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Form Approved
OMB No. 0704-0188

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1. REPORT DATE (DD-MM-YYYY) 11-07-2007		2. REPORT TYPE Conference Paper POSTPRINT		3. DATES COVERED (From - To) 2007	
4. TITLE AND SUBTITLE Cost Cumulant-Based Control for a Class of Linear Quadratic Tracking Problems				5a. CONTRACT NUMBER	
				5b. GRANT NUMBER	
				5c. PROGRAM ELEMENT NUMBER	
6. AUTHOR(S) Khanh D. Pham				5d. PROJECT NUMBER	
				5e. TASK NUMBER	
				5f. WORK UNIT NUMBER	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Air Force Research Laboratory Space Vehicles Directorate 3550 Aberdeen Ave SE Kirtland AFB, NM 87117-5776				8. PERFORMING ORGANIZATION REPORT NUMBER AFRL-VS-PS-TP-2007-1015	
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)				10. SPONSOR/MONITOR'S ACRONYM(S) AFRL/VSSV	
				11. SPONSOR/MONITOR'S REPORT NUMBER(S)	
12. DISTRIBUTION / AVAILABILITY STATEMENT Approved for public release; distribution is unlimited. (Clearance #VS06-0333)					
13. SUPPLEMENTARY NOTES Published in the 2007 American Control Conferences, 11-13 Jul 07, New York, NY					
14. ABSTRACT The topic of cost-cumulant control is currently receiving substantial research from the theoretical community oriented toward stochastic control theory. For instance, the present paper extends the application of cost-cumulant controller design to control of a wide class of linear-quadratic tracking systems where output measurements of a tracker follow as closely as possible a desired trajectory via a complete statistical description of the associated integral-quadratic performance-measure. It is shown that the tracking problem can be solved in two parts: one, a feedback control whose optimization criterion representing a linear combination of finite cumulant indices of an integral-quadratic performance-measure associated to a linear tracking stochastic system over a finite horizon, is determined by a set of Riccati-type differential equations; and two, an affine control which takes into account of dynamics mismatched between a desired trajectory and tracker states, is found by solving an auxiliary set of differential equations (incorporating the desired trajectory) backward from a stable final time.					
15. SUBJECT TERMS Cost-cumulant control; stochastic control; linear-quadratic tracking					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT Unlimited	18. NUMBER OF PAGES 7	19a. NAME OF RESPONSIBLE PERSON Khanh D. Pham
a. REPORT Unclassified	b. ABSTRACT Unclassified	c. THIS PAGE Unclassified			19b. TELEPHONE NUMBER (include area code) 505-846-4823

Cost Cumulant-Based Control for a Class of Linear Quadratic Tracking Problems

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Abstract—The topic of cost-cumulant control is currently receiving substantial research from the theoretical community oriented toward stochastic control theory. For instance, the present paper extends the application of cost-cumulant controller design to control of a wide class of linear-quadratic tracking systems where output measurements of a tracker follow as closely as possible a desired trajectory via a complete statistical description of the associated integral-quadratic performance-measure. It is shown that the tracking problem can be solved in two parts: one, a feedback control whose optimization criterion representing a linear combination of finite cumulant indices of an integral-quadratic performance-measure associated to a linear tracking stochastic system over a finite horizon, is determined by a set of Riccati-type differential equations; and two, an affine control which takes into account of dynamics mismatched between a desired trajectory and tracker states, is found by solving an auxiliary set of differential equations (incorporating the desired trajectory) backward from a stable final time.

I. PRELIMINARIES

An interesting extension of the cost-cumulant control theory [4]-[7] when both perfect and noisy state measurements are available, is to make a linear stochastic system track as closely as possible a desired trajectory via a complete statistical description of the associated finite-horizon integral-quadratic performance-measure. To the best knowledge of the author, this theoretical development appears to be the first of its kind and the optimal control problem being considered herein is actually quite general, and will enable control engineer not only to penalize for variations in, as well as for the levels of, the state variables and control variables, but also to characterize the probabilistic distribution of the performance-measure as needed in post controller-design analysis. Since this problem formulation is parameterized both by the number of cumulants and by the scalar coefficients in the linear combination, it defines a very general Linear-Quadratic-Gaussian (LQG) and Risk Sensitive problem classes. The special cases where only the first cost cumulant is minimized and whereas a denumerable linear combination of cost cumulants is minimized are, of course, the well known minimum-mean LQG problem and the Risk Sensitive control objective, respectively. Some practical applications for this theoretical development can be found in the references [2] and [3] where in tactical and

combat situations, a vehicle with the goal seeking nature initially decides on an appropriate destination and then moves in an optimal fashion toward that destination, and tracking problems in economic stabilization policy, respectively.

Consider a linear stochastic tracking system governed by

$$\begin{aligned} dx(t) &= (A(t)x(t) + B(t)u(t))dt + G(t)dw(t), \quad x(t_0) \quad (1) \\ y(t) &= C(t)x(t) \quad (2) \end{aligned}$$

where the deterministic coefficients $A \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times n})$, $B \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times m})$, $C \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{r \times n})$, and $G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n \times p})$. The system noise $w(t) \in \mathbb{R}^p$ is the p -dimensional stationary Wiener process starting from t_0 , independent of the known $x(t_0) = x_0$, and defined with $\{\mathcal{F}_{t \geq 0}\}$ being its filtration on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_{t \geq 0}\}, \mathcal{P})$ over $[t_0, t_f]$ with the correlation

$$E \{ [w(\tau) - w(\xi)][w(\tau) - w(\xi)]^T \} = W|\tau - \xi|, \quad W > 0.$$

The control input $u \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^m))$ the subset of Hilbert space of \mathbb{R}^m -valued square-integrable process on $[t_0, t_f]$ that are adapted to the σ -field \mathcal{F}_t generated by $w(t)$ to the specified system model is selected so that the measurement output $y \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^r))$ best matches the desired output $z \in L^2([t_0, t_f]; \mathbb{R}^r)$ in the cost cumulant optimization criterion which will be clear shortly. Associated with the initial condition $(t_0, x_0; u) \in [t_0, t_f] \times \mathbb{R}^n \times L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^m))$ is a traditional finite-horizon IQF random cost $J : [t_0, t_f] \times \mathbb{R}^n \times L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^m)) \mapsto \mathbb{R}^+$ such that

$$\begin{aligned} J(t_0, x_0; u) &= [z(t_f) - y(t_f)]^T Q_f [z(t_f) - y(t_f)] \quad (3) \\ &+ \int_{t_0}^{t_f} \{ [z(\tau) - y(\tau)]^T Q(\tau) [z(\tau) - y(\tau)] + u^T(\tau) R(\tau) u(\tau) \} d\tau \end{aligned}$$

in which the terminal penalty error weighting $Q_f \in \mathbb{R}^{r \times r}$, the error weighting $Q \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{r \times r})$, and the control input weighting $R \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times m})$ are deterministic, symmetric, and positive semi-definite with $R(t)$ invertible.

