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GAME THEORY: TOOLKIT AND WORKBOOK FOR DEFENSE ANALYSIS STUDENTS

by

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The purpose of this thesis is to provide a workbook of the game theory topics covered in the course *SO4410 Models of Conflict*. The thesis also provides a software toolkit, which enables students to solve the problems easier and faster, therefore focusing more on analyses of the situation than on the actual mathematical side of the problem.

The workbook gives a basic review of the fundamental concepts and a detailed explanation for solving 'simple' game theory problems by pen and paper. Topics cover two and three person games. Two person games include (1) zero-sum games and their solutions in the pure or mixed strategy, (2) partial-sum games without communication between the players, and (3) communication among players and its effect on the game. Three person games focus on likely coalitions among the players.

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GAME THEORY: TOOLKIT AND WORKBOOK FOR DEFENSE ANALYSIS STUDENTS

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I need to thank Dr. Giordano for showing me that math can be fun and useful, and my wife, Dana, for her patience and support.

– Miroslav Feix

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I. INTRODUCTION

In mathematics, you don't understand things. You just get used to them.

– Johann von Neumann

Conflict is as ancient as humankind, and the extreme end of conflict is war. Military professionals spend a great deal of time studying the nature of conflict, and for that reason, defense analysis students study modeling course sequences that allow them to use mathematical tools in social settings.

Not everyone has a mathematical background; therefore, some concepts used in the sequences are difficult for students to comprehend. In particular, the *SO4410 – Models of Conflict*, which focuses mainly on the basics of game theory, can be difficult to understand.

The purpose of this thesis is to provide a workbook of the game theory topics, which are covered in the course, and to introduce the concepts intuitively. The thesis is not a substitute for the books and articles assigned in *SO4410* but rather a facilitating supplement with explanatory illustrations and computational processes. The Toolkit not only gives the solutions but also adds graphical representation for better comprehension.

Models of conflict assume rational decision makers, trying to maximize decision makers' payoffs. In game theory, the term *rationality* has a different meaning than most people think. Rationality does not mean what we think is best or wise; to be rational, actors have to be able (1) to define their objectives, however foolish they appear to others, (2) to formulate sufficiently different alternative strategies, and (3) to choose a strategy that maximizes their objective.

So the question then becomes: "What should / will the rational value maximizing player do?"

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II. ZERO-SUM GAMES

Zero-sum games entail games where one player wins and the other player loses. There is no room for cooperation and the interests are in total conflict. Each player has a certain set of strategies from which he can choose, and he is unaware of the choices of the other player. The resulting payoff is then determined by the combination of strategies. Since one player wins or the other player loses, we can limit the analysis to the payoffs of one player.

The zero-sum game, where (1) Player 1 (let's call her Rose / Row player) can play strategies A and B, and (2) Player 2 (let's call him Colin / Column player) can play strategies C and D, can be described as:

$$\begin{array}{cc} & \text{Colin} \\ & \begin{array}{cc} \quad C & \quad D \end{array} \\ \text{Rose} & \begin{array}{c} A \\ B \end{array}\begin{pmatrix} AC & AD \\ BC & BD \end{pmatrix} \end{array}$$

with the real numbers:

$$\begin{array}{cc} & \text{Colin} \\ & \begin{array}{cc} \quad C & D \end{array} \\ \text{Rose} & \begin{array}{c} A \\ B \end{array}\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \end{array}$$

The values in the example are the payoffs for Rose. The negatives are the payoffs for Colin. When Rose chooses strategy A, and Colin chooses strategy C, the outcome would be 4.

Hungarian-American mathematician John von Neumann proved that all finite two-person zero-sum games have a solution in either pure or mixed strategies. These strategies give us an expected value of the game. This idea is described in the MiniMax Theorem and serves as a basis for the solution of zero-sum games.

MiniMax Theorem (Rose – maximizing, Colin – minimizing) – for every finite two person zero-sum game there is a solution, (1) and there exists a number V called the value of the game. (2)Rose has a strategy combination such that her average payoff is at least V no matter what Colin does, and (3) Colin has a strategy combination such that his average payoff is no more than V, without regard to Rose's choices.

In real life, the closest we can get to a total conflict is sports or games. And of course, war is the ultimate total conflict endeavor. Therefore, we start here.

Two commanders are facing each other on the battlefield. Rose wants to breach the enemy lines. She has two options. Rose can either attack Colin at the city or through the adjacent mountains. Colin, on the other hand, faces the question of where to prepare the defense.

Payoffs of Rose's forces[1]:

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{cc} \text{City} & \text{Mountains} \end{array} \\ \text{Rose} & \begin{array}{r} \text{City} \\ \text{Mountains} \end{array} & \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \end{array}$$

What can we say about this game? Clearly, the game is in Rose's favor. There are no negative numbers, so Rose can never lose. However, what else? What should a rational player do? Is there an optimal way of playing such a game?

A. MIXED AND PURE STRATEGIES – DISTINCTION

In the game, players can either play only one of their available strategies or they can play some mixture of strategies. If the players' optimal strategy is to play only one strategy, this is called Pure strategy solution or solution in the Pure strategies. If the players' optimal strategy is a combination of the strategies with certain relative frequency, this is called Mixed strategy solution or solution in the Mixed strategy.

The distinction is for convenience, and it helps one solve some games more quickly. Of course, the pure strategy is the mixed strategy where one of the strategies has the probability of playing 100 percent.

[1] Payoffs are arbitrary.

It is important also to note that in the pure strategy solution one can be assured of winning at least the smallest amount in the strategy. However, in the mixed strategy one can only be assured that, over the long haul, the average payoff will be of a certain value. In the short term, it does not have to apply, and one must be careful with the decision or advice of what to do. To illustrate this, it can be thought of as the difference between winning a sure $500 and winning $1000 at 50 percent probability, even though from the expected value point, these are equivalent. The risk tolerance is different from person to person and has to be considered in the interpretation of the insights provided by game theory.

When there is a choice between a pure strategy and mixed, i.e., that yield the same value, it is preferable to choose a pure strategy solution. First, it guarantees the outcome in every play. Second, a pure strategy solution is usually easier to implement. One does not have to care about randomization of the strategies or their coordination. Furthermore, pure strategy solution does not require secrecy as a part of the strategy. The results of the game will not be influenced by the opponent's knowledge of one's next move. However, secrecy is hard to achieve in real life; therefore, pure strategy solutions are preferable.

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III. ZERO-SUM GAMES – PURE STRATEGIES

A. DOMINANT AND DOMINATED STRATEGIES

In order to ensure a clear understanding of the terms, it's important to start with the definitions.

Definitions:

"A strategy S dominates strategy T if every outcome in S is at least as good as the corresponding outcome in T, and at least one outcome in S is strictly better than the corresponding outcome in T."[2]

Dominant strategy is a strategy that **dominates all** the other strategies of the player.

Dominated strategy is a strategy that is **dominated by at least one** of the other strategies of the player.

Why would one care about dominance? The reason is articulated in the dominance principle, "A rational player should never play a dominated strategy."[3] All the outcomes in the dominated strategy are equal or less than the outcomes in some other strategy. Therefore, it would not be beneficial for the player to play a dominated strategy. The player can always get better outcomes by not playing it, independent of what the other player does.

Dixit and Nalebuff in *Thinking Strategically* give similar advice: "If you have a dominant strategy, use it."[4] The dominant strategy outcomes are always better or equal to the outcomes of the player's other strategies. The game theory assumes rational, value

[2] Philip D. Straffin, *Game Theory and Strategy*, New Mathematical Library, Vol. 36, (Washington: Mathematical Association of America, 1993), 8.

[3] Ibid., 8.

[4] Avinash K. Dixit and Barry Nalebuff, *Thinking Strategically: The Competitive Edge in Business, Politics, and Everyday Life*, 1st ed. (New York: Norton, 1991), 86.

maximizing players. The reason for playing a dominant strategy is obvious. If the player maximizes and every other strategy is worse, the player should use a dominant strategy.

1. How to Find Dominant Strategy:

Consider the following game with our commanders. It is a zero-sum game where Rose is trying to maximize and Colin minimize. This time they have three possible courses of action and the game with Rose's payoffs is:

$$\begin{array}{cc} & \text{Colin} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{Rose} \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 2 & 3 & 5 \\ 9 & 6 & 7 \\ 4 & 3 & 4 \end{pmatrix} \end{array}$$

The easiest way for one to find a dominant strategy is to place a mark at the maximum value in each column. When the marks are all in the same row, one can see that this row is Rose's dominant strategy, and she should always play it because all of the other strategies are either worse or indifferent for her.

$$\begin{array}{cc} & \text{Colin} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{Rose} \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 2 & 3 & 5 \\ 9^* & 6^* & 7^* \\ 4 & 3 & 4 \end{pmatrix} \end{array}$$

One can repeat the same process for Colin. Only this time one places the mark at the minimum value in each row. As shown, the marks are not in the same column and Colin does not have a dominant strategy.

$$\begin{array}{cc} & \text{Colin} \\ & \begin{array}{ccc} A & B & C \end{array} \\ \text{Rose} \begin{array}{c} A \\ B \\ C \end{array} & \begin{pmatrix} 2^* & 3 & 5 \\ 9 & 6^* & 7 \\ 4 & 3^* & 4 \end{pmatrix} \end{array}$$

Colin's best response for Rose-B is to play Colin-B, as it gives him the best value that he can achieve. The solution of the game is then Rose-B, Colin-B with the value of the game equal to 6.

Unfortunately, not all of the games are this simple.

B. SADDLE POINT

Definition: Saddle point

"An outcome in a game (with the payoffs to the row player - *maximizing*) is called a **saddle point** if the entry at the outcome is both less than or equal to any entry in its row, and greater than or equal to any entry in its column." [5]

At this point, it is useful to read again von Neumann MiniMax theorem. A saddle point is a special case where both players can reach the value of the game by playing one of the strategies 100 percent of the time.

Saddle point principle: "If a matrix game has a saddle point, both players should play a strategy which contains it."[6]

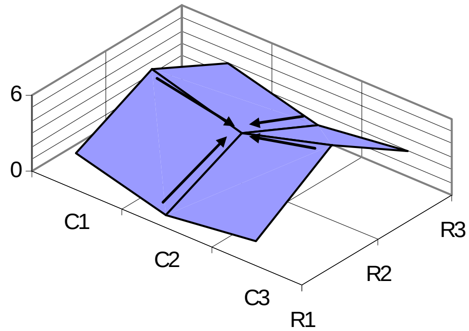
The reason behind the saddle point principle is obvious. The value of the saddle point guarantees the outcome for both players. Switching to any other strategy gives an opponent a chance to respond with the strategy that is beneficial to him. However, when the player uses saddle point strategy, the opponent can only do worse. To sum up, no player has incentive to change a strategy and leave the saddle point strategy combination.

This is illustrated in the 3-D model of the game, where the payoffs are depicted as the height. The saddle point is the lowest point in one direction and the highest from the other. The optimal choice is to stay there. On a chart, one can also see why this point is called a saddle point; with a little bit of imagination, it looks like a saddle.

---

[5] Philip D.Straffin, *Game Theory and Strategy*, New Mathematical Library, Vol. 36, (Washington: Mathematical Association of America, 1993), 9.

[6] Ibid., 9.



Now that it is known why the saddle point is important for the solution of the game, the subsequent question is, “How can one find it?” There are two options. For simple games, one can use the arrow method. As the game gets bigger and more complicated, it is better to use a more general numerical method.

1. Arrow Method

Consider the following game with commanders: it is a zero-sum game where Rose is trying to maximize and Colin minimize. Each commander can focus on three strategies.

$$
\begin{array}{ccc}
 & & \text{Colin} \\
 & & \begin{array}{ccc} C1 & C2 & C3 \end{array} \\
\text{Rose} & \begin{array}{c} R1 \\ R2 \\ R3 \end{array} & \begin{pmatrix} 3 & 1 & 2 \\ 6 & 4 & 6 \\ 3 & 1 & 2 \end{pmatrix}
\end{array}
$$

Working systematically, one starts with Rose as she is trying to maximize the payoffs and prefers the higher values. When Colin is playing C1, Rose prefers to play R2 over R1 and R2 over R3. Accordingly, one places the arrows with the direction from the lower entries (Payoffs) to the higher entries in the same column and continues with the same procedure in the remaining columns.

When the preference arrows for Rose have been completed, one does the same for Colin. However, this time as Colin is trying to minimize, he prefers lower payoffs. Accordingly, the arrow goes in direction from higher to lower entries in the rows.

$$
\begin{array}{cc}
 & \text{Colin} \\
 & \begin{array}{ccccc} \text{C1} & & \text{C2} & & \text{C3} \end{array} \\
\text{Rose} & \begin{array}{c} \text{R1} \\ \\ \text{R2} \\ \\ \text{R3} \end{array}
\begin{pmatrix}
3 & \rightarrow & 1 & \leftarrow & 2 \\
\downarrow & & \downarrow & & \downarrow \\
6 & \rightarrow & \boxed{4} & \leftarrow & 6 \\
\uparrow & & \uparrow & & \uparrow \\
3 & \rightarrow & 1 & \leftarrow & 2
\end{pmatrix}
\end{array}
$$

Saddle point is the entry where no arrow aims out of the saddle point. If we do not depict the indifferent values, it is the value where all of the arrows aim in. In this example, it is the combination Rose-R2, Colin-C2 and the value of the game is 4.

2. General Numerical Method

Finding a saddle point using the arrow method is a little bit confusing when the players have more than three strategies. The game table is easily filled with arrows and the probability of potential error increases. In this case, it is better for one to use a numerical method using the MaxiMin and MiniMax.

Consider the following game with our commanders. It is a zero-sum game where Rose is trying to maximize and Colin minimize, and each commander has five strategies from which to choose.

$$
\begin{array}{cccl}
 & \text{Colin} & & \\
 & \begin{array}{ccccc} \text{C1} & \text{C2} & \text{C3} & \text{C4} & \text{C5} \end{array} & \text{Minimum in Row} & \\
\text{Rose} & \begin{array}{c} \text{R1} \\ \text{R2} \\ \text{R3} \\ \text{R4} \\ \text{R5} \end{array}
\begin{pmatrix}
1 & 0 & 4 & 1 & 5 \\
6 & 2 & 6 & 8 & 0 \\
1 & 3 & 5 & 7 & 1 \\
7 & \boxed{4} & 7 & 5 & 4 \\
3 & 2 & 1 & 4 & 8
\end{pmatrix} & \begin{array}{c} 0 \\ 0 \\ 1 \\ \boxed{4} \\ 1 \end{array} & \begin{array}{c} \\ \\ \\ \text{Maximum of Minimum} \\ \text{(MaxiMin)} \end{array} \\
\text{Maximum in Column} & \begin{array}{ccccc} 7 & \boxed{4} & 7 & 8 & 8 \end{array} & & \\
 & \text{Minimum of Maximum (MiniMax)} & &
\end{array}
$$

Rose Maximin = 4
Colin Minimax = 4
Rose Maximin = Colin Minimax = 4 = Value of the Game

One starts with Rose. Rose wants to maximize her payoff. In this case she chooses the strategy R1; she can get no less than 0 (Colin-C2). It is the minimum payoff for strategy R1. She can do the same for the rest of her strategies. Now she has the worst-case values of her respective strategies. From these strategies, she should pick the strategy that gives her the highest payoff. In other words, she should choose the maximum payoff from the minimum payoffs in row. This strategy/strategies are called MaxiMin strategy. By choosing the MaxiMin strategy, she guarantees herself at least the MaxiMin value. If Colin does not play optimally, she can get more, because all of the payoffs in the MaxiMin strategy are equal or higher.

Now the game is analyzed from Colin's point of view. He wants the payoffs to be as small as possible. Accordingly, one calculates the worst outcomes for each of his strategies. This time, it is the maximum values in columns. His payoff for the chosen strategy cannot be worse than the maximum in the column. Colin is minimizing, so he should choose the strategy with the lowest maximum. These strategies are called the MiniMax strategy.

If the value of Rose's MaxiMin strategy and Colin's MiniMax strategy are the same, the saddle point has been found, and the game is solved. Saddle point lies at the intersection of the MaxiMin and MiniMax strategy. Assuming both players play optimally, no player can get a better outcome by switching to some other strategy. In the example, the saddle point is at Rose-R4 and Colin-C2 with the value of the game at 4.

The process then is for one to first write the minimum values in each row right of the matrix and mark the maximum of the minimum values. Second, one writes the maximum values in each column and marks the maximum of the minimum values. If the marked values are the same, game has a saddle point at the intersection of the corresponding strategies.

a. Games with More than One Saddle Point

The zero-sum game can have more than one saddle point. In this case, all the saddle points have the same value and nicely form a rectangle.

$$
\text{Rose}\quad
\begin{array}{c}
 \\ R1 \\ R2 \\ R3 \\ R4 \\ R5 \\ \text{Maximum in Column}
\end{array}
\begin{array}{c}
\text{Colin} \\
\begin{array}{ccccc} C1 & C2 & C3 & C4 & C5 \end{array} \\
\begin{pmatrix}
3 & 1 & 6 & 2 & 2 \\
7 & \boxed{4} & 5 & \boxed{4} & 7 \\
3 & 3 & 2 & 1 & 3 \\
6 & \boxed{4} & 6 & \boxed{4} & 5 \\
3 & 1 & 2 & 3 & 8
\end{pmatrix} \\
\begin{array}{ccccc} 7 & \boxed{4} & 6 & \boxed{4} & 8 \end{array}
\end{array}
\begin{array}{c}
\text{Minimum in Row} \\ 1 \\ \boxed{4} \\ 1 \\ \boxed{4} \\ 1 \\ \
\end{array}
\quad
\begin{array}{c}
 \\ \\ \\ \\ \text{Maximum of Minimum} \\ \text{(MaxiMin)} \\ \
\end{array}
$$

Minimum of Maximum (MiniMax)

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IV. ZERO-SUM GAMES – MIXED STRATEGIES

Not all of the games have a saddle point and solution in the pure strategies. If the MaxiMin and MiniMax are not the same, the players can still play their MiniMax and MaxiMin strategies, but the outcome is not optimal; they can do better by playing some mix of strategies.

