underline{\Delta x}_7^{[1]} + \underline{\Delta v}_b^{[1]} + \underline{\Delta v}_b^{[1]} + \underline{\Delta v}_s^{[1]} + \underline{\Delta v}_n^{[1]} \qquad (3\text{-}3)$$

Δ : denotes small perturbation about nominal values

Finally, linearising the output equation (2-10) (see also equation (A-1.11)) gives us:

$$\underline{\Delta y}_3^{[1]} = [J_7]\underline{\Delta x}_3^{[1]} + [L_0]\underline{\Delta u}_3^{[1]} \qquad (3\text{-}4)$$

Where:
$[J_7] = [J_0 + K_0 E_4]$: is a 6x6 coefficient matrix.

4. Conclusions

Both the non-linear and linearised autopilot models have been derived in this report. The state-space model of a missile autopilot needs to be validated by comparing the model with the other published results, and through both open and closed-loop systems simulation. The non-linear dynamics model presented as structural quadratic algebraic system is novel and will be used for developing non-linear control techniques suitable for missile systems high g-manoeuvres and operating of a range of aerodynamics conditions. The models developed in this report are useful for applications to precision optimum and adaptive guidance and control. It is hoped that the higher order model with motion and inertial coupling will provide more accurate representation of autopilot dynamics particularly for non axis-symmetric airframes that could be used for adaptive and integrated guidance & control of missiles such as air breathing supersonic and hypersonic vehicles.

5. References

1. **Babister, A.W:** *Aircraft Dynamic Stability and Response*, Pergamon, 1980.
2. **Blakelock, J.H:** *Automatic Control of Aircraft and Missiles*, John Wiley & Sons, Inc., 1965.
3. **Cook, M.V:** *Flight Dynamics Principles*, Arnold, 1997.
4. **Faruqi, F.A:** *On the Algebraic Structure of a Class of Nonlinear Dynamical Systems*, Journal of Applied Mathematics and Computation, 162 (2005).
5. **Garnell, P. and East, D.J:** *Guided Weapon Control Systems*, Pergamon Press, 1977.
6. **Kuo, B.C:** Automatic Control Systems, 6th Edition, Prentice-Hall International Editions, 1991.
7. **Pallett, E.H.J:** *Automatic Flight Control*, 3rd Edition, BSP Professional Books, 1987.
8. **Etkin, B and Reid, L. D:** *Dynamics of Flight – Stability and Control*, 3rd Edition, John Wiley & Sons, Inc. 1996.
9. **Nelson, R. C:** *Flight Stability and Automatic Control*, 2nd Edition, McGraw Hill, 1998.
10. **Bitmead, R. R; Gevers, G; Wertz, V:** *Adaptive Optimal Control (The Thinking Man's GPC)*, Prentice Hall, 1990.
11. **Etkin, B:** *Dynamics of Atmospheric Flight*, John Wiley & Sons, 1972.
12. **Faruqi, F. A and Vu, T.L:** *Mathematical Models for Missile Autopilot Design,* DSTO Rept. DSTO-TN-0449, August 2002.

Appendix A:

A.1. Non-linear (Quadratic) Airframe and IMU Dynamics

Equations (2-1) to (2-3) represent the force equations of a generalised rigid body and describe the translational motion of its centre of gravity (c.g) since the origin of the vehicle body axes is assumed to be co-located with the body c.g. Equations (2-4) to (2-6) represent the moment equations of a generalised rigid body and describe the rotational motion about the body axes through its c.g. Separating the derivative terms and after some algebraic manipulation, Equations (2-1) to (2-3) may be written in a vector form as:

$$\frac{d}{dt}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} uq \\ ur \\ vp \\ vr \\ wp \\ wq \end{bmatrix} + \begin{bmatrix} \tilde{X}+\tilde{T} \\ \tilde{Y} \\ \tilde{Z} \end{bmatrix} + \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} \qquad \text{(A-1.1)}$$

Where:

$$\tilde{X} = \frac{X}{m}; \quad \tilde{Y} = \frac{Y}{m}; \quad \tilde{Z} = \frac{Z}{m}; \quad \tilde{T} = \frac{T}{m}.$$

<u>Note:</u> that in the above equations, the states ***(u, v, w, p, q, r)*** appear as a quadratic form expression.

In matrix notation equations (2-4) to (2-6) (section.2) may be written as:

$$[A_0]\frac{d}{dt}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = [B_0]\begin{bmatrix} p^2 \\ pq \\ pr \\ q^2 \\ qr \\ r^2 \end{bmatrix} + \begin{bmatrix} L \\ M \\ N \end{bmatrix} \qquad \text{(A-1.2)}$$

Here again, the states ***(p,q,r)*** appear as a quadratic form expression.

Where:

$$[A_0]=\begin{bmatrix} I_{xx} & -I_{xy} & -I_{zx} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{yz} & I_{zz} \end{bmatrix}$$: is a 3x3 matrix.

$$[B_0]=\begin{bmatrix} 0 & I_{zx} & -I_{xy} & I_{yz} & (I_{yy}-I_{zz}) & -I_{yz} \\ -I_{zx} & -I_{yz} & (I_{zz}-I_{xx}) & 0 & I_{xy} & I_{zx} \\ I_{xy} & (I_{xx}-I_{yy}) & I_{yz} & -I_{xy} & -I_{zx} & 0 \end{bmatrix}$$: is a 3x6 matrix.

Equation (A-1.2) may also be written as:

$$\frac{d}{dt}\begin{bmatrix} p \\ q \\ r \end{bmatrix}=[A_0]^{-1}[B_0]\begin{bmatrix} p^2 \\ pq \\ pr \\ q^2 \\ qr \\ r^2 \end{bmatrix}+[A_0]^{-1}\begin{bmatrix} L \\ M \\ N \end{bmatrix} \tag{A-1.3}$$

Where:

$$[A_0]^{-1}=\frac{1}{\Delta}\begin{bmatrix} (I_{yy}I_{zz}-I_{yz}{}^2) & (I_{zz}I_{xy}+I_{yz}I_{zx}) & (I_{yz}I_{xy}+I_{yy}I_{zx}) \\ (I_{zz}I_{xy}+I_{yz}I_{zx}) & (I_{xx}I_{zz}-I_{zx}{}^2) & (I_{xx}I_{yz}+I_{zx}I_{xy}) \\ (I_{yz}I_{xy}+I_{yy}I_{zx}) & (I_{xx}I_{yz}+I_{xy}I_{zx}) & (I_{xx}I_{yy}-I_{xy}{}^2) \end{bmatrix}$$: a 3x3 matrix.

$$\Delta=\left(I_{xx}I_{yy}I_{zz}-I_{xx}I_{yz}{}^2-I_{yy}I_{zx}{}^2-I_{zz}I_{xy}{}^2-2I_{yz}I_{zx}I_{xy}\right).$$

The selection of the particular order of the terms in the 'quadratic-state' vectors $[uq\ ur\ vp\ vr\ wp\ wq]^T$ of Equation (A-1.1) and $[p^2\ pq\ pr\ q^2 qr\ r^2]^T$ of Equation (A-1.2) is made on the basis of retaining lexicographic order of the variables. Combining Equations (A-1.1) and (A-1.3), we obtain the full 6th order rigid body dynamics state equations as:

$$\frac{d}{dt}\begin{bmatrix} \underline{x}_1^{[1]} \\ -- \\ \underline{x}_2^{[1]} \end{bmatrix}=\begin{bmatrix} [C_0] & | & [0_{3\times6}] \\ -- & | & ------ \\ [0_{3\times6}] & | & [A_0]^{-1}[B_0] \end{bmatrix}\begin{bmatrix} \underline{x}_1^{[2]} \\ -- \\ \underline{x}_2^{[2]} \end{bmatrix}+\begin{bmatrix} [I_{3\times3}] & | & [0_{3\times3}] \\ --- & | & --- \\ [0_{3\times3}] & | & [A_0]^{-1} \end{bmatrix}\begin{bmatrix} \underline{u}_1^{[1]} \\ -- \\ \underline{u}_2^{[1]} \end{bmatrix}+\begin{bmatrix} \underline{g} \\ -- \\ \underline{0} \end{bmatrix}$$

(A-1.4)

Where:

$$[C_0] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}$$: is a 3x6 matrix.

$\underline{x}_1^{[1]} = [u \quad v \quad w]^T$: is a 3x1 linear state vector.

$\underline{x}_2^{[1]} = [p \quad q \quad r]^T$: is a 3x1 linear state vector.

$\underline{x}_1^{[2]} = [uq \quad ur \quad vp \quad vr \quad wp \quad wq]^T$: is a 6x1 quadratic state vector.

$\underline{x}_2^{[2]} = \left[p^2 \quad pq \quad pr \quad q^2 \quad qr \quad r^2\right]^T$: is a 6x1 quadratic state vector.

$\underline{u}_1^{[1]} = \left[\tilde{X} + \tilde{T} \quad \tilde{Y} \quad \tilde{Z}\right]^T$: is a 3x1 'force' input vector.

$\underline{u}_2^{[1]} = [L \quad M \quad N]^T$: is a 3x1 'moment' input vector.

$\underline{g} = \left[g_x \quad g_y \quad g_z\right]^T$: is a 3x1 gravity vector.

<u>Note:</u> superscript $^{[1]}$ is used to indicate that the state vector is linear; while superscript $^{[2]}$ is used to indicate that the state vector is a quadratic/bilinear form.

