

Stability Analysis of a Tensioned String with Periodic Supports

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PREFACE

The work described in this report was prepared under Project 07PR01253-00, "Broadband Acoustic Window Designs for Heavyweight and Lightweight Torpedoes," principal investigator Andrew J. Hull (Code 8212). The sponsoring activity is the Office of Naval Research, program manager David Drumheller (ONR 333).

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13. ABSTRACT (Maximum 200 words) This report analyzes the zero-pole locations of an infinite length of tensioned string that has attached periodic supports. The dynamic response of the system is derived for distributed wavenumber forcing and discrete point forcing acting on the string. These wavenumber-frequency transfer functions are then written in zero-pole format by a mathematical transformation of their infinite series. Once this is accomplished, the locations of the system's poles and zeros become apparent, and they can be plotted in the wavenumber-frequency plane. It is shown that there are specific regions where an infinite number of poles can exist and specific regions where poles cannot exist. For the system with wavenumber forcing, the system zeros correspond very closely to the system poles except in the area of the fundamental unsupported string resonance. For the system with point forcing, the zeros can exist in the entire wavenumber-frequency plane except at the fundamental resonance. A numerical example is included, and the different zones of the system are demonstrated.			
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TABLE OF CONTENTS

Section		Page
	LIST OF TABLES.....	ii
1	INTRODUCTION	1
2	EQUATION OF MOTION.....	3
3	SOLUTION WITH A WAVENUMBER-FORCING FUNCTION.....	5
4	SOLUTION WITH A POINT-FORCING FUNCTION	7
5	TRANSFORMATION INTO ZERO-POLE FORM.....	9
6	NUMERICAL EXAMPLE.....	15
7	CONCLUSIONS.....	25
8	REFERENCES	27

LIST OF ILLUSTRATIONS

Figure		Page
1	Tensioned, Reinforced String with Coordinate System.....	3
2	Regions of Pole Existence (Gray) and Nonexistence (White).....	12
3	Location of the System Poles (Black Lines) in the Wavenumber-Frequency Plane	16
4	Response of the String Versus Wavenumber and Frequency for an Applied Wavenumber Load.....	17
5	Response of the Wavenumber-Forced System at 225 Hz.....	18
6	Response of the Wavenumber-Forced System at 275 Hz.....	18
7	Location of the System Zeros (Black Lines) in the Wavenumber-Frequency Plane for Continuous Wavenumber Forcing	20
8	Response of the String Versus Wavenumber and Frequency for an Applied Point Load...	22
9	Response of the Point-Forced System at 225 Hz.....	23
10	Response of the Point-Forced System at 275 Hz.....	23
11	Location of the System Zeros (Black Lines) in the Wavenumber-Frequency Plane for Discrete Forcing.....	24

LIST OF TABLES

Table		Page
1	Location of Poles in Wavenumber for a Frequency of 275 Hz	19
2	Location of Zeros in Wavenumber for the System with a Wavenumber Forcing	21
3	Location of Zeros in Wavenumber for the System with a Point Forcing	25

STABILITY ANALYSIS OF A TENSIONED STRING WITH PERIODIC SUPPORTS

1. INTRODUCTION

The motion of tensioned strings is typically studied to understand the dynamic response of more complicated systems. Using a number of assumptions, the underlying differential equation of motion is typically a second-order wave equation written as a function of space and time. The problem of an unreinforced string is a classical continuous media problem and has been discussed by numerous authors.^{1,2,3} The equation of motion of a tensioned string with multiple sets of supports has been derived.⁴ The tensioned string with periodic stiffeners has been analyzed in the frequency domain with a moving harmonic force⁵ and a suddenly applied concentrated force.⁶ In these papers, the stability of the system is discussed, particularly with respect to the value of the stiffeners. There is no extension of the problem into the wavenumber domain. Other continuous systems, specifically thin plates,^{7,8} thick plates,⁹ and beams¹⁰ have been modeled with periodic masses, dampers, and/or springs attached to the medium. These studies typically involve modeling a continuous system with one or more differential equations, using various boundary conditions to represent the mechanical elements attached to the system, and then deriving a solution technique to find the corresponding displacements of the system.

This report presents a zero-pole analysis of a tensioned string in wavenumber-frequency space when it is loaded with a continuous wavenumber-forcing function and a discrete point-forcing function. The equations of motion and corresponding solution to both mechanical loads are found using a previously derived analytical method. These equations are then transferred from an infinite summation series into a continuous analytical expression using a series-to-trigonometric mathematical formula. Once this is accomplished, the dynamic response of the system is in zero-pole format, allowing an examination of the stability of the system from a parametric standpoint. Additionally, the locations of the transfer function poles and zeros are apparent. A numerical example is included to illustrate these effects. The dynamic response of this system is discussed.

2. EQUATION OF MOTION

The system model is that of an infinite-length, tensioned string attached to periodically spaced discrete stiffeners, as shown in figure 1. The string is under a tension of T (N), has a constant mass per unit length ρ (kg/m), and an external load per unit length of $f(x,t)$ (N/m). The stiffeners are equally spaced at a distance of L (m) in the x -direction and each has a stiffness of K (N/m). The model uses the following assumptions: (1) the forcing function acting on the string is at a definite wavenumber and frequency or is a discrete point function at a definite frequency, (2) motion is normal to the string in the transverse direction (one-dimensional system), (3) the string has infinite spatial extent in the x -direction, (4) the particle motion is linear, (5) the string offers no resistance to bending, (6) the string is perfectly elastic, and (7) there is no damping present in the system.

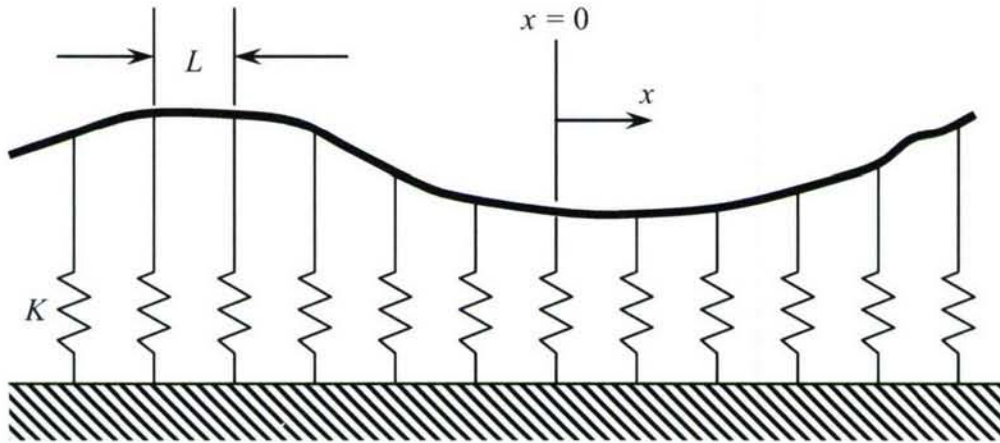


Figure 1. Tensioned, Reinforced String with Coordinate System

The motion of the system is governed by the equation⁴

$$T \frac{\partial^2 w(x,t)}{\partial x^2} - \rho \frac{\partial^2 w(x,t)}{\partial t^2} - K \sum_{n=-\infty}^{n=\infty} w(x,t) \delta(x - nL) = f(x,t), \quad (1)$$

where $w(x,t)$ is the transverse displacement of the string (m), x is the position on the string (m), t is time (s), and $\delta(x - nL)$ is the spatial Dirac delta function (1/m).

