

# Control of Uncertain Systems under Constraints: Switching Horizon Predictive Control of Persistently Disturbed Input-Saturated Plants

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## Abstract

*In order to provide computationally affordable predictive control algorithms, predictive switching logic schemes are considered whereby a feedback-gain is switched-on at any time from a family of candidate feedback-gains so as to control a discrete-time input-saturated LTI system possibly subject to persistent bounded disturbances of unknown arbitrary magnitude. It is constructively shown that such schemes do exist which ensure, along with good tracking performance, global asymptotic and semi-global exponential stability in the noiseless case, as well as finite  $l_\infty$ -induced gain to the disturbance-to-state map, whenever the structure of the disturbed plant can make such properties conceptually achievable.*

*Key words:* Switching control; Predictive control; Control of input-saturated systems; Anti-windup control; Nonlinear control.

## 1 Introduction

In recent years, control of input-saturated dynamical systems – a subject of ever-lasting fundamental interest in control engineering – has attracted significant research efforts. These contributions have been basically of a two-fold nature: on one side (Sussmann, Sontag & Yang, 1994; Hou, Saberi, Lin & Sannuti, 1998) the major emphasis has been on basic issues, *e.g.*, characterization of state domains whose elements can be asymptotically regulated to zero by bounded inputs; on the other side, the main interest has been on anti-windup control schemes (Hanus, Kinnaert & Henrotte, 1987; Glattfelder, Ohta, Mosca & Wieland, 2000) whereby suitable corrections to a pre-designed compensator are generated whenever input saturations take place. A common shortcoming of these contributions is that, though at times robustness

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against model inaccuracies and asymptotically vanishing disturbances are considered, little is known on how to deal efficiently with persistent disturbances of unknown arbitrary magnitude.

A third source of contributions to the subject has been originated within model-based predictive control (MBPC) (Mosca, 1995; Mayne, Rawlings, Rao & Scokaert, 2000; Maciejowski, 2002). MBPC, though inherently tailored to handle input and/or state-related constraints, has been so far hampered in having a significant impact on applications other than slow-process control, mainly because of its high computational load. Moreover, an extension of MBPC to the case of persistent disturbances (Scokaert & Mayne, 1998) appears to be so computationally demanding that it can be yet considered in practice an unsolved problem.

The lecture related to this paper aims at describing a computationally affordable solution to the regulation problem of discrete-time input-saturated linear time-invariant (LTI) systems subject to persistent bounded disturbances of unknown arbitrary magnitude. The solution that will be elaborated hereafter enjoys the following features:

1. It is realized via a supervisory switching control scheme whereby a feedback-gain matrix, selected from a finite family of pre-designed candidate feedback-gains, is at any time switched-on in feedback to the plant according to the previous gain matrix and the information, either complete or partial, on the current plant state;
2. No disturbance upper-bound need to be known;
3. The feedback-gain selection is made in accordance with a predictive control philosophy, and each candidate feedback-gain is tuned on to a different control-horizon;
4. The supervisory switching logic is flexible enough so as to enable the designer to simplify the scheme by trading off performance vs. memory and/or computational complexity, while retaining guaranteed stability properties.

For an overview on hybrid systems - the general framework which encompasses the developments of the present paper - the reader is referred to (Antsaklis, 2000).

## 2 Problem Formulation

Some notations used throughout the paper are listed hereafter for the reader's convenience:

$$\overline{m} := \{1, 2, \dots, m\}$$

with  $m$  an integer greater than or equal to 1;  $\|x\|_h$  is a shorthand notation for  $[x'P(h)x]^{1/2}$  where the prime denotes transpose and  $P(h)$  is as in (17); finally,  $\|x\| := [x'x]^{1/2}$  denotes the usual (un-weighted) Euclidean norm.

The paper deals with the regulation problem of a discrete-time input-saturated LTI plant

$$\begin{cases} x(t+1) = \Phi x(t) + G\sigma(u(t)) + \xi(t) \\ \hat{x}(t) = x(t) + \tilde{x}(t) \end{cases} \quad (1)$$

where:  $t \in \mathbb{Z}_+ := \{0, 1, \dots\}$ ;  $x \in \mathbb{R}^n$  is the plant state;  $u = [u_1, \dots, u_m]' \in \mathbb{R}^m$  the control input; the prime denotes transpose;

$$\sigma(u) = \begin{cases} u & , \quad u_i \in [-\underline{u}_i, \overline{u}_i] \quad , \quad \forall i \in \overline{m} \\ n(u) & , \quad \text{elsewhere} \end{cases} \quad (2)$$

$\underline{u}_i, \bar{u}_i > 0$ ; and  $n(u) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ , an arbitrary bounded nonlinear function of  $u$ . In (1), both  $\xi$  and  $\tilde{x}$  are bounded and possibly persistent disturbances of unknown arbitrary magnitude. The problem is to find, based on  $\hat{x}$ , a partial state-information vector, feedback controls

$$u = f(\hat{x}) \quad (3)$$

so as to ensure, under suitable conditions, stability in the noiseless case as well as finite  $l_\infty$ -induced gain of the disturbance-to-state map from  $\xi$  and  $\tilde{x}$  to  $x$  embedded in (1)-(3). The adopted approach consists of selecting a discrete family  $\mathcal{F} = \{F_h\}_{h=1}^\infty$  of linear state-feedback gains  $F_h$  and a switching logic

$$h(t) = \ell(\hat{x}(t), h(t-1)) \quad (4)$$

in such a way that the regulated plant

$$x(t+1) = \Phi x(t) + G\sigma(F_{h(t)}\hat{x}(t)) + \xi(t) \quad (5)$$

have the stated stability properties.