Equation (A-1.4) may be written in a compact form as:

$$\frac{d}{dt}\underline{x}_3^{[1]} = [F_0]\,\underline{x}_3^{[2]} + [G_0]\,\underline{u}_3^{[1]} + \underline{g}^{[1]} \qquad \text{(A-1.5)}$$

Where:

$$\underline{x}_3^{[1]} = \begin{bmatrix} \underline{x}_1^{[1]} \\ -- \\ \underline{x}_2^{[1]} \end{bmatrix} = [u \quad v \quad w \quad | \quad p \quad q \quad r]^T$$: is a 6x1 linear-state vector.

$$\underline{x}_3^{[2]} = \begin{bmatrix} \underline{x}_1^{[2]} \\ -- \\ \underline{x}_2^{[2]} \end{bmatrix} = \begin{bmatrix} uq & ur & vp & vr & wp & wq & | & p^2 & pq & pr & q^2 & qr & r^2 \end{bmatrix}^T$$: is a 12x1 quadratic-state vector.

$$\underline{u}_3^{[1]} = \begin{bmatrix} \underline{u}_1^{[1]} \\ -- \\ \underline{u}_2^{[1]} \end{bmatrix} = \begin{bmatrix} \tilde{X} + \tilde{T} & \tilde{Y} & \tilde{Z} & | & L & M & N \end{bmatrix}^T$$: is 6x1 a vector function of control inputs, forces and moments.

$$\underline{g}^{[1]} = \begin{bmatrix} \underline{g} \\ -- \\ \underline{0}_{3\times 1} \end{bmatrix} = \begin{bmatrix} g_x & g_y & g_z & 0 & 0 & 0 \end{bmatrix}^T$$: is the 6x1 gravity (or disturbance) vector.

$$[F_0] = \begin{bmatrix} [C_0] & | & [0_{3\times 6}] \\ --- & | & ------ \\ [0_{3\times 6}] & | & [A_0]^{-1}[B_0] \end{bmatrix}$$: is a 6x12 state-coefficient matrix.

$$[G_0] = \begin{bmatrix} [I_{3\times 3}] & | & [0_{3\times 3}] \\ --- & | & --- \\ [0_{3\times 3}] & | & [A_0]^{-1} \end{bmatrix}$$: is a 6x6 input-coefficient matrix.

Subscripts under **[*I*]** and **[*0*]** matrices denote the matrix dimensions.

A.1.1 Body Acceleration Model

Generally, not all state variables in the state equation are accessible or measurable. The common accessible measurement variables, in most missiles or airplanes, are the angular rate components (roll rate p, pitch rate q, and yaw rate r) and the acceleration components (a_x, a_y, a_z). The acceleration components at point O (where O is at a distance of d_x, d_y and d_z from the central point of gravity, c.g., along x-, y- and z-axis, respectively), may be written as [11]:

$$\begin{aligned} a_x &= \dot{u} + qw - rv - d_x(q^2 + r^2) + d_y(pq - \dot{r}) + d_z(pr + \dot{q}) \\ &= \tilde{X} + \tilde{T} + g_x - d_x(q^2 + r^2) + d_y(pq - \dot{r}) + d_z(pr + \dot{q}) \end{aligned} \qquad \text{(A-1.6)}$$

$$\begin{aligned} a_y &= \dot{v} + ru - pw + d_x(pq + \dot{r}) - d_y(p^2 + r^2) + d_z(qr - \dot{p}) \\ &= \tilde{Y} + g_y + d_x(pq + \dot{r}) - d_y(p^2 + r^2) + d_z(qr - \dot{p}) \end{aligned} \qquad \text{(A-1.7)}$$

$$a_z = \dot{w} + pv - qu + d_x(pr - \dot{q}) + d_y(qr + \dot{p}) - d_z(p^2 + q^2)$$
$$= \tilde{Z} + g_z + d_x(pr - \dot{q}) + d_y(qr + \dot{p}) - d_z(p^2 + q^2) \qquad \text{(A-1.8)}$$

Or in matrix form:➔

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \underline{y}_1^{[1]} = [D_0]\frac{d}{dt}\begin{bmatrix} \underline{x}_1^{[1]} \\ -- \\ \underline{x}_2^{[1]} \end{bmatrix} + [D_1]\begin{bmatrix} \underline{x}_1^{[2]} \\ -- \\ \underline{x}_2^{[2]} \end{bmatrix} \qquad \text{(A-1.9)}$$

Where:

$\underline{y}_1^{[1]} = [a_x \quad a_y \quad a_z]^T$: is a 3X1 body acceleration vector.

$[D_0] = \begin{bmatrix} 1 & 0 & 0 & 0 & d_z & -d_y \\ 0 & 1 & 0 & -d_z & 0 & d_x \\ 0 & 0 & 1 & d_y & -d_x & 0 \end{bmatrix}$: is a 3x6 accelerometer 'offset' matrix.

$[D_1] = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & d_y & d_z & -d_x & 0 & -d_x \\ 0 & 1 & 0 & 0 & -1 & 0 & -d_y & d_x & 0 & 0 & d_z & -d_y \\ -1 & 0 & 1 & 0 & 0 & 0 & -d_z & 0 & d_x & -d_z & d_y & 0 \end{bmatrix}$: is a 3x12 accelerometer 'offset' matrix.

Substituting from equation (A-1.5) for the $\frac{d}{dt}[..]$ term in the above equation gives us:

$$\underline{y}_1^{[1]} = [D_0F_0 + D_1]\begin{bmatrix} \underline{x}_1^{[2]} \\ -- \\ \underline{x}_2^{[2]} \end{bmatrix} + [D_0G_0]\begin{bmatrix} \underline{u}_1^{[1]} \\ -- \\ \underline{u}_2^{[1]} \end{bmatrix} + [D_0]\begin{bmatrix} \underline{g} \\ -- \\ \underline{0} \end{bmatrix}$$

➔

$$\underline{y}_1^{[1]} = [D_0F_0 + D_1]\underline{x}_3^{[2]} + [D_0G_0]\underline{u}_3^{[1]} + [D_0]\underline{g}^{[1]} \qquad \text{(A-1.10)}$$

Now, the expression for the body rates is:

$\underline{y}_2^{[1]} = \underline{x}_2^{[1]} = [p \quad q \quad r]^T$: is a 3X1 body rate vector.

Combining this with equation (A-1.10) gives us:

$$\begin{bmatrix} \underline{y}_1^{[1]} \\ -- \\ \underline{y}_2^{[1]} \end{bmatrix} = \begin{bmatrix} [0_{3\times6}] \\ ---------- \\ [0_{3\times3} \quad | \quad I_{3\times3}] \end{bmatrix} \underline{x}_3^{[1]} + \begin{bmatrix} [D_0F_0 + D_1] \\ -------- \\ [0_{3\times12}] \end{bmatrix} \underline{x}_3^{[2]} + \begin{bmatrix} [D_0G_0] \\ ---- \\ [0_{3\times6}] \end{bmatrix} \underline{u}_3^{[1]} + \begin{bmatrix} [D_0] \\ --- \\ [0_{3\times6}] \end{bmatrix} \underline{g}^{[1]}$$

$\Rightarrow$

$$\underline{y}_3^{[1]} = [J_0]\,\underline{x}_3^{[1]} + [K_0]\,\underline{x}_3^{[2]} + [L_0]\,\underline{u}_3^{[1]} + [M_0]\,\underline{g}^{[1]} \qquad \text{(A-1.11)}$$

Where:

$\underline{y}_3^{[1]} = \begin{bmatrix} \underline{y}_1^{[1]} \\ -- \\ \underline{y}_2^{[1]} \end{bmatrix}$: is a 6x1 output vector.

$[J_0] = \begin{bmatrix} [0_{3\times6}] \\ ---------- \\ [0_{3\times3} \quad | \quad I_{3\times3}] \end{bmatrix}$: is a 6x6 linear-state output coefficient matrix.

$[K_0] = \begin{bmatrix} [D_0F_0 + D_1]_{3\times12} \\ ---------- \\ [0_{3\times12}] \end{bmatrix}$: is a 6x12 quadratic-state output coefficient matrix.

$[L_0] = \begin{bmatrix} [D_0G_0]_{3\times6} \\ ------ \\ [0_{3\times6}] \end{bmatrix}$: is a 6x6 coefficient matrix.

$[M_0] = \begin{bmatrix} [D_0]_{3\times6} \\ ---- \\ [0_{3\times6}] \end{bmatrix}$: is a 6x6 output coefficient matrix.

<u>Note:</u> equation (A-1.11) represents the actual accelerations and body rates outputs; these have to be measured using a body fixed IMU (accelerometers and gyros).

A.1.2 Accelerometer and Gyro Dynamics

Let us assume a second order dynamics for the accelerometers and gyros respectively; thus:

$$\frac{a_{\alpha o}}{a_\alpha} = \frac{\omega_\alpha^2}{\left(s^2 + 2\zeta_\alpha\omega_\alpha s + \omega_\alpha^2\right)}; \quad \alpha = x, y, z \qquad \text{(A-1.12)}$$

$$\frac{\mu_o}{\mu} = \frac{\omega_\mu^2}{\left(s^2 + 2\zeta_\mu \omega_\mu s + \omega_\mu^2\right)}; \quad \mu = p, q, r \tag{A-1.13}$$

(ω, ζ): denote the sensor natural frequency and the damping factor respectively. Subscript *'o'* denote output values, and subscripts (x, y, z) and (p, q, r) denote accelerations and body rates measured by accelerometer and gyro orthogonal triads respectively.

A.1.3 A.1.3 Accelerometer Dynamics

In state space form equations for the accelerometer may be written as:

$$\frac{d}{dt}\begin{bmatrix} a_{xo} \\ \dot{a}_{xo} \\ a_{yo} \\ \dot{a}_{yo} \\ a_{zo} \\ \dot{a}_{zo} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_x^2 & -2\zeta_x\omega_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_y^2 & -2\zeta_y\omega_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_z^2 & -2\xi_z\omega_z \end{bmatrix}\begin{bmatrix} a_{xo} \\ \dot{a}_{xo} \\ a_{yo} \\ \dot{a}_{yo} \\ a_{zo} \\ \dot{a}_{zo} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \omega_x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_y^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_z^2 \end{bmatrix}\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \tag{A-1.14}$$

Where: $\dot{a}_{\alpha o} = \frac{d}{dt} a_{\alpha o}; \quad \alpha = x, y, z$

This equation is of the form:

$$\frac{d}{dt}\underline{x}_4^{[1]} = [W_0]\,\underline{x}_4^{[1]} + [W_1]\,\underline{y}_1^{[1]} \tag{A-1.15}$$

Where:

$$[W_0] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_x^2 & -2\zeta_x\omega_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_y^2 & -2\zeta_y\omega_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_z^2 & -2\zeta_z\omega_z \end{bmatrix}$$: is a 6x6 coefficient matrix containing accelerometer parameters.

$$[W_1] = \begin{bmatrix} 0 & 0 & 0 \\ \omega_x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_y^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_z^2 \end{bmatrix}$$: is a 6x3 coefficient matrix containing accelerometer parameters.

$$\underline{x}_4^{[1]} = \begin{bmatrix} a_{xo} & \dot{a}_{xo} & a_{yo} & \dot{a}_{yo} & a_{zo} & \dot{a}_{zo} \end{bmatrix}^T$$: is a 6x1 accelerometer state vector.