Mixed strategy is "a strategy that involves the random choice of pure strategies, according to particular probabilities. A mixed strategy of a player is optimal if it guarantees the value of the game."[7]

The definition has a couple of key points deserving explanation. First, the choice has to be truly random. Unlike the games with the saddle point where secrecy is not required for successful play, in the mixed strategies secrecy is crucial. The opponent who knows in advance, what the choice of strategy would be, can take advantage of the knowledge and respond with his best counter strategy. A better way to surprise the opponent is to surprise oneself and entrust the choice to some random generator or choice picker. In nature, there are plenty of random events or technical means that can be used.

Next, what are these particular probabilities? Probability indicates the relative frequency of playing a strategy. In theory, a player can choose the probabilities at will, and, of course, the sum of the probabilities has to be equal to one. Any particular probabilities can be played. However, in the game theory arena, one is assumed to be a rational, value-maximizing player. Therefore, one hopes to find a probability mix that can guarantee the player his best achievable outcome against the optimally playing opponent. From von Neumann's MiniMax theorem, it is clear that the mix guarantees the value of the game.

[7] Consortium for Mathematics and Its Applications, *For all Practical Purposes: Mathematical Literacy in Today's World*, 6th ed. (New York: W.H. Freeman, 2003), 582.

A. HOW TO SOLVE OPTIMAL MIXED STRATEGIES

In the following, methods for finding the optimal strategy mix are described, and the thesis covers the following: the method of using the graphical description of the game, the expected value method, and Williams's method of oddments. Additionally, the thesis will explain how to convert the game into a linear program and leave the numerically difficult process to the software solvers.

As an illustrative example, two volleyball teams will try to decide what plays to make. In volleyball, basically, there are two options for the offensive team. It can either try to overcome the opponent by attacking from the center or from the side. The opponent faces the question of where to prepare the blocks. Rose is captain of the offensive team, and Colin is the captain of blocking team. Success percentages of the offensive team follows.

$$\begin{array}{ccc} & & \text{Colin - blocking} \\ & & \begin{array}{cc} \text{center} & \text{side} \end{array} \\ \text{Rose - attacking} & \begin{array}{r} \text{side} \\ \text{center} \end{array} & \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \end{array}$$

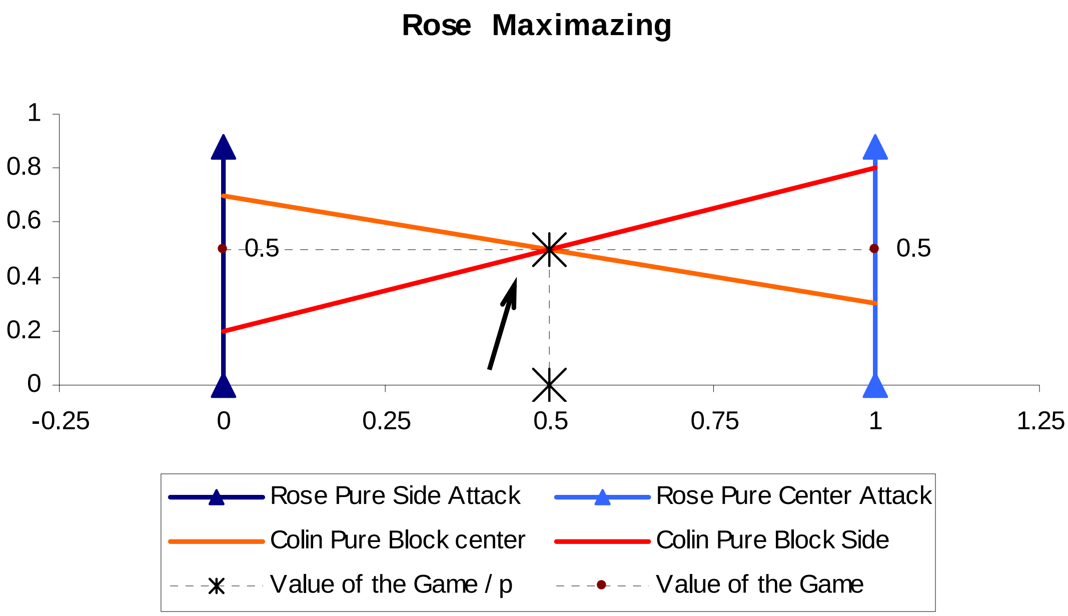
Neither team has a dominant strategy, nor is there a saddle point. What should they decide to play?

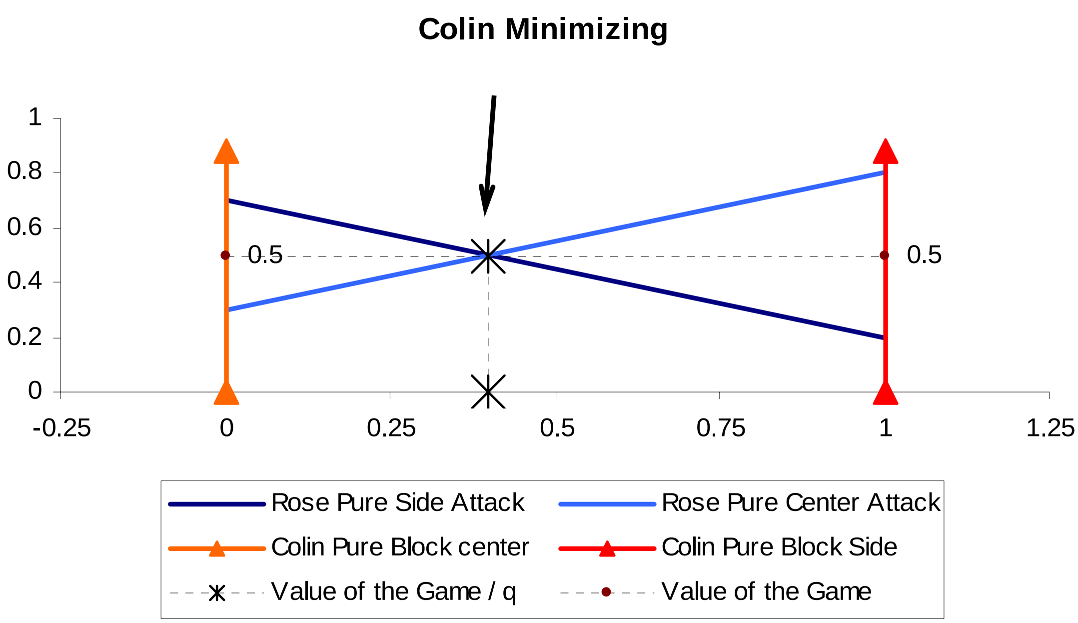
1. Graphical Method

The Graphical method is a basic and easy way of solving the game. However, the results are not precise enough, so one must be careful with the interpretations. One must question the sensitivity of the resulting mix of strategies; however, sometimes it is good enough.

A graphical depiction of the game[8] with a separate graph for each team is shown below.

[8] See Appendix 2: how to create the graphical representation.





The values from the graphs can be readily understood. In the example, Rose's optimal mixed strategy is to attack 50 percent of the time from the side and 50 percent of the time from the center. Over time, this assures her success at 0.5 (value of the game). Colin's optimal strategy is to prepare the block at the center 60 percent of the time and 40 percent from the side. The value of the game is the same, a 50 percent success for the offensive side.

2. Expected Value Method

The Expected value method is a general method for solving a game. Equalizing the expected value for an opponent's strategies is a universal method for any zero-sum game. In principle, one is trying to find a mix of strategies that removes the opponent's decisions from consideration. If the expected value for any mix of opponent's strategies is equal, it does not matter what the opponent plays, the value of the game stays the same.

If p is assigned as the probability of strategy for Rose-Center, the probability of strategy for Rose-Side is then 1-p (sum of probabilities has to be 1). Accordingly, for Colin-Side the probability is q and Colin-Center is 1-q.

$$\begin{array}{cccc} & & \text{Colin - blocking} & \\ & & \begin{array}{cc}\text{center} & \text{side}\end{array} & \\ \text{Rose - attacking} & \begin{array}{r}\text{side}\\ \text{center}\end{array} & \begin{pmatrix}0.7 & 0.2\\ 0.3 & 0.8\end{pmatrix} & \begin{array}{c}1-p\\ p\end{array} \\ & & \begin{array}{cc}1-q & q\end{array} & \end{array}$$

Now, the game is analyzed from Rose's point of view:

If Colin would play his Pure-Center, Rose can expect the payoff to be:

$$EV(\text{Colin-Center}) = 0.7*(1-p)+0.3*p$$

If Colin would play his Pure-Side, Rose can expect the payoff to be:

$$EV(\text{Colin-Side}) = 0.2*(1-p)+0.8*p$$

As stated earlier, the goal is to try to remove the opponent's decision from consideration. If these two expected values are equal, it does not matter what Colin plays. One can solve the following equation with one unknown variable.

$$
\begin{aligned}
EV(\text{Colin-Center}) &= EV(\text{Colin-Side}) \\
0.7*(1-p)+0.3*p &= 0.2*(1-p)+0.8*p \\
0.7-0.7p+0.3p &= 0.2-0.2p+0.8p \\
0.7-0.4p &= 0.2+0.6p \\
-1.0p &= -0.5 \\
p &= 0.5 \\
1-p &= 0.5 \\
\text{Value of the Game} &= 0.7*0.5+0.3*0.5 = 0.35+0.15 = \underline{\underline{0.5}}
\end{aligned}
$$

The solution for Rose: Rose's optimal strategy is to attack 50 percent at the center and 50 percent at the side. She can expect to have a success ratio of 0.5.

Similarly, from Colin's point of view:

If Rose would play her Pure-Side, Colin can expect the payoff to be:

$$EV(\text{Rose-Side}) = 0.7*(1-q)+0.2*q$$

If Rose would play her Pure-Center, Colin can expect the payoff to be:

$$EV(\text{Rose-Center}) = 0.3*(1-q)+0.8*q$$

Equalizing and solving:

$$
\begin{aligned}
EV(\text{Rose-Side}) &= EV(\text{Rose-Center}) \\
0.7*(1-q)+0.2*q &= 0.3*(1-q)+0.8*q \\
0.7-0.7q+0.2q &= 0.3-0.3q+0.8q \\
0.7-0.5q &= 0.3+0.5q \\
-1.0q &= -0.4 \\
q &= 0.4 \\
1-q &= 0.6 \\
\text{Value of the game} &= 0.7*0.6+0.2*0.4 = 0.42+0.08 = \underline{\underline{0.5}}
\end{aligned}
$$

The solution for Colin: Colin's optimal strategy is to prepare 60 percent at the center and 40 percent at the side. He can expect to lower Rose's success ratio to 0.5.

The expected value principle is used in linear programming techniques for solving games with many strategies.

3. William’s Method of Oddments

William’s method of oddments is used for solving 2x2 zero-sum games without a saddle point (check for pure strategy solution first). Some people find it easier than the expected value method. The weakness of the method is in its limited usability; it is valid only for games with two players and two strategies each.

$$
\begin{array}{lcccccc}
 & & \multicolumn{2}{c}{\text{Colin - blocking}} & & & \\
 & & \text{center} & \text{side} & & & \\
 & \text{side} & 0.7 & 0.2 & |0.7-0.2| = |0.5| & 0.5 & 0.5/(0.5+0.5) = 0.5 \\
\text{Rose - attacking} & & & & & & \\
 & \text{center} & 0.3 & 0.8 & |0.3-0.8| = |0.5| & 0.5 & 0.5/(0.5+0.5) = 0.5 \\
 & & |0.7-0.3| & |0.2-0.8| & & & \\
 & & |0.4| & |0.6| & & & \\
 & & 0.6 & 0.4 & & & \\
 & & 0.6/(0.6+0.4) & 0.4/(0.6+0.4) & & & \\
 & & 0.6 & 0.4 & & &
\end{array}
$$

The method is shown above. For Rose, one must take absolute value of the difference between the payoffs for her respective strategy. If one interchanges the absolute values, the ratio with which she should play the strategies is then discovered. The same procedure is repeated for Colin. Simple but limited.

B. REDUCTION OF THE GAME USING ELIMINATION OF THE DOMINATED STRATEGIES

When one tries to solve the game using pen and paper, it is useful to simplify the game by trying to reduce the number of strategies of each player. Assuming that no rational player would play the dominated strategy, the payoffs in this strategy can be deleted from the matrix. After deleting a dominated strategy, one forms a new matrix and checks for any other dominated strategy. The process is repeated until there are no dominated strategies. Even when players do not have the dominant strategy in the original game, after the reduction there can be a dominant strategy. At the very least, any irrelevant strategies have been eliminated.

Example:

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{ccccc} C1 & C2 & C3 & C4 & C5 \end{array} \\ \text{Rose} & \begin{array}{c} R1 \\ R2 \\ R3 \\ R4 \\ R5 \end{array} & \begin{pmatrix} 4 & 2 & 5 & 2 & 3 \\ 2 & 1 & 0 & 2 & 2 \\ 3 & 2 & 4 & 2 & 4 \\ 5 & 0 & 6 & 1 & 1 \\ 1 & 3 & 7 & 8 & 7 \end{pmatrix} \end{array}$$

Rose-R2 is dominated by Rose-R3➔ Delete

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{ccccc} C1 & C2 & C3 & C4 & C5 \end{array} \\ \text{Rose} & \begin{array}{c} R1 \\ R2 \\ R3 \\ R4 \\ R5 \end{array} & \begin{pmatrix} 4 & 2 & 5 & 2 & 3 \\ & & & & \\ 3 & 2 & 4 & 2 & 4 \\ 5 & 0 & 6 & 1 & 1 \\ 1 & 3 & 7 & 8 & 7 \end{pmatrix} \end{array}$$

Colin-C3, C4 and C5 are dominated by Colin-C2 ➔ Delete

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{ccccc} C1 & C2 & C3 & C4 & C5 \end{array} \\ \text{Rose} & \begin{array}{c} R1 \\ R2 \\ R3 \\ R4 \\ R5 \end{array} & \begin{pmatrix} 4 & 2 & & & \\ & & & & \\ 3 & 2 & & & \\ 5 & 0 & & & \\ 1 & 3 & & & \end{pmatrix} \end{array}$$

Rose-R3 is dominated by Rose-R1 ➔ Delete R3. The game has been reduced from 5x5 to 3x2.

Resulting game after the reduction of irrelevant (dominated) strategies

$$
\begin{array}{ll}
 & \text{Colin} \\
 & \begin{array}{cc} \text{C1} & \text{C2} \end{array} \\
\text{Rose} & \begin{array}{c} \text{R1} \\ \text{R4} \\ \text{R5} \end{array} \begin{pmatrix} 4 & 2 \\ 5 & 0 \\ 1 & 3 \end{pmatrix}
\end{array}
$$

V. PARTIAL-SUM GAMES

In the partial-sum (non zero-sum) game, the payoffs of the players are not strictly opposed. The success of one player does not always mean failure for the other. As the interests of the players are not totally in conflict, such a game offers the opportunities for cooperation in order to achieve mutually advantageous outcomes. These opportunities do not rule out the competitive side of the game. Players still want to achieve their best possible outcome.

Cooperation requires communication in order to achieve some coordinated strategy. As communication is the key component, partial-sum will be analyzed with three different assumptions. The game can be played:

1) Without Communication

2) With Communication before the game

3) With Cooperation

A. PARETO PRINCIPLE

As is shown in the previous chapters, it is rational for the player to play a dominant strategy (if he has one) in the zero-sum game. Does this principle apply in the partial-sum games? The following game attempts to answer this question.

$$
\begin{array}{cc}
 & \text{Colin} \\
 & \begin{array}{cc} \quad C & \quad\quad D \end{array} \\
\text{Rose} & \begin{array}{c} A \\ B \end{array}\begin{pmatrix} (3, 3) & (1, 4) \\ (4, 1) & (2, 2) \end{pmatrix}
\end{array}
$$

Both players have a dominant strategy. Rose gets better outcomes by playing Rose-B and Colin by playing Colin-D. When both players use their dominant strategy the result is BD [2,2]. However, looking at the game, outcome AC [3,3] is better for both of them. In this game, use of the dominant strategy leads to the less preferable outcome.

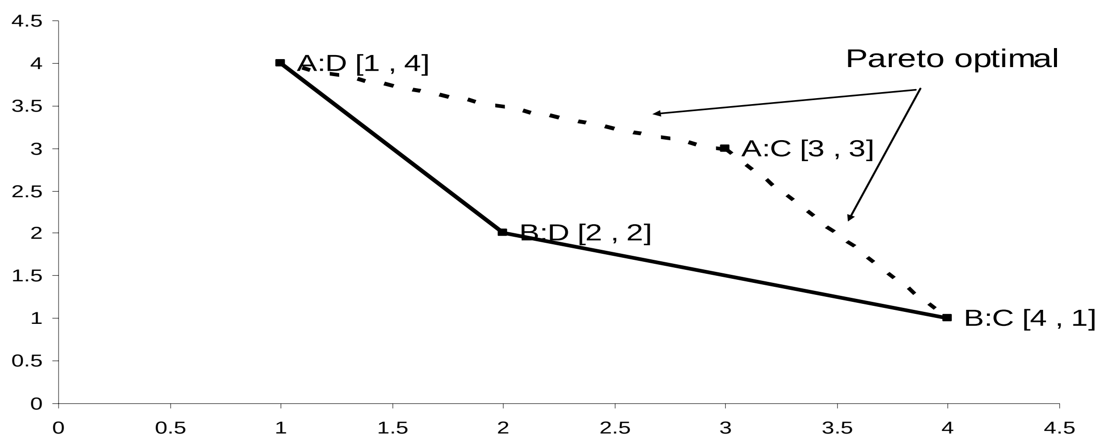
Economist Wilfredo Pareto proposed that one should not accept the solution when there is some other possibility, which is better for everybody involved.

Pareto Principle: "To be acceptable as a solution of the game, an outcome should be Pareto Optimal." [9]

Pareto Optimal: The outcome where neither player can improve payoff without hurting (decreasing the payoff) of the other player.

As in this case, group rationality (Pareto) is sometimes in conflict with the individual rationality (dominant). The eventual outcome depends on the players. Obtaining a Pareto optimal outcome usually requires some sort of communication and cooperation among the players.