The paper is organized as follows. Sect. 3 describes the specific type of feedback-gain matrices that are adopted to realize possible control actions. Sect. 4 shows that, thanks to the type of candidate feedback-gain matrices, stability under *arbitrary* switching control can be ensured to an LTI system with no input saturations by only imposing a simple but essential admissibility condition on the switching sequences. Sect. 5 exploits the stability results in Sect. 4 so as to extend them, via the adoption of appropriate switching logic supervisors, to LTI systems subject to input saturations and persistent bounded disturbances of unknown arbitrary magnitude. Sect. 6 reports an examples, while Sect. 7 ends the paper with conclusive remarks.

### 3 Candidate Receding Horizon Feedback-Gains

Consider temporarily the noiseless linear variant of (1)

$$x(t+1) = \Phi x(t) + Gu(t) \quad (6)$$

Assume

$$\Sigma = (\Phi, G) \quad \text{reachable} \quad (7)$$

Note that (7) entails no loss of generality in that, if  $\Sigma$  is stabilizable but not reachable, all subsequent developments apply to the reachable subsystem of the plant obtained via a Gilbert-Kalman reachability decomposition. Because of properties that are motivated next in some detail, the candidate feedback-gains are chosen as follows

$$F_h := -\Psi_u^{-1}G'(\Phi^{h-1})'\mathcal{G}_h^{-1}\Phi^h \quad (8)$$

where:  $h$  is a positive integer,  $h \geq \nu$ ,  $\nu$  the reachability index of  $\Sigma$ ;  $\Psi_u = \Psi_u' > 0$ ; and  $\mathcal{G}_h$  is the  $h$ -order reachability Gramian

$$\mathcal{G}_h := \sum_{k=1}^h \Phi^{k-1}G\Psi_u^{-1}G'(\Phi^{k-1})' \quad (9)$$

An explanation on where (8) stems from is in order. To this end, we consider the following control problem  $\mathcal{P}_h$ .

**Zero terminal-state minimum energy control problem  $\mathcal{P}_h$ .** Let  $x$  be the state of (6) at time 0, and  $\mathcal{U}_h(x)$  the set of all controls  $\omega$  of length  $h$ ,  $\omega = [u'(0) \cdots u'(h-1)]'$  which drive the system state  $x$  to the zero-state  $0_X$  in  $h$  time-steps

$$\mathcal{U}_h(x) := \left\{ \omega \in (\mathbb{R}^m)^h : x(h) = 0_X \right\} \quad (10)$$

where  $x(h) = \Phi^h x + \sum_{k=0}^{h-1} \Phi^{h-1-k} G u(k)$ . Note that  $\mathcal{U}_h(x) \neq \emptyset, \forall x \in \mathbb{R}^n$ , if  $h \geq \nu$ .  $\mathcal{P}_h$  consists of finding  $u_h(x)$ , the element in  $\mathcal{U}_h(x)$  of minimum energy

$$\sum_{k=0}^{h-1} u'(k) \Psi_u u(k) = \omega' \widehat{\Psi}_u \omega \quad (11)$$

where  $\Psi_u = \Psi'_u > 0$  and  $\widehat{\Psi}_u := \text{Diag} \{ \Psi_u, \dots, \Psi_u \}$ -( $h$ -times). For  $h \geq \nu$ , such an element  $u_h(x)$  is as follows

$$\begin{aligned} u_h(x) &:= [u'_h(0|x), \dots, u'_h(h-1|x)]' \\ &= [\mathcal{F}'_h(0) \cdots \mathcal{F}'_h(h-1)]' x \\ &= \mathcal{F}_h x \end{aligned} \quad (12)$$

$$\mathcal{F}_h := -\widehat{\Psi}_u^{-1} \mathcal{R}'_h \mathcal{G}_h^{-1} \Phi^h \quad (13)$$

where  $\mathcal{R}_h$  is the  $h$ -order reachability matrix

$$\mathcal{R}_h := [\Phi^{h-1} G | \dots | \Phi G | G] \quad (14)$$

The integer  $h$  is referred to as the *control horizon* associated to  $\mathcal{P}_h$ . ■

Note that  $F_h$  in (8) is given in terms of  $\mathcal{F}_h$  as follows

$$F_h = [I_m \quad 0_{m \times m(h-1)}] \mathcal{F}_h \quad (15)$$

Hence,  $F_h$  is recognized to be the feedback-gain matrix of the receding horizon regulator (Kwon & Pearson, 1975) associated to problem  $\mathcal{P}_h$ .