Substituting for $\underline{y}_1^{[1]}$ from equation (A-1.10) into equation (A-1.15), we get:

$$\frac{d}{dt}\underline{x}_4^{[1]} = [W_0]\ \underline{x}_4^{[1]} + [W_1 D_0 F_0 + W_1 D_1]\underline{x}_3^{[2]} + [W_1 D_0 G_0]\underline{u}_3^{[1]} + [W_1 D_0]\underline{g^{[1]}} \qquad \text{(A-1.16)}$$

➔

$$\frac{d}{dt}\underline{x}_4^{[1]} = [W_0]\ \underline{x}_4^{[1]} + [W_2]\underline{x}_3^{[2]} + [W_3]\underline{u}_3^{[1]} + [W_4]\underline{g}^{[1]} \qquad \text{(A-1.17)}$$

Where:

$[W_2] = [W_1 D_0 F_0 + W_1 D_1]$: is a 6x12 coefficient matrix.

$[W_3] = [W_1 D_0 G_0]$: is a 6x6 coefficient matrix.

$[W_4] = [W_1 D_0]$: is a 6x6 coefficient matrix.

The measurement model is given by:

$$\underline{z}_4^{[1]} = [J_1]\underline{x}_4^{[1]} + \underline{v}_{a_b}^{[1]} + \underline{v}_{a_d}^{[1]} + \underline{v}_{a_s}^{[1]} + \underline{v}_{a_n}^{[1]} \qquad \text{(A-1.18)}$$

Where:

$\underline{z}_4^{[1]} = [a_{x_m} \quad a_{y_m} \quad a_{z_m}]^T$: is a 3x1 accelerometer measurement vector.

$\underline{v}_{a_b}^{[1]} = [v_{x_b} \quad v_{y_b} \quad v_{z_b}]^T$: is a 3x1 accelerometer bias error vector.

$\underline{v}_{a_d}^{[1]} = [v_{x_d} \quad v_{y_d} \quad v_{z_d}]^T$: is a 3x1 accelerometer drift error vector.

$\underline{v}_{a_s}^{[1]} = [v_{x_s} \quad v_{y_s} \quad v_{z_s}]^T$: is a 3x1 accelerometer scale factor error vector.

$\underline{v}_{a_n}^{[1]} = \left[v_{x_n} \quad v_{y_n} \quad v_{z_n} \right]^T$: is a 3x1 accelerometer noise error vector.

$[J_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$: is a 3x6 matrix.

A.1.4 Gyro Dynamics

Similarly, state space form equations for the gyros may be written as:

$$\frac{d}{dt}\begin{bmatrix} p_o \\ \dot{p}_o \\ q_o \\ \dot{q}_o \\ r_o \\ \dot{r}_o \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_p^2 & -2\zeta_p\omega_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_q^2 & -2\zeta_q\omega_q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_q^2 & -2\zeta_q\omega_q \end{bmatrix}\begin{bmatrix} p_o \\ \dot{p}_o \\ q_o \\ \dot{q}_o \\ r_o \\ \dot{r}_o \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \omega_p^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_q^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_r^2 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{A-1.19}$$

Where:

$\dot{\mu}_o = \frac{d}{dt}\mu_o; \quad \mu = p,q,r$

Equation (A-1.19) is of the form:

$$\frac{d}{dt}\underline{x}_5^{[1]} = [W_5]\underline{x}_5^{[1]} + [W_6]\underline{x}_2^{[1]} \tag{A-1.20}$$

$\Rightarrow$

$$\frac{d}{dt}\underline{x}_5^{[1]} = [W_5]\underline{x}_5^{[1]} + [0_{6\times3} \quad | \quad W_6]\underline{x}_3^{[1]}$$

$\Rightarrow$

$$\frac{d}{dt}\underline{x}_5^{[1]} = [W_5]\underline{x}_5^{[1]} + [W_7]\ \underline{x}_3^{[1]} \tag{A-1.21}$$

Where:

$\underline{x}_5^{[1]} = \begin{bmatrix} p_o & \dot{p}_o & q_o & \dot{q}_o & r_o & \dot{r}_o \end{bmatrix}^T$: is a 6x1 gyro state vector.

$\underline{x}_2^{[1]} = \underline{y}_2^{[1]} = \underline{\omega} = [p \quad q \quad r]^T$

$$[W_5]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_p^2 & -2\zeta_p\omega_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_q^2 & -2\zeta_q\omega_q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_r^2 & -2\zeta_r\omega_r \end{bmatrix}$$: is a 6x6 coefficient matrix containing gyro parameters.

$$[W_6]=\begin{bmatrix} 0 & 0 & 0 \\ \omega_p^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_q^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_r^2 \end{bmatrix}$$: is a 6x3 coefficient matrix containing gyro parameters.

$[W_7]=[0_{6\times3} \quad | \quad W_6]$: is a 6x6 coefficient matrix.

The measurement model is given by:

$$\underline{z}_5^{[1]}=[J_2]\,\underline{x}_5^{[1]}+\underline{v}_{g_b}^{[1]}+\underline{v}_{g_d}^{[1]}+\underline{v}_{g_s}^{[1]}+\underline{v}_{g_n}^{[1]} \tag{A-1.22}$$

Where:

$\underline{z}_4^{[1]}=[p_m \quad q_m \quad r_m]^T$: is a 3x1 gyro measurement vector.

$\underline{v}_{g_b}^{[1]}=[v_{p_b} \quad v_{q_b} \quad v_{r_b}]^T$: is a 3x1 gyro bias error vector.

$\underline{v}_{g_d}^{[1]}=[v_{p_d} \quad v_{q_d} \quad v_{r_d}]^T$: is a 3x1 gyro drift error vector.

$\underline{v}_{g_s}^{[1]}=[v_{p_s} \quad v_{q_s} \quad v_{r_s}]^T$: is a 3x1 gyro scale factor error vector.

$\underline{v}_{g_n}^{[1]}=[v_{p_n} \quad v_{q_n} \quad v_{r_n}]^T$: is a 3x1 gyro noise error vector.

$$[J_2]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$: is a 3x6 matrix.

A.1.5 Actuation Servo Model

Let us assume a second order dynamics for the actuators, that is:

$$\frac{\alpha_o}{\alpha_i} = \frac{\omega_\alpha^2}{\left(s^2 + 2\zeta_\alpha \omega_\alpha s + \omega_\alpha^2\right)}; \quad \alpha = \xi, \eta, \zeta \tag{A-1.23}$$

$(\omega_\alpha, \zeta_\alpha)$: denote the sensor natural frequency and the damping factor respectively; subscript 'i' denotes the servo (control) input value (servo demand), and subscripts (ξ, η, ζ) denote roll, pitch, and yaw outputs from the servo.

In state space form equations for the actuation system model may be written as:

$$\frac{d}{dt}\begin{bmatrix} \xi_o \\ \dot{\xi}_o \\ \eta_o \\ \dot{\eta}_o \\ \zeta_o \\ \dot{\zeta}_o \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_\xi^2 & -2\zeta_\xi \omega_\xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_\eta^2 & -2\zeta_\eta \omega_\eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_\zeta^2 & -2\zeta_\zeta \omega_\zeta \end{bmatrix} \begin{bmatrix} \xi_o \\ \dot{\xi}_o \\ \eta_o \\ \dot{\eta}_o \\ \zeta_o \\ \dot{\zeta}_o \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \omega_\xi^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_\eta^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_\zeta^2 \end{bmatrix} \begin{bmatrix} \xi_i \\ \eta_i \\ \zeta_i \end{bmatrix} \tag{A-1.24}$$

Where:

$$\dot{\alpha} = \frac{d}{dt}\alpha; \quad \alpha = \xi, \eta, \zeta$$

This equation is of the form:

$$\frac{d}{dt}\underline{x}_6^{[1]} = [V_0]\,\underline{x}_6^{[1]} + [V_1]\,\underline{u}_4^{[1]} \tag{A-1.25}$$

Where:

$\underline{x}_6^{[1]} = \begin{bmatrix} \xi_o & \dot{\xi}_o & \eta_o & \dot{\eta}_o & \zeta_o & \dot{\zeta}_o \end{bmatrix}^T$: is a 6x1 state vector.

$\underline{u}_4^{[1]} = \underline{\alpha}_i = [\xi_i \quad \eta_i \quad \zeta_i]^T$: is a 3x1 control (servo actuator) input vector.

$$[V_0]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega_\xi^2 & -2\zeta_\xi\omega_\xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_\eta^2 & -2\zeta_\eta\omega_\eta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_\zeta^2 & -2\zeta_\zeta\omega_\zeta \end{bmatrix}$$: is a 6x6 servo actuator coefficient matrix.

$$[V_1]=\begin{bmatrix} 0 & 0 & 0 \\ \omega_\xi^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega_\eta^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_\zeta^2 \end{bmatrix}$$: is a 6x3 servo input coefficient matrix.

If the actuator system noise is included in the model then the actual output from the actuator servo may be written as:

$$\frac{d}{dt}\underline{x}_6^{[1]}=[V_0]\underline{x}_6^{[1]}+[V_1]\underline{u}_4^{[1]}+\underline{v}_{s_n}^{[1]} \qquad \text{(A-1.26)}$$

We may also write for the actuator output:

$$\underline{y}_6^{[1]}=[\xi_o \quad \eta_o \quad \varsigma_o]^T=[V_2]\underline{x}_6^{[1]} \qquad \text{(A-1.27)}$$

Where:

$\underline{v}_{s_n}^{[1]}=\begin{bmatrix} v_{\xi_n} & v_{\dot{\xi}_n} & v_{\eta_n} & v_{\dot{\eta}_n} & v_{\zeta_n} & v_{\dot{\zeta}_n} \end{bmatrix}^T$: is a 6x1 actuator servo noise error vector.

$$[V_2]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$: is a 3x6 matrix.