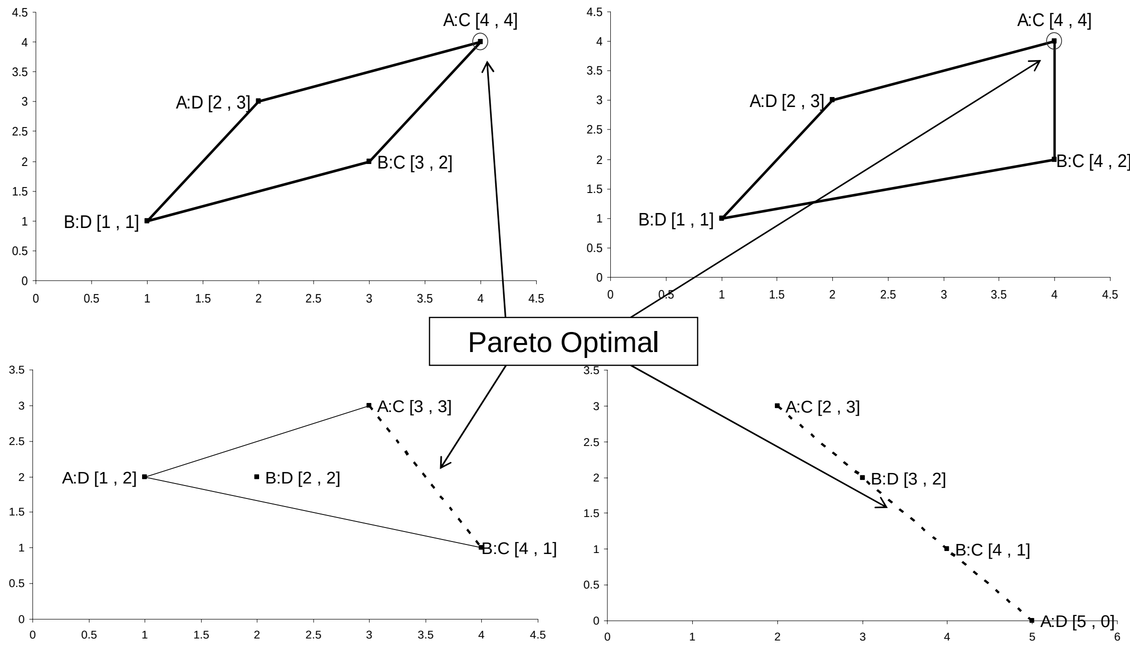
With the assumption that the outcome should be Pareto optimal, the next question is, "What is Pareto optimal, and what is it not (Pareto inferior)?" The simplest way for this to be understood is to draw a **payoff polygon** of the game. On the chart, the X-axis depicts the payoffs of Rose, and the Y-axis depicts the payoffs of Colin. By plotting the pure strategy solutions on the chart, one can see that the convex (everything inside) polygon enclosing the pure strategy solutions is then the **payoff polygon** or the **feasible region**. Therefore, the points inside the polygon are the possible solutions of the game. Graphically:



[9] Philip D.Straffin, *Game Theory and Strategy*, New Mathematical Library, Vol. 36, (Washington: Mathematical Association of America, 1993), 69.

The solution point is Pareto optimal when there is not some other possible solution point north / east / north-east of this point. If one were to imagine (draw) lines heading north and east from this point and find that there are no possible solutions in this quadrant, the point is Pareto optimal. Pareto optimal points form the northeastern boundary of the payoff polygon. In the chart above, it is line AD-AC, and AC-BC.

Pareto optimal can be just a single point, line segment, or several line segments. Examples of some polygons and their Pareto optimal outcomes:



The lower right game is actually a constant sum game, which can be converted into a zero-sum. In the zero-sum game, all outcomes are Pareto optimal.

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VI. PARTIAL-SUM GAMES WITHOUT COMMUNICATION

A. EQUILIBRIUMS

As the zero-sum games have a saddle point, the partial-sum games have equilibriums. Correspondingly, once the players play the strategies forming equilibrium, they cannot unilaterally improve their payoffs.

For finding the equilibrium outcomes in the pure strategy, the same idea of movement diagram, as in the zero-sum games, is used. Only this time each player has a separate set of payoffs. It is necessary to compare appropriate values.

$$\begin{array}{cc} & \text{Colin} \\ & \begin{array}{ccc} C & \quad & D \end{array} \\ \text{Rose} \begin{array}{c} A \\ \\ B \end{array} & \begin{pmatrix} (3, 3) & \rightarrow & (1, 4) \\ \downarrow & & \downarrow \\ (4, 1) & \rightarrow & \boxed{(2, 2)} \end{pmatrix} \end{array} \qquad \begin{array}{cc} & \text{Colin} \\ & \begin{array}{ccc} C & \quad & D \end{array} \\ \text{Rose} \begin{array}{c} A \\ \\ B \end{array} & \begin{pmatrix} \boxed{(4, 4)} & \leftarrow & (1, 3) \\ \uparrow & & \downarrow \\ (3, 1) & \rightarrow & \boxed{(2, 2)} \end{pmatrix} \end{array}$$

In the left game, both players have a dominant strategy and BD [2,2] is the probable outcome without communication. Rose-B and Colin-D is also an equilibrium outcome, as all arrows are heading in.[10] The right game has two equilibriums, AC [4,4] and BD [2,2].

In the zero-sum games, saddle points do not always exist. The same applies for partial-sum games. To illustrate, one should consider the game below. This game does not have pure strategy equilibrium. There is no point where all arrows are heading in.

$$\begin{array}{cc} & \text{Colin} \\ & \begin{array}{ccc} C & \quad & D \end{array} \\ \text{Rose} \begin{array}{c} A \\ \\ B \end{array} & \begin{pmatrix} (3,4) & \leftarrow & (2,3) \\ \downarrow & & \uparrow \\ (4,1) & \rightarrow & (1,2) \end{pmatrix} \end{array}$$

[10] More precisely, none of the arrows is heading out.

John F. Nash proved that every two-person game has at least one equilibrium either in Pure or in Mixed strategies.[11] The equilibriums are also called Nash Equilibriums. Nash Equilibrium in the mixed strategy is formed by equalizing strategies of the respective players.

B. EQUALIZING STRATEGIES

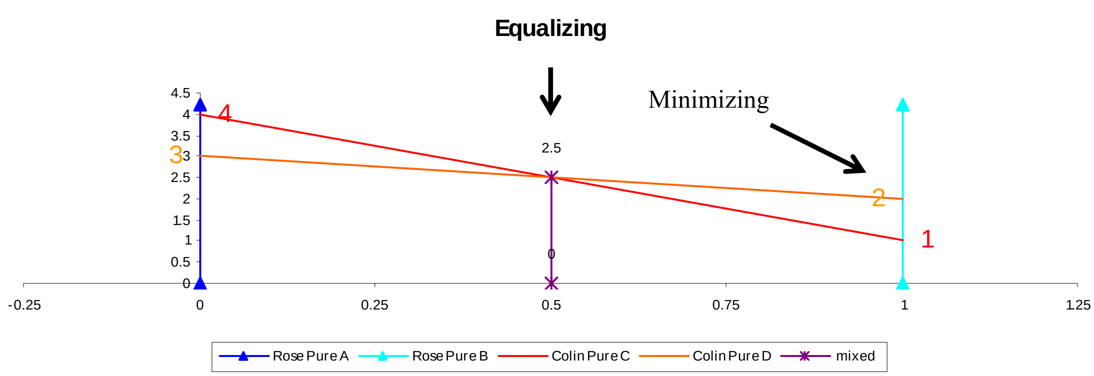
Equalizing strategies, if adopted, assure that neither player can gain by switching to some other strategy. The use of equalizing strategy 'stymies' the other player's position and removes his choices from consideration.

In the game above, Rose can equalize the payoffs of Colin when she plays Colin's game and Colin's game is the zero-sum game with Colin's payoffs.

Colin's Game / Rose - Equalizing / Colin Maximizing

$$
\begin{array}{ccc}
 & & \text{Colin} \\
 & & \begin{array}{ccc} C & & D \end{array} \\
\text{Rose} & \begin{array}{c} A \\ \\ B \end{array} & \begin{pmatrix} 4 & \leftarrow & 3 \\ \downarrow & & \downarrow \\ 1 & \rightarrow & 2 \end{pmatrix}
\end{array}
$$

This game has a saddle point at BD. However, B and D are not equalizing strategies. They do not equalize the Colin Payoffs. The graph shows the solution.



[11] John F. Nash, "Equilibrium Points in n-Person Games," *Proceedings of the National Academy of Sciences of the United States of America* 36, no. 1 (January 15, 1950): 48-49.

In this case, Rose's Equalizing strategy ½ A and ½ B is not Rose's optimal strategy. However, it equalizes Colin's payoffs. It does not matter what Colin does; his payoff will be 2.5. Rose can stymie Colin's position when she plays the intersection of the lines Colin-Pure C and Colin-Pure D. For calculation of the probabilities one can use either the expected value method or method of oddments. One would do well to remember, however, that probabilities have to comply with:

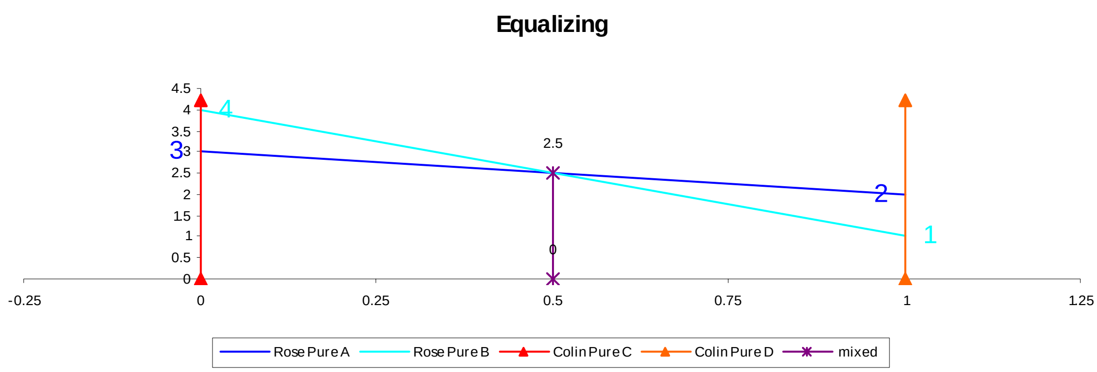
$$p_1, p_2 \in [0,1]$$
$$p_1 + p_2 = 1$$

In other words, they have to intersect between Rose-Pure A and Rose-Pure B**;** otherwise, the equalizing strategy does not exist.

Similarly, Colin can equalize Rose's payoffs by playing an equalizing strategy in Rose's game.

Rose's game / Colin - Equalizing / Rose - Maximizing

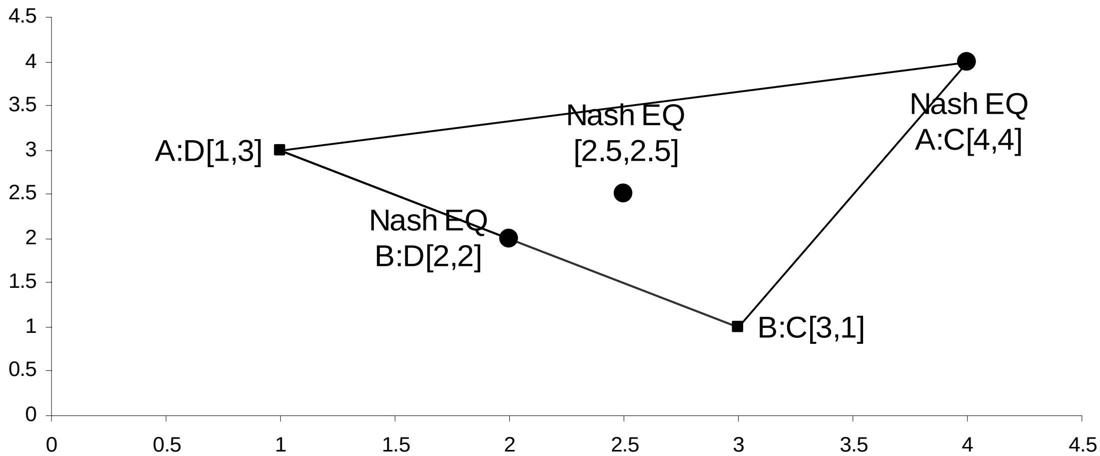
$$\begin{array}{cc} & \text{Colin} \\ & \begin{array}{ccc} \text{C} & & \text{D} \end{array} \\ \text{Rose} & \begin{array}{c} \text{A} \\ \\ \text{B} \end{array} \begin{pmatrix} 3 & \rightarrow & 2 \\ \downarrow & & \uparrow \\ 4 & \rightarrow & 1 \end{pmatrix} \end{array}$$



Colin's equalizing strategy is ½ C and ½ D. This will give Rose the payoff 2.5. If both players play equalizing strategy**,** the result is the Nash Equilibrium point [2.5,2.5].

Nash equilibriums in partial-sum games have some problematic properties. In the zero-sum games, saddle points were equivalent and interchangeable. They were also Pareto optimal. None of these carry over to partial-sum games.

$$\begin{array}{c|cccc} & & & \text{Colin} & \\ & & C & & D \\ \hline \text{Rose} & A & \boxed{(4, 4)} & \leftarrow & (1, 3) \\ & & \uparrow & & \downarrow \\ & B & (3, 1) & \rightarrow & \boxed{(2, 2)} \end{array}$$



The game above has three Nash Equilibriums**:** two in pure strategies and one in mixed strategy. Each one of the Nash Equilibriums, once adopted, fixes the players' payoffs, and players can not unilaterally improve their position. Each equilibrium is different, so for which one should they try? In this game, the answer is simple because point AC[4,4] is Pareto optimal. However, this is not always the case.

As the outcomes for Nash Equilibrium in a mixed strategy are obtained from the other player's game without regard to one's own payoffs, the resulting outcome for the players is usually low.

C. PRUDENTIAL STRATEGIES – STATUS QUO

The player's worst-case scenario in the partial-sum game happens when the opponent turns hostile. The opponent's goal is no longer to maximize his own payoffs, but to minimize the payoffs of the other. What should the player do in such a case? What is his optimal strategy?

A consideration of the following game will help illustrate:

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{cc} C & D \end{array} \\ \text{Rose} & \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} (2,1) & (3,2) \\ (4,3) & (1,4) \end{pmatrix} \end{array}$$

When Colin turns hostile and tries to minimize Rose's payoffs, he disregards his own payoffs. Now it is a zero-sum game with Rose's payoffs, and Rose should play her MaxiMin strategy in order to assure herself at least the value of game.

Rose's Game / Rose - Maximizing / Colin - Minimizing

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{cc} C & D \end{array} \\ \text{Rose} & \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \end{array}$$

Rose's optimal strategy in this game is to play ¾ A and ¼ B with the value of the game 2.5. In a partial-sum game, this strategy is called Rose's **prudential strategy** and the corresponding value of the game is called Rose's **security level**. No matter what Colin does, Rose can guarantee herself at least the security level.

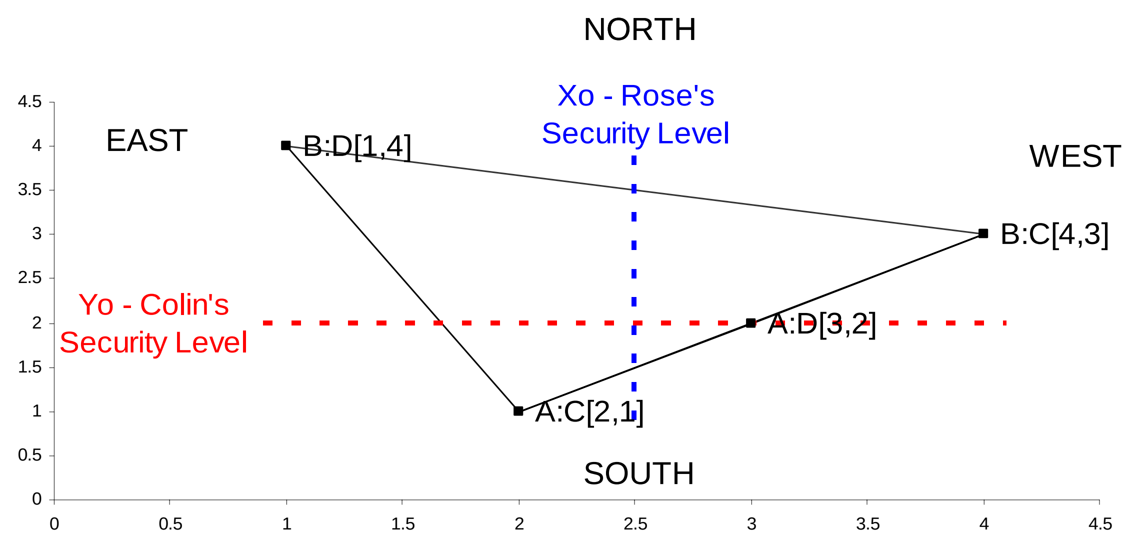
Similarly for Colin, when Rose turns hostile, Colin should respond with his prudential strategy in his game.

Colin's Game / Colin - Maximizing / Rose - Minimizing

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{cc} C & D \end{array} \\ \text{Rose} & \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \end{array}$$

Colin's game has a saddle point at AD**,** and his prudential strategy is to play D all the time with the security level (value of the game) equal to 2.

The intersection of the security levels is called a **status quo.** The players can not do worse even in the worst possible scenario. From the payoff polygon of the game below, it is obvious that the outcome is not Pareto Optimal. The outcome also is not equilibrium. Both players can do better by playing some other strategy mix. There is still a possibility of moving in the Northeast direction.



Minimax strategies in the zero-sum games produce stability. This does not apply in the partial-sum games and "there is no cogent general solution concept for non-zero-sum games."[12] However, games where there exists at least one Pareto optimal equilibrium outcome or there are more such outcomes**,** which are equivalent and interchangeable**,** are solvable in the strict sense. These Pareto optimal equilibriums are reasonable outcomes of the game[13] in the games without communication.

[12] Philip D.Straffin, *Game Theory and Strategy*, New Mathematical Library, Vol. 36, (Washington: Mathematical Association of America, 1993), 70.

[13] Ibid., 70.

VII. PARTIAL-SUM GAMES – COMMUNICATION BEFORE THE GAME

Up to this point, the games were played without communication between the players. This chapter gives an overview of what can happen when the players can communicate and how communication influences the outcomes of the game.

In this chapter, players will try to unilaterally improve their position by making conditional or unconditional commitments called **strategic moves**. They will not cooperate. The problem of communication lies in credibility. The purpose of this text is not to offer a guide for achieving credibility of communication. Rather, the thesis focuses on analyzing what options players have, and the credibility of their commitment is taken as a given.