From (12) and (13) it follows that

$$u'_h(x) \widehat{\Psi}_u u_h(x) = \|x\|_h^2 := x' P(h) x \quad (16)$$

where

$$\begin{aligned} P(h) &= (\Phi^h)' \mathcal{G}_h^{-1} \Phi^h \\ &= P'(h) \geq 0 \end{aligned} \quad (17)$$

It can be seen (Mosca 1995) that, if  $P(\nu)$  is as in (17) for  $h = \nu$ , then for  $h \geq \nu$ ,  $P(h+1)$  satisfies the Riccati difference equation

$$P(h+1) = \Phi' P(h) \Phi - \Phi' P(h) G (\Psi_u + G' P(h) G)^{-1} G' P(h) \Phi \quad (18)$$

and

$$F_{h+1} = -[\Psi_u + G' P(h) G]^{-1} G' P(h) \Phi \quad (19)$$

In addition,

$$P(h+1) \leq P(h) \quad (20)$$

From (17) and (20), it follows that the following limit exists

$$\lim_{h \rightarrow \infty} P(h) =: P(\infty) \geq 0 \quad (21)$$

Further, if (6) is ANCBI (asymptotically null-controllable with bounded input), *viz.* (Sussmann *et al.*, 1994), (6) is stabilizable and has no exponentially unstable eigenvalue,

$$P(\infty) = 0_{n \times n} \quad (22)$$

with rate of convergence faster than or equal to  $1/h$ .

Let

$$M_h(x) := \max \left\{ \frac{[u_h(k|x)]_i}{v_i} : v_i = -\underline{u}_i, \bar{u}_i; k+1 \in \overleftarrow{h}; i \in \overleftarrow{m} \right\} \quad (23)$$

where  $[u]_i$  denotes the  $i$ -th component of the vector  $u$ . Note that the whole sequence  $u_h(x)$  stays within the linear range of  $\sigma(\cdot)$  if and only if  $M_h(x) \leq 1$ . It follows from (16) and (22) that for an ANCBI system for any  $x \in \mathbb{R}^n$  it is possible to find a large enough horizon  $h$  so as to satisfy  $M_h(x) \leq 1$ .

The considerations in the remark that follows play a significant role in the proofs of Theorem 3 and Theorem 4.

**Remark 1** - Consider the orthogonal decomposition

$$\mathbb{R}^n = \mathcal{R}((\Phi^n)') \oplus \mathcal{N}(\Phi^n) \quad (24)$$

where  $\mathcal{N}(\Phi^n) = \mathcal{N}(\Phi^h)$ ,  $\forall h \geq n$ , and  $\mathcal{R}(\cdot)$  and  $\mathcal{N}(\cdot)$  denote range-space and, respectively, null-space. Accordingly, let  $x = x_r + x^\perp$ , with  $x_r \in \mathcal{R}((\Phi^n)')$  and  $x_r' x^\perp = 0$ . Notice that  $\|x\|_h = \|x_r\|_h$ . If  $\Phi = S$ ,  $S$  a stability matrix, as  $h \rightarrow \infty$  one has

$$\underline{U} \underline{\lambda}_S^h \|x_r\| \leq M_h(x) \leq \overline{U} \bar{\lambda}_S^h \|x_r\| \quad (25)$$

where  $\underline{U}, \overline{U} \in (0, \infty)$ , and  $\bar{\lambda}_S$  ( $\underline{\lambda}_S$ ) denotes the maximum (the nonzero minimum) among the moduli of the  $S$ -eigenvalues.

Let  $\Phi = Q$ , where  $Q$  has all its eigenvalues on the unit circle. From (13) and the fact that  $Q$  is similar to a upper triangular matrix of Jordan blocks associated to eigenvalues of unitary modulus, it follows that, as  $h \rightarrow \infty$ ,

$$M_h(x) = V(x)[1 + O(h^{-1})]h^{-p(x)}\|x\| \leq \overline{M} h^{-1}\|x\| \quad (26)$$

where  $O(h^{-1})$  stands for a quantity of the order of  $h^{-1}$  or that vanishes at a faster rate, while both  $V(x)$  and  $p(x)$  only depend on  $x/\|x\|$ . Further,  $V(x)$  and  $\overline{M}$  are positive reals, and  $p(x)$  an integer which takes values in  $\overline{\mu}$ , where  $\mu$  denotes the maximum among the dimensions of the Jordan-blocks associated to the  $Q$ -eigenvalues. ■

In the remaining part of this paper, particular attention will be devoted to systems algebraically equivalent to the ones with transition-matrices having the following block-diagonal structure

$$x(t+1) = \begin{bmatrix} S & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} s(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} G_s \\ G_q \end{bmatrix} \sigma(u(t)) + \begin{bmatrix} \xi_s(t) \\ \xi_q(t) \end{bmatrix} \quad (27)$$

where:  $x = [s' q']'$ ;  $dim s = dim S$ ;  $\xi_s(\cdot)$  and  $\xi_q(\cdot)$  are bounded disturbances;  $S$  a stability matrix; and  $Q$  has all its eigenvalues of unit modulus with arbitrary geometric multiplicities. The relevance of (27) is that a generic ANCBI system is algebraically equivalent to a system with a block-diagonal structure as (27) with  $(Q, G_q)$  reachable. A class of ANCBI systems, sometimes considered in the literature (Hou *et al.*, 1998; Saberi *et al.*, 2000), is the one wherein the stable modes are affected by the neutrally stable ones, while the latter are only affected by the controls, and the disturbances only enter the stable ones

$$z(t+1) = \begin{bmatrix} A & \tilde{A} \\ 0 & \mathcal{A} \end{bmatrix} \begin{bmatrix} z_s(t) \\ z_q(t) \end{bmatrix} + \begin{bmatrix} B \\ \mathcal{B} \end{bmatrix} \sigma(u(t)) + \begin{bmatrix} \omega_s(t) \\ 0 \end{bmatrix} \quad (28)$$

where:  $z = [z'_s z'_q]'$ ;  $dim z_s = dim A$ ;  $\omega_s(t)$  a disturbance;  $A$  a stability matrix; and  $\mathcal{A}$  has all its eigenvalues of unit modulus with arbitrary geometric multiplicities. Systems of the form (28) can be shown to be algebraically equivalent to systems as in (27) with  $dim S = dim A$  and  $\xi_q(t) \equiv 0$ : a particular instance considered in Theorem 3.