3. SOLUTION WITH A WAVENUMBER-FORCING FUNCTION

The problem is first solved with a wavenumber-forcing function, which is a continuous exponential harmonic function in space and time, written as

$$f(x,t) = F \exp(ikx) \exp(-i\omega t), \quad (2)$$

where ω is frequency (rad/s), k is wavenumber with respect to the x -axis (rad/m), and F is the magnitude of the distributed force (N/m). The solution to the problem is written in series form where the string displacement is equal to a sum of unknown functions of wavenumber and frequency multiplied by an exponential spatially harmonic function in the x -direction multiplied by an exponential harmonic function in time. The displacement becomes

$$w(x,t) = \sum_{n=-\infty}^{n=+\infty} W_n(k, \omega) \exp(ik_n x) \exp(-i\omega t), \quad (3)$$

where

$$k_n = k + \frac{2\pi n}{L}, \quad (4)$$

Using the principle of virtual work¹¹ results in the governing equation, which is written as

$$(-k_m^2 T + \rho \omega^2) W_m(k_m, \omega) - \frac{K}{L} \sum_{n=-\infty}^{n=+\infty} W_n(k_n, \omega) = \begin{cases} F & m = 0 \\ 0 & m \neq 0 \end{cases}, \quad (5)$$

or

$$A(k_m, \omega) W_m(k_m, \omega) - \frac{K}{L} \sum_{n=-\infty}^{n=+\infty} W_n(k_n, \omega) = \begin{cases} F & m = 0 \\ 0 & m \neq 0 \end{cases}, \quad (6)$$

where

$$A(k_m, \omega) = -Tk_m^2 + \rho\omega^2, \quad (7)$$

and k is the wavenumber of excitation (rad/m). Solving for the coefficients, and noting that the solution in the wavenumber-frequency domain is the zeroth order coefficient, yields the displacement divided by the input force as

$$\frac{W_0(k, \omega)}{F} = A^{-1}(k, \omega) \left[\frac{\left(\frac{K}{L}\right) A^{-1}(k, \omega)}{1 - \left(\frac{K}{L}\right) \sum_{m=-\infty}^{m=+\infty} A^{-1}(k_m, \omega)} \right] + A^{-1}(k, \omega), \quad (8)$$

which, after some rearranging, becomes

$$\frac{W_0(k, \omega)}{F} = \frac{1}{(-Tk^2 + \rho\omega^2)} \left[\frac{\frac{(K/L)}{(-Tk^2 + \rho\omega^2)} + 1 - \sum_{n=-\infty}^{n=+\infty} \frac{(K/L)}{(-Tk_n^2 + \rho\omega^2)}}{1 - \sum_{n=-\infty}^{n=+\infty} \frac{(K/L)}{(-Tk_n^2 + \rho\omega^2)}} \right]. \quad (9)$$

4. SOLUTION WITH A POINT-FORCING FUNCTION

The problem is next solved with a point-forcing function, which is a continuous exponential harmonic function in time and a delta function in space, written as

$$f(x,t) = F\delta(x - x_0)\exp(-i\omega t), \quad (10)$$

where x_0 is the spatial location of the point force (m) and F is the magnitude of the point force (N). To differentiate this solution from the solution found in section 3, the displacement variable is changed from $w(x,t)$ to $u(x,t)$. The system equation in the wavenumber-frequency domain, when $x_0 = 0$, becomes

$$(-k^2T + \rho\omega^2)U(k, \omega) = \frac{K}{L} \sum_{n=-\infty}^{n=+\infty} U(k_n, \omega) + \frac{F}{L}. \quad (11)$$

This equation is solved in the wavenumber domain using previously developed analytical methods,^{7,8} and the result is

$$\frac{U(k, \omega)}{(F/L)} = \frac{1}{(-Tk^2 + \rho\omega^2)} \left[\frac{1}{1 - \sum_{n=-\infty}^{n=+\infty} \frac{(K/L)}{(-Tk_n^2 + \rho\omega^2)}} \right]. \quad (12)$$

5. TRANSFORMATION INTO ZERO-POLE FORM

To facilitate an understanding of this system, it is desirable to transform equations (9) and (12) into a single analytical expression that does not contain summations. This transformation is accomplished by rewriting the infinite series solution term as¹²

$$\sum_{n=-\infty}^{n=+\infty} \frac{1}{(-Tk_n^2 + \rho\omega^2)} = \frac{-L \cos\left(\frac{\omega L}{2c}\right) \sin\left(\frac{\omega L}{2c}\right)}{2\omega\sqrt{T\rho} \left[\sin^2\left(\frac{kL}{2}\right) - \sin^2\left(\frac{\omega L}{2c}\right) \right]}, \quad (13)$$

where $c = \sqrt{T/\rho}$.

Inserting equation (13) into equation (9) results in

$$\frac{W_0(k, \omega)}{F} = \frac{z_w(k, \omega)}{p(k, \omega)}, \quad (14)$$

where $z_w(k, \omega)$ is an expression that contains the system zeros for a continuous forcing function and $p(k, \omega)$ is an expression that contains the system poles in the wavenumber-frequency plane. These expressions are

$$z_w(k, \omega) = 2\omega\sqrt{T\rho} \left[\frac{K}{L(-Tk^2 + \rho\omega^2)} + 1 \right] \left[\sin^2\left(\frac{kL}{2}\right) - \sin^2\left(\frac{\omega L}{2c}\right) \right] + K \cos\left(\frac{\omega L}{2c}\right) \sin\left(\frac{\omega L}{2c}\right), \quad (15)$$

and

$$p(k, \omega) = (-Tk^2 + \rho\omega^2) \left\{ (2\omega\sqrt{T\rho}) \left[\sin^2\left(\frac{kL}{2}\right) - \sin^2\left(\frac{\omega L}{2c}\right) \right] + K \cos\left(\frac{\omega L}{2c}\right) \sin\left(\frac{\omega L}{2c}\right) \right\}. \quad (16)$$

Inserting equation (13) into equation (12) results in

$$\frac{U(k, \omega)}{(F/L)} = \frac{z_u(k, \omega)}{p(k, \omega)}, \quad (17)$$

where $z_u(k, \omega)$ is an expression that contains the system zeros for a point-forcing function and $p(k, \omega)$ is an expression that contains the system poles in the wavenumber-frequency plane. The expression for $z_u(k, \omega)$ is

$$z_u(k, \omega) = 2\omega\sqrt{T\rho} \sin\left(\frac{kL}{2} + \frac{\omega L}{2c}\right) \sin\left(\frac{kL}{2} - \frac{\omega L}{2c}\right). \quad (18)$$

The system pole expression for the discrete point-loaded string is identical to the system pole expression for the continuous wavenumber-loaded string.