<u>*Note that:*</u>

$$\begin{aligned}\underline{u}_3^{[1]} &= \left[\tilde{X}(..) \quad \tilde{Y}(..) \quad \tilde{Z}(..) \quad L(..) \quad M(..) \quad N(..)\right]^T \\ &= \underline{f}(u,v,w,p,q,r,\dot{u},\dot{v},\dot{w},\xi_o,\eta_o,\zeta_o)=\underline{f}\left(\underline{x}_3^{[1]},\underline{\dot{x}}_3^{[1]},[V_2]\underline{x}_6^{[1]}\right)\end{aligned} \qquad \text{(A-1.28)}$$

A.1.6 Non-linear Airframe, IMU and Actuator Dynamic

We can now combine equations (A-1.5), (A-1.17), (A-1.21) and (A-1.26) to give us an overall airframe, IMU and actuator dynamic model; this equation is of the Form:

$$\frac{d}{dt}\underline{x}_7^{[1]} = [F_1]\underline{x}_7^{[1]} + [F_2]\underline{x}_3^{[2]} + [G_1]\underline{u}_3^{[1]}(..) + [G_2]\underline{u}_4^{[1]} + [H_1]\underline{g}^{[1]} + [H_2]\underline{v}_{s_n}^{[1]} \qquad \text{(A-1.29)}$$

Where:

$\underline{x}_7^{[1]} = \left[\underline{x}_3^{[1]T} \quad | \quad \underline{x}_4^{[1]T} \quad | \quad \underline{x}_5^{[1]T} \quad | \quad \underline{x}_6^{[1]T} \right]^T$: is a 24 x1 state vector.

$$[F_1] = \begin{bmatrix} [0_{6\times6}] & | & [0_{6\times6}] & | & [0_{6\times6}] & | & [0_{6\times6}] \\ --- & | & --- & | & --- & | & --- \\ [0_{6\times6}] & | & [W_0] & | & [0_{6\times6}] & | & [0_{6\times6}] \\ --- & | & --- & | & --- & | & --- \\ [W_7] & | & [0_{6\times6}] & | & [W_5] & | & [0_{6\times6}] \\ --- & | & --- & | & --- & | & --- \\ [0_{6\times6}] & | & [0_{6\times6}] & | & [0_{6\times6}] & | & [V_0] \end{bmatrix}$$: is a 24x24 coefficient matrix.

$[F_2] = \left[[F_0]^T \quad | \quad [W_2]^T \quad | \quad [0_{12\times6}] \quad | \quad [0_{12\times6}] \right]^T$: is a 24x12 coefficient matrix.

$[G_1] = \left[[G_0]^T \quad | \quad [W_3]^T \quad | \quad [0_{6\times6}] \quad | \quad [0_{6\times6}] \right]^T$: is a 24x6 coefficient matrix.

$[G_2] = \left[[0_{3\times6}] \quad | \quad [0_{3\times6}] \quad | \quad [0_{3\times6}] \quad | \quad [V_1]^T \right]^T$: is a 24x3 coefficient matrix.

$[H_1] = \left[[I_{6\times6}] \quad | \quad [W_4]^T \quad | \quad [0_{6\times6}] \quad | \quad [0_{6\times6}] \right]^T$: is a 24x6 coefficient matrix.

$[H_2] = \left[[0_{6\times6}] \quad | \quad [0_{6\times6}] \quad | \quad [0_{6\times6}] \quad | \quad [I_{6\times6}] \right]^T$: is a 24x6 coefficient matrix.

A block diagram of the decomposed version (derived by considering the sub-matrices) of the overall model is given in Figure A-1.1. See appendix section 3.

A.1.7 Measurement Model

Equations (A-1.18) and (A.1.22) may be combined to give the overall measurement model:

$$\underline{z}_7^{[1]} = [J_6]\underline{x}_7^{[1]} + \underline{v}_b^{[1]} + \underline{v}_d^{[1]} + \underline{v}_s^{[1]} + \underline{v}_n^{[1]} \qquad \text{(A-1.30)}$$

Where:

$\underline{z}_7^{[1]} = \left[\underline{z}_4^{[1]T} \quad | \quad \underline{z}_5^{[1]T} \right]^T = \left[a_{x_m} \quad a_{y_m} \quad a_{z_m} \quad p_m \quad q_m \quad r_m \right]^T$: is a 6x1 IMU (gyro + accelerometer) measurement vector.

$\underline{v}_b^{[1]} = \left[\underline{v}_{a_b}^{[1]T} \quad | \quad \underline{v}_{g_b}^{[1]T} \right]^T = \left[v_{x_b} \quad v_{y_b} \quad v_{z_b} \quad v_{p_b} \quad v_{q_b} \quad v_{r_b} \right]^T$: is a 6x1 IMU bias error vector.

$\underline{v}_d^{[1]} = \left[\underline{v}_{a_d}^{[1]T} \quad | \quad \underline{v}_{g_d}^{[1]T} \right]^T = \left[v_{x_d} \quad v_{y_d} \quad v_{z_d} \quad v_{p_d} \quad v_{q_d} \quad v_{r_d} \right]^T$: is a 6x1 IMU drift error vector.

$\underline{v}_s^{[1]} = \left[\underline{v}_{a_s}^{[1]T} \quad | \quad \underline{v}_{g_s}^{[1]T} \right]^T = \left[v_{x_s} \quad v_{y_s} \quad v_{z_s} \quad v_{p_s} \quad v_{q_s} \quad v_{r_s} \right]^T$: is a 6x1 IMU scale factor error vector.

$\underline{v}_n^{[1]} = \left[\underline{v}_{a_n}^{[1]T} \quad | \quad \underline{v}_{g_n}^{[1]T} \right]^T = \left[v_{x_n} \quad v_{y_n} \quad v_{z_n} \quad v_{p_n} \quad v_{q_n} \quad v_{r_n} \right]^T$: is a 6x1 IMU noise error vector.

$\left[J_6 \right] = \begin{bmatrix} 0_{3\times6} & | & J_1 & | & 0_{3\times6} & | & 0_{3\times6} \\ --- & | & --- & | & --- & | & --- \\ 0_{3\times6} & | & 0_{3\times6} & | & J_2 & | & 0_{3\times6} \end{bmatrix}$: is a 6x24 matrix.

A.2. Linearised Airframe, Actuation and IMU Dynamics

Equation (A1.30) defines the complete non-linear description of the full 6-DOF airframe model. These equations contain quadratic terms in states and will be classed as the quadratic dynamic model. This type of model is required when autopilot design is undertaken for a missile executing high g- or high angle of attack manoeuvres, and ***(u, v, w, p, q, r)*** are not small. A more detailed consideration of the algebraic structure of this type of dynamic systems is given in [4].

A.2.1 Linearised Aerodynamic Forces and Moments

Assuming that $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$, L , M and N are functions of $u, v, w, p, q, r, \dot{u}, \dot{v}, \dot{w} \xi, \eta, \varsigma$ and using first order linearisation about the nominal values u_0, v_0, w_0, p_0, q_0, r_0, ξ_0, η_0 and ς_0, we get:

$$\Delta\underline{u}_3^{[1]} = \begin{bmatrix} \Delta\tilde{X}+\Delta\tilde{T} \\ \Delta\tilde{Y} \\ \Delta\tilde{Z} \\ \Delta L \\ \Delta M \\ \Delta N \end{bmatrix} = \begin{bmatrix} \tilde{X}_u & \tilde{X}_v & \tilde{X}_w & \tilde{X}_p & \tilde{X}_q & \tilde{X}_r \\ \tilde{Y}_u & \tilde{Y}_v & \tilde{Y}_w & \tilde{Y}_p & \tilde{Y}_q & \tilde{Y}_r \\ \tilde{Z}_u & \tilde{Z}_v & \tilde{Z}_w & \tilde{Z}_p & \tilde{Z}_q & \tilde{Z}_r \\ L_u & L_v & L_w & L_p & L_q & L_r \\ M_u & M_v & M_w & M_p & M_q & M_r \\ N_u & N_v & N_w & N_p & N_q & N_r \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \\ \Delta w \\ \Delta p \\ \Delta q \\ \Delta r \end{bmatrix} + \begin{bmatrix} \tilde{X}_\xi & \tilde{X}_\eta & \tilde{X}_\varsigma \\ \tilde{Y}_\xi & \tilde{Y}_\eta & \tilde{Y}_\varsigma \\ \tilde{Z}_\xi & \tilde{Z}_\eta & \tilde{Z}_\varsigma \\ L_\xi & L_\eta & L_\varsigma \\ M_\xi & M_\eta & M_\varsigma \\ N_\xi & N_\eta & N_\varsigma \end{bmatrix} \begin{bmatrix} \Delta\xi_o \\ \Delta\eta_o \\ \Delta\zeta_o \end{bmatrix}$$

$$+ \begin{bmatrix} \tilde{X}_{\dot{u}} & \tilde{X}_{\dot{v}} & \tilde{X}_{\dot{w}} & 0 & 0 & 0 \\ \tilde{Y}_{\dot{u}} & \tilde{Y}_{\dot{v}} & \tilde{Y}_{\dot{w}} & 0 & 0 & 0 \\ \tilde{Z}_{\dot{u}} & \tilde{Z}_{\dot{v}} & \tilde{Z}_{\dot{w}} & 0 & 0 & 0 \\ L_{\dot{u}} & L_{\dot{v}} & L_{\dot{w}} & 0 & 0 & 0 \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & 0 & 0 & 0 \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{v} \\ \Delta\dot{w} \\ \Delta\dot{p} \\ \Delta\dot{q} \\ \Delta\dot{r} \end{bmatrix} \qquad \text{(A-2.1)}$$