**Remark 2** - Assume that the linear (unsaturated) variant of (27) be reachable by the input  $\sigma(u) = u$ . Because (12) is linear in  $x$ ,

$$u_h(x) = u_h^s(s) + u_h^q(q) \quad (29)$$

and, by (27),

$$u_h^s(k|s) := -\Psi_u^{-1} G' (\Phi^{h-k-1})' \mathcal{G}_h^{-1} S^h s \quad (30)$$

where  $k+1 \in \overleftarrow{h}$ ,  $\Phi$  denotes the state-transition matrix of (27), and similar definition for  $u_h^q(q)$ .

Let  $\|q\| = 0$ . In such a case, lower and upper-bounds similar to the ones in (25) can be established, *viz.*,

$$\underline{U} \underline{\lambda}_S^h \|s_r\| \leq M_h([s' 0']') \leq \overline{U} \overline{\lambda}_S^h \|s_r\| \quad (31)$$

where  $s_r$  is the orthogonal projection of  $s$  onto  $\mathcal{R}((S^n)')$ .

Let  $\|q\| > 0$ . By extending to the present case (26), one finds that, as  $h \rightarrow \infty$ ,

$$M_h(x) = V(q)[1 + O(h^{-1})]h^{-p(q)}\|q\| \leq \overline{M} h^{-1}\|q\| \quad (32)$$

where both  $V(q)$  and  $p(q)$  only depend on  $q/\|q\|$ . Further,  $V(q)$  and  $\overline{M}$  are positive reals and  $p(q)$  an integer which takes values in  $\overleftarrow{\mu}$ , where  $\mu$  denotes the maximum dimension of the Jordan-blocks associated to the  $Q$ -eigenvalues. ■

In the sequel, our attention will be focused on the family of state-feedback gain-matrices

$$\mathcal{F} := \{F_h\}_{h=\underline{h}}^\infty, \quad \underline{h} \geq \nu \quad (33)$$

along with the system (6) under a time-varying state-feedback control  $u(t) = F_{h(t)}x(t)$ ,  $F_{h(t)} \in \mathcal{F}$ ,

$$\begin{aligned} x(t+1) &= \Phi_{h(t)}x(t) \\ \Phi_{h(t)} &:= \Phi + GF_{h(t)} \end{aligned} \quad (34)$$

## 4 Exponential stability under arbitrary admissible switching

A control horizon sequence  $\{h(t)\}_{t \in \mathbb{Z}_+}$ ,  $h(t) \in \mathbb{Z}_+$ ,  $h(t) \geq \nu$ , is called *admissible* if

$$h(t+1) \geq h(t) - 1 \quad (35)$$

Let  $\mathcal{S}$  denote the set of all such admissible sequences, and  $(\Sigma, \mathcal{S})$  the system (34) under an arbitrary admissible switching sequence in  $\mathcal{S}$ . The statements that close the present section are proved in (Mosca, 2005).

**Lemma 1** *Along all possible trajectories of  $(\Sigma, \mathcal{S})$ , one has*

$$\|x(t+h(t))\|_{h(t+h(t))} < \|x(t)\|_{h(t)} \quad (36)$$

for all  $x(t)$  such that  $\|x(t)\|_{h(t)} > 0$ . ■

The theorem that follows is the main result of this section and fundamental for the subsequent developments of this paper.

**Theorem 1** *Consider the control system  $(\Sigma, \mathcal{S})$  composed of the LTI reachable plant (6) under a time-varying state-feedback control  $u(t) = F_{h(t)}x(t)$  realized by arbitrary admissible switching sequences in  $\mathcal{S}$ . Then, provided that  $\underline{h} \geq n$  and  $\{h(t)\}_{t \in \mathbb{Z}_+}$  be bounded, viz.*

$$n \leq \underline{h} \leq h(t) \leq \bar{h} \quad (37)$$

with  $\bar{h} < \infty$ ,  $(\Sigma, \mathcal{S})$  is exponentially stable

$$\|x(t)\| \leq M\lambda^t \|x(0)\| \quad (38)$$

with  $M \in (0, \infty)$ , and decaying rate  $\lambda$  depending on  $\bar{h}$  and  $\underline{h}$ . ■

**Remark 3** - It is to be pointed out that (37) encompasses the case of a fixed regulation horizon  $h$ ,  $\underline{h} = \bar{h} = h \geq n$ , for which, to the best of the author's knowledge, exponential stability of (34) for an arbitrary  $h$  has remained so far an open problem (Mosca, 1995). ■