The terms for the poles are now examined. The denominator in equation (14) has two distinct terms and both are set equal to zero. The first term is

$$-Tk^2 + \rho\omega^2 = 0, \quad (19)$$

and this corresponds to the pole location of the string without stiffeners. This is referred to as the fundamental pole or resonance of the (unstiffened) string. When K is not equal to zero, the numerator *and* the denominator in equations (14) and (17) both approach zero when the relationship between wavenumber and frequency in equation (19) is satisfied and, thus, the term given by equation (19) does *not* correspond to a pole of the stiffened system. The second term is

$$(2\omega\sqrt{T\rho}) \left[\sin^2\left(\frac{kL}{2}\right) - \sin^2\left(\frac{\omega L}{2c}\right) \right] + K \cos\left(\frac{\omega L}{2c}\right) \sin\left(\frac{\omega L}{2c}\right) = 0, \quad (20)$$

which can be rewritten as

$$\sin\left(\frac{kL}{2}\right) = \sqrt{\sin^2\left(\frac{\omega L}{2c}\right) - \frac{K}{2\omega\sqrt{T\rho}} \cos\left(\frac{\omega L}{2c}\right) \sin\left(\frac{\omega L}{2c}\right)}. \quad (21)$$

Note that for the equality in equation (21) to hold true, it is required that

$$0 \leq \sin^2\left(\frac{\omega L}{2c}\right) - \frac{K}{2\omega\sqrt{T\rho}} \cos\left(\frac{\omega L}{2c}\right) \sin\left(\frac{\omega L}{2c}\right) \leq 1, \quad (22)$$

and this occurs when

$$\tan\left(\frac{\omega L}{2c} - \frac{m\pi}{2}\right) \leq \frac{K}{2\omega\sqrt{T\rho}}, \quad (23)$$

where m is the largest integer, so that

$$0 \leq \frac{\omega L}{2c} - \frac{m\pi}{2} \leq \frac{\pi}{2}. \quad (24)$$

As a result, the inequality given by equation (23) is a necessary condition for poles to exist in the wavenumber-frequency plane for the stiffened system. This expression is not a function of wavenumber. Figure 2 is a plot of equation (23) where the x -axis is the term $K/(2\omega\sqrt{T\rho})$ and the y -axis is nondimensional frequency $\omega L/2c$. The gray area of the plot is a region that corresponds to a location where poles may be present in the system response and the white area is the region where poles cannot exist. When the inequality in equation (23) is satisfied, the poles of the stiffened system will reside at

$$\hat{k}_n^p = \pm \frac{2}{L} \left| \arcsin(\sqrt{\theta}) + n\pi \right| \quad n = \dots, -3, -2, -1, 0, 1, 2, 3, \dots, \quad (25)$$

where

$$\theta = \sin^2\left(\frac{\omega L}{2c}\right) - \frac{K}{2\omega\sqrt{T\rho}} \cos\left(\frac{\omega L}{2c}\right) \sin\left(\frac{\omega L}{2c}\right), \quad (26)$$

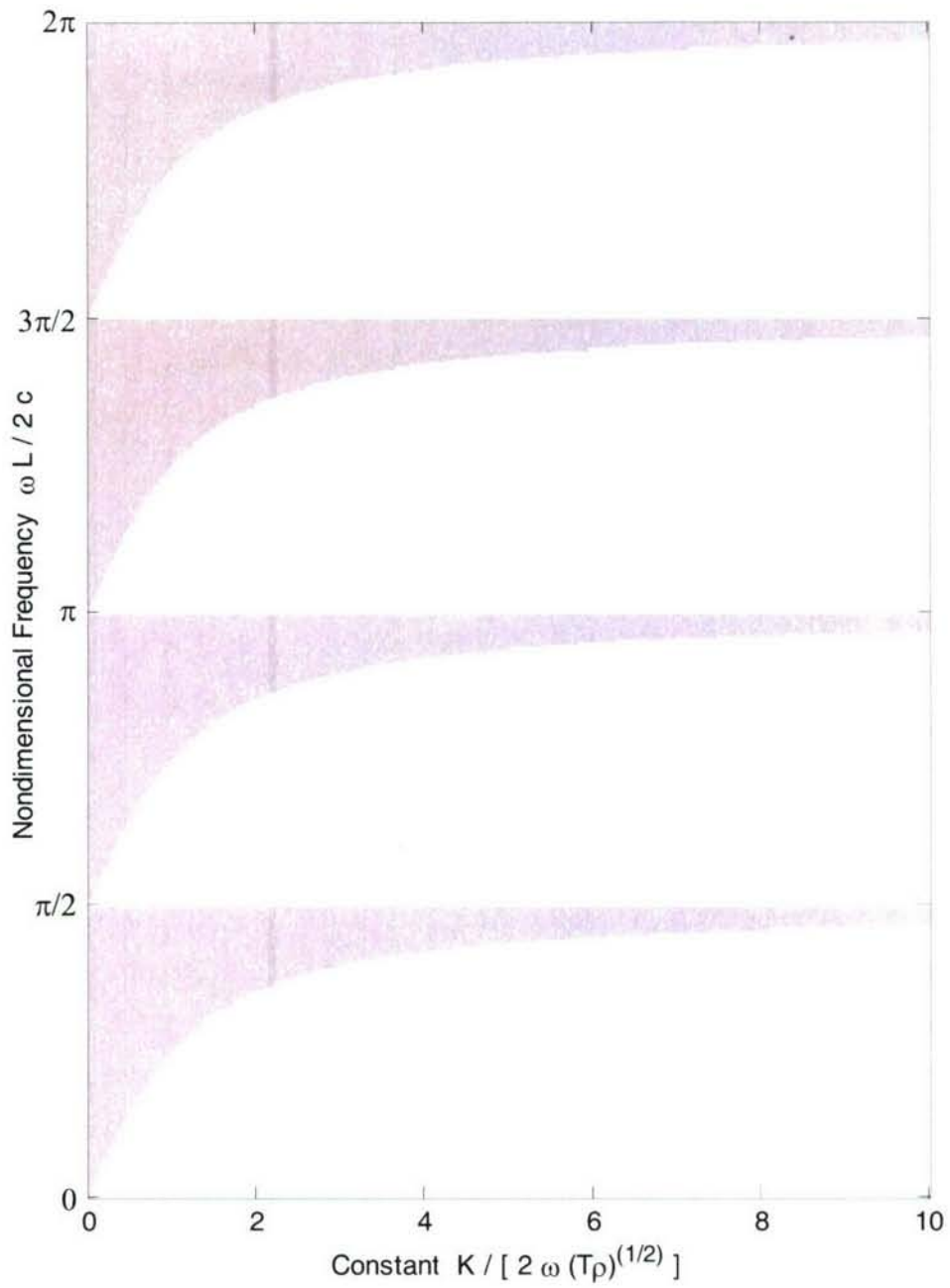


Figure 2. Regions of Pole Existence (Gray) and Nonexistence (White)

and \hat{k}_n^p is the location of the n indexed poles (rad/m).