This equation may be written as:

$$\Delta\underline{u}_3^{[1]} = [E_0]\Delta\underline{x}_3^{[1]} + [E_1V_2]\Delta\underline{x}_6^{[1]} + [E_3]\frac{d}{dt}\Delta\underline{x}_3^{[1]}$$

$\Rightarrow$

$$\Delta\underline{u}_3^{[1]} = [E_0]\Delta\underline{x}_3^{[1]} + [E_2]\Delta\underline{x}_6^{[1]} + [E_3]\frac{d}{dt}\Delta\underline{x}_3^{[1]} \qquad \text{(A-2.2)}$$

Where:

$\Delta(..)$: denotes 'small' deviations from the normal steady-state condition (see Tables A-1.1 – A1-3):

$$\tilde{X}_\alpha = \frac{\partial\tilde{X}}{\partial\alpha}; \quad \alpha = u,v,w,p,q,r,\dot{u},\dot{v},\dot{w}\,\xi,\eta,\varsigma\,; \; \tilde{Y}_\alpha = \frac{\partial\tilde{Y}}{\partial\alpha}; \quad \alpha = u,v,w,p,q,r,\dot{u},\dot{v},\dot{w}\,\xi,\eta,\varsigma$$

$$\tilde{Z}_\alpha = \frac{\partial\tilde{Z}}{\partial\alpha}; \quad \alpha = u,v,w,p,q,r,\dot{u},\dot{v},\dot{w}\,\xi,\eta,\varsigma\,; \; L_\alpha = \frac{\partial L}{\partial\alpha}; \quad \alpha = u,v,w,p,q,r,\dot{u},\dot{v},\dot{w}\,\xi,\eta,\varsigma$$

$$M_\alpha = \frac{\partial M}{\partial\alpha}; \quad \alpha = u,v,w,p,q,r,\dot{u},\dot{v},\dot{w}\,\xi,\eta,\varsigma\,; \; N_\alpha = \frac{\partial N}{\partial\alpha}; \quad \alpha = u,v,w,p,q,r,\dot{u},\dot{v},\dot{w}\,\xi,\eta,\varsigma$$

$[E_2] = [E_1V_2]$: is a 6x6 matrix.

$\Delta\tilde{T} = 0; \quad \Delta\underline{g}^{[1]} = \underline{0}$

$$[E_0]=\begin{bmatrix} \tilde{X}_u & \tilde{X}_v & \tilde{X}_w & \tilde{X}_p & \tilde{X}_q & \tilde{X}_r \\ \tilde{Y}_u & \tilde{Y}_v & \tilde{Y}_w & \tilde{Y}_p & \tilde{Y}_q & \tilde{Y}_r \\ \tilde{Z}_u & \tilde{Z}_v & \tilde{Z}_w & \tilde{Z}_p & \tilde{Z}_q & \tilde{Z}_r \\ L_u & L_v & L_w & L_p & L_q & L_r \\ M_u & M_v & M_w & M_p & M_q & M_r \\ N_u & N_v & N_w & N_p & N_q & N_r \end{bmatrix}$$: is a 6x6 aero-derivative matrix.

$$[E_1]=\begin{bmatrix} \tilde{X}_\xi & \tilde{X}_\eta & \tilde{X}_\varsigma \\ \tilde{Y}_\xi & \tilde{Y}_\eta & \tilde{Y}_\varsigma \\ \tilde{Z}_\xi & \tilde{Z}_\eta & \tilde{Z}_\varsigma \\ L_\xi & L_\eta & L_\varsigma \\ M_\xi & M_\eta & M_\varsigma \\ N_\xi & N_\eta & N_\varsigma \end{bmatrix}$$: is a 6x3 control-derivative matrix.

$$[E_3]=\begin{bmatrix} \tilde{X}_{\dot{u}} & \tilde{X}_{\dot{v}} & \tilde{X}_{\dot{w}} & 0 & 0 & 0 \\ \tilde{Y}_{\dot{u}} & \tilde{Y}_{\dot{v}} & \tilde{Y}_{\dot{w}} & 0 & 0 & 0 \\ \tilde{Z}_{\dot{u}} & \tilde{Z}_{\dot{v}} & \tilde{Z}_{\dot{w}} & 0 & 0 & 0 \\ L_{\dot{u}} & L_{\dot{v}} & L_{\dot{w}} & 0 & 0 & 0 \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & 0 & 0 & 0 \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & 0 & 0 & 0 \end{bmatrix}$$: is a 6x6 aero-derivative matrix.

A.2.2 Linearisation of the Quadratic State Vector

It is easily verified the first order linearization of the quadratic state vector defined in section 2.1. :

$$\underline{x}_3^{[2]}=\left[\underline{x}_1^{[2]T} \quad | \quad \underline{x}_2^{[2]T}\right]^T=\left[uq \quad ur \quad vp \quad vr \quad wp \quad wq \quad | \quad p^2 \quad pq \quad pr \quad q^2 \quad qr \quad r^2\right]^T$$

may be written as:

$$\Delta\underline{x}_3^{[2]}=[E_4]\,\Delta\underline{x}_3^{[1]} \tag{A-2.3}$$

Where:

$$[E_4]=\begin{bmatrix} q_0 & 0 & 0 & 0 & u_0 & 0 \\ r_0 & 0 & 0 & 0 & 0 & u_0 \\ 0 & p_0 & 0 & v_0 & 0 & 0 \\ 0 & r_0 & 0 & 0 & 0 & v_0 \\ 0 & 0 & p_0 & w_0 & 0 & 0 \\ 0 & 0 & q_0 & 0 & w_0 & 0 \\ 0 & 0 & 0 & 2p_0 & 0 & 0 \\ 0 & 0 & 0 & q_0 & p_0 & 0 \\ 0 & 0 & 0 & r_0 & 0 & p_0 \\ 0 & 0 & 0 & 0 & 2q_0 & 0 \\ 0 & 0 & 0 & 0 & r_0 & q_0 \\ 0 & 0 & 0 & 0 & 0 & 2r_0 \end{bmatrix}$$: is a 12x6 matrix of steady state values.

$$\Delta\underline{x}_3^{[1]}=\left[\Delta\underline{x}_1^{[1]T} \;\; | \;\; \Delta\underline{x}_2^{[1]T}\right]^T=[\Delta u \;\; \Delta v \;\; \Delta w \;\; | \;\; \Delta p \;\; \Delta q \;\; \Delta r]^T$$: is a 6x1 linear Δ -state vector.

A.2.3 Linearised Airframe, IMU, Actuator Model

The 'small' perturbation model for the quadratic dynamic model of equation (A-1.29) may be written as:

$$\frac{d}{dt}\Delta\underline{x}_7^{[1]}=[F_1]\Delta\underline{x}_7^{[1]}+[F_2]\Delta\underline{x}_3^{[2]}+[G_1]\Delta\underline{u}_3^{[1]}+[G_2]\Delta\underline{u}_4^{[1]}+[H_1]\Delta\underline{g}^{[1]}+[H_2]\Delta\underline{v}_{s_n}^{[1]} \tag{A-2.4}$$

Substituting for $\Delta\underline{x}_3^{[2]}$ and $\Delta\underline{u}_3^{[1]}$ from equations (A-2.2) and (A-2.3), gives us:

$$\begin{aligned}\frac{d}{dt}\Delta\underline{x}_7^{[1]}=&[F_1]\Delta\underline{x}_7^{[1]}+[F_2E_4+G_1E_0]\Delta\underline{x}_3^{[1]}+[G_1E_2]\;\Delta\underline{x}_6^{[1]}+[G_1E_3]\frac{d}{dt}\Delta\underline{x}_3^{[1]}\\ &+[G_2]\Delta\underline{u}_4^{[1]}+[H_2]\Delta\underline{v}_{s_n}^{[1]}\end{aligned} \tag{A-2.5}$$

Where:

$$\Delta\underline{g}^{[1]}=\underline{0}\;;\;[F_2E_4+G_1E_0]=\begin{bmatrix}[F_0E_4+G_0E_0]\\ \hline [W_2E_4+W_3E_0]\\ \hline [0_{6\times6}]\\ \hline [0_{6\times6}]\end{bmatrix};[G_1E_2]=\begin{bmatrix}[G_0E_2]\\ \hline [W_3E_2]\\ \hline [0_{6\times6}]\\ \hline [0_{6\times6}]\end{bmatrix};[G_1E_3]=\begin{bmatrix}[G_0E_3]\\ \hline [W_3E_3]\\ \hline [0_{6\times6}]\\ \hline [0_{6\times6}]\end{bmatrix}$$

These above matrices are all of dimension 24X6. See also the block diagram Figure A-1.2. Now, Equation (A-2.5)➔

$$\frac{d}{dt}\Delta \underline{x}_7^{[1]} = [F_1]\Delta \underline{x}_7^{[1]} + \left[[F_2E_4 + G_1E_0] \mid [0_{24\times6}] \mid [0_{24\times6}] \mid [G_1E_2] \right]\Delta \underline{x}_7^{[1]}$$
$$+ [G_2]\Delta \underline{u}_4^{[1]} + \left[[G_1E_3] \mid [0_{24\times6}] \mid [0_{24\times6}] \mid [0_{24\times6}] \right]\frac{d}{dt}\Delta \underline{x}_7^{[1]} \qquad \text{(A-2.6)}$$
$$+ [H_2]\Delta \underline{v}_{s_n}^{[1]}$$

➔

$$\frac{d}{dt}[F_3]\Delta \underline{x}_7^{[1]} = [F_4]\Delta \underline{x}_7^{[1]} + [G_2]\Delta \underline{u}_4^{[1]} + [H_2]\Delta \underline{v}_{s_n}^{[1]} \qquad \text{(A-2.7)}$$

➔

$$\frac{d}{dt}\Delta \underline{x}_6^{[1]} = [F_3]^{-1}[F_4]\Delta \underline{x}_7^{[1]} + [F_3]^{-1}[G_2]\Delta \underline{u}_4^{[1]} + [F_3]^{-1}[H_2]\Delta \underline{v}_{s_n}^{[1]} \qquad \text{(A-2.8)}$$

➔

$$\frac{d}{dt}\Delta \underline{x}_7^{[1]} = [F_5]\Delta \underline{x}_7^{[1]} + [G_3]\Delta \underline{u}_4^{[1]} + [H_3]\Delta \underline{v}_{s_n}^{[1]} \qquad \text{(A-2.9)}$$

Where:

$[F_3] = [I_{24\times24}] - [G_1E_3 \mid 0_{24\times6} \mid 0_{24\times6} \mid 0_{24\times6}]$: is a 24x24 coefficient matrix.