A. FIRST MOVE

The first move can be described as the ability of the player to either make a move (play a strategy) before the other player or make a commitment to play some strategy under all circumstances. These options are considered interchangeable during an analysis of the game. The critical question remains: is it preferable for the player to play first or force the other player to move first?

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{cc} C & \quad D \end{array} \\ \text{Rose} & \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} (4,2) & (1,1) \\ (2,3) & (3,4) \end{pmatrix} \end{array}$$

In this game, neither player has a dominant strategy. The likely outcome, without communication, can be BC [2,3] as the intersection of maximin strategies Rose-B, Colin-C. Can the players improve their outcome by playing the first move or forcing the other to play?

The illustration begins with Rose: What will happen when Rose plays A and what will happen when she plays B? If Rose plays Rose-A, then Colin, looking at his outcome, would choose Colin-C, as it gives him a higher payoff than Colin-D. The result is AC [4,2]. If Rose plays Rose-B, then Colin replies with D, and the outcome is BD [3,4]. By comparing these two outcomes, one can see that it is better for Rose to play A, as it gives Rose her best outcome with the payoff 4. Still further one can question whether the outcome is better than the likely outcome without communication. In this case the answer is yes**;** therefore**,** it is preferable for Rose to play first (Rose-A) in order to get her best outcome.

$$\begin{array}{ll} \text{If Rose A then Colin C} & [\boxed{4},2] \\ \text{If Rose B then Colin D} & [\boxed{3},4] \\ \text{Better for Rose} & [\boxed{4},2] \end{array}$$

Better than Rose's likely outcome? Yes

The same can be done for Colin. Colin has a first move to play D with Rose responding B. This gives Colin his best outcome DB [3,4].

$$\begin{array}{ll} \text{If Colin C then Rose A} & [4,\boxed{2}] \\ \text{If Colin D then Rose B} & [3,\boxed{4}] \\ \text{Better for Colin} & [3,\boxed{4}] \end{array}$$

Better than Colin's likely outcome? Yes

To illustrate further, another example is given. Again the players do not have dominant strategies so the likely outcome, when Rose and Colin play maximin strategies, is AC [2,3]

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{cc} \text{C} & \text{D} \end{array} \\ \text{Rose} & \begin{array}{c} \text{A} \\ \text{B} \end{array} & \begin{pmatrix} (2,3) & (3,1) \\ (4,2) & (1,4) \end{pmatrix} \end{array}$$

First move Rose:

$$\begin{array}{ll} \text{If Rose A then Colin C} & [\boxed{2},3] \\ \text{If Rose B then Colin D} & [\boxed{1},4] \\ \text{Better for Rose} & [\boxed{2},3] \end{array}$$

Better than Rose's likely outcome? Equal

Rose has a first move, but the result is not better than the likely outcome without communication. Nevertheless, Rose can secure her likely outcome by playing Rose-A.

First move Colin:

$$\begin{array}{lll} \text{If Colin C then Rose B} & [4, & \boxed{2}] \\ \text{If Colin D then Rose A} & [3, & \boxed{1}] \\ \quad\text{Better for Colin} & [4, & \boxed{2}] \end{array}$$

Better than Colin's likely outcome? No

Colin does not have a first move. In both cases, Rose can respond with a strategy that is worse for Colin than the likely outcome. Looking at Rose's payoffs one can see that it is beneficial for Rose to force Colin to move first. If Colin has to move, he would likely choose to play C (better than D). Rose would then play Rose-A and get her best outcome.

The Game of Chicken is an example of a game where both players can get their best outcome by making the first move.

B. THREAT

Threat is one type of conditional commitment. It is a commitment to play a certain strategy as a reaction to the opponent's choice of strategy. In the case of threat, it **hurts both players**. If the other player believes it (the threat is credible), one of the pure strategy solutions is taken out of consideration. How can one know whether the players have the option of making a threat?

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{cc} \text{C} & \text{D} \end{array} \\ \text{Rose} & \begin{array}{c} \text{A} \\ \text{B} \end{array} & \begin{pmatrix} (1, 1) & (3, 2) \\ (4, 3) & (2, 4) \end{pmatrix} \end{array}$$

In this game, Colin has a dominant strategy Colin-D and the likely outcome without communication can be AD [3,2]. The example begins with Colin. Does Colin have a threat?

Without communication, Rose plays A. However, Colin would like to force Rose to play B, so his threat is against A. He says, “If you (Rose) play A, I (Colin) will sacrifice my payoffs in order to hurt you. Normally I would play D, but if you play A, I will respond with C.”

The following analyzes whether this is a workable threat:

Normally:
If Rose A then Colin D [3 ,2]
Threat:
If Rose A then Colin C [1],1] It hurts Colin, hurts Rose it is a threat
If Rose B then Colin D [2],4]
Better for Rose [2],4]
Better than Colin's likely outcome? Yes

Colin would normally respond with D at Rose-A. His threat is to play C as a response to Rose-A. The threat hurts both players, as their payoffs are lower than an outcome without communication. If Rose plays B, Colin will play D. Now Rose has to decide what is better for her. She chooses between AC [1,1] and BD[2,4]. Therefore it is better for Rose to play B**,** as it gives her a higher payoff (1<2). The outcome of the game is then BD, and Colin gets his best outcome. **Colin has a threat, and it works alone**.

In the same vein, Rose is analyzed. She would like to force Colin to play C and she threatens D.

Normally:
If Colin D then Rose A [3 ,2]
Threat:
If Colin D then Rose B [2 ,4] It hurts Rose, but it is benficial to Colin - Not a threat

Rose does not have a threat, as her conditional commitment would be beneficial to Colin.

The next game is an example where **Rose has a threat, but it does not work independently**. Colin has a dominant strategy C and the likely outcome, without communication, would be AC [2,4].

$$\begin{array}{cc} & \text{Colin} \\ & \begin{array}{cc} C & D \end{array} \\ \text{Rose} & \begin{array}{c} A \\ B \end{array} \begin{pmatrix} (2, 4) & (3, 3) \\ (1, 2) & (4, 1) \end{pmatrix} \end{array}$$

Rose would like to force Colin to play D, and her threat is focused on C.

Normally:
If Colin C then Rose A [2 , 4]
Threat:
If Colin C then Rose B [1, $\boxed{2}$] It hurts Colin, hurts Rose; it is a threat
If Colin D then Rose B [4, $\boxed{1}$]
Better for Colin [1, $\boxed{2}$]

Even with Rose's threat, it is still better for Colin to play C. By playing C, Colin gets 2 which is a better outcome than complying with the threat and getting 1. Sometimes when a player has a threat which does not work by itself, the player can combine it with some other conditional move.

C. PROMISE

Another type of conditional move is called the promise. The **promise is hurtful for a player and beneficial to the opponent**. As in the case of the threat, a promise has an ability to remove one pure strategy solution from consideration. Again, it is necessary to first explore whether the player has the option to make a promise and then how the game would evolve.

$$\begin{array}{cc} & \text{Colin} \\ & \begin{array}{cc} C & D \end{array} \\ \text{Rose} & \begin{array}{c} A \\ B \end{array} \begin{pmatrix} (2, 2) & (3, 1) \\ (4, 3) & (1, 4) \end{pmatrix} \end{array}$$

In this game, neither player has a dominant strategy. The players would probably play their maximin strategies. The likely outcome, without communication, is AC [2,2].

Colin would like to persuade Rose to play B. His promise then focuses on this strategy.[14] Normally, if Rose plays B, Colin responds with D and the resulting payoff is BD[1,4]. However, Colin promises to hurts himself and plays Colin-C. It would look like the following:

Normally:

If Rose B then Colin D $[1\ ,4]$

Promise:

If Rose B then Colin C $[\boxed{4},3]$ It hurts Colin, beneficial to Rose; it is a promise

If Rose A then Colin C $[\boxed{2},2]$

Better for Rose $[\boxed{4},3]$

Better than Colin's likely outcome? Yes

The conditions for the existence of promise have been met. The promise hurts Colin and is beneficial to Rose. For Rose it is advantageous to comply**;** she can get her best outcome. By doing so, she allows Colin to get his second best outcome, which is better than the likely outcome without communication. Colin has a threat, which works independently.

Now to consider a previous game where Rose has a threat which does not work alone; does she have a promise? As a reminder, Colin has a dominant strategy C and the likely outcome without communication would be AC[2,4].

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{cc} \text{C} & \text{D} \end{array} \\ \text{Rose} & \begin{array}{c} \text{A} \\ \text{B} \end{array} & \begin{pmatrix} (2, 4) & (3, 3) \\ (1, 2) & (4, 1) \end{pmatrix} \end{array}$$

Rose would like Colin to play D. Her promise focuses on D.

[14] Threat is focused (threatens) on the strategy which the player would like to eliminate and promise is focused on strategy he would like the other player to play.

Normally:

If Colin D then Rose B $[4\ \ ,1]$

Promise:

If Colin D then Rose A $[3,\boxed{3}]$ It hurts Rose, beneficial to Colin; it is a promise

If Colin C then Rose A $[2,\boxed{4}]$

Better for Colin $[2,\boxed{4}]$

Rose has a promise to play A in case Colin plays D. It hurts her and is beneficial to Colin. However, it is still better for Colin to play C and get his best outcome.

D. COMBINATION OF THREATS AND PROMISES

In the last game, Rose has a threat and promise and neither one works independently. What if Rose were to make the threat and promise together?

$$
\begin{array}{ccc}
 & & \text{Colin} \\
 & & \begin{array}{cc} C & \quad D \end{array} \\
\text{Rose} & \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} (2,4) & (3,3) \\ (1,2) & (4,1) \end{pmatrix}
\end{array}
$$

Threat:

If Colin C then Rose B $[1,\boxed{2}]$

Promise:

If Colin D then Rose A $[3,\boxed{3}]$

Better for Colin $[3,\boxed{3}]$

In this case, when threat eliminates outcome AC [2,4] and promise eliminates BD [4,1], Colin has to choose between the remaining two options. It is better for him to play D with the result AD [3,3]. The result is second best for Rose, and the combination of threat and promise works for her.

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VIII. PARTIAL-SUM GAMES – COOPERATIVE SOLUTION

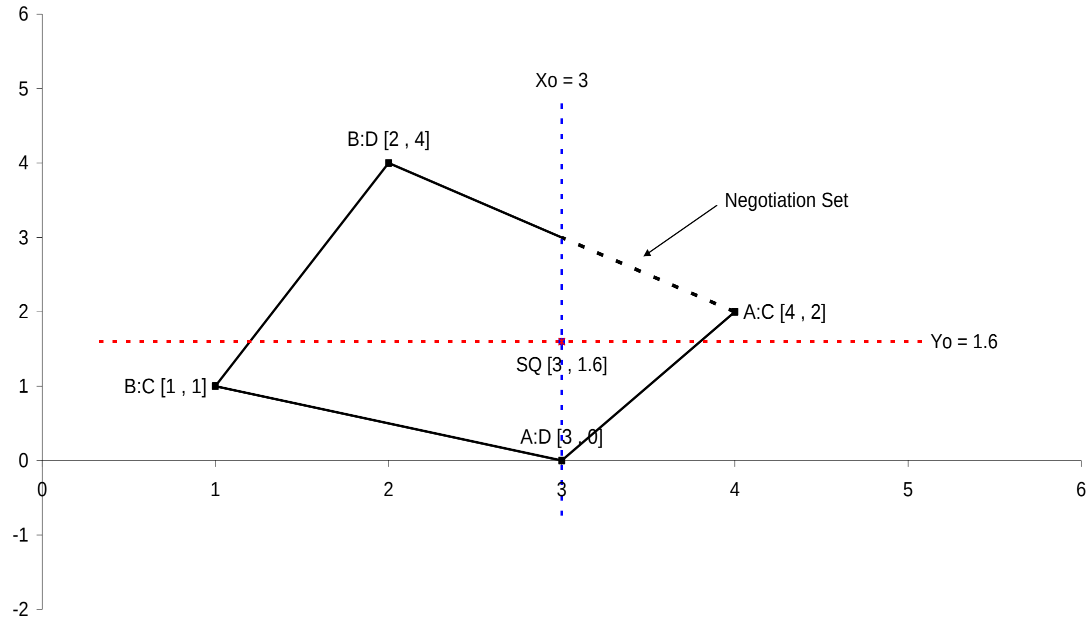
Up to this point, this thesis has analyzed the partial-sum game with the assumption of non-cooperation between the players. The players either did not communicate at all, or tried to improve their position unilaterally by using some sort of strategic moves. The results were not always Pareto optimal; this means that there was room for improvement for at least one of the players without hurting the other. Now the thesis looks at the game from a different strategic perspective.

Assumption: the players decide to cooperate in order to improve their payoffs. They can sit behind the negotiation table and make an agreement or they can call for the help of an outside arbiter to solve their problem.

Von Neumann and Morgensten proposed that the arbitrated solution to the partial-sum game should be (1) Pareto optimal and (2) at or above the security level of the players.[15] This appears reasonable. Pareto optimal outcome tells us that there is no other outcome better for both players or better for one player without hurting the other. The at-or-above the security level condition ensures that no player is forced to accept the solution that is worse for him than the solution of the game played without communication. These two combined conditions give the range of solutions from which to choose. They are called the negotiation set. Look at an example.

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{cc} \text{C} & \text{D} \end{array} \\ \text{Rose} & \begin{array}{c} \text{A} \\ \text{B} \end{array} & \begin{pmatrix} (4, 2) & (3, 0) \\ (1, 1) & (2, 4) \end{pmatrix} \end{array}$$

[15] John Von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behavior*, 60th anniversary ed. (Princeton, N.J.; Woodstock: Princeton University Press, 2004).



The next question that needs consideration involves the point from the negotiation set the arbiter should choose. Considering that there are many points in the negotiation set, one must discern which one is fair.

A. NASH ARBITRATION SCHEME

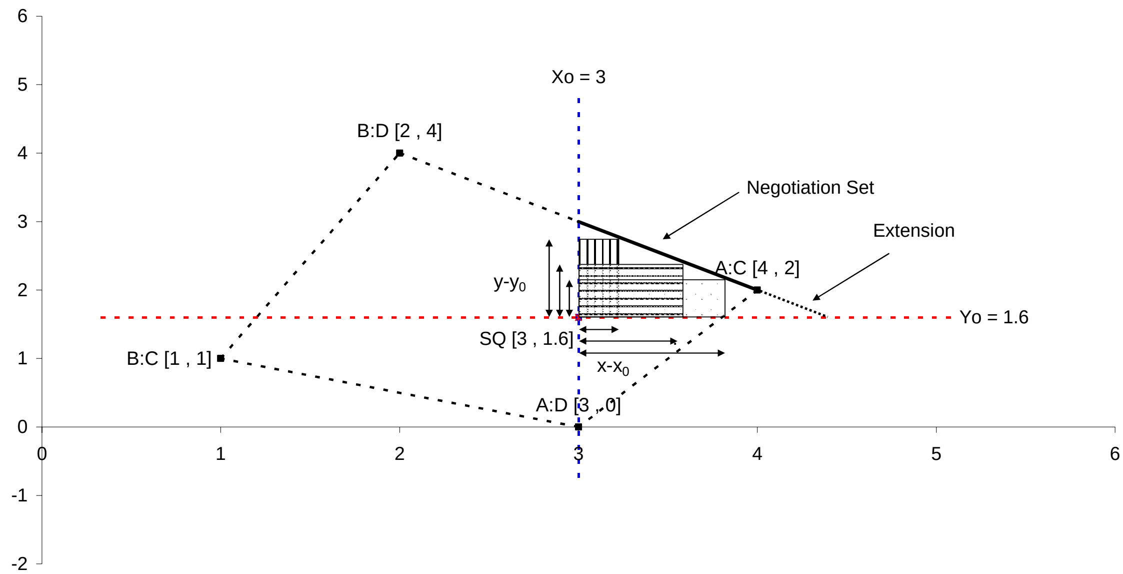
Definition:

"If SQ(status quo)=(x_0,y_0), then the arbitrated solution point N is the point (x,y) in the polygon with $x \geq x_0$ and $y \geq y_0$ which maximizes the product $(x- x_0)*(y- y_0)$."[16]

Status quo point in the definition is the likely outcome of the game when the negotiation fails. An arbitrated solution should be better for both players than the status quo; this is incorporated in the definition by $x \geq x_0$ and $y \geq y_0$. Status quo is the minimum the players can get. Everything above is improvement of their gain. The solution has to

[16] Philip D.Straffin, *Game Theory and Strategy*, New Mathematical Library, Vol. 36, (Washington: Mathematical Association of America, 1993), 105; John F. Nash, "The Bargaining Problem," *Econometrica* 18, no. 2 (April, 1950): 155.

maximize their joint utility. The objective function $(x - x_0)*(y - y_0)$, maximizes these 'above security level' utilities. In other words, it has to maximize the area of the rectangle.



Solution N is called the Nash Point in honor of John Nash. It turns out that a solution lies at ½ of the height (y-axis player) and ½ of the basis (x-axis player) of the triangle formed by the Pareto optimal line segment (sometimes has to be extended) and status quo lines. Such a solution is fair in some sense. It maximizes the available space and gives both players an equal part of what they can expect.

When the calculated Nash Point is outside of the Pareto optimal line, the solution of the game is the closest pure strategy solution (corner point). One could argue that this does not give both players an equal share**;** however, it is still the best (fairest) possible solution.

The arbitrated solution of the game in the Nash arbitration scheme depends on the status quo point. Again, one must consider what is likely to happen when the negotiation fails. The answer is beyond the scope of game theory, as it has to include many outside considerations. One possible choice is the status quo formed by the players' security levels, as this is the worst-case scenario. It can also be the status quo after some strategic moves were used. Nash argued for optimal threat status quo.[17]

Calculating the Nash Point as maximization of $(x - x_0)*(y - y_0)$ is quite complicated. An easier approach is described next. However, this equation is useful when more than one Pareto optimal line exists and one has to decide upon which line the Nash Point lies. This can be resolved by calculation of the payoff for each Pareto optimal pure strategy solution. The Nash point lies at the Pareto optimal lines with the highest pure strategy payoff.