In view of the linear system (1)-(2) and the quadratic cost (3), it is reasonable to assume the control input being generated from a class of linear-memoryless state-feedback strategies $\gamma : [t_0, t_f] \times L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^n)) \mapsto L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^m))$, has a form of

$$u(t) = \gamma(t, x(t)) = K(t)x(t) + u_z(t), \quad (4)$$

where $u_z \in \mathcal{C}([t_0, t_f]; \mathbb{R}^m)$ is an additional control signal which takes into consideration for dynamics mismatched between the tracking states $x(t)$ and the desired trajectory

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$z(t)$ on $[t_0, t_f]$ and $K \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times n})$ is an admissible feedback gain in a sense to be specified later. Hence, for the given initial condition $(t_0, x_0) \in [t_0, t_f] \times \mathbb{R}^n$ and subject to the control policy (4), the dynamics of the tracking problem are governed by

$$dx(t) = [A(t) + B(t)K(t)]x(t)dt + B(t)u_z(t)dt + G(t)dw(t), \quad x(t_0) = x_0, \quad (5)$$

$$y(t) = C(t)x(t), \quad (6)$$

and the IQF random cost

$$J(t_0, x_0; K, u_z) = [z(t_f) - y(t_f)]^T Q_f [z(t_f) - y(t_f)] + \int_{t_0}^{t_f} \left\{ [z(\tau) - y(\tau)]^T Q(\tau) [z(\tau) - y(\tau)] + [K(\tau)x(\tau) + u_z(\tau)]^T R(\tau) [K(\tau)x(\tau) + u_z(\tau)] \right\} d\tau. \quad (7)$$

It is now necessary to develop a procedure for generating some cost cumulants for the tracking problem. These cost cumulants are then used to form a performance index in the cost-cumulant control optimization. In general, it is suggested that the initial condition (t_0, x_0) should be replaced by any arbitrary pair (α, x_α) . Then, for the given u_z and admissible feedback gain K , the random cost (7) is seen as the ‘‘cost-to-go’’, $J(\alpha, x_\alpha)$. The moment-generating function of the vector-valued random process (5) is defined by

$$\varphi(\alpha, x_\alpha; \theta) \triangleq E \{ \exp(\theta J(\alpha, x_\alpha)) \}, \quad (8)$$

where the scalar $\theta \in \mathbb{R}^+$ is a small parameter. Thus, the cumulant-generating function immediately follows

$$\psi(\alpha, x_\alpha; \theta) \triangleq \ln \{ \varphi(\alpha, x_\alpha; \theta) \}, \quad (9)$$

in which $\ln\{\cdot\}$ denotes the natural logarithmic transformation of an enclosed entity.

Theorem 1: Cost-Cumulant Generating Equations.

For $\alpha \in [t_0, t_f]$ and $\theta \in \mathbb{R}^+$, define $\varphi(\alpha, x_\alpha; \theta) \triangleq \varrho(\alpha, \theta) \exp \{ x_\alpha^T \Upsilon(\alpha, \theta) x_\alpha + 2x_\alpha^T \eta(\alpha, \theta) \}$ and $v(\alpha, \theta) \triangleq \ln \{ \varrho(\alpha, \theta) \}$. Then, the cost cumulant-generating function can be expressed as follows

$$\psi(\alpha, x_\alpha; \theta) = x_\alpha^T \Upsilon(\alpha, \theta) x_\alpha + 2x_\alpha^T \eta(\alpha, \theta) + v(\alpha, \theta) \quad (10)$$

where $\Upsilon(\alpha, \theta)$, $\eta(\alpha, \theta)$, and $v(\alpha, \theta)$ solve the backward-in-time differential equations

$$\begin{aligned} \frac{d}{d\alpha} \Upsilon(\alpha, \theta) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T \Upsilon(\alpha, \theta) \\ &\quad - \Upsilon(\alpha, \theta)[A(\alpha) + B(\alpha)K(\alpha)] \\ &\quad - 2\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha, \theta) \\ &\quad - \theta C^T(\alpha)Q(\alpha)C(\alpha) - \theta K^T(\alpha)R(\alpha)K(\alpha), \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{d}{d\alpha} \eta(\alpha, \theta) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T \eta(\alpha, \theta) \\ &\quad - \Upsilon(\alpha, \theta)B(\alpha)u_z(\alpha) \\ &\quad - \theta K^T(\alpha)R(\alpha)u_z(\alpha) + \theta C^T(\alpha)Q(\alpha)z(\alpha), \end{aligned} \quad (12)$$

$$\begin{aligned} \frac{d}{d\alpha} v(\alpha, \theta) &= -\text{Tr} \{ \Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha) \} \\ &\quad - 2\eta^T(\alpha, \theta)B(\alpha)u_z(\alpha) \\ &\quad - \theta u_z^T(\alpha)R(\alpha)u_z(\alpha) - \theta z^T(\alpha)Q(\alpha)z(\alpha) \end{aligned} \quad (13)$$

with the terminal conditions $\Upsilon(t_f, \theta) = \theta C^T(t_f)Q_f C(t_f)$, $\eta(t_f, \theta) = \theta C^T(t_f)Q_f z(t_f)$, $v(t_f, \theta) = \theta z^T(t_f)Q_f z(t_f)$.

Remark. The expression for cost cumulants (10) in the tracking problem indicates that additional second and third affine terms are taking into account of dynamics mismatched in their trajectory-governing equations.

By definition, cost cumulants for the tracking problem can be generated by employing the MacLaurin series expansion for the cumulant-generating function

$$\begin{aligned} \psi(\alpha, x_\alpha; \theta) &= \sum_{i=1}^{\infty} \kappa_i(\alpha, x_\alpha) \frac{\theta^i}{i!}, \\ &= \sum_{i=1}^{\infty} \frac{\partial^{(i)}}{\partial \theta^{(i)}} \psi(\alpha, x_\alpha; \theta) \Big|_{\theta=0} \frac{\theta^i}{i!}, \end{aligned} \quad (14)$$

in which $\kappa_i(\alpha, x_\alpha)$ are called cost cumulants. Furthermore, the series coefficients of the expansion are computed by (10)

$$\begin{aligned} \frac{\partial^{(i)}}{\partial \theta^{(i)}} \psi(\alpha, x_\alpha; \theta) \Big|_{\theta=0} &= x_\alpha^T \frac{\partial^{(i)}}{\partial \theta^{(i)}} \Upsilon(\alpha, \theta) \Big|_{\theta=0} x_\alpha \\ &\quad + 2x_\alpha^T \frac{\partial^{(i)}}{\partial \theta^{(i)}} \eta(\alpha, \theta) \Big|_{\theta=0} + \frac{\partial^{(i)}}{\partial \theta^{(i)}} v(\alpha, \theta) \Big|_{\theta=0}. \end{aligned} \quad (15)$$