$[F_4] = [F_1] + [F_2E_4 + G_1E_0 \mid 0_{24\times6} \mid 0_{24\times6} \mid G_1E_2]$: is a 24x24 coefficient matrix.

$[F_5] = [F_3]^{-1}[F_4]$: is a 24x24 coefficient matrix.

$[G_3] = [F_3]^{-1}[G_2]$: is a 24x3 coefficient matrix.

$[H_3] = [F_3]^{-1}[H_2]$: is a 24x3 coefficient matrix.

Small perturbation model of the measurement model (see equation (A-1.30) may be written as:

$$\underline{\Delta z}_7^{[1]} = [J_6]\underline{\Delta x}_7^{[1]} + \underline{\Delta v}_b^{[1]} + \underline{\Delta v}_b^{[1]} + \underline{\Delta v}_s^{[1]} + \underline{\Delta v}_n^{[1]} \qquad \text{(A-2.10)}$$

Δ : denotes small perturbation about nominal values.

Finally, linearising the output equation (A-1.11) gives us:

$$\underline{\Delta y}_3^{[1]} = \left[J_0 + K_0 E_4\right] \underline{\Delta x}_3^{[1]} + \left[L_0\right] \underline{\Delta u}_3^{[1]} \tag{A-2.11}$$

$\Rightarrow$

$$\underline{\Delta y}_3^{[1]} = \left[J_7\right] \underline{\Delta x}_3^{[1]} + \left[L_0\right] \underline{\Delta u}_3^{[1]} \tag{A-2.12}$$

Where:
$\left[J_7\right] = \left[J_0 + K_0 E_4\right]$**:** is a 6x6 coefficient matrix.

A.3. Decomposed State Space Models

A.3.1 Nonlinear Model:

Using the relationship established in the appendix section 1.4, we may write:

$$\frac{d}{dt}\begin{bmatrix} \underline{x}_3^{[1]} \\ \hline \underline{x}_4^{[1]} \\ \hline \underline{x}_5^{[1]} \\ \hline \underline{x}_6^{[1]} \end{bmatrix} = \left[\begin{array}{c|c|c|c} [0_{6\times6}] & [0_{6\times6}] & [0_{6\times6}] & [0_{6\times6}] \\ \hline [0_{6\times6}] & [W_0] & [0_{6\times6}] & [0_{6\times6}] \\ \hline [W_7] & [0_{6\times6}] & [W_5] & [0_{6\times6}] \\ \hline [0_{6\times6}] & [0_{6\times6}] & [0_{6\times6}] & [V_0] \end{array}\right] \begin{bmatrix} \underline{x}_3^{[1]} \\ \hline \underline{x}_4^{[1]} \\ \hline \underline{x}_5^{[1]} \\ \hline \underline{x}_6^{[1]} \end{bmatrix} + \begin{bmatrix} [F_0] \\ \hline [W_2] \\ \hline [0_{6\times12}] \\ \hline [0_{6\times12}] \end{bmatrix} \underline{x}_3^{[2]} + \begin{bmatrix} [G_0] \\ \hline [W_3] \\ \hline [0_{6\times6}] \\ \hline [0_{6\times6}] \end{bmatrix} \underline{u}_3^{[1]} + \begin{bmatrix} [0_{6\times3}] \\ \hline [0_{6\times3}] \\ \hline [0_{6\times3}] \\ \hline [V_1] \end{bmatrix} \underline{u}_4^{[1]} + \begin{bmatrix} [I_{6\times6}] \\ \hline [W_4] \\ \hline [0_{6\times6}] \\ \hline [0_{6\times6}] \end{bmatrix} \underline{g}^{[1]} + \begin{bmatrix} [0_{6\times6}] \\ \hline [0_{6\times6}] \\ \hline [0_{6\times6}] \\ \hline [I_{6\times6}] \end{bmatrix} \underline{v}_{s_n}^{[1]} \qquad \text{(A3-1)}$$

A.3.2 Linearised Model:

Using the relationship established in the appendix section 1.4 and equations (A-2.2), (A-2.3) and (A-2.5), we may write:

$$\frac{d}{dt}\begin{bmatrix} \underline{\Delta x}_3^{[1]} \\ \hline \underline{\Delta x}_4^{[1]} \\ \hline \underline{\Delta x}_5^{[1]} \\ \hline \underline{\Delta x}_6^{[1]} \end{bmatrix} = \left[\begin{array}{c|c|c|c} [0_{6\times6}] & [0_{6\times6}] & [0_{6\times6}] & [0_{6\times6}] \\ \hline [0_{6\times6}] & [W_0] & [0_{6\times6}] & [0_{6\times6}] \\ \hline [W_7] & [0_{6\times6}] & [W_5] & [0_{6\times6}] \\ \hline [0_{6\times6}] & [0_{6\times6}] & [0_{6\times6}] & [V_0] \end{array}\right] \begin{bmatrix} \underline{\Delta x}_3^{[1]} \\ \hline \underline{\Delta x}_4^{[1]} \\ \hline \underline{\Delta x}_5^{[1]} \\ \hline \underline{\Delta x}_6^{[1]} \end{bmatrix} + \begin{bmatrix} [F_0 E_4 + G_0 E_0] \\ \hline [W_2 E_4 + W_3 E_0] \\ \hline [0_{6\times6}] \\ \hline [0_{6\times6}] \end{bmatrix} \underline{\Delta x}_3^{[1]} + \begin{bmatrix} [G_0 E_2] \\ \hline [W_3 E_2] \\ \hline [0_{6\times6}] \\ \hline [0_{6\times6}] \end{bmatrix} \underline{\Delta x}_6^{[1]} + \begin{bmatrix} [0_{6\times3}] \\ \hline [0_{6\times3}] \\ \hline [0_{6\times3}] \\ \hline [V_1] \end{bmatrix} \underline{\Delta u}_4^{[1]} + \begin{bmatrix} [G_0 E_3] \\ \hline [W_3 E_3] \\ \hline [0_{6\times6}] \\ \hline [0_{6\times6}] \end{bmatrix} \frac{d}{dt}\underline{\Delta x}_3^{[1]} + \begin{bmatrix} [0_{6\times6}] \\ \hline [0_{6\times6}] \\ \hline [0_{6\times6}] \\ \hline [I_{6\times6}] \end{bmatrix} \underline{\Delta v}_{s_n}^{[1]}$$

(A3-2)

Block diagrams for the decomposed versions for the nonlinear and the linearised models are given in Figures A-1.1 and A-1.2 respectively

Table A-1.1. Longitudinal (Roll) Aerodynamic Derivatives and Coefficients:

Derivatives		Derivatives		Normalised Coefficients	
Symbol	**Units**	**Symbols**	**Units**	**Symbols**	**Units**
X_u	$Nm^{-1}\ sec$	-	-	$C_{x_u}\left(= X_u / QS_x\right)$	$m^{-1}\ sec$
X_v	$Nm^{-1}\ sec$	$X_\beta\left(= UX_v\right)$	N	$C_{x_\beta}\left(= X_\beta / QS_x\right)$	-
X_w	$Nm^{-1}\ sec$	$X_\alpha\left(= UX_w\right)$	N	$C_{x_\alpha}\left(= X_\alpha / QS_x\right)$	-
X_p	$N\ sec$	-	-	$C_{x_p}\left(= X_p / QS_x\right)$	sec
X_q	$N\ sec$	-	-	$C_{x_q}\left(= X_q / QS_x\right)$	sec
X_r	$N\ sec$	-	-	$C_{x_r}\left(= X_r / QS_x\right)$	sec
X_ξ	N	-	-	$C_{x_\xi}\left(= X_\xi / QS_x\right)$	-
X_η	N	-	-	$C_{x_\eta}\left(= X_\eta / QS_x\right)$	-
X_ζ	N	-	-	$C_{x_\zeta}\left(= X_\zeta / QS_x\right)$	-
$X_{\dot{u}}$	$Nm^{-1}\ sec^2$	-	-	$C_{x_{\dot{u}}}\left(= X_{\dot{u}} / QS_x\right)$	$m^{-1}\ sec^2$
$X_{\dot{v}}$	$Nm^{-1}\ sec^2$	$X_{\dot{\beta}}\left(= UX_{\dot{v}}\right)$	$N\ sec$	$C_{x_{\dot{v}}}\left(= X_{\dot{v}} / QS_x\right)$	$m^{-1}\ sec^2$
$X_{\dot{w}}$	$Nm^{-1}\ sec^2$	$X_{\dot{\alpha}}\left(= UX_{\dot{w}}\right)$	$N\ sec$	$C_{x_{\dot{w}}}\left(= X_{\dot{w}} / QS_x\right)$	$m^{-1}\ sec^2$
L_u	$N\ sec$	-	-	$C_{l_u}\left(= L_u / QS_x c_x\right)$	$m^{-1}\ sec$
L_v	$N\ sec$	-	-	$C_{l_v}\left(= L_v / QS_x c_x\right)$	$m^{-1}\ sec$
L_w	$N\ sec$	-	-	$C_{l_w}\left(= L_w / QS_x c_x\right)$	$m^{-1}\ sec$
L_p	$Nm\ sec$	-	-	$C_{l_p}\left(= L_p / QS_x c_x\right)$	sec
L_q	$Nm\ sec$	-	-	$C_{l_q}\left(= L_q / QS_x c_x\right)$	sec
L_r	$Nm\ sec$	-	-	$C_{l_r}\left(= L_r / QS_x c_x\right)$	sec
L_ξ	Nm	-	-	$C_{l_\xi}\left(= L_\xi / QS_x c_x\right)$	-
L_η	Nm	-	-	$C_{l_\eta}\left(= L_\eta / QS_x c_x\right)$	-
L_ζ	Nm	-	-	$C_{l_\zeta}\left(= L_\zeta / QS_x c_x\right)$	-
$L_{\dot{u}}$	$N\ sec^2$	-	-	$C_{l_{\dot{u}}}\left(= L_{\dot{u}} / QS_x c_x\right)$	$m^{-1}\ sec^2$
$L_{\dot{v}}$	$N\ sec^2$	$L_{\dot{\beta}}\left(= UL_{\dot{v}}\right)$	$Nm\ sec$	$C_{l_{\dot{v}}}\left(= L_{\dot{v}} / QS_x c_x\right)$	$m^{-1}\ sec^2$
$L_{\dot{w}}$	$N\ sec^2$	$L_{\dot{\alpha}}\left(= UL_{\dot{w}}\right)$	$Nm\ sec$	$C_{l_{\dot{w}}}\left(= L_{\dot{w}} / QS_x c_x\right)$	$m^{-1}\ sec^2$