B. NASH ARBITRATION – HOW TO SOLVE

$$\begin{array}{cc} & \text{Colin} \\ & \begin{array}{cc} \quad C & \qquad D \end{array} \\ \text{Rose} & \begin{array}{c} A \\ B \end{array}\begin{pmatrix} (4, 2) & (2, 0) \\ (1, 3) & (3, 4) \end{pmatrix} \end{array}$$

Rose and Colin are not satisfied with the results of their game above and decide to cooperate in order to increase their payoff. They bring their case to the arbiter, and he suggests using the Nash arbitration scheme for their problem. Rose and Colin know that in case their negotiation fails they would use prudential strategies and end up at the status quo SQ[2.5,2].

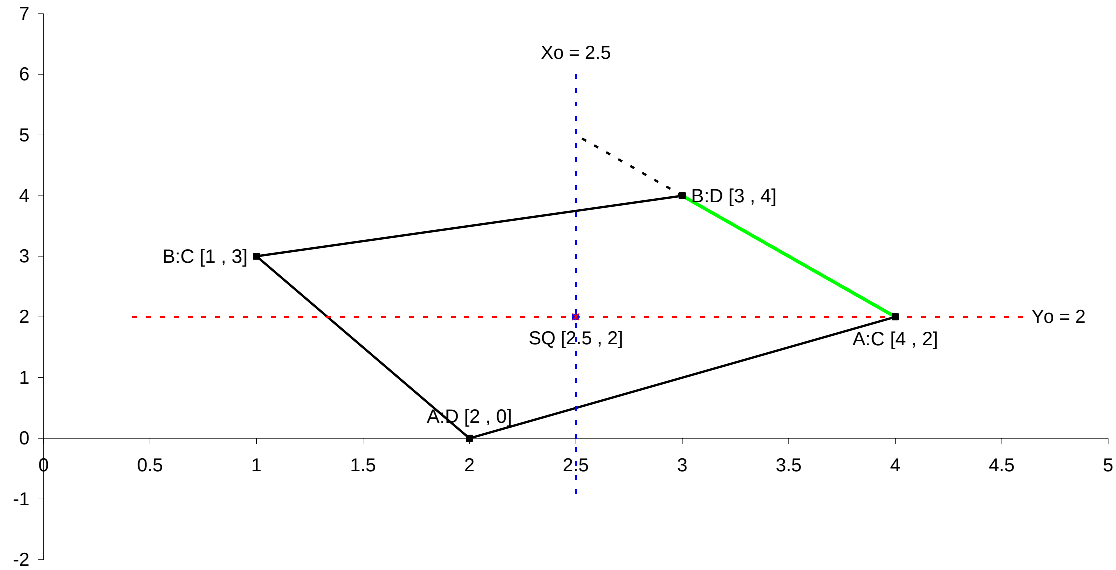
There are two options for solving the new problem. It can be done graphically or numerically. It is obvious that a graphical option is easier but less precise. The process is illustrated by using a graphical solution, which shows the interdependency of variables.

1. Graphical Solution of Nash Arbitration

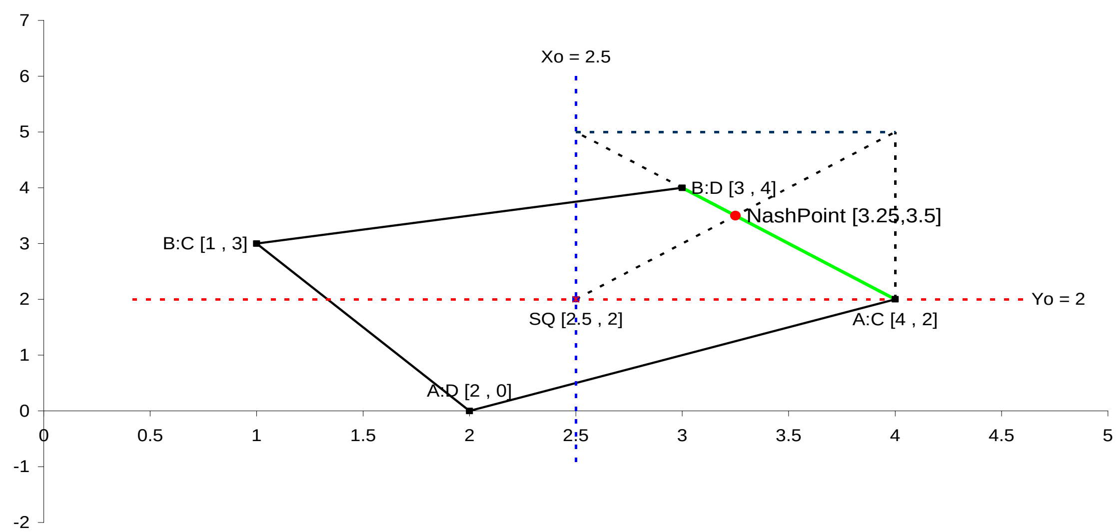
To begin, one draws a payoff polygon and security levels and determines the Pareto optimal line. If the line does not intercept both security levels, then the line must be lengthened in order to form a triangle.

The result looks like this.

[17] John F. Nash, "Two-Person Cooperative Games," *Econometrica* 21, no. 1 (January, 1953): 128.



The next step includes drawing a rectangle and its diagonal line in order to divide the height and basis in two**.** The Nash point is located at the intersection of the Pareto optimal line and the diagonal line. It is possible to measure its coordinates. If it is outside the negotiation set, the Nash Point is the closest strategy solution.



Once the Nash Point is found, one has to decide what the players should do in order to achieve an arbitrated outcome. The solution lies at the line segment BD-AC**;** therefore**,** they have to play these pure strategy options with certain relative frequency (probabilities). To discover how often the point should be played, one must measure the distance from one of the points to the Nash point and divide it with the overall length of the BD-AC line segment. One minus the result gives the answer. The same process applies for the second point or just the addition to one.

The results are that Rose and Colin should play BD 75% of the time and AC 25% of the time. They also have to agree upon some system that ensures that they will play their strategy in coordination. The game is the most difficult case for coordination. Because when Rose plays A, Colin has to play C, and when Rose plays B, Colin has to play D. Sometimes it can be easier when only one player has to change the strategy in order to get Nash Point.

2. Numerical Solution of Nash Arbitration

A simple formula for Nash Point**:**

$$\text{Nash Point}\left(X_0+\frac{b}{2|m|},Y_0+\frac{b}{2}\right)$$

Where *m* is the slope of the Pareto optimal line, X_0,Y_0 are the security levels of the respective players and *b* is the height of the triangle formed by Security Levels and Pareto optimal line.

To begin, one must calculate (enumerate) the **equation of the Pareto optimal line segment** from two known points:

$$BD=[3,4], AC=[4,2]$$
$$y=mx+b_{line}$$
$$m=\frac{Y_2-Y_1}{X_2-X_1}=\frac{2-4}{4-3}=\frac{-2}{1}=-2$$
$$b_{line}=Y_1-mX_1=4-(-2*3)=4-(-6)=10$$
$$y=-2x+10$$

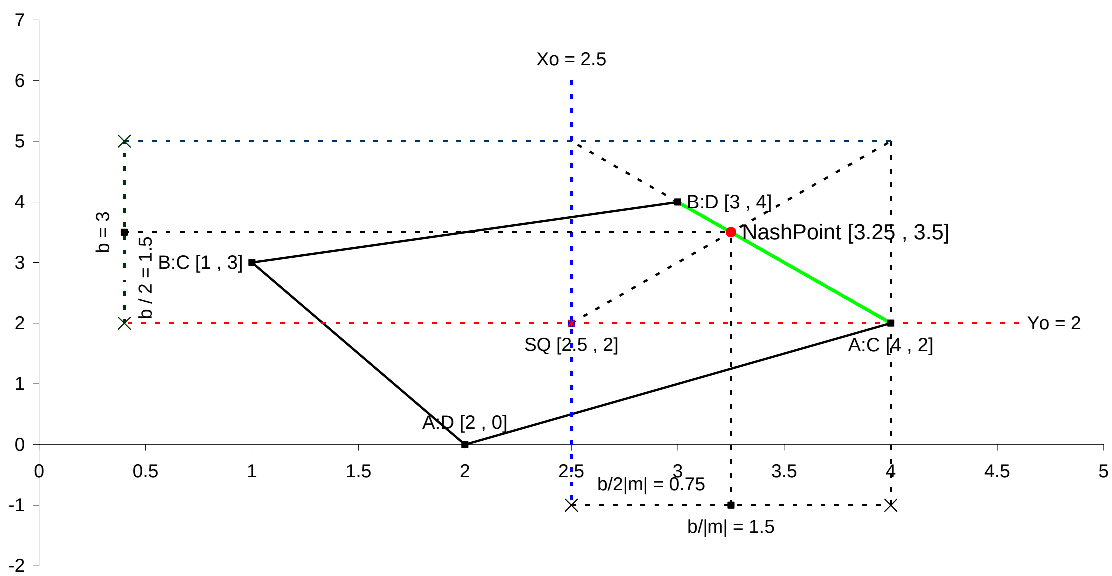
Next, one calculates the **height of the triangle,** which is the intercept of the Pareto optimal line and X Security Level and subtract Y_0**.**

$$b = \text{Height of the triangle}$$
$$Y_i = mX_0 + b_{line}$$
$$Y_i = -2 * 2.5 + 10 = -5 + 10 = 5$$
$$b = Y_i - Y_0$$
$$b = 5 - 2 = 3$$

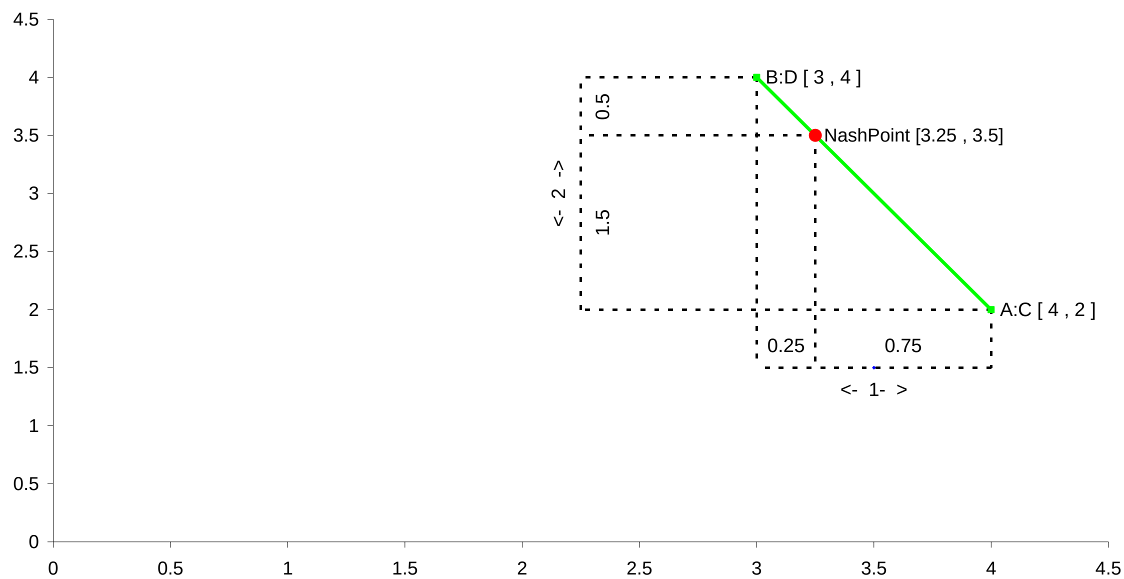
Finally, one has everything necessary to calculate the coordinates of the Nash Point.

$$\text{Nash Point}\left(X_0 + \frac{b}{2|m|}, Y_0 + \frac{b}{2}\right)$$
$$X_{Nash} = X_0 + \frac{b}{2|m|} = 2.5 + \frac{3}{2|-2|} = 2.5 + \frac{3}{4} = \underline{\underline{3.25}}$$
$$Y_{Nash} = Y_0 + \frac{b}{2} = 2 + \frac{3}{2} = \underline{\underline{3.5}}$$
$$\text{Nash Point } (3.25, 3.5)$$

A graphical depiction of the calculation follows below.



What remains is to calculate the relative frequency of the pure strategy options. For clearer understanding, the process is depicted in the graph below.



In the graphical method, the length of the line segments must be measured. It is possible to calculate the length of the line segment but there is an easier way: one can calculate the difference on the X-axis between the Nash Point and the Pure Strategy solution and divide it by the difference between the Pure Strategy solutions.

$1 - \frac{|X_{Nash} - X_{AC}|}{|X_{BD} - X_{AC}|}$ is the relative frequency of AC [18]

$1 - \frac{|X_{Nash} - X_{BD}|}{|X_{BD} - X_{AC}|}$ is the relative frequency of BD

Absolute value in the equation is there for convenience; it eliminates from consideration which number is higher and avoids the confusion of negative numbers. Using the equation**:**

[18] AC and BD are from the example, substitute for appropriate strategy combination.

AC[4,2] ; BD [3,4]; Nash Point[3.25,3.5]

$$1-\frac{|3.25-4|}{|4-3|}=1-\frac{|-0.75|}{|1|}=1\text{-}0.75=0.25\ \%\ \text{of AC}$$

and

$$1-\frac{|3.25-3|}{|4-3|}=1-\frac{|0.25|}{|1|}=1\text{-}0.25=0.75\ \%\ \text{BD}$$

The end result is that the players should cooperate and play pure strategy combination AC 25% of the time and pure strategy combination BD 75% of the time.

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IX. 3-PERSON GAMES

Up to this point, the thesis has analyzed games involving two players. However, most games in the real world involve more than two players, and as the number of players increases, the complexity of the game increases enormously. This thesis, however, limits the analysis to 3-person zero-sum (constant sum) games.

This chapter is an introduction into the vast area of n-person[19] games. It deals with the description of the 3-person game and equilibriums. Furthermore, the chapter covers the analysis of likely coalitions among the players and the idea of sidepayments.

To illustrate, a game with three players is considered. They will be called Rose, Colin and Larry. Each player has two strategies from which to choose; therefore, the game has 2x2x2=8 possible outcomes. This is described in the form of a tree diagram.

```
                          Larry-E  = ACE
              Colin-C
                          Larry-F  = ACF
Rose-A
                          Larry-E  = ADE
              Colin-D
                          Larry-F  = ADF

                          Larry-E  = BCE
              Colin-C
                          Larry-F  = BCF
Rose-B
                          Larry-E  = BDE
              Colin-D
                          Larry-F  = BDF
```

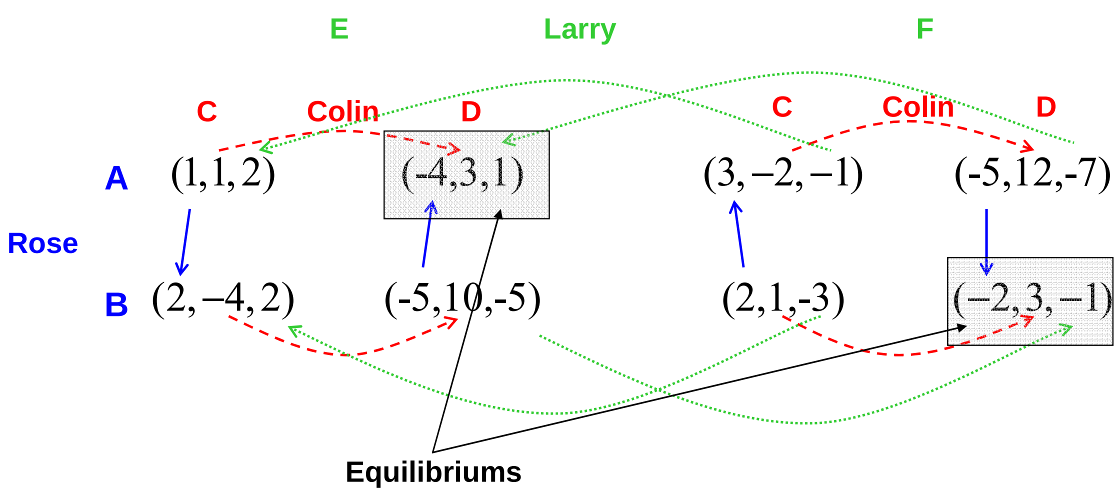
[19] n is three and more

Rose can pick from strategies A and B. Colin responds with C or D. Larry can play E or F for each strategy combination of the previous players. The solution and payoffs are then determined by a combination of the three pure strategies of the respective players.

A. EQUILIBRIUMS

As in two person games, it is possible to find equilibriums by using a movement diagram. It is just slightly more complicated than in two person games. It is necessary to compare appropriate values. For example, Larry's payoffs for BDE and BDF. The logic is as follows: when Rose plays B, and Colin plays D, what is better for Larry?

The following movement diagram analyzes a zero-sum game among Rose, Colin and Larry.



In this game, Colin has a dominant strategy, Colin-D. The idea of the dominant strategy is applicable even in the n-person games. Similarly as in partial-sum, a game playing dominant strategy does not assure the best outcome. Nevertheless, without the use of communication, if the player has a dominant strategy, it is preferable to use it.

This game has two equilibriums, ADE and BDF. It can be seen that all arrows are heading inward. It is possible to see that in 3-person games, equilibriums are not equivalent and interchangeable even in a zero-sum game. They are indifferent for Colin, but Rose would prefer BDF[-2,3,-1] and Larry ADE[-4,3,1]. If Larry tries for his favorite equilibrium and plays Larry-E, and Rose tries her favorite Rose-B, the result would be

BDE(-5,10,-5)[20]. It is worse for both of them and good news for Colin. There is no simple solution theory for what the players should do.