**Remark 4** - The admissibility condition (35) turns out to be not only sufficient for the stability property stated in Theorem 1, but, in a wide sense, also necessary, if no other condition is imposed. In fact, there are cases wherein, if (35) is violated, stability is lost even if  $\bar{h}$  remains finite. There are, of course, ways other than (35) to ensure stability under arbitrary state-gain switching, *e.g.*, the system state can be suitably processed at any switching time as indicated in (Hespanha & Morse, 2002). ■

## 5 Hysteresis switching regulation

**Noiseless case** - Consider the noiseless variant of (1)

$$x(t+1) = \Phi x(t) + G\sigma(u(t)) \quad (39)$$

along with (2) and (7). Given the system state  $x$ ,  $u_h(x)$  will denote, as in Sect. 3, the minimum energy input sequence in  $\mathcal{U}_h(x)$ . Consider the following horizon switching logic ( $\underline{h} \geq n$ )

$$h(t) = \begin{cases} \tilde{h}(t), & \text{if } M_{\tilde{h}(t)}(x(t)) \leq 1 \\ \hat{h}(t), & \text{otherwise.} \end{cases} \quad (40)$$

$$\tilde{h}(t) := \max\{\underline{h}, h(t-1) - 1\}$$

$$\hat{h}(t) := \min\{h \in \mathbb{Z}_+ : h \geq h(t-1); M_h(x(t)) \leq 1\}$$

with  $t = 1, 2, \dots$ ,  $h(-1) = \underline{h}$ , and  $M_h(x)$  as in (23). Then, the following theorem, whose proof can be found in (Mosca, 2005), follows.

**Theorem 2** *Consider the noiseless input-saturated system (39), (2) and (7) with  $u(t) = F_{h(t)}x(t)$  subject to the feedback switching logic (40). Assume that the initial system state  $x(0)$  at time 0 be such that  $h(0)$  exist finite. Then, logic (40) yields the admissible switching sequence  $h(t+1) = h(t) - 1$ ,  $t = 0, \dots, h(0) - \underline{h}$ ,  $h(t) = \underline{h}$ ,  $\forall t \geq h(0) - \underline{h}$ , the resulting switched system  $x(t+1) = \Phi_{h(t)}x(t)$  satisfies the input saturation constraints, is asymptotically stable, and  $\|x(t)\|$  decays to zero exponentially fast. In particular, if  $\Sigma$  is ANCBI, the resulting switched system is globally asymptotically stable and semi-globally exponentially stable.*

■

**Noisy case** - It is known (Hou *et al.*, 1998; Saberi *et al.*, 2000) that (27) with  $\xi_q(\cdot) \equiv 0$  has the most general structure of an input-saturated LTI system for which it makes sense to consider stability and boundedness under arbitrary  $l_\infty$ -disturbances. Under such circumstances, the argument that follows is used to render plausible the conjecture that  $h(t)$ , chosen by (a suitably modified version of) (40), cannot get unbounded. If this is the case, according to Theorem 1, the contribution of any past noisy sample vanishes exponentially fast, and hence  $x(t)$  stays bounded.

Consider first that, by the switching logic (40),  $h(0) < \infty$  for any  $x(0) \in \mathbb{R}^n$ , and  $\{h(t)\}_{t=0}^\infty$  is in  $\mathcal{S}$ . Next, assume, by contradiction, that  $h(\cdot)$  grows unbounded. This implies that  $\|x(\cdot)\|$  does the same. As  $\|s(\cdot)\|$  is bounded because  $S$  is stable and  $\sigma(u)$  and  $\xi_s$  are both bounded, there are times  $t$  large enough at which  $\|x(t)\|^2 = \|s(t)\|^2 + \|q(t)\|^2 \simeq \|q(t)\|^2$ . Under these circumstances,  $h(t)$  is essentially chosen according to the restricted noiseless system with state  $q(t)$ . Hence, according to Theorem 2, at subsequent times,  $h(t+k) = h(t) - k$  till  $\|q(t+k)\|$  is reduced so much that the effect of  $\|s(t+k)\|$  becomes significant again for the selection of subsequent horizons. Thereafter,  $h(\cdot)$  may start to increase till the condition  $\|x(t)\|^2 \simeq \|q(t)\|^2$  is possibly restored. Consequently, the regulation horizon begins again to decrease by one at each subsequent time. In words, a “horizon resetting” mechanism is inherently enforced. The conjecture is that such a mechanism prevents  $h(\cdot)$ , and hence the plant state, from becoming unbounded. However, in order to prove that the horizon resetting property holds, it is required to replace the switching logic (40) with its variant (41) equipped with a “hysteresis” facility. The latter consists of the hysteresis constant  $\epsilon$  in (41) which ensures that the values taken on by the control input components stay strictly inside the saturation levels whenever the control horizon is greater than  $\underline{h}$  and required not to decrease. Consequently,  $\epsilon > 0$  in (41) makes  $h$  “easier” to decrease than to increase or stay constant.