The zeros of the system with a wavenumber-forcing function are found by setting the numerator of equation (14) equal to zero. This results in

$$z_w(k, \omega) = 2\omega\sqrt{T\rho} \left[\frac{K}{L(-Tk^2 + \rho\omega^2)} + 1 \right] \left[\sin^2\left(\frac{kL}{2}\right) - \sin^2\left(\frac{\omega L}{2c}\right) \right] + K \cos\left(\frac{\omega L}{2c}\right) \sin\left(\frac{\omega L}{2c}\right) = 0, \quad (27)$$

which is transcendental in wavenumber and frequency. The zeros of the system with a discrete forcing function are found by setting the numerator of equation (17) equal to zero. This results in

$$\hat{k}_n^u = \pm \left| \frac{\omega}{c} + \frac{2n\pi}{L} \right| \quad n = \dots, -3, -2, -1, 1, 2, 3, \dots, \quad (28)$$

where \hat{k}_n^u is the location of the n indexed zeros (rad/m) for the system with point forcing. The value of $n = 0$ is not a zero because the fundamental pole exists at this location, which negates the effects of this specific zero. Note that the zero location given in equation (28) is independent of the value of the stiffener K .

6. NUMERICAL EXAMPLE

The theory and equations that were developed in the previous sections are now illustrated using a numerical example. The system parameters are as follows: the string tension T is 100 N, the density per unit length ρ is 1 kg/m, the spring constant K is 3×10^4 N/m, the spring spacing L is 0.05 m, and the computed wavespeed c is 10 m/s. Figure 3 is a plot of the location of the system poles, plotted in black, in the wavenumber-frequency plane. This figure is valid for both the wavenumber load and the discrete load. The gray region corresponds to the gray region from figure 2 mapped into the wavenumber-frequency space. These gray regions are where poles can exist, and the white areas are the regions where poles cannot exist. The dashed line in the plot is the function $k = \omega / c$ and corresponds to the pole location of the unstiffened (i.e., no supports) system. Figure 4 is a plot of the response of the string versus wavenumber and frequency with a wavenumber load. Figure 5 is a constant frequency slice of figure 4 at 225 Hz and corresponds to an area where poles cannot exist, and figure 6 is a constant frequency slice at 275 Hz and corresponds to an area where poles can exist. Table 1 lists the first eight pole locations of the system in the positive wavenumber space with the value of n used to calculate the pole location at a frequency of 275 Hz. Figure 3 and table 1 were determined using the positive values of equation (24). The stability of the system based on the values of the stiffeners is discussed by others,⁵ although their analysis is only in the frequency domain. Figure 7 is a plot of the location of the system zeros for wavenumber forcing in the wavenumber-frequency plane. These values were determined by applying a root finder to equation (27). The roots that correspond to $k = \omega / c$ were discarded due to the fundamental pole also residing at that location. Table 2 lists the first seven zero locations of the system in the positive wavenumber space at a frequency of 275 Hz. A zero with a wavenumber value of 127.3 rad/m exists at 225 Hz. Numerical simulations suggest that for wavenumber forcing, the zeros are almost collocated with the poles at all locations in the wavenumber-frequency plane except around the region where $k = \pm\omega / c$. This concept can be visualized by comparing figure 3 and figure 7, where the system poles and zeros are in almost the same location except around the region of $k = \pm\omega / c$. Note that in figure 6 the poles and the zeros are almost at the identical location in wavenumber. Figure 8 is a plot of the response of the string versus wavenumber and frequency with a discrete point load. Figure 9 is a constant frequency slice of figure 8 at 225 Hz and figure 10 is a constant frequency slice at 275 Hz. Figure 11 is a plot of the location of the system zeros for point forcing in the

wavenumber-frequency plane. Table 3 lists of the first seven zero locations of the system in the positive wavenumber space with the value of n used to calculate the pole location at frequencies of 225 and 275 Hz. Figure 11 and table 3 were determined using the positive values of equation (28).