In view of the results (14) and (15), cost cumulants for the tracking problem are obtained as follows

$$\begin{aligned} \kappa_i(\alpha, x_\alpha) &= x_\alpha^T \frac{\partial^{(i)}}{\partial \theta^{(i)}} \Upsilon(\alpha, \theta) \Big|_{\theta=0} x_\alpha \\ &\quad + 2x_\alpha^T \frac{\partial^{(i)}}{\partial \theta^{(i)}} \eta(\alpha, \theta) \Big|_{\theta=0} + \frac{\partial^{(i)}}{\partial \theta^{(i)}} v(\alpha, \theta) \Big|_{\theta=0}, \end{aligned} \quad (16)$$

for any finite $1 \leq i < \infty$. For notational convenience, it is necessary to denote $H(\alpha, i) \triangleq \frac{\partial^{(i)}}{\partial \theta^{(i)}} \Upsilon(\alpha, \theta) \Big|_{\theta=0}$, $\check{D}(\alpha, i) \triangleq \frac{\partial^{(i)}}{\partial \theta^{(i)}} \eta(\alpha, \theta) \Big|_{\theta=0}$, and $D(\alpha, i) \triangleq \frac{\partial^{(i)}}{\partial \theta^{(i)}} v(\alpha, \theta) \Big|_{\theta=0}$.

Theorem 2: Cost Cumulants in Tracking Problems.

The tracker dynamics governed by (5)-(6) attempt to track the desired trajectory $z(t)$ with the IQF cost (7). For $k \in \mathbb{Z}^+$ fixed, the k th-cost cumulant of the Chi-square type random cost (7) is given by

$$\begin{aligned} \kappa_k(t_0, x_0; K, u_z) &= x_0^T H(t_0, k) x_0 \\ &\quad + 2x_0^T \check{D}(t_0, k) + D(t_0, k), \end{aligned} \quad (17)$$

where $\{H(\alpha, i)\}_{i=1}^k$, $\{\check{D}(\alpha, i)\}_{i=1}^k$, and $\{D(\alpha, i)\}_{i=1}^k$ evaluated at $\alpha = t_0$ satisfy the matrix- and vector-valued differential equations (with the dependence of $H(\alpha, i)$, $\check{D}(\alpha, i)$, and $D(\alpha, i)$ upon u_z and K suppressed)

$$\begin{aligned} \frac{d}{d\alpha} H(\alpha, 1) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T H(\alpha, 1) \\ &\quad - H(\alpha, 1)[A(\alpha) + B(\alpha)K(\alpha)] \\ &\quad - C^T(\alpha)Q(\alpha)C(\alpha) - K^T(\alpha)R(\alpha)K(\alpha), \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{d}{d\alpha} H(\alpha, i) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T H(\alpha, i) \\ &\quad - H(\alpha, i)[A(\alpha) + B(\alpha)K(\alpha)] \\ &\quad - \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} H(\alpha, j)G(\alpha)WG^T(\alpha)H(\alpha, i-j), \end{aligned} \quad (19)$$

together with

$$\begin{aligned} \frac{d}{d\alpha} \check{D}(\alpha, 1) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T \check{D}(\alpha, 1) \\ &\quad - H(\alpha, 1)B(\alpha)u_z(\alpha) \\ &\quad - K^T(\alpha)R(\alpha)u_z(\alpha) + C^T(\alpha)Q(\alpha)z(\alpha), \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{d}{d\alpha} \check{D}(\alpha, i) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T \check{D}(\alpha, i) \\ &\quad - H(\alpha, i)B(\alpha)u_z(\alpha), \quad 2 \leq i \leq k \end{aligned} \quad (21)$$

and

$$\begin{aligned} \frac{d}{d\alpha} D(\alpha, 1) &= -\text{Tr} \{H(\alpha, 1)G(\alpha)WG^T(\alpha)\} \\ &\quad - 2\check{D}^T(\alpha, 1)B(\alpha)u_z(\alpha) \\ &\quad - u_z^T(\alpha)R(\alpha)u_z(\alpha) - z^T(\alpha)Q(\alpha)z(\alpha), \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{d}{d\alpha} D(\alpha, i) &= -\text{Tr} \{H(\alpha, i)G(\alpha)WG^T(\alpha)\} \\ &\quad - 2\check{D}^T(\alpha, i)B(\alpha)u_z(\alpha), \quad 2 \leq i \leq k \end{aligned} \quad (23)$$

where the terminal conditions $H(t_f, 1) = C^T(t_f)Q_f C(t_f)$, $H(t_f, i) = 0$ for $2 \leq i \leq k$; $\check{D}(t_f, 1) = -C^T(t_f)Q_f z(t_f)$, $\check{D}(t_f, i) = 0$ for $2 \leq i \leq k$ and $D(t_f, 1) = z^T(t_f)Q_f z(t_f)$, $D(t_f, i) = 0$ for $2 \leq i \leq k$.

II. PROBLEM STATEMENTS

In preparing for the control statements of the tracking problem, let k -tuple variables \mathcal{H} , $\check{\mathcal{D}}$, and \mathcal{D} be defined as $\mathcal{H}(\cdot) \triangleq (\mathcal{H}_1(\cdot), \dots, \mathcal{H}_k(\cdot))$, $\check{\mathcal{D}}(\cdot) \triangleq (\check{D}_1(\cdot), \dots, \check{D}_k(\cdot))$, $\mathcal{D}(\cdot) \triangleq (\mathcal{D}_1(\cdot), \dots, \mathcal{D}_k(\cdot))$ for each element $\mathcal{H}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^{n \times n})$ of \mathcal{H} , $\mathcal{D}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^n)$ of $\check{\mathcal{D}}$ and $\mathcal{D}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R})$ of \mathcal{D} having the representations $\mathcal{H}_i(\cdot) \equiv H(\cdot, i)$, $\mathcal{D}_i(\cdot) \equiv \check{D}(\cdot, i)$, and $\mathcal{D}_i(\cdot) \equiv D(\cdot, i)$ with the right members satisfying the dynamic equations (18)-(23) on the horizon $[t_0, t_f]$. The problem formulation is greatly simplified if the convenient mappings are introduced

$$\begin{aligned} \mathcal{F}_i &: [t_0, t_f] \times (\mathbb{R}^{n \times n})^k \times \mathbb{R}^{m \times n} \mapsto \mathbb{R}^{n \times n} \\ \check{\mathcal{G}}_i &: [t_0, t_f] \times (\mathbb{R}^{n \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^{m \times n} \times \mathbb{R}^m \mapsto \mathbb{R}^n \\ \mathcal{G}_i &: [t_0, t_f] \times (\mathbb{R}^{n \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^m \mapsto \mathbb{R} \end{aligned}$$