* $Q = \frac{1}{2}\rho U^2 : \left(Nm^{-2}\right);\ S_x : \left(m^2\right);\ c_x : (m)$

Table A-2.1. Lateral (Pitch) Aerodynamic Derivatives and Coefficients:

Derivatives		Derivatives		Normalised Coefficients	
Symbol	**Units**	**Symbols**	**Units**	**Symbols**	**Units**
Y_u	$Nm^{-1}\ sec$	-	-	$C_{y_u}\left(=Y_u / QS_y\right)$	$m^{-1}\ sec$
Y_v	$Nm^{-1}\ sec$	$Y_\beta\left(=UY_v\right)$	N	$C_{y_\beta}\left(=Y_\beta / QS_y\right)$	-
Y_w	$Nm^{-1}\ sec$	$Y_\alpha\left(=UY_w\right)$	N	$C_{Y_\alpha}\left(=Y_\alpha / QS_y\right)$	-
Y_p	$N\ sec$	-	-	$C_{y_p}\left(=Y_p / QS_y\right)$	sec
Y_q	$N\ sec$	-	-	$C_{y_q}\left(=Y_q / QS_y\right)$	sec
Y_r	$N\ sec$	-	-	$C_{y_r}\left(=Y_r / QS_y\right)$	sec
Y_ξ	N	-	-	$C_{y_\xi}\left(=Y_\xi / QS_y\right)$	-
Y_η	N	-	-	$C_{y_\eta}\left(=Y_\eta / QS_y\right)$	-
Y_ζ	N	-	-	$C_{y_\zeta}\left(=Y_\zeta / QS_y\right)$	-
$Y_{\dot{u}}$	$Nm^{-1}\ sec^2$	-	-	$C_{y_{\dot{u}}}\left(=Y_{\dot{u}} / QS_y\right)$	$m^{-1}\ sec^2$
$Y_{\dot{v}}$	$Nm^{-1}\ sec^2$	$Y_{\dot{\beta}}\left(=UY_{\dot{v}}\right)$	$N\ sec$	$C_{y_{\dot{v}}}\left(=Y_{\dot{v}} / QS_y\right)$	$m^{-1}\ sec^2$
$Y_{\dot{w}}$	$Nm^{-1}\ sec^2$	$Y_{\dot{\alpha}}\left(=UY_{\dot{w}}\right)$	$N\ sec$	$C_{y_{\dot{w}}}\left(=Y_{\dot{w}} / QS_y\right)$	$m^{-1}\ sec^2$
M_u	$N\ sec$	-	-	$C_{m_u}\left(=M_u / QS_y c_y\right)$	$m^{-1}\ sec$
M_v	$N\ sec$	-	-	$C_{m_v}\left(=M_v / QS_y c_y\right)$	$m^{-1}\ sec$
M_w	$N\ sec$	-	-	$C_{m_w}\left(=M_w / QS_y c_y\right)$	$m^{-1}\ sec$
M_p	$Nm\ sec$	-	-	$C_{m_p}\left(=M_p / QS_y c_y\right)$	sec
M_q	$Nm\ sec$	-	-	$C_{m_q}\left(=M_q / QS_y c_y\right)$	sec
M_r	$Nm\ sec$	-	-	$C_{m_r}\left(=M_r / QS_y c_y\right)$	sec
M_ξ	Nm	-	-	$C_{m_\xi}\left(=M_\xi / QS_y c_y\right)$	-
M_η	Nm	-	-	$C_{m_\eta}\left(=M_\eta / QS_y c_y\right)$	-
M_ζ	Nm	-	-	$C_{m_\zeta}\left(=M_\zeta / QS_y c_y\right)$	-
$M_{\dot{u}}$	$N\ sec^2$	-	-	$C_{m_{\dot{u}}}\left(=M_{\dot{u}} / QS_y c_y\right)$	$m^{-1}\ sec^2$
$M_{\dot{v}}$	$N\ sec^2$	$M_{\dot{\beta}}\left(=UM_{\dot{v}}\right)$	$Nm\ sec$	$C_{m_{\dot{v}}}\left(=M_{\dot{v}} / QS_y c_y\right)$	$m^{-1}\ sec^2$
$M_{\dot{w}}$	$N\ sec^2$	$M_{\dot{\alpha}}\left(=UM_{\dot{w}}\right)$	$Nm\ sec$	$C_{m_{\dot{w}}}\left(=M_{\dot{w}} / QS_y c_y\right)$	$m^{-1}\ sec^2$

Table A-3.1. Lateral (Yaw) Aerodynamic Derivatives and Coefficients:

Derivatives		**Derivatives**		**Normalised Coefficients**	
Symbol	**Units**	**Symbols**	**Units**	**Symbols**	**Units**
Z_u	$Nm^{-1}\ sec$	-	-	$C_{z_u}\left(=Z_u/QS_z\right)$	$m^{-1}\ sec$
Z_v	$Nm^{-1}\ sec$	$Z_\beta\left(=UZ_v\right)$	N	$C_{z_\beta}\left(=Z_\beta/QS_z\right)$	-
Z_w	$Nm^{-1}\ sec$	$Z_\alpha\left(=UZ_w\right)$	N	$C_{z_\alpha}\left(=Z_\alpha/QS_z\right)$	-
Z_p	$N\ sec$	-	-	$C_{z_p}\left(=Z_p/QS_z\right)$	sec
Z_q	$N\ sec$	-	-	$C_{z_q}\left(=Z_q/QS_z\right)$	sec
Z_r	$N\ sec$	-	-	$C_{z_r}\left(=Z_r/QS_z\right)$	sec
Z_ξ	N	-	-	$C_{z_\xi}\left(=Z_\xi/QS_z\right)$	-
Z_η	N	-	-	$C_{z_\eta}\left(=Z_\eta/QS_z\right)$	-
Z_ζ	N	-	-	$C_{z_\zeta}\left(=Z_\zeta/QS_z\right)$	-
$Z_{\dot{u}}$	$Nm^{-1}\ sec^2$	-	-	$C_{z_{\dot{u}}}\left(=Z_{\dot{u}}/QS_z\right)$	$m^{-1}\ sec^2$
$Z_{\dot{v}}$	$Nm^{-1}\ sec^2$	$Z_{\dot{\beta}}\left(=UZ_{\dot{v}}\right)$	$N\ sec$	$C_{z_{\dot{v}}}\left(=Z_{\dot{v}}/QS_z\right)$	$m^{-1}\ sec^2$
$Z_{\dot{w}}$	$Nm^{-1}\ sec^2$	$Z_{\dot{\alpha}}\left(=UZ_{\dot{w}}\right)$	$N\ sec$	$C_{z_{\dot{w}}}\left(=Z_{\dot{w}}/QS_z\right)$	$m^{-1}\ sec^2$
N_u	$N\ sec$	-	-	$C_{n_u}\left(=N_u/QS_zc_z\right)$	$m^{-1}\ sec$
N_v	$N\ sec$	-	-	$C_{n_v}\left(=N_v/QS_zc_z\right)$	$m^{-1}\ sec$
N_w	$N\ sec$	-	-	$C_{n_w}\left(=N_w/QS_zc_z\right)$	$m^{-1}\ sec$
N_p	$Nm\ sec$	-	-	$C_{n_p}\left(=N_p/QS_zc_z\right)$	sec
N_q	$Nm\ sec$	-	-	$C_{n_q}\left(=N_q/QS_zc_z\right)$	sec
N_r	$Nm\ sec$	-	-	$C_{n_r}\left(=N_r/QS_zc_z\right)$	sec
N_ξ	Nm	-	-	$C_{n_\xi}\left(=N_\xi/QS_zc_z\right)$	-
N_η	Nm	-	-	$C_{n_\eta}\left(=N_\eta/QS_zc_z\right)$	-
N_ζ	Nm	-	-	$C_{n_\zeta}\left(=N_\zeta/QS_zc_z\right)$	-
$N_{\dot{u}}$	$N\ sec^2$	-	-	$C_{n_{\dot{u}}}\left(=N_{\dot{u}}/QS_zc_z\right)$	$m^{-1}\ sec^2$
$N_{\dot{v}}$	$N\ sec^2$	$N_{\dot{\beta}}\left(=UN_{\dot{v}}\right)$	$Nm\ sec$	$C_{n_{\dot{v}}}\left(=N_{\dot{v}}/QS_zc_z\right)$	$m^{-1}\ sec^2$
$N_{\dot{w}}$	$N\ sec^2$	$N_{\dot{\alpha}}\left(=UL_{\dot{w}}\right)$	$Nm\ sec$	$C_{n_{\dot{w}}}\left(=N_{\dot{w}}/QS_zc_z\right)$	$m^{-1}\ sec^2$