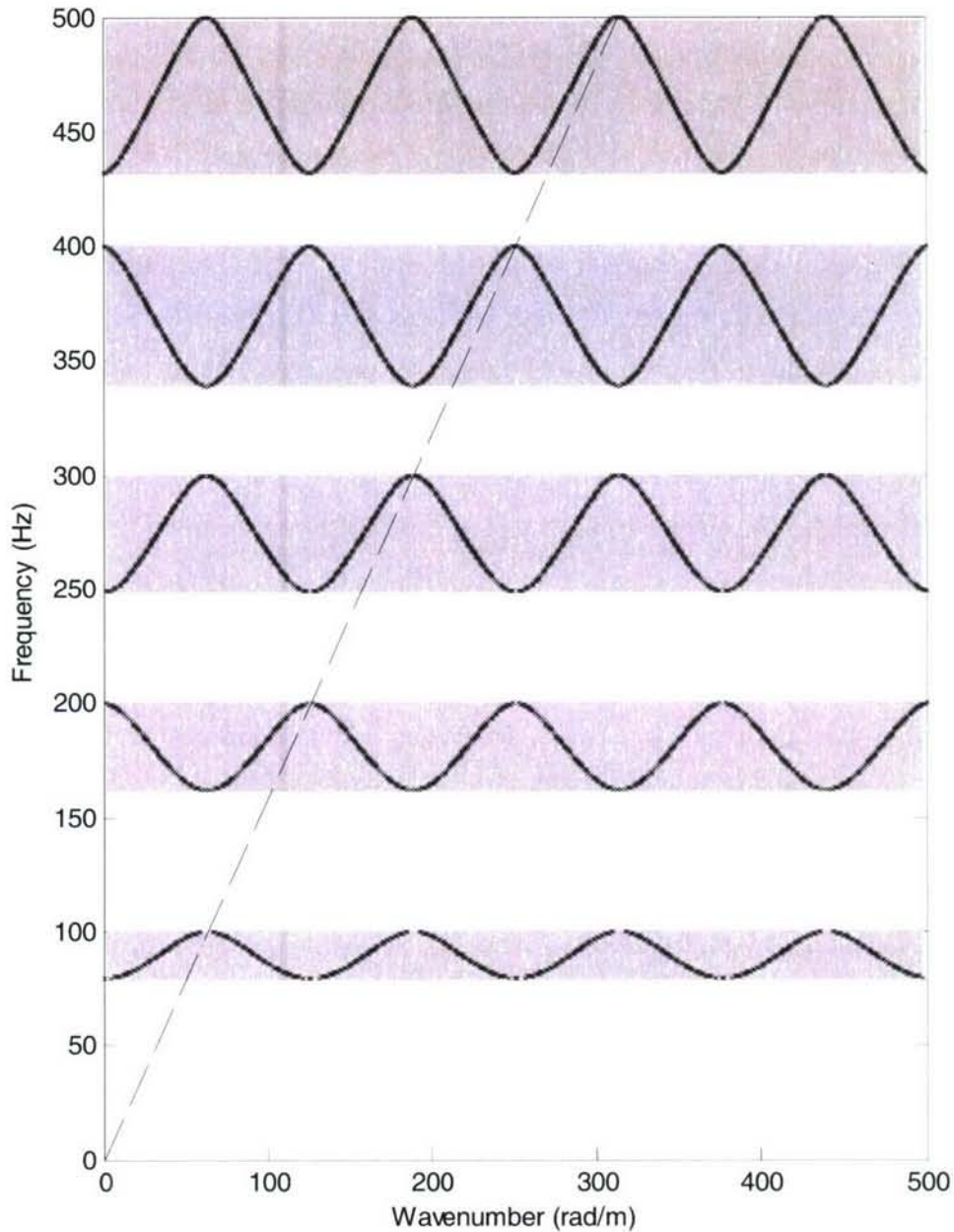


Figure 3. Location of the System Poles (Black Lines) in the Wavenumber-Frequency Plane
 (Gray areas are regions where poles can exist. The dashed line is the function $k = \omega / c$.)

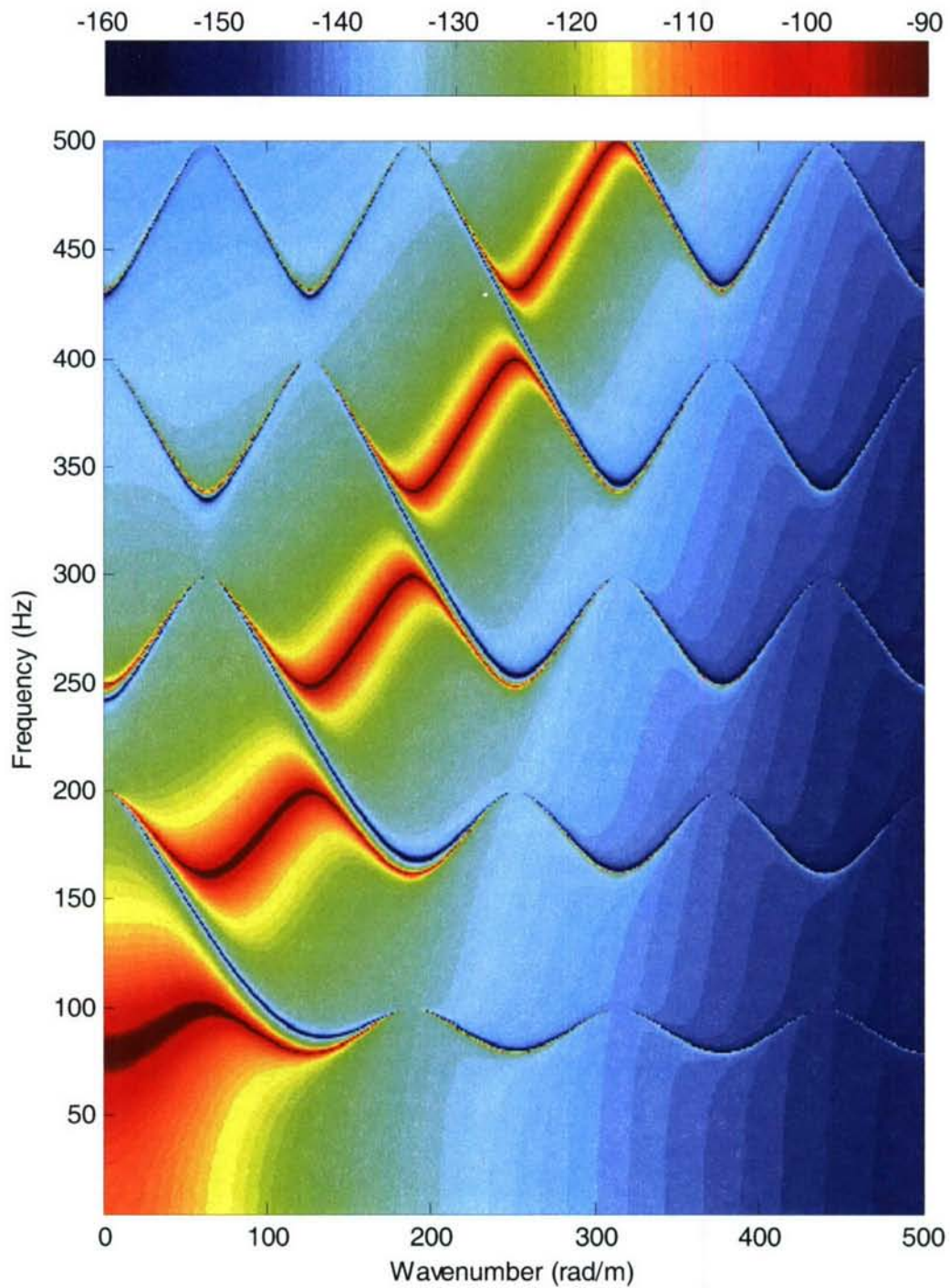


Figure 4. Response of the String Versus Wavenumber and Frequency for an Applied Wavenumber Load
 (Units are in dB re m/(N/m))