B. LIKELY COALITIONS

When the players can communicate, it is natural that they try to form coalitions. An illustration of how it would look if Larry and Colin decided to form a coalition against Rose is given below. The game is zero-sum, so it is possible to focus only on Rose's payoffs. The resulting sub game is then:

Rose v. Colin and Larry

		Larry-E Colin-C	Larry-E Colin-D	Larry-F Colin-C	Larry-F Colin-D
Rose	A	1	-4	3	-5
	B	2	-5	2	-2

The best way to play a zero-sum game is to play Maximin/Minimax strategy. This prudent strategy assures the value of the game for the players. Using the software toolkit provided in the thesis, the optimal strategy of the players is:

Rose alone: 75% of A and 25% of B

Coalition Larry & Colin: 75% of DE and 25% of DF

Value of the game: -4.25 for Rose / +4.25 for Coalition

As a result, the coalition wins 4.25. However, this is only half of the answer. It is also important for one to know how the payoffs are divided between members of the coalition. The payoffs of the players for each strategy combination are different, so it is necessary to treat them separately. One begins by calculating the relative frequency of playing of each strategy combination. For example, using the ADE strategy combination: Rose plays A 75 percent of the time and the coalition plays DE 75 percent of the time. If these are multiplied, one gets a relative frequency of ADE that is 56.25 percent. Now, if one were to multiply each original payoff for ADE

[20] Assuming that Colin plays dominant strategy Colin-D

[-4,3,1] by this relative frequency, the adjusted payoffs would be [-2.25, 1.6875, 0.5325]. When this is done for all outcomes, the values of each player are added. An illustration of this follows:

Strategy	Frequency of playing			Original payoffs	Adjusted payoffs
A x CE	.75	0	=0	[1,1,2]	$[0,0,0]$
A x DE	.75	.75	=0.5625	[-4,3,1]	$[-2.25, 1.6875, 0.5325]$
A x CF	.75	0	=0	[3,-2,-1]	$[0,0,0]$
A x DF	.75	.25	=0.1875	[-5,12,-7]	$[-0.9375, 2.25, -1.3125]$
B x CE	.25	0	=0	[2,-4,2]	$[0,0,0]$
B x DE	.25	.75	=0.1875	[-5,10,-5]	$[-0.9375, 1.875, -0.9375]$
B x CF	.25	0	=0	[2,1,-3]	$[0,0,0]$
B x DF	.25	.25	=0.0625	[-2,3,-1]	$[-0.125, 0.1875, -0.0625]$
Sum Up Rose/Colin/Larry					[-4.25,6,-1.75]

Rose receives her value of the game -4.25. Colin receives 6, and Larry receives -1.75. Therefore, it is Colin, who is doing well in this coalition. One could consider other possible coalitions:

Colin v. Rose and Larry

		Larry-E Rose-A	Larry-E Rose-B	Larry-F Rose-A	Larry-F Rose-B
Colin	C	1	-4	-2	1
	D	3	10	12	3

Results:

Colin alone: 100% of D

Coalition Rose and Larry: 100% of AE

Value of the game: 3 for Colin / -3 for Coalition

Division of payoffs: [-4,3,1]

Larry v. Rose and Colin

		Colin-C Rose-A	Colin-C Rose-B	Colin-D Rose-A	Colin-D Rose-B
Larry	A	-2	2	1	-5
	B	-1	-3	-7	-1

Results:

Larry alone: 50% of E and 50% of F

Coalition Rose and Colin: 33.33 % of AD and 66.66 of BD

Value of the game: 3 for Larry / -3 for Coalition

Division of payoffs: [-3.833, 6.833, -3]

It is possible to imply what coalition is likely to occur by comparing the results for all three possible coalitions. Larry would prefer to play in coalition with Rose. Colin would prefer to be with Rose, and Rose would prefer to be with Colin. When two players prefer to play with each other, a coalition is likely to occur. Therefore, for this game, the most likely coalition is Rose and Colin, as both players can receive their best outcome playing together. However, sometimes the players do not have a preference of another player, and so it is not possible to say what coalition is likely to form.

Rose	alone	-4.25	
	with Colin	-3.83	Most preferred
	with Larry	-4	
Colin	alone	3	
	with Rose	6.83	Most preferred
	with Larry	6	
Larry	alone	-3	
	with Rose	1	Most preferred
	with Colin	-1.75	

C. SIDEPAYMENTS

Larry is certainly not happy with the likely coalition between Rose and Colin. He tries to find a way to persuade Colin to play with him. If Larry would give Colin, for the sake of argument, one unit, Colin's total payoff would then be seven, and he would have an incentive to play with Larry. Larry's payoff in this case would be -2.75 (-1.75-1), which is still better than playing alone and ending up with -3. This idea is called **sidepayment**.

Sidepayments are not possible in all games because it's necessary for payoffs to be transferable. It is hard to transfer one's good feelings or, for example, health. Generally, it is difficult to place a value on nonmaterial things. Also, payoffs have to be comparable. One unit of utility has to mean the same for both players. Fortunately, it is possible to express most games in the economic area in terms of money, with the caveat that sometimes the value of one dollar is not the same for all the players.

A further difficulty with the sidepayments is that nothing can stop Rose from offering Colin 0.25 of unit and beat Colin's offer. The players then can place bids and counter bids in a circular manner. There is a whole field of game theory that deals with which coalition can form and how the payoffs should be divided among the players.

APPENDIX 1 – TOOLKIT MANUAL

This manual, for the software toolkit included in the thesis, uses the following format.

File name: Name of the file containing the template.

Lpslink.dll: Yes/No – tells whether it is necessary to install *lpslink.dll*. It is an open source lpsolve solver using the simplex method. One can copy the file in user path or into C:\WINDOWS\system32. One can also double click on *CopyLink.bat*. It copies the dynamic library automatically into the appropriate space.

Description: Brief description of the software tool and its purpose.

Assumptions: Any necessary assumptions used in the model.

Instructions: Instructions tell the user how and where to input the data, which and in what order to push the command buttons.

Screenshots: Annotated screenshots facilitate understanding of the templates.

Additional information for all parts of the toolkit:

- Excel spreadsheets are protected against accidental damage. There is no password, so the templates and macros can be adjusted according to user needs.

- The names of the players and their strategies can be changed; this change is then reflected in the answers for easier readability and understanding.

- Macros have to be enabled in order to function properly.

A. ZERO-SUM GAME: MAXIMIN & MINIMAX, SADDLE POINT

File name: *ZeroSum_MaxiMin_MiniMax_SaddlePoint.xls*

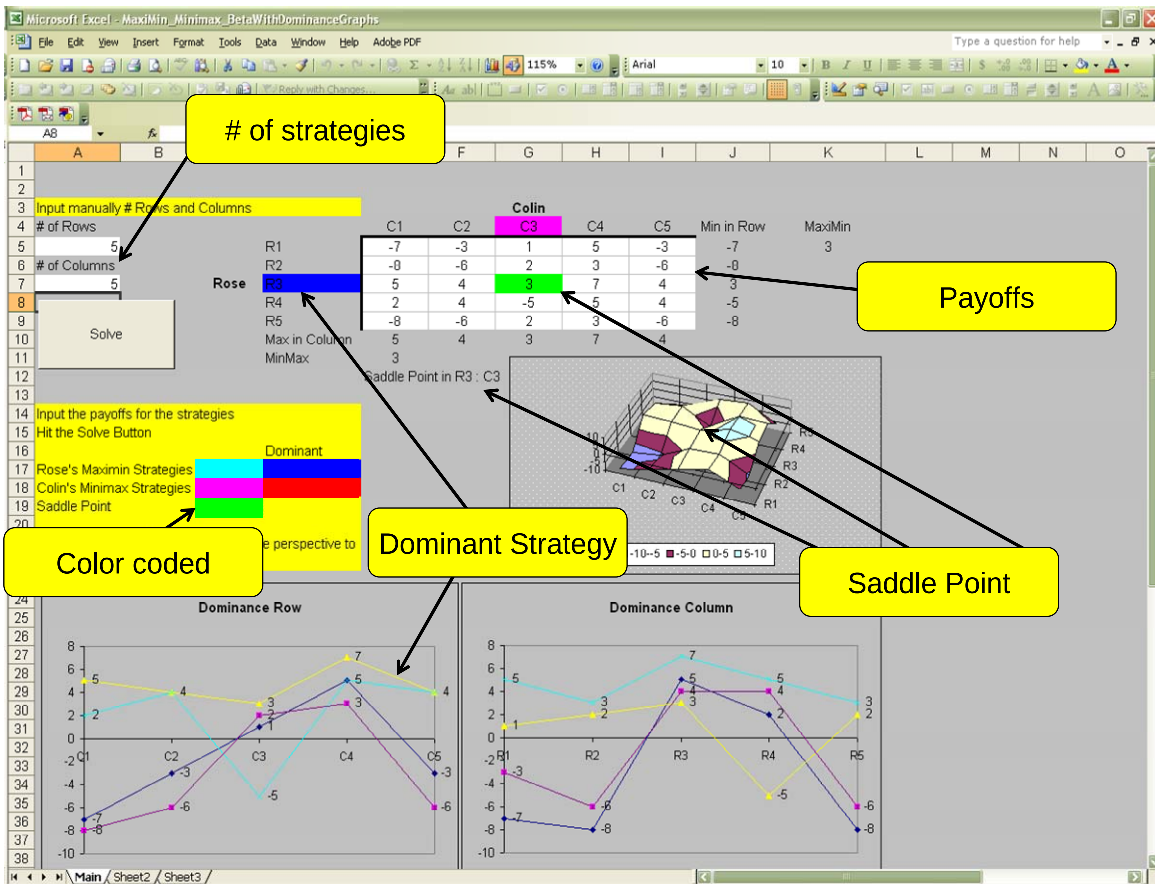
Lpslink.dll: No

Description: Solver finds the MaxiMin strategies of the Row player and MiniMax strategies of the Column player. If the values are equal, it marks the intersection as Saddle point and writes it out. The color legend also tells whether the strategy is dominant or not. Graphs are helpful for visualization of the problem.

Assumptions: Row player is maximizing. Column player is minimizing. Row player's payoffs.

Instructions: Write the payoffs in the appropriate cells, starting from left upper corner. Write the number of strategies for each player and hit ***solve*** button.

Screenshots:



B. ZERO-SUM GAME: TWO PLAYERS / TWO STRATEGIES

File name: *ZeroSum_2x2.xls*

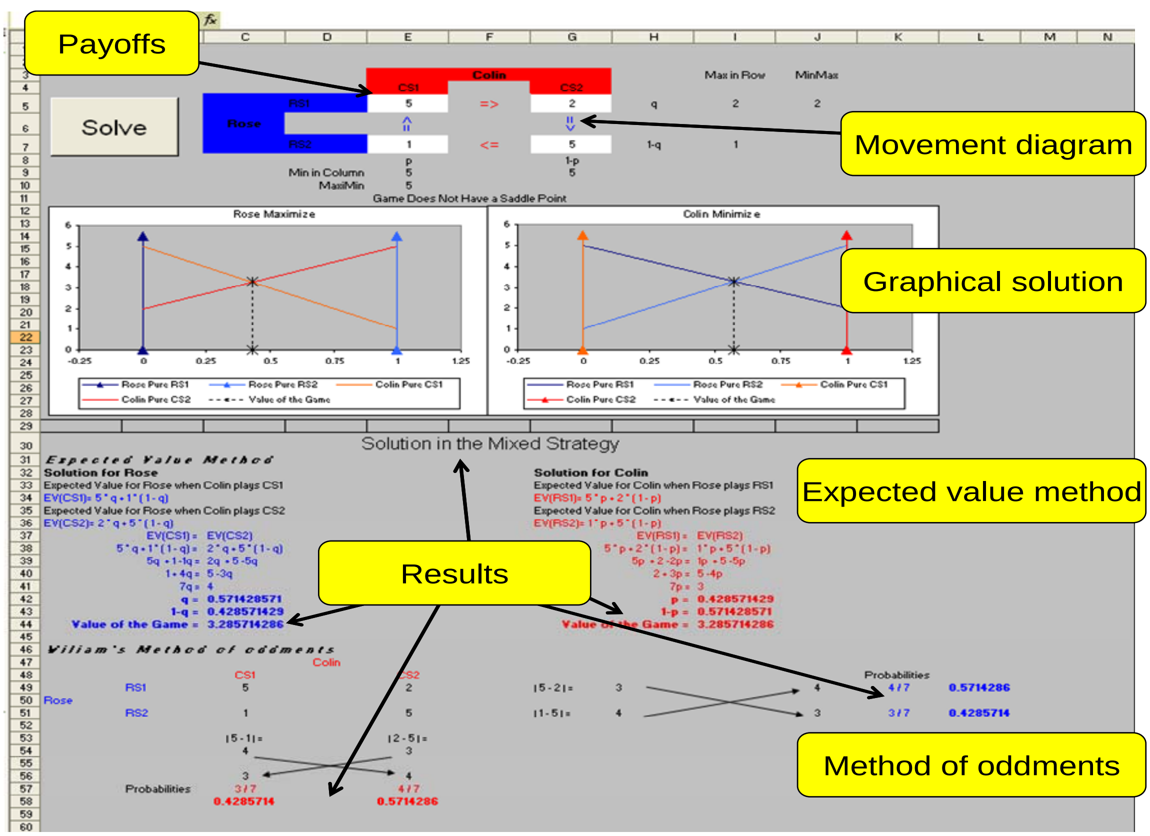
Lpslink.dll: No

Description: Solver finds a value of the game and optimal strategies of the players. It tells whether the solution is in pure strategy or mixed strategy. Solver shows a movement diagram of the game and graphical solution. In the numerical part, it uses and shows both methods covered in the thesis (expected value method and method of oddments). The process of reaching the solution is intentionally detailed to help with the understanding of the problem.

Assumptions: Row player is maximizing. Column player is minimizing. Row player's payoffs.

Instructions: Write the payoffs and hit ***solve*** button.

Screenshots:



C. ZERO-SUM GAME: TWO PLAYERS / UP TO TEN STRATEGIES

File name: *ZeroSum_UpTo10x10.xls*

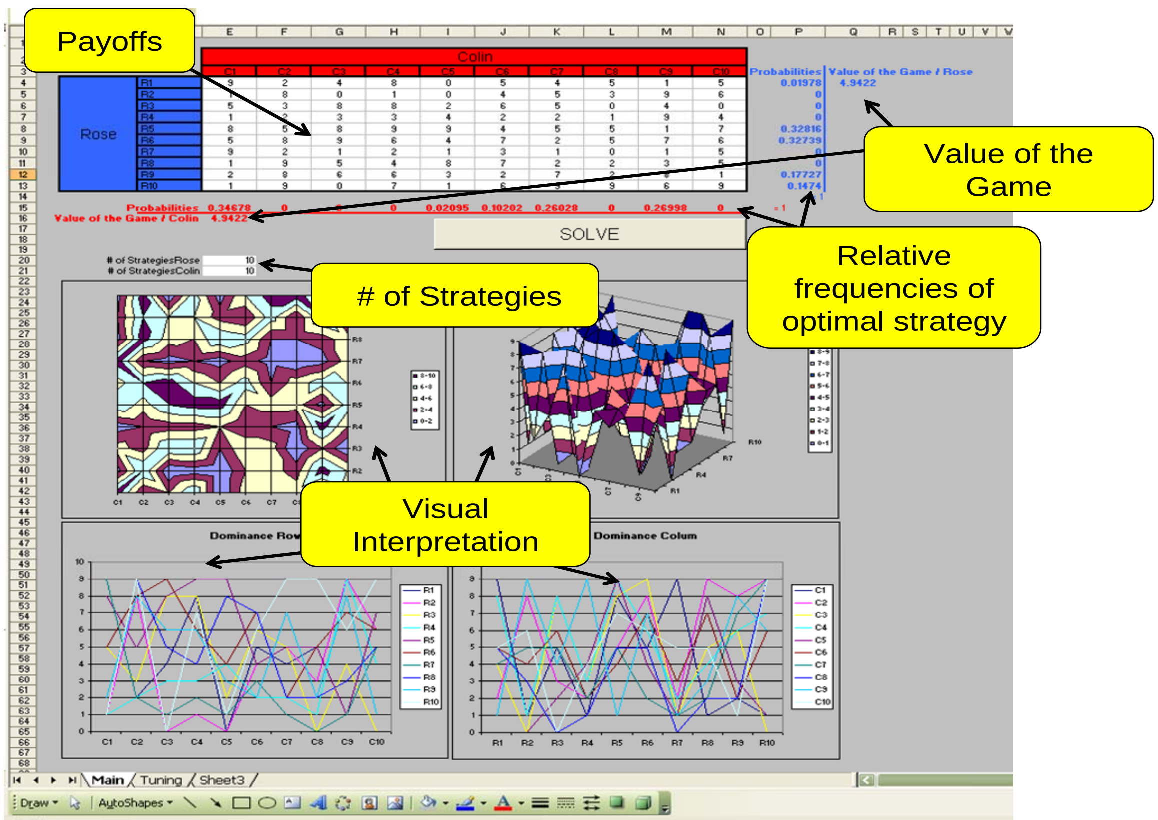
Lpslink.dll: Yes

Description: Solver finds a value of the game and optimal strategies of the players. Solver calculates the results for each player separately. It allows for the comparison of the values of the game (should be equal) and checks whether it works properly. Solver also incorporates a graphical depiction of the game. Sometimes it offers interesting insights; sometimes it is just a mess.

Assumptions: Row player is maximizing. Column player is minimizing. Row player's payoffs.

Instructions: Write the payoffs in the appropriate cells, starting from left upper corner. Write the number of strategies for each player and hit ***solve*** button.