This page left intentionally blank

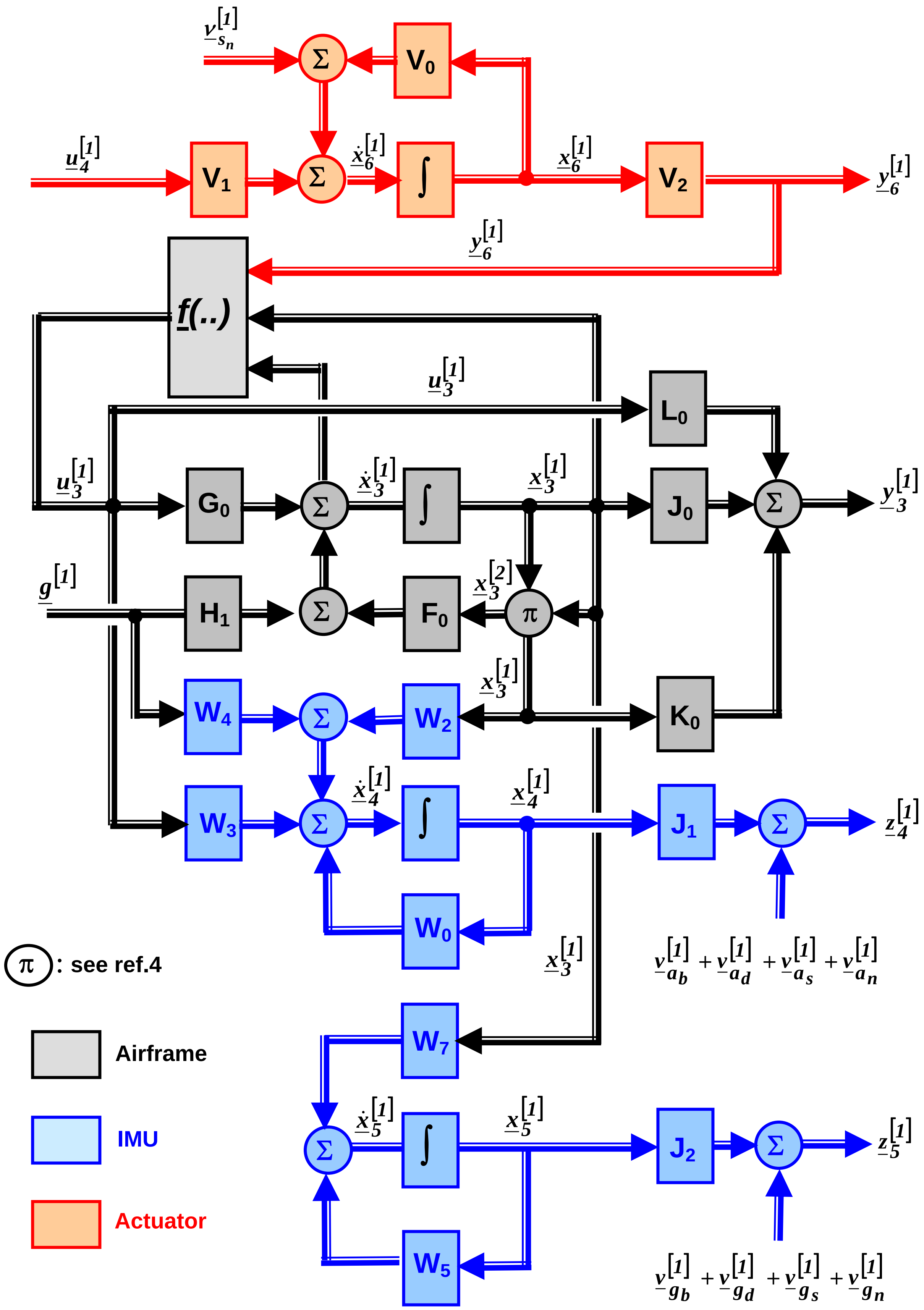

Figure A-1.1 Nonlinear (Quadratic) Airframe Model including Actuator and IMU

This page left intentionally blank

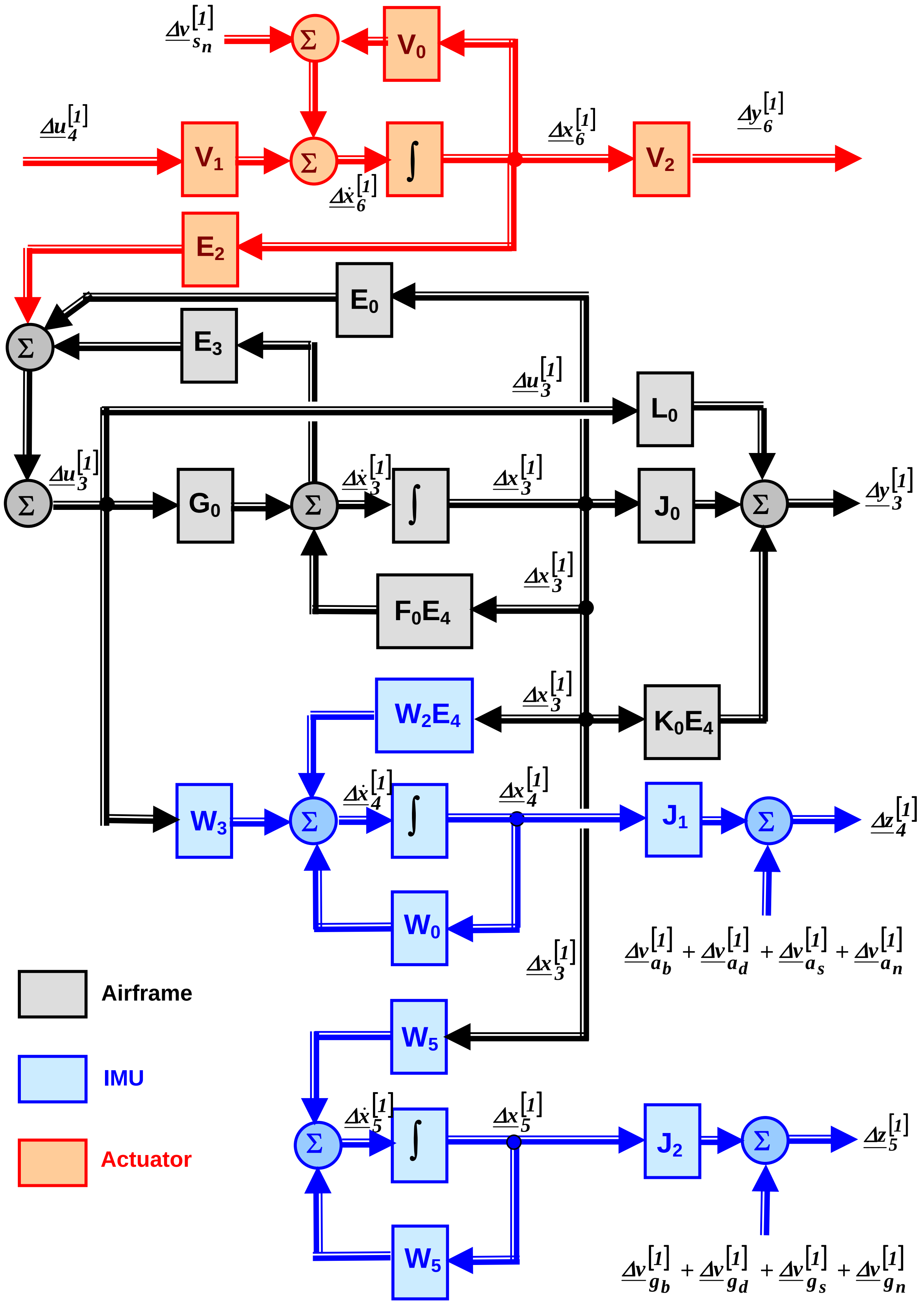

Figure A-1.2 Linearised Airframe Model Including Actuator

DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION DOCUMENT CONTROL DATA					
			1. PRIVACY MARKING/CAVEAT (OF DOCUMENT)		
2. TITLE State Space Model for Autopilot Design of Aerospace Vehicles			3. SECURITY CLASSIFICATION (FOR UNCLASSIFIED REPORTS THAT ARE LIMITED RELEASE USE (L) NEXT TO DOCUMENT CLASSIFICATION) Document (U) Title (U) Abstract (U)		
4. AUTHOR(S) Dr. Farhan A. Faruqi			5. CORPORATE AUTHOR DSTO Defence Science and Technology Organisation PO Box 1500 Edinburgh South Australia 5111 Australia		
6a. DSTO NUMBER DSTO-TR-1990	6b. AR NUMBER AR-013-894		6c. TYPE OF REPORT Technical Report	7. DOCUMENT DATE March 2007	
8. FILE NUMBER 2007/1006135/1	9. TASK NUMBER 05/303	10. TASK SPONSOR CWSD	11. NO. OF PAGES 39	12. NO. OF REFERENCES 12	
13. URL on the World Wide Web http://dspace.dsto.defence.gov.au/dspace/handle/1947/8679			14. RELEASE AUTHORITY Chief, Weapons Systems Division		
15. SECONDARY RELEASE STATEMENT OF THIS DOCUMENT *Approved for public release* OVERSEAS ENQUIRIES OUTSIDE STATED LIMITATIONS SHOULD BE REFERRED THROUGH DOCUMENT EXCHANGE, PO BOX 1500, SALISBURY, SA 5108					
16. DELIBERATE ANNOUNCEMENT No Limitations					
17. CASUAL ANNOUNCEMENT Yes					
18. DEFTEST DESCRIPTORS Guidance, automatic control, optimal control, modelling, mathematical models					

19. ABSTRACT

This report is a follow on to the report given in DSTO-TN-0559 and considers the derivation of the mathematical model for a aerospace vehicles and missile autopilots in state space form. The basic equations defining the airframe dynamics are non-linear, however, since the non-linearities are "structured" (in the sense that the states are of quadratic form) a novel approach of expressing this non-linear dynamics in state space form is given. This should provide a useful way to implement the equations in a computer simulation program and possibly for future application of non-linear analysis and synthesis techniques, particularly for autopilot design of aerospace vehicles executing high g-manoeuvres.

This report also considers a locally linearised state space model that lends itself to better known linear techniques of the modern control theory. A coupled multi-input multi-output (MIMO) model is derived suitable for both the application of the modern control techniques as well as the classical time-domain and frequency domain techniques. The models developed are useful for further research on precision optimum guidance and control. It is hoped that the model will provide more accurate presentations of missile autopilot dynamics and will be used for adaptive and integrated guidance & control of agile missiles.