where the actions are given by

$$\begin{aligned} \mathcal{F}_1(\alpha, \mathcal{H}, K) &\triangleq -[A(\alpha) + B(\alpha)K(\alpha)]^T \mathcal{H}_1(\alpha) \\ &\quad - \mathcal{H}_1(\alpha) [A(\alpha) + B(\alpha)K(\alpha)] \\ &\quad - C^T(\alpha)Q(\alpha)C(\alpha) - K^T(\alpha)R(\alpha)K(\alpha), \\ \mathcal{F}_i(\alpha, \mathcal{H}, K) &\triangleq -[A(\alpha) + B(\alpha)K(\alpha)]^T \mathcal{H}_i(\alpha) \\ &\quad - \mathcal{H}_i(\alpha) [A(\alpha) + B(\alpha)K(\alpha)] \\ &\quad - \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{H}_j(\alpha)G(\alpha)WG^T(\alpha)\mathcal{H}_{i-j}(\alpha), \\ \check{\mathcal{G}}_1(\alpha, \mathcal{H}, \check{\mathcal{D}}, K, u_z) &\triangleq -[A(\alpha) + B(\alpha)K(\alpha)]^T \check{D}_1(\alpha) \\ &\quad - \mathcal{H}_1(\alpha)B(\alpha)u_z(\alpha) - K^T(\alpha)R(\alpha)u_z(\alpha) + C^T(\alpha)Q(\alpha)z(\alpha), \\ \check{\mathcal{G}}_i(\alpha, \mathcal{H}, \check{\mathcal{D}}, K, u_z) &\triangleq -[A(\alpha) + B(\alpha)K(\alpha)]^T \check{D}_i(\alpha) \\ &\quad - \mathcal{H}_i(\alpha)B(\alpha)u_z(\alpha), \end{aligned}$$

$$\begin{aligned} \mathcal{G}_1(\alpha, \mathcal{H}, \check{\mathcal{D}}, u_z) &\triangleq -\text{Tr} \{ \mathcal{H}_1(\alpha)G(\alpha)WG^T(\alpha) \} \\ &\quad - 2\check{D}_1^T(\alpha)B(\alpha)u_z(\alpha) - u_z^T(\alpha)R(\alpha)u_z(\alpha) - z^T(\alpha)Q(\alpha)z(\alpha), \\ \mathcal{G}_i(\alpha, \mathcal{H}, \check{\mathcal{D}}, u_z) &\triangleq -\text{Tr} \{ \mathcal{H}_i(\alpha)G(\alpha)WG^T(\alpha) \} \\ &\quad - 2\check{D}_i^T(\alpha)B(\alpha)u_z(\alpha). \end{aligned}$$

Now there is no difficulty to establish the product mappings

$$\begin{aligned} \mathcal{F}_1 \times \dots \times \mathcal{F}_k &: [t_0, t_f] \times (\mathbb{R}^{n \times n})^k \times \mathbb{R}^{m \times n} \mapsto (\mathbb{R}^{n \times n})^k \\ \check{\mathcal{G}}_1 \times \dots \times \check{\mathcal{G}}_k &: [t_0, t_f] \times (\mathbb{R}^{n \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^{m \times n} \times \mathbb{R}^m \mapsto (\mathbb{R}^n)^k \\ \mathcal{G}_1 \times \dots \times \mathcal{G}_k &: [t_0, t_f] \times (\mathbb{R}^{n \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^m \mapsto \mathbb{R}^k \end{aligned}$$

along with the corresponding notations $\mathcal{F} \triangleq \mathcal{F}_1 \times \dots \times \mathcal{F}_k$, $\check{\mathcal{G}} \triangleq \check{\mathcal{G}}_1 \times \dots \times \check{\mathcal{G}}_k$, and $\mathcal{G} \triangleq \mathcal{G}_1 \times \dots \times \mathcal{G}_k$. Thus, the dynamic equations of motion (18)-(23) can be rewritten as

$$\begin{aligned} \frac{d}{d\alpha} \mathcal{H}(\alpha) &= \mathcal{F}(\alpha, \mathcal{H}(\alpha), K(\alpha)), \quad \mathcal{H}(t_f) = \mathcal{H}_f, \\ \frac{d}{d\alpha} \check{\mathcal{D}}(\alpha) &= \check{\mathcal{G}}(\alpha, \mathcal{H}(\alpha), \check{\mathcal{D}}(\alpha), K(\alpha), u_z(\alpha)), \quad \check{\mathcal{D}}(t_f) = \check{\mathcal{D}}_f, \\ \frac{d}{d\alpha} \mathcal{D}(\alpha) &= \mathcal{G}(\alpha, \mathcal{H}(\alpha), \check{\mathcal{D}}(\alpha), u_z(\alpha)), \quad \mathcal{D}(t_f) = \mathcal{D}_f \end{aligned}$$

where k -tuple values $\mathcal{H}_f = (C^T(t_f)Q_f C(t_f), 0, \dots, 0)$, $\check{\mathcal{D}}_f = (-C^T(t_f)Q_f z(t_f), 0, \dots, 0)$, and $\mathcal{D}_f = (0, \dots, 0)$.

Note that the product system uniquely determines \mathcal{H} , $\check{\mathcal{D}}$ and \mathcal{D} once the admissible affine control u_z and feedback gain K are specified. Hence, they are considered as $\mathcal{H} \equiv \mathcal{H}(\cdot, K)$, $\check{\mathcal{D}} \equiv \check{\mathcal{D}}(\cdot, K, u_z)$, and $\mathcal{D} \equiv \mathcal{D}(\cdot, K, u_z)$. The performance index in the cost-cumulant control problem can now be formulated in u_z and K .

Definition 1: Performance Index.