Screenshots:



D. PARTIAL-SUM GAME: PRUDENTIAL STRATEGIES

File name: *PartialSum_Prudential_SecurityLevels.xls*

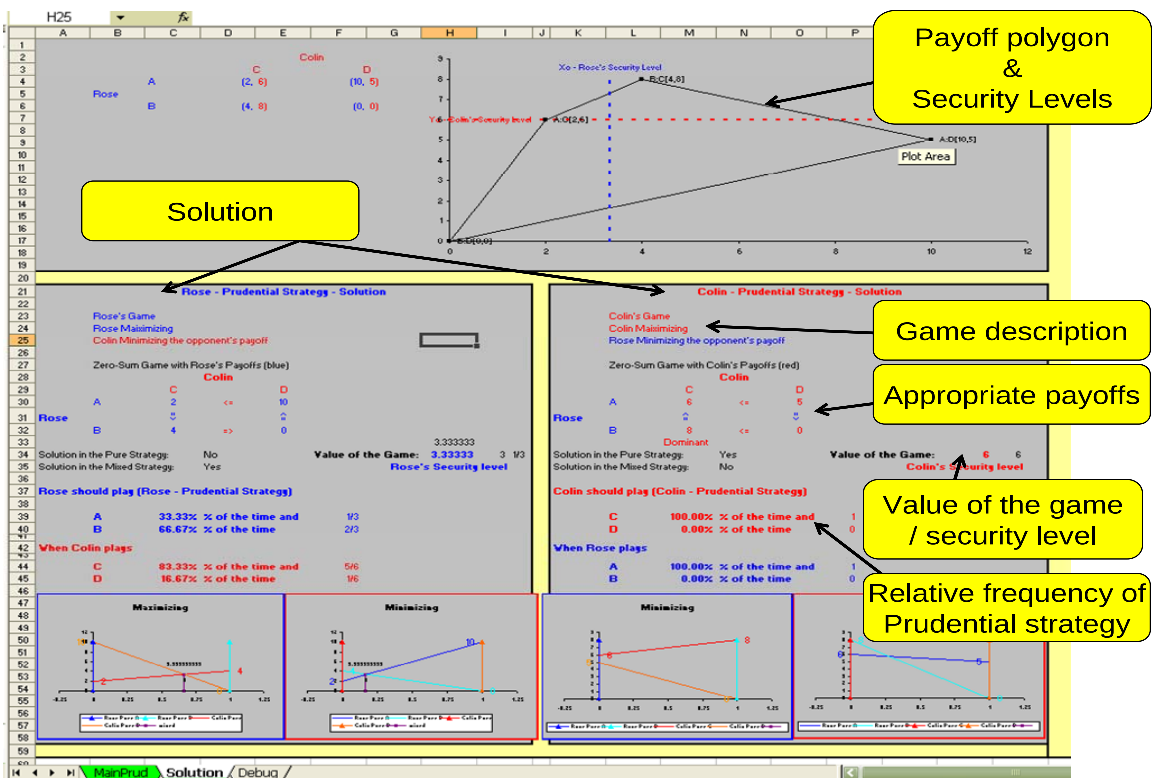
Lpslink.dll: Yes

Description: Solver finds prudential strategies of the players in the partial-sum game, when the opponent turns hostile. The value of the game, when prudential strategy is used, is the player's security level. The solution is divided into two annotated sub-games, where one player is maximizing, and the other is minimizing. The graph shows the payoff polygon and the security levels of the players.

Assumptions: Rose's prudential strategy – In Rose's game (Rose's payoffs), Rose is maximizing and Colin is minimizing. Colin's prudential strategy – In Colin's game (Colin's payoffs), Colin is maximizing and Rose is minimizing.

Instructions: Write the payoffs of the players in the ***MainPrud*** sheet and hit the ***solve*** button. It solves the game and switches to the ***Solution*** sheet.

Screenshots:



E. PARTIAL-SUM GAME: EQUALIZING STRATEGIES AND NASH EQUILIBRIUMS

File name: *PartialSum_Equalizing_NashEquilibriums.xls*

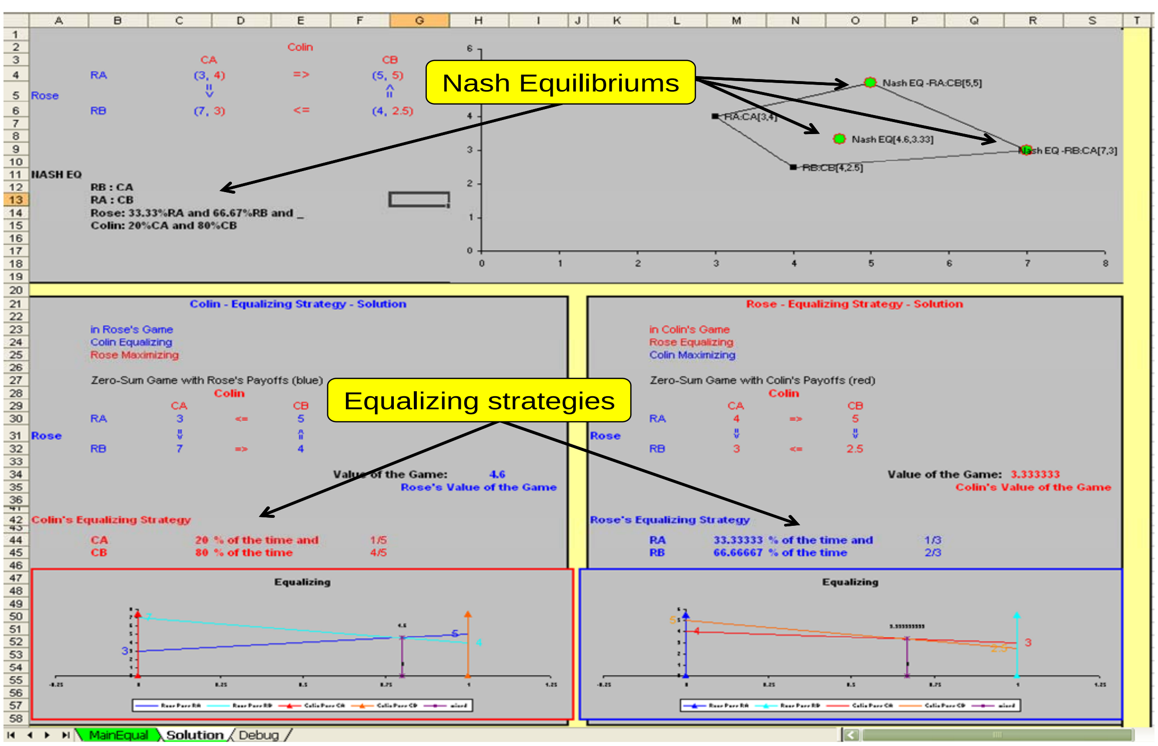
Lpslink.dll: No

Description: The solver finds the Nash Equilibriums and equalizing strategies of the players in the partial-sum game. The values of the game, when the players use equalizing strategy, form Nash Equilibrium in the mixed strategy. The solution is divided into two annotated sub-games, where one player is maximizing, and the other is equalizing. The graph shows the payoff polygon and Nash Equilibriums.

Assumptions: Rose's equalizing strategy – In Colin's game (Colin's payoffs), Colin is maximizing, and Rose is equalizing. Colin's equalizing strategy – In Rose's game (Rose's payoffs), Rose is maximizing, and Colin is equalizing.

Instructions: Write the payoffs of the players in the ***MainEqual*** sheet and hit the ***solve*** button.

Screenshots:



F. PARTIAL-SUM GAME: NASH ARBITRATION

File name: *PartialSum_NashArbitration.xls*

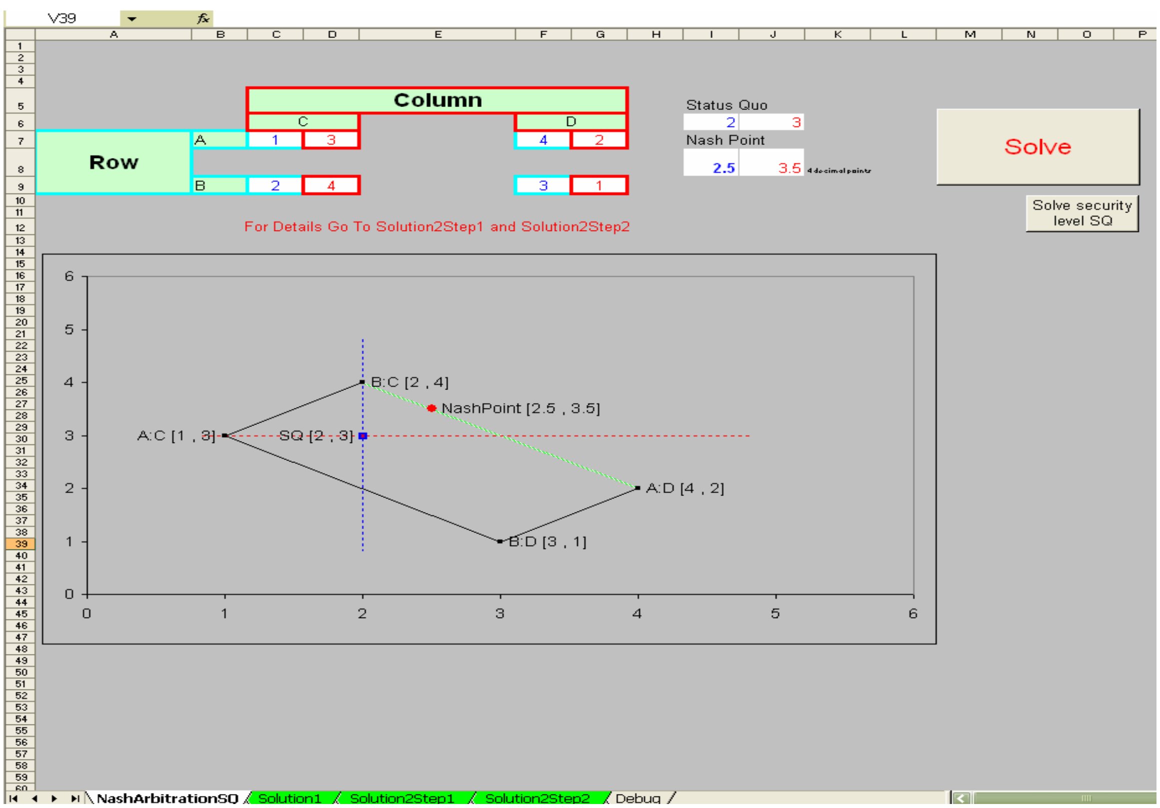
Lpslink.dll: Yes

Description: The solver solves the Nash arbitration and finds a Nash point. It solves a game from a Status Quo point. The solver provides a detailed graphical solution. A status quo, based on the security levels of the players, is built in. It is also possible to choose a different starting point.

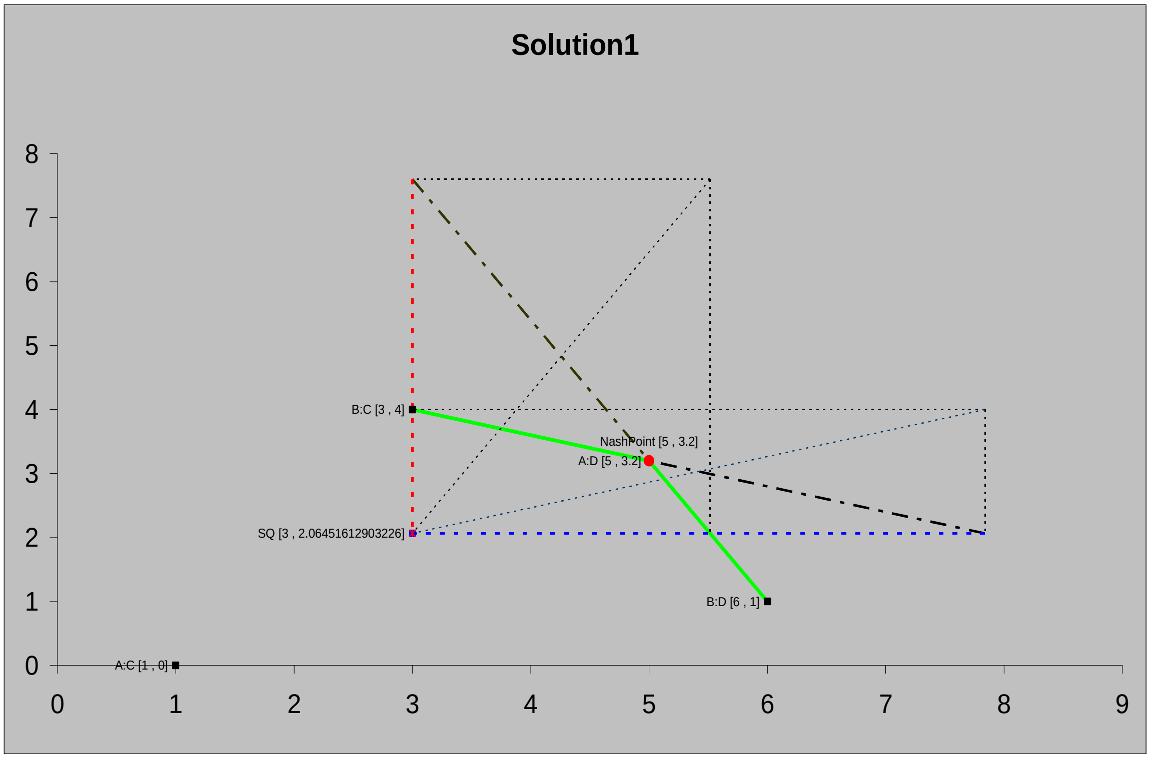
Assumptions: Both players are maximizing. Status Quo lies inside the payoff polygon. Fair solution maximizes $(x-x_0)*(y-y_0)$.

Instructions: Write payoffs of the players in the ***NashArbitrationSQ*** sheet. Set a status quo by pushing ***Solve Security levels SQ*** button or write the required status quo. Solve the game → ***Solve*** button.

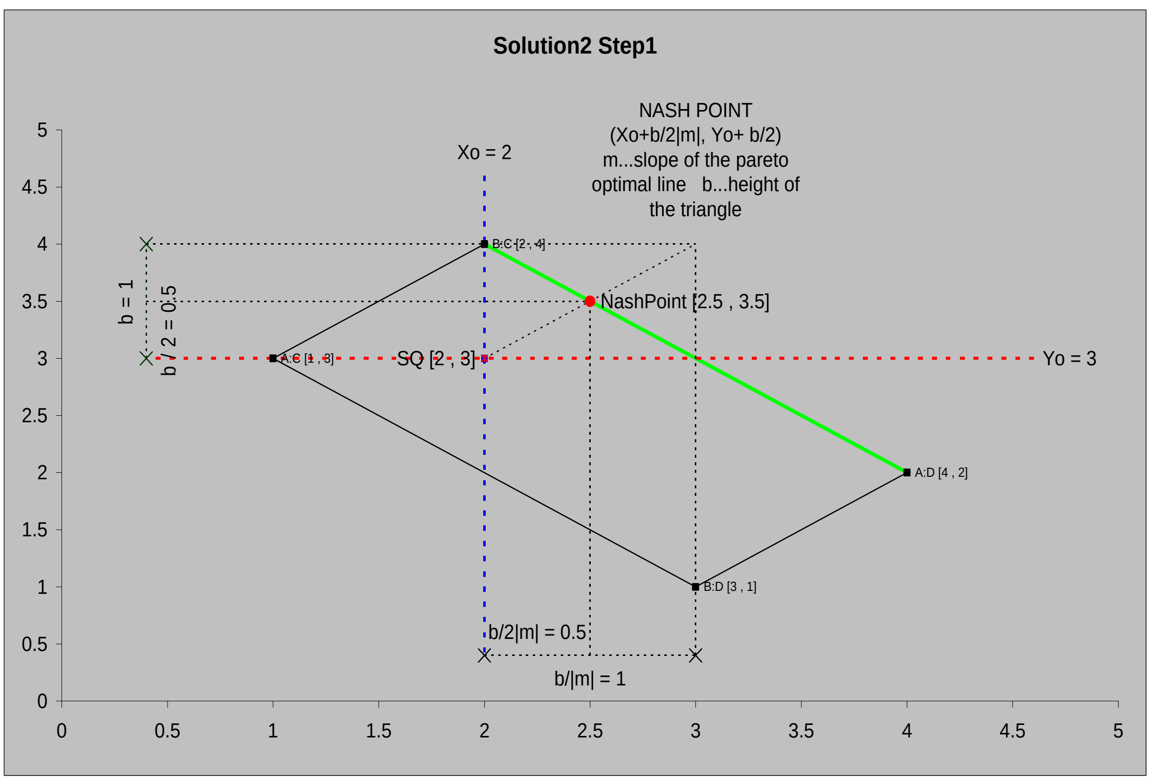
Screenshots:



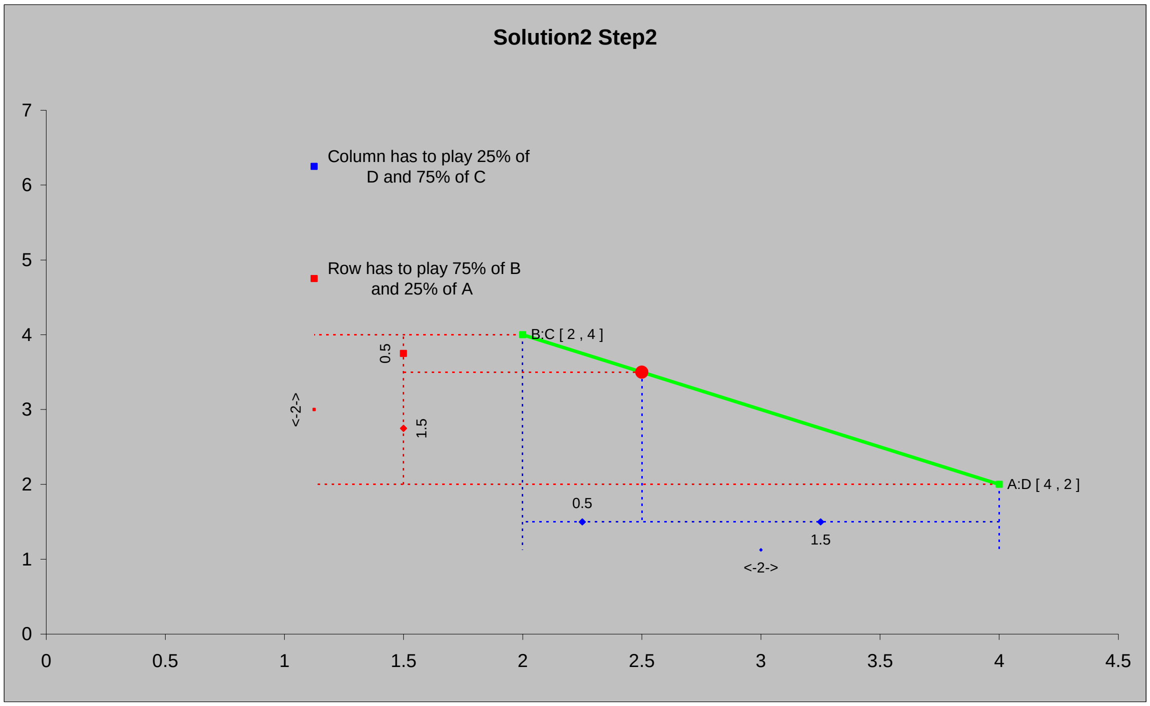
Sheet – ***Solution1***, solution in the pure strategy