**Theorem 3** (Mosca, 2005) Consider a generic input-saturated noisy ANCBI system algebraically equivalent to (27) with  $\xi_q(\cdot) \equiv 0$ . Let the control  $u(t) = F_{h(t)}x(t)$  with  $F_h$  as in (8) and  $h(t)$  chosen according to the following hysteresis switching logic ( $\underline{h} \geq n$ )

$$h(t) = \begin{cases} \tilde{h}(t), & \text{if } M_{\tilde{h}(t)}(x(t)) \leq 1 \\ \hat{h}(t), & \text{otherwise.} \end{cases} \quad (41)$$

$$\tilde{h}(t) := \max\{\underline{h}, h(t-1) - 1\}$$

$$\hat{h}(t) := \min\{h \in \mathbb{Z}_+ : h \geq h(t-1); M_h(x(t)) \leq 1 - \epsilon\}$$

where:  $t = 1, \dots$ ;  $\epsilon \in (0, 1)$ ;  $h(0) = \hat{h}(0)$  with  $h(-1) = \underline{h}$ ; and  $M_h(z)$  as in (23). Then, the resulting closed-loop hysteresis switched system is bounded-noise bounded-state  $l_\infty$ -stable irrespective of both the initial state  $x(0) \in \mathbb{R}^n$  and the magnitude of  $\xi_s(\cdot)$ .

Proof. See the Appendix. ■

Theorem 3 ensures  $l_\infty$ -stability irrespective of the magnitude of the state-disturbance provided that the latter does not affect directly the unstable modes of the ANCBI plant. In the case of a generic state-disturbance which can affect directly both the stable and the unstable modes of an ANCBI plant, next Theorem 4 shows that the "horizon resetting" mechanism, and hence  $l_\infty$ -stability, holds true provided that the magnitude of the state-disturbance, directly injected on the critically unstable modes be smaller than a bound dependent on the hysteresis constant  $\epsilon$ .

**Theorem 4** (Mosca, 2005) Consider a generic input-saturated noisy ANCBI system algebraically equivalent to (27) with  $\xi_q(\cdot) \neq 0$  and  $\bar{u} = \underline{u}_i = \bar{u}_i, \forall i \in \bar{m}$ . Let the control  $u(t) = F_{h(t)}x(t)$  with  $F_h$  as in (8) and  $h(t)$  chosen according to the hysteresis switching logic (41). Then, the resulting closed-loop hysteresis switched system is bounded-noise bounded-state  $l_\infty$ -stable irrespective of the initial state  $x(0) \in \mathbb{R}^n$  provided that,  $\forall t \in \mathbb{Z}_+$ ,

$$\|\xi_q(t)\| < \epsilon(1 - \epsilon)/\bar{M} \quad (42)$$

where  $\epsilon$  is the hysteresis constant in (41) and  $\bar{M}$  is as in (26).

Proof. See the Appendix. ■

**Remark 5** - The switching logic (41) ((40)) can be simplified with no change in the conclusions of Theorem 3 and Theorem 4 (Theorem 2) by trading-off performance vs. memory and/or computational complexity, e.g., by replacing  $\hat{h}(t)$  in (41) [(40)] with

$$\check{h}(t) := \min\{h \in \mathbb{H} \cup \{h(t-1)\} : h \geq h(t-1); M_h(x(t)) \leq 1 - \epsilon\} \quad (43)$$

where  $\mathbb{H}$  is an assigned subset of  $\mathbb{Z}_+$ , e.g.,  $\mathbb{H} = \{\underline{h}, \underline{h} + \Delta h, \underline{h} + 2\Delta h, \dots\}$ , with  $\Delta h$  a positive integer. With such a choice,  $h(\cdot)$  can only increase by a multiple of  $\Delta h$  at a time. On the other hand, according to (35),  $h(\cdot)$  can only decrease by one unit at a time. In this connection, one could avoid to hold in the computer memory all feedback-gain matrices  $\mathcal{F}_h, \forall h \in \{\underline{h}, \dots, \bar{h}\}$ , by exploiting the fact that, as can be shown,  $u_{h-1}(k|\Phi_h x) = u_h(k+1|x)$  or equivalently

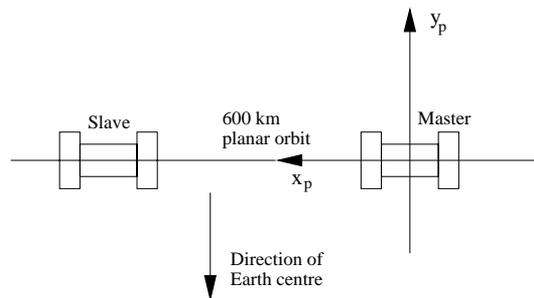
$$\mathcal{F}_{h-1}(k) = \mathcal{F}_h(k+1)\Phi_h^{-1} \quad (44)$$

$k = 0, 1, \dots, h-2$ , provided that  $\Phi_h$  be nonsingular. ■

**Partial state information -** If in the hysteresis switching logic (41) the true plant state  $x(t)$  is replaced by the vector  $x(t) + [\zeta'(t) 0']'$  where  $\zeta(t) \in \mathbb{R}^{n_s}$  is a bounded sensor-noise acting on the stable component of the  $x$ -state, it is immediate to see that the conclusions of Theorem 3 hold true. This implies that the statement of Theorem 3 can be extended to an ANCB plant algebraically equivalent to (27) under the hysteresis switching logic (41) based on a partial state information,  $\hat{x}(t) = [\hat{s}'(t) q'(t)]'$  where  $\hat{s}(t)$  is a filtered-estimate of  $s(t)$  based on observations  $y(t) = [\gamma'(t) q'(t)]'$  with  $\gamma(t) = Hs(t) + n(t) \in \mathbb{R}^p$  with  $n(\cdot)$  a bounded noise. More generally, if a bounded sensor-noise acts also on the unstable component of the  $z$ -state, Theorem 4 can be used so as to estimate the noise magnitude below which boundedness can be guaranteed.