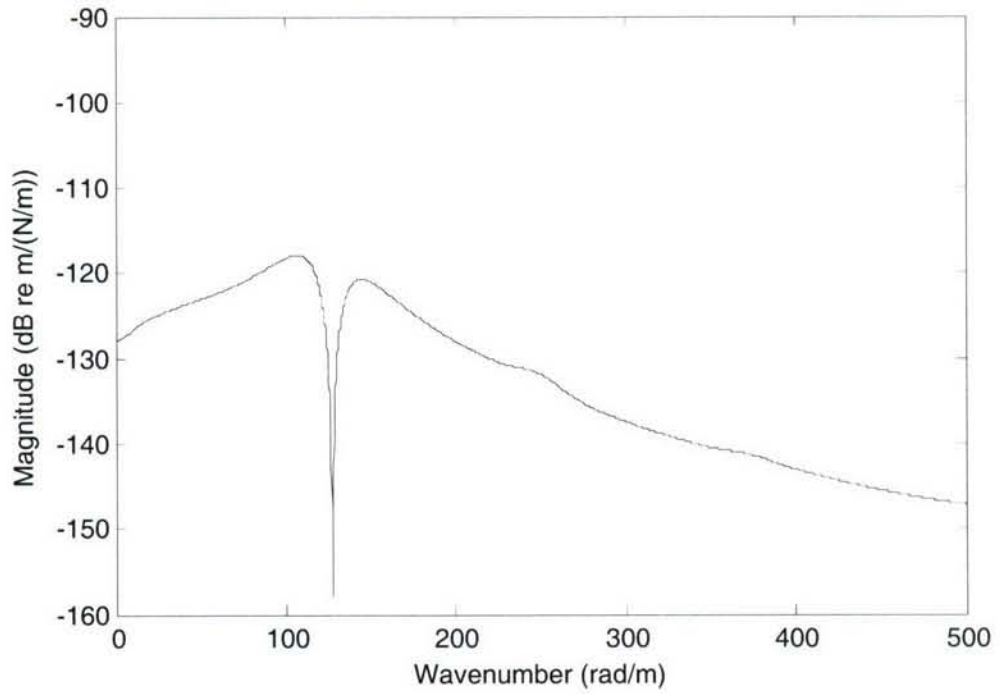


Figure 5. Response of the Wavenumber-Forced System at 225 Hz

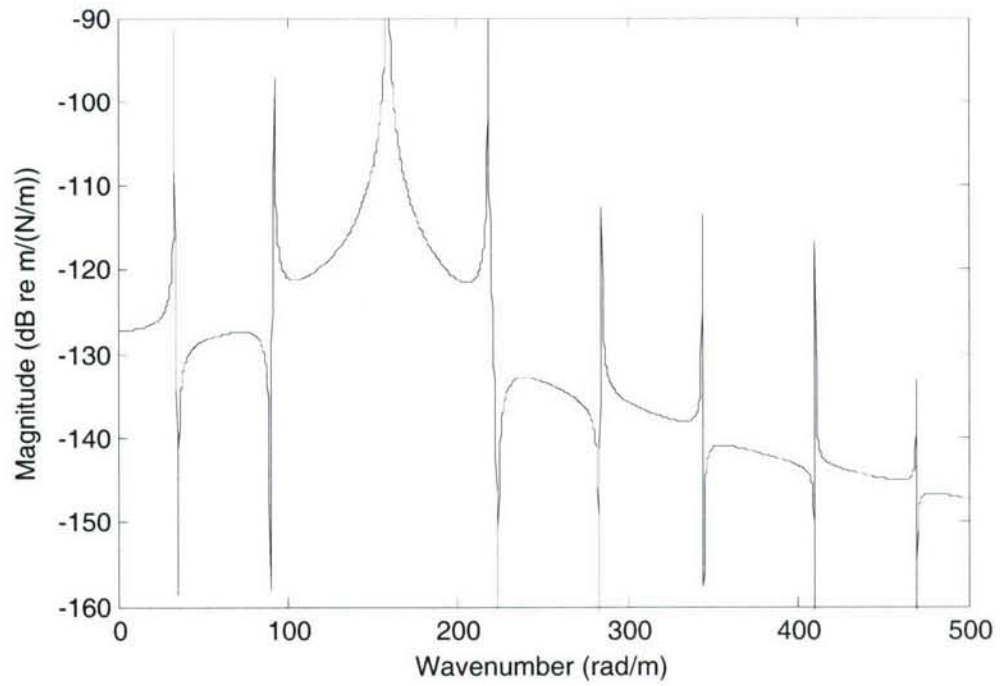


Figure 6. Response of the Wavenumber-Forced System at 275 Hz

Table 1. Location of Poles in Wavenumber for a Frequency of 275 Hz

Pole Location (rad/m)	Value of n
33.3	0
92.4	1
158.9	-1
218.0	2
284.6	-2
343.7	3
410.3	-3
469.4	4

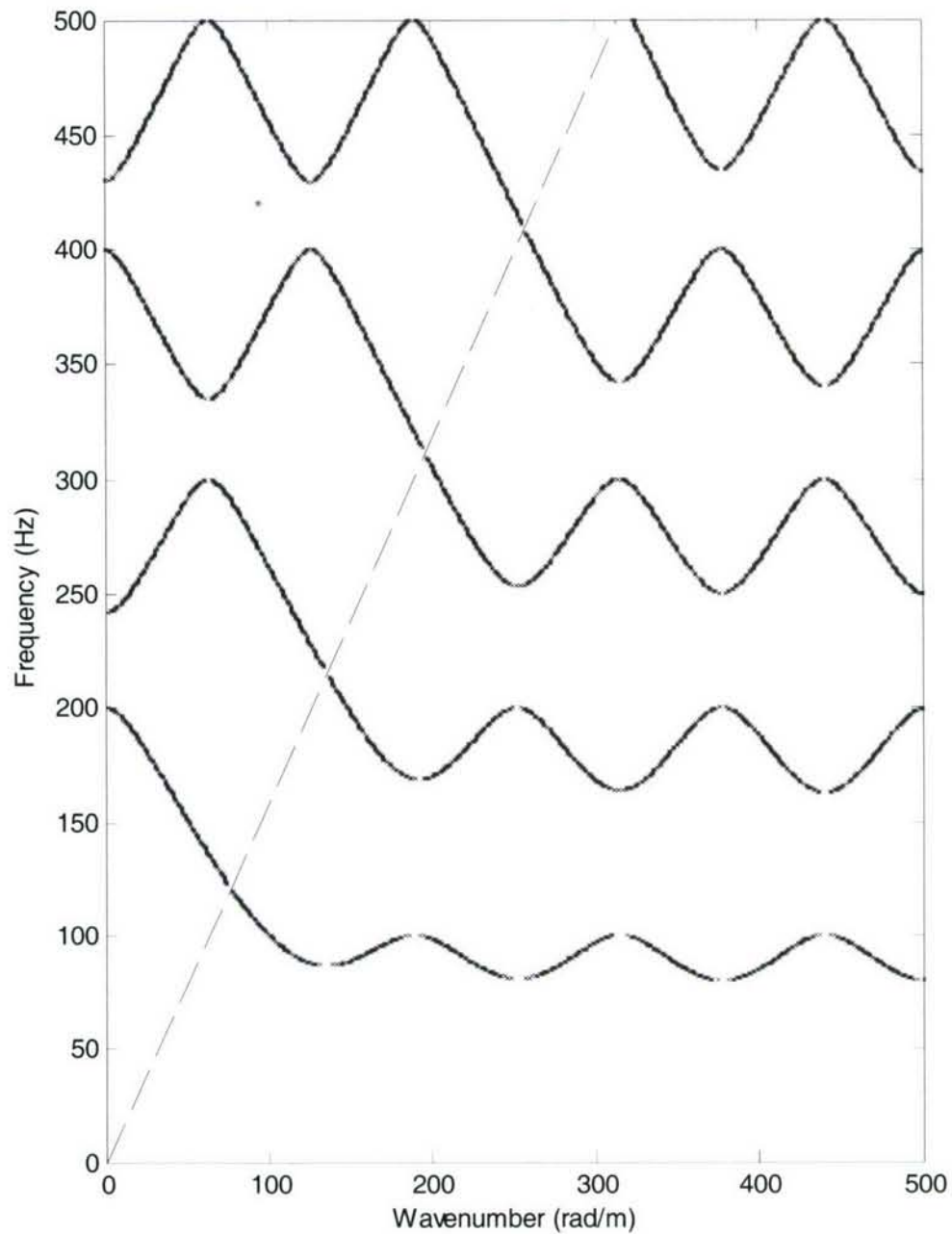


Figure 7. Location of the System Zeros (Black Lines) in the Wavenumber-Frequency Plane for Continuous Wavenumber Forcing
 (The dashed line is the function $k = \omega/c$.)