Fix $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$. Then, for the given (t_0, x_0) , the performance index

$$\phi_{tk} : [t_0, t_f] \times (\mathbb{R}^{n \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^k \mapsto \mathbb{R}^+$$

in cost-cumulant control for the tracking problem is defined as follows

$$\begin{aligned} \phi_{tk} &\left(t_0, \mathcal{H}(t_0), \check{\mathcal{D}}(t_0), \mathcal{D}(t_0) \right) \\ &\triangleq \sum_{i=1}^k \mu_i \left[x_0^T \mathcal{H}_i(t_0) x_0 + 2x_0^T \check{\mathcal{D}}_i(t_0) + \mathcal{D}_i(t_0) \right] \end{aligned} \quad (24)$$

where the scalar, real constants μ_i represent parametric design freedom and levels of influence on the overall cost distribution. The solutions $\{\mathcal{H}_i(t_0) \geq 0\}_{i=1}^k$, $\{\check{\mathcal{D}}_i(t_0)\}_{i=1}^k$ and $\{\mathcal{D}_i(t_0)\}_{i=1}^k$ evaluated at $\alpha = t_0$ satisfy the dynamic equations of motion

$$\begin{aligned} \frac{d}{d\alpha} \mathcal{H}(\alpha) &= \mathcal{F}(\alpha, \mathcal{H}(\alpha), K(\alpha)), \quad \mathcal{H}(t_f) = \mathcal{H}_f, \\ \frac{d}{d\alpha} \check{\mathcal{D}}(\alpha) &= \check{\mathcal{G}}(\alpha, \mathcal{H}(\alpha), \check{\mathcal{D}}(\alpha), K(\alpha), u_z(\alpha)), \quad \check{\mathcal{D}}(t_f) = \check{\mathcal{D}}_f, \\ \frac{d}{d\alpha} \mathcal{D}(\alpha) &= \mathcal{G}(\alpha, \mathcal{H}(\alpha), \check{\mathcal{D}}(\alpha), u_z(\alpha)), \quad \mathcal{D}(t_f) = \mathcal{D}_f. \end{aligned}$$

Definition 2: Affine Control and Feedback Gains.

Let compact subsets $\bar{U} \subset \mathbb{R}^m$ and $\bar{K} \subset \mathbb{R}^{m \times n}$ be the sets of allowable affine inputs and gain values. For the given $k \in \mathbb{Z}^+$

and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$, the set of admissible affine controls $\mathcal{U}_{t_f, \mathcal{H}_f, \check{\mathcal{D}}_f, \mathcal{D}_f; \mu}$ and feedback gains $\mathcal{K}_{t_f, \mathcal{H}_f, \check{\mathcal{D}}_f, \mathcal{D}_f; \mu}$ are respectively assumed to be the classes of $\mathcal{C}([t_0, t_f]; \mathbb{R}^m)$ and $\mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times n})$ with values $u_z(\cdot) \in \bar{\mathcal{U}}$ and $K(\cdot) \in \bar{\mathcal{K}}$ for which solutions to the dynamic equations with $\mathcal{H}(t_f) = \mathcal{H}_f$, $\check{\mathcal{D}}(t_f) = \check{\mathcal{D}}_f$, and $\mathcal{D}(t_f) = \mathcal{D}_f$

$$\frac{d}{d\alpha} \mathcal{H}(\alpha) = \mathcal{F}(\alpha, \mathcal{H}(\alpha), K(\alpha)), \quad (25)$$

$$\frac{d}{d\alpha} \check{\mathcal{D}}(\alpha) = \check{\mathcal{G}}(\alpha, \mathcal{H}(\alpha), \check{\mathcal{D}}(\alpha), K(\alpha), u_z(\alpha)), \quad (26)$$

$$\frac{d}{d\alpha} \mathcal{D}(\alpha) = \mathcal{G}(\alpha, \mathcal{H}(\alpha), \check{\mathcal{D}}(\alpha), u_z(\alpha)) \quad (27)$$

exist on the interval of optimization $[t_0, t_f]$.

Definition 3: Optimization Problem.

Suppose that $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$ are fixed. Then the control optimization problem is defined as the minimization of (24) over $u_z(\cdot) \in \mathcal{U}_{t_f, \mathcal{H}_f, \check{\mathcal{D}}_f, \mathcal{D}_f; \mu}$, $K(\cdot) \in \mathcal{K}_{t_f, \mathcal{H}_f, \check{\mathcal{D}}_f, \mathcal{D}_f; \mu}$ and subject to the dynamic equations of motion (25)-(27) for $\alpha \in [t_0, t_f]$.

Definition 4: Reachable Set.

Let reachable set \mathcal{Q} be defined $\mathcal{Q} \triangleq \left\{ (\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}) \in [t_0, t_f] \times (\mathbb{R}^{n \times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^k \right\}$ such that $\mathcal{U}_{\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}; \mu} \neq \emptyset$ and $\mathcal{K}_{\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}; \mu} \neq \emptyset$.

By adapting to the initial cost problem and the terminologies present in cost-cumulant control, the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the value function is motivated by the excellent treatment [1] and is given below.

Theorem 3: HJB Equation-Mayer Problem.

Let $(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z})$ be any interior point of the reachable set \mathcal{Q} at which the value function $\mathcal{V}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z})$ is differentiable. If there exist optimal affine control $u_z^* \in \mathcal{U}_{\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}; \mu}$ and feedback gain $K^* \in \mathcal{K}_{\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}; \mu}$, then the partial differential equation of dynamic programming

$$\begin{aligned} 0 = & \min_{u_z \in \bar{\mathcal{U}}, K \in \bar{\mathcal{K}}} \left\{ \frac{\partial}{\partial \varepsilon} \mathcal{V}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}) \right. \\ & + \frac{\partial}{\partial \text{vec}(\mathcal{Y})} \mathcal{V}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}) \text{vec}(\mathcal{F}(\varepsilon, \mathcal{Y}, K)) \\ & + \frac{\partial}{\partial \text{vec}(\check{\mathcal{Z}})} \mathcal{V}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}) \text{vec}(\check{\mathcal{G}}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, K, u_z)) \\ & \left. + \frac{\partial}{\partial \text{vec}(\mathcal{Z})} \mathcal{V}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}) \text{vec}(\mathcal{G}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, u_z)) \right\} \quad (28) \end{aligned}$$

is satisfied together with the boundary value condition $\mathcal{V}(t_0, \mathcal{H}_0, \check{\mathcal{D}}_0, \mathcal{D}_0) = \phi_{tk}(t_0, \mathcal{H}_0, \check{\mathcal{D}}_0, \mathcal{D}_0)$.

Theorem 4: Verification Theorem.

Fix $k \in \mathbb{Z}^+$ and let $\mathcal{W}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z})$ be a continuously differentiable solution of the HJB equation (28) which satisfies the boundary condition $\mathcal{W}(t_0, \mathcal{H}_0, \check{\mathcal{D}}_0, \mathcal{D}_0) = \phi_{tk}(t_0, \mathcal{H}_0, \check{\mathcal{D}}_0, \mathcal{D}_0)$. Let $(t_f, \mathcal{H}_f, \check{\mathcal{D}}_f, \mathcal{D}_f)$ be in \mathcal{Q} ; (u_z, K) in $\mathcal{U}_{t_f, \mathcal{H}_f, \check{\mathcal{D}}_f, \mathcal{D}_f; \mu} \times \mathcal{K}_{t_f, \mathcal{H}_f, \check{\mathcal{D}}_f, \mathcal{D}_f; \mu}$; \mathcal{H} , $\check{\mathcal{D}}$ and \mathcal{D} the corresponding solutions of (25)-(27). Then