## 6 An Example

The reader is referred to (Bacconi, Casavola & Mosca, 2003) for more details on the example reported hereafter concerning a multiple-input satellite formation problem. Formation flying of satellites is currently an active area of research. The simplest problem in formation flying control is the formation keeping in which the formation is composed by two spacecrafts. In such a case, the so called “master/slave” approach can be used to maintain a desired position of the slave satellite with respect to the master satellite. Because the input forces are subject to saturation, MBPC strategies can be considered particularly effective to that end. A standard approach to the problem of formation flying control is to linearize the dynamics of the satellites around some reference orbit. In the case that the reference orbit is approximately circular and for small distances (less than 1 km) between the two vehicles, the orbit and position of a pair of nano-satellites can be schematized as in Fig. 6. From the related linearized equations of motion we get



**Figure 1: Position of the slave satellite with respect to the master.**

the following state space equations of the relative motion of the slave spacecraft:

$$\dot{x} = \Phi_c x + G_c(u + d)$$

where

$$x = [x_p \quad \dot{x}_p \quad y_p \quad \dot{y}_p]'$$

$$\Phi_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2\omega \\ 0 & 0 & 0 & 1 \\ 0 & 2\omega & 3\omega^2 & 0 \end{bmatrix} \quad G_c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Here:  $x_p, y_p$  are the relative-coordinates of the slave satellite;  $u = [u_{x_p} \ u_{y_p}]'$  are the actuator forces in the  $x_p$  and  $y_p$  positive axis directions;  $d = [d_{x_p} \ d_{y_p}]'$  are the relative disturbances. Further,  $\omega = 0.001086 \text{ rad/s}$  is the assumed angular velocity of the master satellite in the inertial frame. With a sampling period  $T$ , the ZOH sampled dynamical model of the slave satellite takes the form

$$x(t+1) = \Phi x(t) + Gu(t) + \xi(t) \quad (45)$$

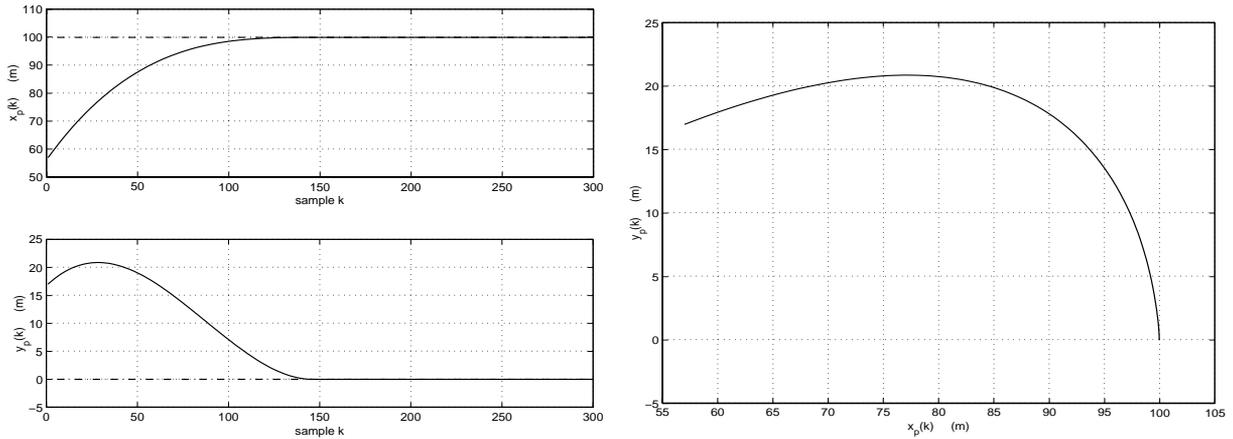
In (45),  $\xi(t)$  represents relative disturbance forces accumulated during a sampling period of duration  $T$ . At any sample time, a measure  $\hat{x}(t)$  of the state is available:  $\hat{x}(t) = x(t) + \tilde{x}(t)$  with  $\tilde{x}(t)$  the measurement error. The objective of the control is to drive the slave vehicle to the desired relative position  $(\bar{x}_p, \bar{y}_p)$ . Subsequently, the relative position has to be maintained in spite of the presence of disturbances and state measurement errors, with an accuracy of at least 1 cm. Manoeuvres are made thanks to a combination of small jet actuators able to produce any force components within a given square rectangular region  $U = [-u_m, u_m] \times [-u_m, u_m]$ . Hence, equation (45) has to be replaced by

$$x(t+1) = \Phi x(t) + G\sigma(u(t)) + \xi(t)$$

We apply the control procedure of this paper to the problem of formation flying control of a pair of nano-satellites of masses equal to 10 kg. We consider a sampling period  $T = 0.3 \text{ s}$ . Note that the discrete-time system (45) is ANCB. The disturbance set is assumed to be spherical with radius  $\xi_0 = 0.0025$ . The objective of the control is to keep the slave craft in a ball set centered on  $(\bar{x}_p, \bar{y}_p) = (100, 0)$  with radius 0.01 m. The input set  $U$  is assumed to be an axis-aligned rectangle with opposite corners at  $[-0.15, -0.15]$ ,  $[0.15, 0.15]$ .