Table 2. Location of Zeros in Wavenumber for the System with a Wavenumber Forcing

Zero Location (rad/m) for 275 Hz
35.8
90.0
223.8
283.6
345.3
410.0
470.4

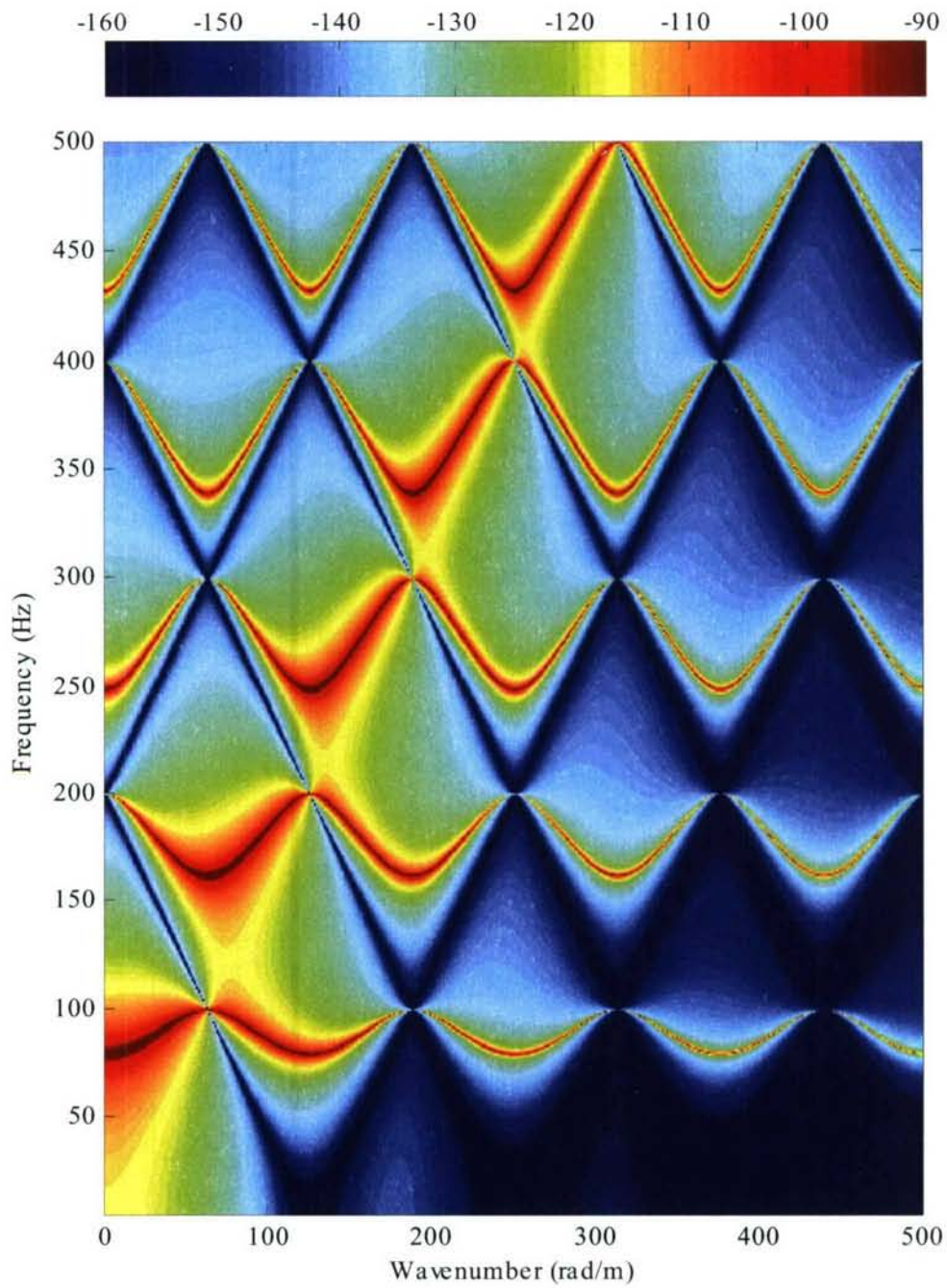


Figure 8. Response of the String Versus Wavenumber and Frequency for an Applied Point Load
 (Units are in dB re $m/(N/m)$)