$\mathcal{W}(\alpha, \mathcal{H}(\alpha), \check{\mathcal{D}}(\alpha), \mathcal{D}(\alpha))$ is a non-increasing function of α . If (u_z^*, K^*) is in $\mathcal{U}_{t_f, \mathcal{H}_f, \check{\mathcal{D}}_f, \mathcal{D}_f; \mu} \times \mathcal{K}_{t_f, \mathcal{H}_f, \check{\mathcal{D}}_f, \mathcal{D}_f; \mu}$ defined on $[t_0, t_f]$ with corresponding solutions, \mathcal{H}^* , $\check{\mathcal{D}}^*$, and \mathcal{D}^* of (25)-(27) such that for $\alpha \in [t_0, t_f]$

$$\begin{aligned} 0 = & \frac{\partial}{\partial \varepsilon} \mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \check{\mathcal{D}}^*(\alpha), \mathcal{D}^*(\alpha)) \\ & + \frac{\partial}{\partial \text{vec}(\mathcal{Y})} \mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \check{\mathcal{D}}^*(\alpha), \mathcal{D}^*(\alpha)) \\ & \quad \cdot \text{vec}(\mathcal{F}(\alpha, \mathcal{H}^*(\alpha), K^*(\alpha))) \\ & + \frac{\partial}{\partial \text{vec}(\check{\mathcal{Z}})} \mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \check{\mathcal{D}}^*(\alpha), \mathcal{D}^*(\alpha)) \\ & \quad \cdot \text{vec}(\check{\mathcal{G}}(\alpha, \mathcal{H}^*(\alpha), \check{\mathcal{D}}^*(\alpha), K^*(\alpha), u_z^*(\alpha))) \\ & + \frac{\partial}{\partial \text{vec}(\mathcal{Z})} \mathcal{W}(\alpha, \mathcal{H}^*(\alpha), \check{\mathcal{D}}^*(\alpha), \mathcal{D}^*(\alpha)) \\ & \quad \cdot \text{vec}(\mathcal{G}(\alpha, \mathcal{H}^*(\alpha), \check{\mathcal{D}}^*(\alpha), u_z^*(\alpha))), \quad (29) \end{aligned}$$

then u_z^* and K^* are optimal. Moreover,

$$\mathcal{W}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}) = \mathcal{V}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}) \quad (30)$$

where $\mathcal{V}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z})$ is the value function.

III. OPTIMAL SOLUTION OF k CC CONTROL

The treatment of HJB approach to obtaining a solution to the cost-cumulant control problem requires to parameterize the terminal time and states of the dynamical equations as $(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z})$ rather than $(t_f, \mathcal{H}_f, \check{\mathcal{D}}_f, \mathcal{D}_f)$. That is, for $\varepsilon \in [t_0, t_f]$ and $1 \leq i \leq k$, the states of the system (25)-(27) defined on the interval $[t_0, \varepsilon]$ have the terminal values denoted by $\mathcal{H}(\varepsilon) = \mathcal{Y}$, $\check{\mathcal{D}}(\varepsilon) = \check{\mathcal{Z}}$, and $\mathcal{D}(\varepsilon) = \mathcal{Z}$. It is observed that the performance index (24) is quadratic affine in terms of the arbitrarily fixed x_0 . This suggests a solution to the HJB equation (28) may be sought as below.

Theorem 5: Fix $k \in \mathbb{Z}^+$ and let $(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z})$ be any interior point of \mathcal{Q} at which the scalar-valued function

$$\begin{aligned} \mathcal{W}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}) = & x_0^T \sum_{i=1}^k \mu_i (\mathcal{Y}_i + \mathcal{E}_i(\varepsilon)) x_0 \\ & + 2x_0^T \sum_{i=1}^k \mu_i (\check{\mathcal{Z}}_i + \check{\mathcal{T}}_i(\varepsilon)) + \sum_{i=1}^k \mu_i (\mathcal{Z}_i + \mathcal{T}_i(\varepsilon)) \quad (31) \end{aligned}$$

is differentiable. The time-varying parametric functions $\mathcal{E}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^{n \times n})$, $\check{\mathcal{T}}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^n)$ and $\mathcal{T}_i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R})$ are yet to be determined. The derivative of $\mathcal{W}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z})$ with respect to ε is given

$$\begin{aligned} \frac{d}{d\varepsilon} \mathcal{W}(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z}) = & x_0^T \sum_{i=1}^k \mu_i \left(\mathcal{F}_i(\varepsilon, \mathcal{Y}, K) + \frac{d}{d\varepsilon} \mathcal{E}_i(\varepsilon) \right) x_0 \\ & + 2x_0^T \sum_{i=1}^k \mu_i \left(\check{\mathcal{G}}_i(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, K, u_z) + \frac{d}{d\varepsilon} \check{\mathcal{T}}_i(\varepsilon) \right) \\ & + \sum_{i=1}^k \mu_i \left(\mathcal{G}_i(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, u_z) + \frac{d}{d\varepsilon} \mathcal{T}_i(\varepsilon) \right), \quad (32) \end{aligned}$$

provided $u_z \in \bar{U}$ and $K \in \bar{K}$.

Replacing (31) into the HJB equation (28), it follows that

$$0 = \min_{u_z \in \bar{U}, K \in \bar{K}} \left\{ x_0^T \sum_{i=1}^k \mu_i \left(\mathcal{F}_i(\varepsilon, \mathcal{Y}, K) + \frac{d}{d\varepsilon} \mathcal{E}_i(\varepsilon) \right) x_0 \right. \\ \left. + 2x_0^T \sum_{i=1}^k \mu_i \left(\check{\mathcal{G}}_i(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, K, u_z) + \frac{d}{d\varepsilon} \check{\mathcal{T}}_i(\varepsilon) \right) \right. \\ \left. + \sum_{i=1}^k \mu_i \left(\mathcal{G}_i(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, u_z) + \frac{d}{d\varepsilon} \mathcal{T}_i(\varepsilon) \right) \right\}. \quad (33)$$

Note that

$$\sum_{i=1}^k \mu_i \mathcal{F}_i(\varepsilon, \mathcal{Y}, K) = -[A(\varepsilon) + B(\varepsilon)K]^T \sum_{i=1}^k \mu_i \mathcal{Y}_i \\ - \sum_{i=1}^k \mu_i \mathcal{Y}_i [A(\varepsilon) + B(\varepsilon)K] - \mu_1 C^T(\varepsilon) Q(\varepsilon) C(\varepsilon) \\ - \mu_1 K^T R(\varepsilon) K - \sum_{i=2}^k \mu_i \sum_{j=1}^{i-1} \frac{2!}{j!(i-j)!} \mathcal{Y}_j G(\varepsilon) W G^T(\varepsilon) \mathcal{Y}_{i-j}, \\ \sum_{i=1}^k \mu_i \check{\mathcal{G}}_i(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, K, u_z) = -[A(\varepsilon) + B(\varepsilon)K]^T \sum_{i=1}^k \mu_i \check{\mathcal{Z}}_i \\ - \sum_{i=1}^k \mu_i \mathcal{Y}_i B(\varepsilon) u_z - \mu_1 K^T R(\varepsilon) u_z + \mu_1 C^T(\varepsilon) Q(\varepsilon) z(\varepsilon), \\ \sum_{i=1}^k \mu_i \mathcal{G}_i(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, u_z) = - \sum_{i=1}^k \mu_i \text{Tr} \{ \mathcal{Y}_i G(\varepsilon) W G^T(\varepsilon) \} \\ - 2 \sum_{i=1}^k \mu_i \check{\mathcal{Z}}_i^T B(\varepsilon) u_z - \mu_1 u_z^T R(\varepsilon) u_z - \mu_1 z^T(\varepsilon) Q(\varepsilon) z(\varepsilon).$$