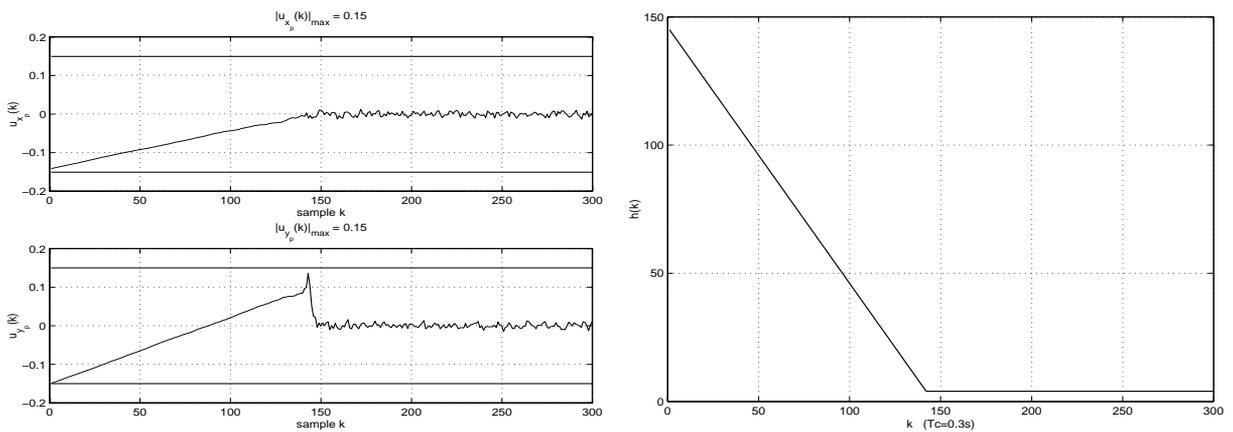
The following figures show simulation results for an initial condition  $x(0) = [57 \ 3 \ 17 \ 1]'$ . Stability and positioning accuracy is good in spite of the large ( $\leq 0.002$ ) random errors in the state vector estimates. We observe that the desired position of the slave satellite is obtained in less than 150 time steps, that is about 45 s (Fig. 3(a)). Thereafter, the formation keeping is achieved, with an accuracy of less than 1 cm. Fulfillment of the input saturation constraints is highlighted by Fig. 4(a). The results are comparable or better than others obtained by alternative techniques. The evolution of the regulation horizon is depicted in Fig. 4(b). Notice that, because of the initial state and the input saturation constraints, the initial value of the required regulation horizon is large: about 150 time steps. It guarantees that all the predicted samples of the input sequence satisfy the constraints. Afterwards, the value of  $h(t)$  decreases by one at each sampling time and reaches the dimension of the system.

**Figure 2:**



(a) Evolution of the relative position of the slave satellite with respect to the master. (b) Trajectory of the slave satellite with respect to the master.

**Figure 3:**



(a) Input forces to the slave spacecraft.

(b) Control horizon evolution.

## 7 Conclusions

The paper provides, relatively to alternative approaches, a computationally affordable solution to the regulation problem of discrete-time input-saturated LTI systems subject to persistent bounded disturbances of unknown arbitrary magnitude. The proposed solution enjoys the following features: It consists of a supervisory switching control logic whereby a feedback-gain, selected at any time from a family of pre-designed candidate feedback-gains, is switched-on in feedback to the plant according to the information, either complete or partial, on the current plant state; No disturbance upper-bound need to be known; The controller selection is made in accordance with a predictive control philosophy, and each candidate feedback-gain is tuned on to a different horizon in a receding-horizon control sense; The supervisory switching logic is flexible enough so as to enable the designer to simplify the scheme by trading off performance vs. memory and/or computational complexity while retaining the required stability properties. It is proved that the adopted switching logic ensures global asymptotic and semi-global exponential stability in the ideal noiseless case, and finite  $l_\infty$ -induced gain to the disturbance-to-state map, whenever the structure of the disturbed plant can make such properties conceptually achievable, *viz.*, the disturbance which enters an ANCBI system does not act directly on the critically unstable modes, the latter being only indirectly affected by the disturbance via the feedback controls. If this is not the case, a hysteresis facility in the switching logic becomes essential for handling disturbances of small enough magnitude which directly affect the critically unstable modes of the plant.

For unstable (non-ANCBI) noisy systems with input saturations, finite-gain stabilization is a rather intricate matter. In such a case, it is known that no global or semi-global finite-gain stabilization can be achieved. In particular, suitability of our approach for that case appears yet unexplored as in the present paper our methods of proof mainly rely on property (22) whose validity is limited to ANCBI systems. The approach adopted in this paper lies at the crossroad of different control methods: constrained MBPC whereby an optimization problem is numerically solved in real-time within each sampling interval; constraint management via a reference-governor unit (Gilbert & Kolmanovsky, 2002) whereby a fixed feedback compensator is used, and, hence, the on-line optimization is restricted to only re-modulate the shapes of the set-point changes; and switching supervisory control. In this paper robustness against persistent bounded disturbances of unknown arbitrary magnitude is acquired, in contrast with standard MBPC, at the cost of giving up on-line optimization (and, hence, full performance), while circumventing the bottleneck, present in reference governor methods, of a fixed preselected compensator, by switching among precomputed candidate linear feedback-gains.

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