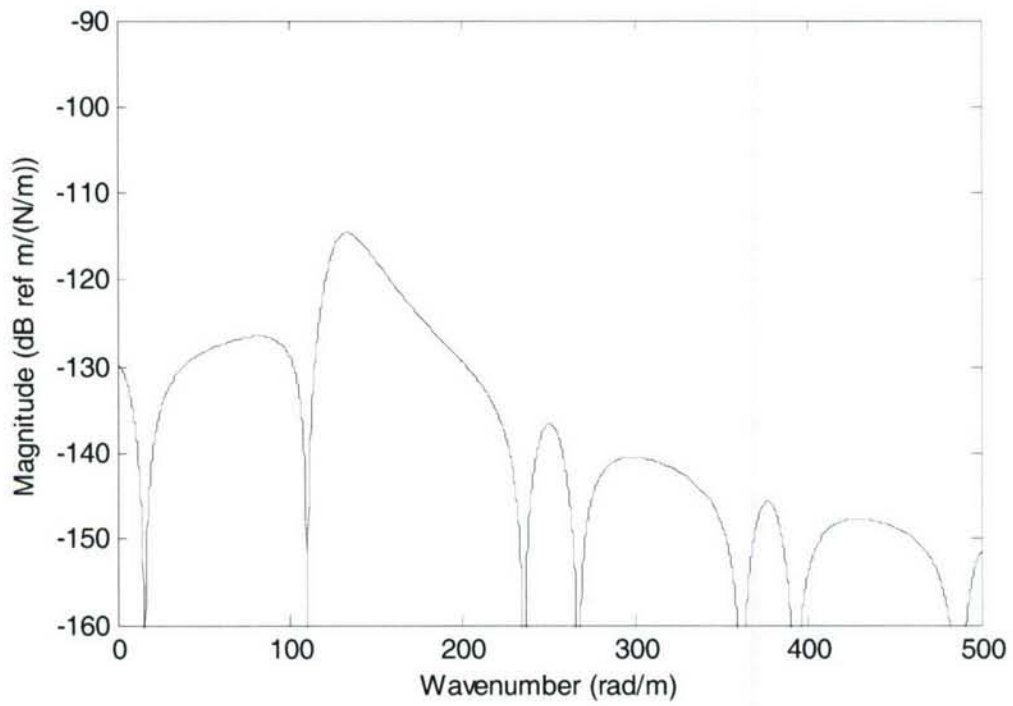


Figure 9. Response of the Point-Forced System at 225 Hz

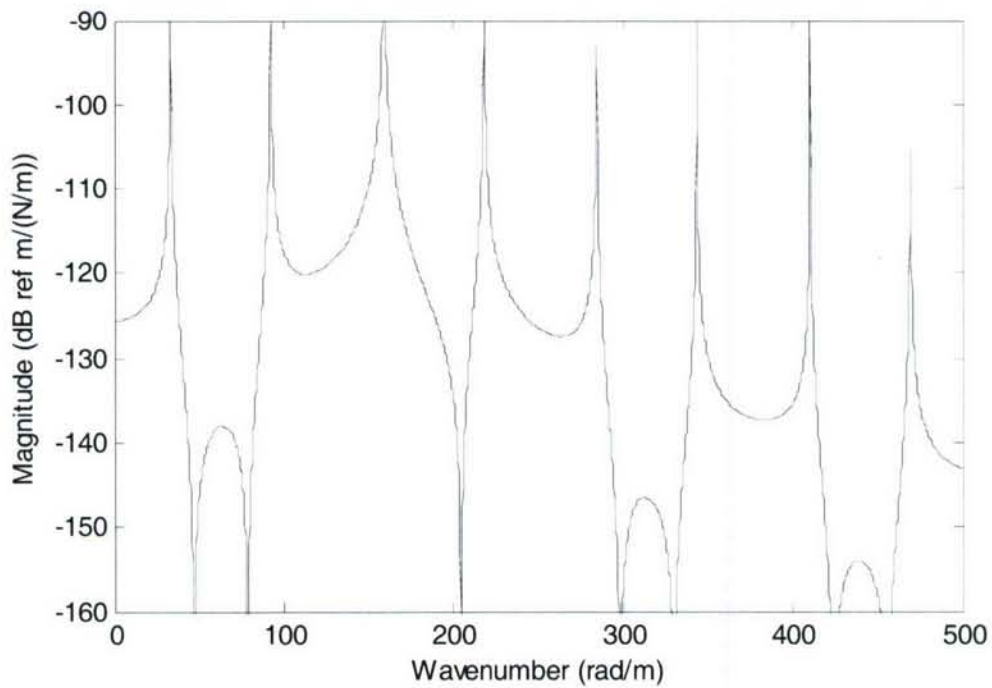


Figure 10. Response of the Point-Forced System at 275 Hz

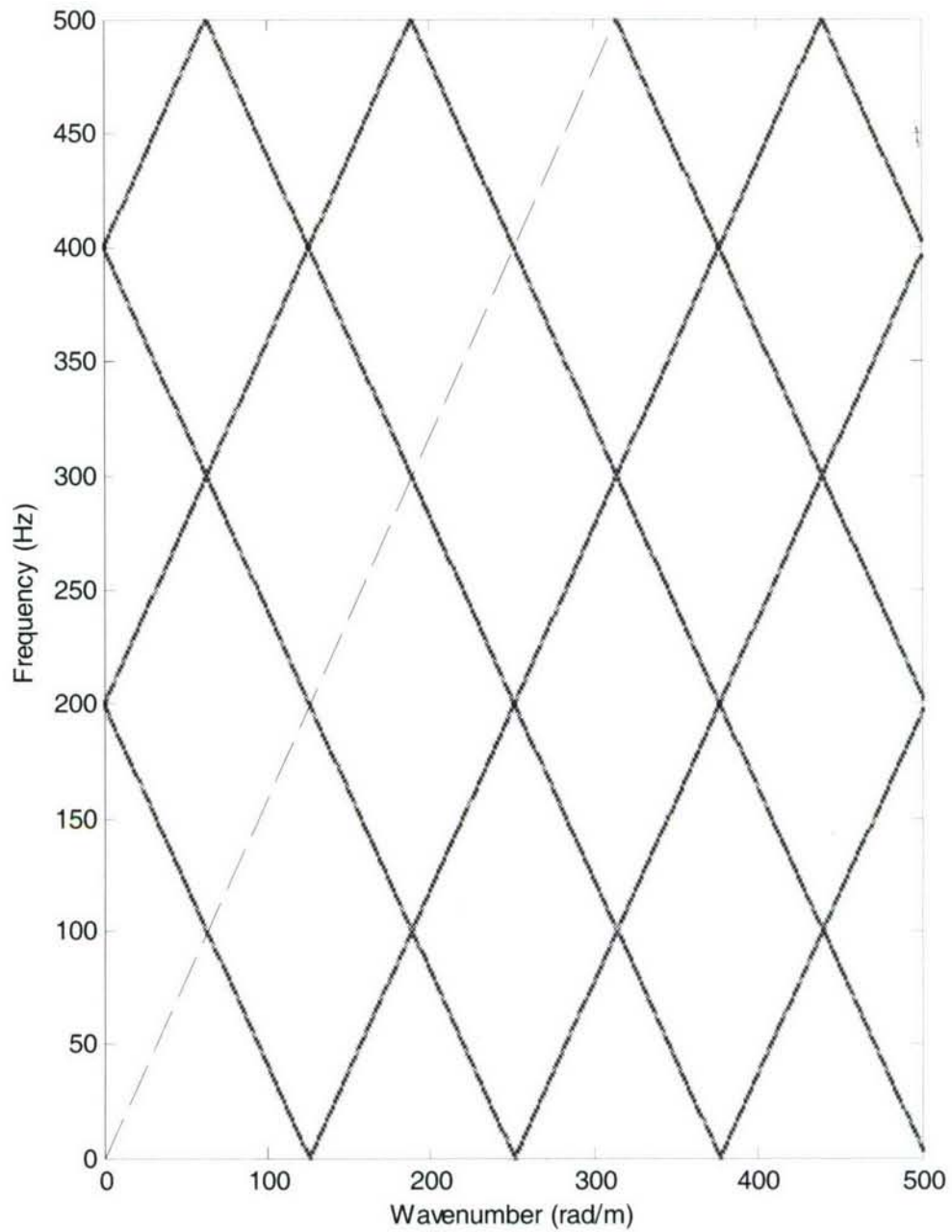


Figure 11. Location of the System Zeros (Black Lines) in the Wavenumber-Frequency Plane for Discrete Forcing
 (The dashed line is the function $k = \omega / c$.)

Table 3. Location of Zeros in Wavenumber for the System with a Point Forcing

Zero Location (rad/m) for 225 Hz	Zero Location (rad/m) or 275 Hz	Value of n
15.7	47.1	-1
110.0	78.5	-2
235.6	204.2	-3
267.0	298.5	1
361.3	329.9	-4
392.7	424.1	2
486.9	455.5	-5

7. CONCLUSIONS

The zero-pole response of a tensioned, reinforced string has been derived in the wavenumber-frequency domain for wavenumber-forcing and point-forcing functions. The stability of the system for various stiffener values has been demonstrated. A numerical example was included to illustrate where the poles and zeros are located in the wavenumber-frequency plane. It was shown that there are regions where the poles can and cannot exist. It was also shown that zeros for wavenumber forcing are almost collocated with the poles, and the zeros for point forcing can exist almost everywhere.

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