Since x_0 and M_0 are arbitrary vector and rank-one matrix, the necessary condition for an extremum of (24) on $[t_0, \varepsilon]$ is obtained by differentiating (33) with respect to u_z and K

$$u_z(\varepsilon, \check{\mathcal{Z}}) = -R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \check{\mathcal{Z}}_r, \quad (34)$$

$$K(\varepsilon, \mathcal{Y}) = -R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \mathcal{Y}_r, \quad (35)$$

where $\hat{\mu}_r \triangleq \mu_r / \mu_1$ and $\mu_1 > 0$. Substituting (34) and (35) into (33) leads to the value function

$$x_0^T \left[\sum_{i=1}^k \mu_i \frac{d}{d\varepsilon} \mathcal{E}_i(\varepsilon) - A^T(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_i - \sum_{i=1}^k \mu_i \mathcal{Y}_i A(\varepsilon) \right. \\ \left. - \mu_1 C^T(\varepsilon) Q(\varepsilon) C(\varepsilon) \right. \\ \left. + \sum_{r=1}^k \hat{\mu}_r \mathcal{Y}_r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_i \right. \\ \left. + \sum_{i=1}^k \mu_i \mathcal{Y}_i B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{Y}_s \right. \\ \left. - \mu_1 \sum_{r=1}^k \hat{\mu}_r \mathcal{Y}_r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{Y}_s \right]$$

$$- \sum_{i=2}^k \mu_i \sum_{j=1}^{i-1} \frac{2!}{j!(i-j)!} \mathcal{Y}_j G(\varepsilon) W G^T(\varepsilon) \mathcal{Y}_{i-j} \Big] x_0 \\ + 2x_0^T \left[\sum_{i=1}^k \mu_i \frac{d}{d\varepsilon} \check{\mathcal{T}}_i(\varepsilon) - A^T(\varepsilon) \sum_{i=1}^k \mu_i \check{\mathcal{Z}}_i + \mu_1 C^T(\varepsilon) Q(\varepsilon) z(\varepsilon) \right. \\ \left. + \sum_{r=1}^k \mu_r \mathcal{Y}_r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{i=1}^k \mu_i \check{\mathcal{Z}}_i \right. \\ \left. + \sum_{i=1}^k \mu_i \mathcal{Y}_i B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \check{\mathcal{Z}}_r \right. \\ \left. - \mu_1 \sum_{r=1}^k \hat{\mu}_r \mathcal{Y}_r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \check{\mathcal{Z}}_s \right] \\ + \sum_{i=1}^k \mu_i \frac{d}{d\varepsilon} \mathcal{T}_i(\varepsilon) - \sum_{i=1}^k \mu_i \text{Tr} \{ \mathcal{Y}_i G(\varepsilon) W G^T(\varepsilon) \} \\ + 2 \sum_{i=1}^k \mu_i \check{\mathcal{Z}}_i^T B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \check{\mathcal{Z}}_r - \mu_1 z^T(\varepsilon) Q(\varepsilon) z(\varepsilon) \\ - \mu_1 \sum_{r=1}^k \hat{\mu}_r \check{\mathcal{Z}}_r^T B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \check{\mathcal{Z}}_s. \quad (36)$$

The remaining task is to display time-dependent functions $\{\mathcal{E}_i(\cdot)\}_{i=1}^k$, $\{\check{\mathcal{T}}_i(\cdot)\}_{i=1}^k$, and $\{\mathcal{T}_i(\cdot)\}_{i=1}^k$, which yield a sufficient condition to have the left-hand side of (36) being zero for any $\varepsilon \in [t_0, t_f]$, when $\{\mathcal{Y}_i\}_{i=1}^k$ and $\{\check{\mathcal{Z}}_i\}_{i=1}^k$ are evaluated along solutions to the cumulant-generating equations. A careful observation of (36) suggests that $\{\mathcal{E}_i(\cdot)\}_{i=1}^k$, $\{\check{\mathcal{T}}_i(\cdot)\}_{i=1}^k$ and $\{\mathcal{T}_i(\cdot)\}_{i=1}^k$ can be chosen to satisfy certain differential equations whose explicit representations are omitted herein due to the space limitation. The affine control and feedback gain specified in (34) and (35) are now applied along the solution trajectories of the equations (25)-(27)

$$\frac{d}{d\varepsilon} \mathcal{H}_1(\varepsilon) = -A^T(\varepsilon) \mathcal{H}_1(\varepsilon) - \mathcal{H}_1(\varepsilon) A(\varepsilon) - C^T(\varepsilon) Q(\varepsilon) C(\varepsilon) \\ + \mathcal{H}_1(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) \\ + \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \mathcal{H}_1(\varepsilon) \\ - \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon), \quad (37) \\ \frac{d}{d\varepsilon} \mathcal{H}_i(\varepsilon) = -A^T(\varepsilon) \mathcal{H}_i(\varepsilon) - \mathcal{H}_i(\varepsilon) A(\varepsilon) \\ + \mathcal{H}_i(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \mathcal{H}_s(\varepsilon) \\ + \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \mathcal{H}_i(\varepsilon) \\ - \sum_{j=1}^{i-1} \frac{2!}{j!(i-j)!} \mathcal{H}_j(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \mathcal{H}_{i-j}(\varepsilon), \quad (38)$$

$$\begin{aligned}
\frac{d}{d\varepsilon} \check{\mathcal{D}}_1(\varepsilon) &= -A^T(\varepsilon)\check{\mathcal{D}}_1(\varepsilon) + C^T(\varepsilon)Q(\varepsilon)z(\varepsilon) \\
&+ \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\check{\mathcal{D}}_1(\varepsilon) \\
&+ \mathcal{H}_i(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \check{\mathcal{D}}_r(\varepsilon) \\
&- \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \check{\mathcal{D}}_s(\varepsilon), \quad (39)
\end{aligned}$$

$$\frac{d}{d\varepsilon} \check{\mathcal{D}}_i(\varepsilon) = \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\check{\mathcal{D}}_i(\varepsilon) \quad (40)$$

$$- A^T(\varepsilon)\check{\mathcal{D}}_i(\varepsilon) + \mathcal{H}_i(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \check{\mathcal{D}}_r(\varepsilon),$$

$$\frac{d}{d\varepsilon} \mathcal{D}_1(\varepsilon) = -\text{Tr} \{ \mathcal{H}_1(\varepsilon)G(\varepsilon)WG^T(\varepsilon) \} - z^T(\varepsilon)Q(\varepsilon)z(\varepsilon)$$

$$+ 2\check{\mathcal{D}}_1^T(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \check{\mathcal{D}}_r(\varepsilon)$$

$$- \sum_{r=1}^k \hat{\mu}_r \check{\mathcal{D}}_r^T(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{s=1}^k \hat{\mu}_s \check{\mathcal{D}}_s(\varepsilon), \quad (41)$$

$$\frac{d}{d\varepsilon} \mathcal{D}_i(\varepsilon) = -\text{Tr} \{ \mathcal{H}_i(\varepsilon)G(\varepsilon)WG^T(\varepsilon) \}$$

$$+ 2\check{\mathcal{D}}_i^T(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \check{\mathcal{D}}_r(\varepsilon) \quad (42)$$

where the terminal conditions $\mathcal{H}_1(t_f) = C^T(t_f)Q_f C(t_f)$, $\mathcal{H}_i(t_f) = 0$ for $2 \leq i \leq k$; $\check{\mathcal{D}}_1(t_f) = -C^T(t_f)Q_f z(t_f)$, $\check{\mathcal{D}}_i(t_f) = 0$ for $2 \leq i \leq k$ and $\mathcal{D}_1(t_f) = z^T(t_f)Q_f z(t_f)$, $\mathcal{D}_i(t_f) = 0$ for $2 \leq i \leq k$. The boundary condition of $W(\varepsilon, \mathcal{Y}, \check{\mathcal{Z}}, \mathcal{Z})$ implies that

$$\begin{aligned}
&x_0^T \sum_{i=1}^k \mu_i (\mathcal{H}_{i0} + \mathcal{E}_i(t_0)) x_0 \\
&+ 2x_0^T \sum_{i=1}^k \mu_i (\check{\mathcal{D}}_{i0} + \check{\mathcal{T}}_i(t_0)) + \sum_{i=1}^k \mu_i (\mathcal{D}_{i0} + \mathcal{T}_i(t_0)) \\
&= x_0^T \sum_{i=1}^k \mu_i \mathcal{H}_{i0} x_0 + 2x_0^T \sum_{i=1}^k \mu_i \check{\mathcal{D}}_{i0} + \sum_{i=1}^k \mu_i \mathcal{D}_{i0}.
\end{aligned}$$

Therefore, the extremizing affine control (34) and state-feedback gain (35) minimizing (24) become optimal

$$\begin{aligned}
u_z^*(\varepsilon) &= -R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \check{\mathcal{D}}_r^*(\varepsilon), \\
K^*(\varepsilon) &= -R^{-1}(\varepsilon)B^T(\varepsilon) \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r^*(\varepsilon).
\end{aligned}$$

Theorem 6: Cost-Cumulant Control Solution.

The tracker dynamics governed by (5)-(6) attempt to track the desired trajectory $z(t)$ with the Chi-square random cost (7). Assume both $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$ are fixed. Then, the control solution for the

multi-cumulant tracking problem is implemented by

$$u^*(t) = K^*(t)x^*(t) + u_z^*(t), \quad (43)$$

$$K^*(\alpha) = -R^{-1}(\alpha)B^T(\alpha) \sum_{r=1}^k \hat{\mu}_r \mathcal{H}_r^*(\alpha), \quad (44)$$

$$u_z^*(\alpha) = -R^{-1}(\alpha)B^T(\alpha) \sum_{r=1}^k \hat{\mu}_r \check{\mathcal{D}}_r^*(\alpha), \quad (45)$$

where $\hat{\mu}_r \triangleq \mu_i/\mu_1$ represent different levels of influence as they deem important to the overall cost distribution and $\{\mathcal{H}_r^*(\alpha)\}_{r=1}^k$, and $\{\check{\mathcal{D}}_r^*(\alpha)\}_{r=1}^k$ are the solutions of the backward-in-time Riccati-type matrix differential equations

$$\frac{d}{d\alpha} \mathcal{H}_1^*(\alpha) = -[A(\alpha) + B(\alpha)K^*(\alpha)]^T \mathcal{H}_1^*(\alpha) \quad (46)$$

$$\begin{aligned}
&- \mathcal{H}_1^*(\alpha) [A(\alpha) + B(\alpha)K^*(\alpha)] \\
&- C^T(\alpha)Q(\alpha)C(\alpha) - K^{*T}(\alpha)R(\alpha)K^*(\alpha),
\end{aligned}$$

$$\frac{d}{d\alpha} \mathcal{H}_r^*(\alpha) = -[A(\alpha) + B(\alpha)K^*(\alpha)]^T \mathcal{H}_r^*(\alpha) \quad (47)$$

$$\begin{aligned}
&- \mathcal{H}_r^*(\alpha) [A(\alpha) + B(\alpha)K^*(\alpha)] \\
&- \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!} \mathcal{H}_s^*(\alpha)G(\alpha)WG^T(\alpha)\mathcal{H}_{r-s}^*(\alpha),
\end{aligned}$$

and the auxiliary backward-in-time vector-valued differential equations

$$\frac{d}{d\alpha} \check{\mathcal{D}}_1^*(\alpha) = -[A(\alpha) + B(\alpha)K^*(\alpha)]^T \check{\mathcal{D}}_1^*(\alpha) \quad (48)$$

$$\begin{aligned}
&- \mathcal{H}_1(\alpha)B(\alpha)u_z^*(\alpha) \\
&- K^{*T}(\alpha)R(\alpha)u_z^*(\alpha) + C^T(\alpha)Q(\alpha)z(\alpha),
\end{aligned}$$

$$\frac{d}{d\alpha} \check{\mathcal{D}}_r^*(\alpha) = -[A(\alpha) + B(\alpha)K^*(\alpha)]^T \check{\mathcal{D}}_r^*(\alpha) \quad (49)$$

$$- \mathcal{H}_r(\alpha)B(\alpha)u_z^*(\alpha)$$

with the terminal boundary conditions $\mathcal{H}_1^*(t_f) = C^T(t_f)Q_f C(t_f)$, $\mathcal{H}_r^*(t_f) = 0$ for $2 \leq r \leq k$ and $\check{\mathcal{D}}_1^*(t_f) = -C^T(t_f)Q_f z(t_f)$, $\check{\mathcal{D}}_r^*(t_f) = 0$ for $2 \leq r \leq k$.

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