Sheet – ***Solution2Step1***, solution in the mixed strategy



Sheet – ***Solution2Step2***, solution in the mixed strategy



G. PARTIAL-SUM GAME: STRATEGIC MOVES

File name: *PartialSum_StrategicMoves.xls*

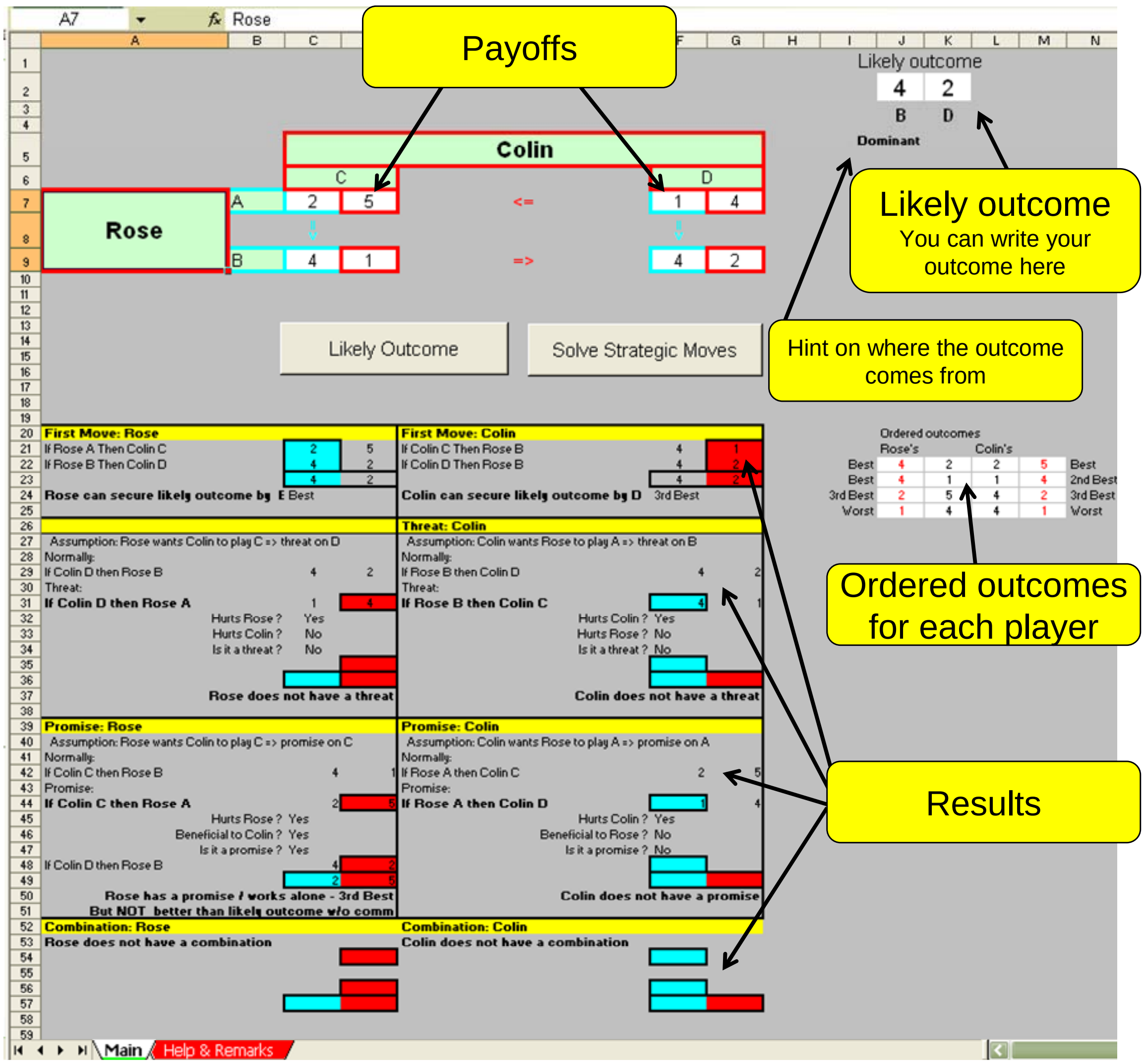
Lpslink.dll: No

Description: The solver analyzes the options available to the players. The Toolkit solves the first move, threat, promise, and their combinations. The solver finds whether a player has a strategic move, if the move works and whether the outcome is better than a likely outcome without communication.

Assumptions: The solver assumes that the moves are credible and feasible. The solver assumes that player wants the opponent to play opposite strategy than the opponent's strategy in likely outcome without communication. The likely outcome in this solver is determined as follows: (1) The solver checks for dominant strategies. If at least one player has a dominant strategy, the opponent chooses the better outcome for him and the combination is the likely outcome. If not, (2) the solver uses the intersection of MaxiMin strategies of the players.

Instructions: First, calculate the likely outcome (***likely outcome*** button). Then, analyze the game (***solve strategic moves*** button). If it's desirable to have a different likely outcome, it is possible to write the payoffs and strategies in the appropriate place and hit ***solve strategic moves***. In this case, it is necessary to write the resulting payoffs and precise strategy combination. Any misspelling causes failure and wrong results.

Screenshots:



H. 3 – PERSON: COALITIONS

File name: *3Person.xls*

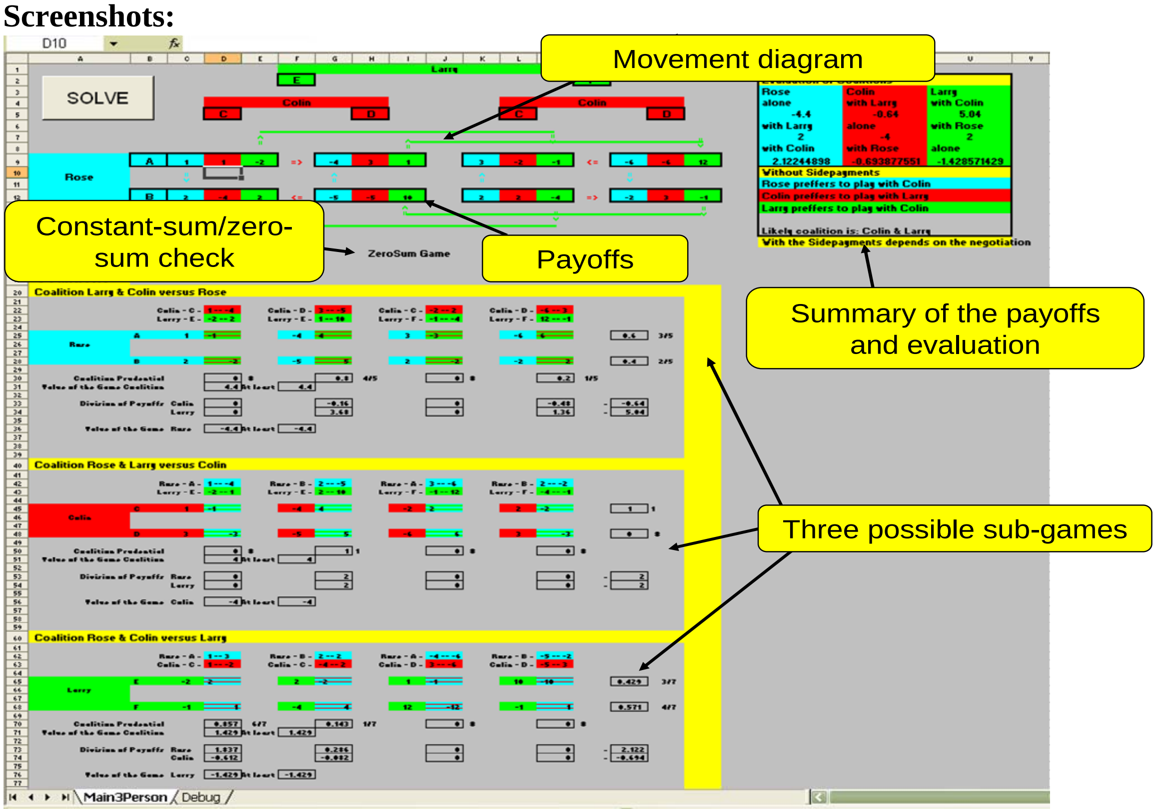
Lpslink.dll: Yes

Description: The solver analyzes possible coalitions and which coalition is likely to form. The solver checks whether it is a constant sum-game and analyzes three possible coalitions. It tells the value of the game in the sub-games, optimal prudential strategies, and how the payoffs are divided among the members of the coalition. A summary of the payoffs for each option shows what the players would like to do. Arrows can be used for determination of the Nash equilibrium.

Assumptions: The game has to be either constant-sum or zero-sum. In the sub-game player v. coalition, both sides play prudential strategy. The game is without sidepayments.

Instructions: Write payoffs of the players in the ***Main3Person*** sheet and hit ***Solve*** button.

Screenshots:



I. LINEAR PROGRAMMING

File name: *LP_lpsolve_5Variables.xls* **and** *LP_lpsolve_10Variables.xls*

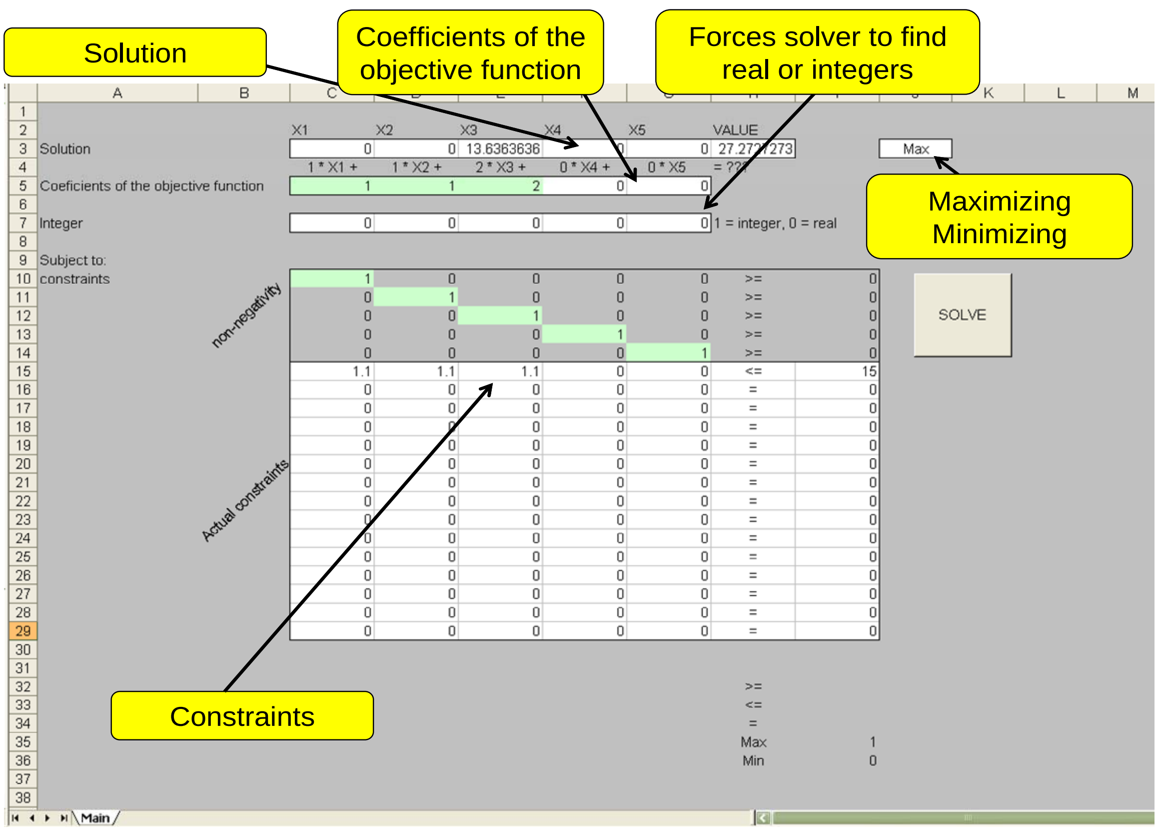
Lpslink.dll: Yes

Description: This is a template for solving LP. The solver uses the open source lpsolve library for the solution (simplex method). By using this template, one does not have to worry about the proper structure of the input data required by lpsolve.

Assumptions: The problem has to be bounded; otherwise, the software does not work (error code 3).

Instructions: Always start from the left. First, write the coefficient of the objective function. Second, write whether the results should be integer (write 1) or real (write 0). Third, determine whether it is a maximizing or minimizing problem. Next, write the constraints and finally hit ***solve*** button. For inequality and Max/Min use scroll down menus.

Screenshots:



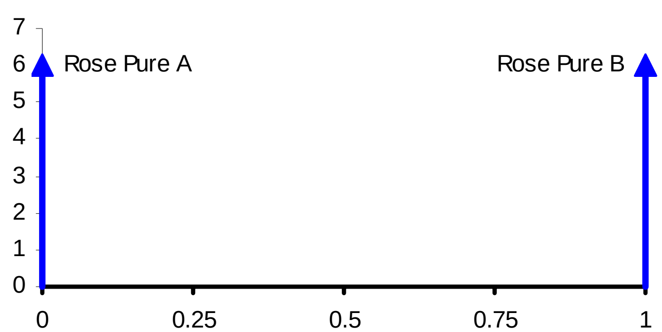
APPENDIX 2 – HOW TO MAKE A CHART FOR A 2X2 ZERO-SUM GAME

A Zero-Sum game between Rose and Colin with these payoffs will be considered.

$$\begin{array}{ccc} & & \text{Colin} \\ & & \begin{array}{cc} C & D \end{array} \\ \text{Rose} & \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix} \end{array}$$

It is simple for one to make a graphical depiction of the game. It is necessary to create two graphs, one for each player. The following illustration begins with Rose.

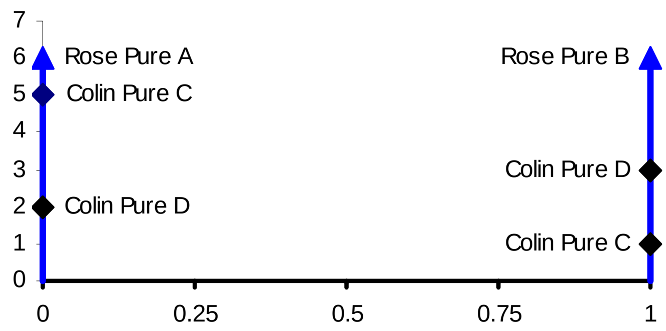
Step 1. Make a graph with the values on the x-axis 0 to 1 and on the y-axis large enough to accommodate the highest and lowest payoffs. In the graph on the x-axis, erect a perpendicular line intersecting the axis in the 0 and 1. These lines will represent Rose's Pure strategies.



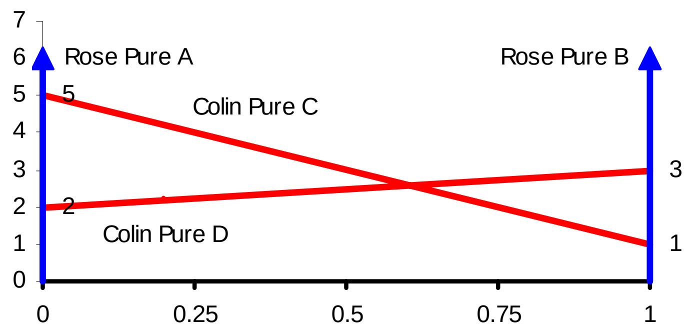
Step 2. Rose has options to play either strategy A, or strategy B, or some combination of these two strategies.

If Rose would play only strategy A (pure Strategy) her payoffs would be 5 (Colin-C) and 2 (Colin-D).Points [0, 5], [0, 2]. Place the values on the graph.

The same can be done for Rose's strategy B. If Rose would play only strategy 2 her payoffs would be 1(Colin-C) and 3 (Colin-D) Points [1, 1], [1, 3].



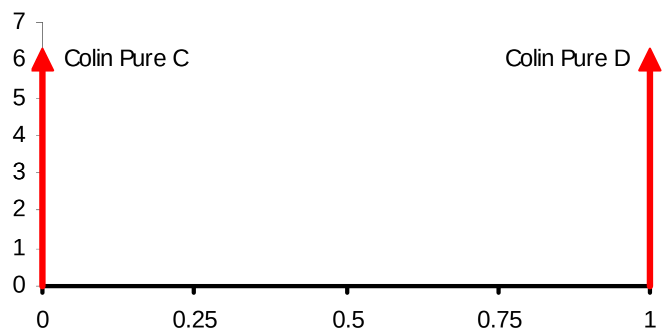
Step 3. Now connect points depicting the values of Colin's strategies. Colin Pure Strategy 1 [0, 5] and [1, 1].Colin Pure Strategy 2 [0, 2] and [1, 3].



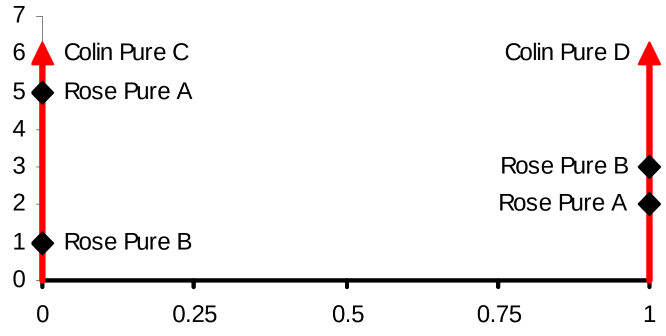
The resulting graph depicts the game from Rose's point of view.

The same process is used for creating the graph for Colin.

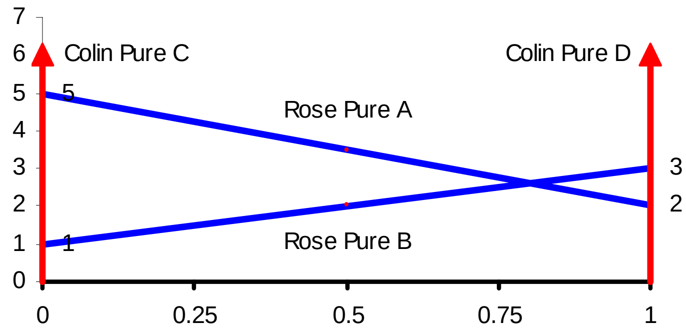
Step 1.



Step 2.



Step 3.



The graphical depiction shows very clearly whether the game has saddle point. In the 2x2 zero-sum game, the game has the saddle point only when at least one player has a dominant strategy. In the graph, it is easy to see: the lines do not intersect each other on the open interval (0,1).

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APPENDIX 3 – HOW TO CONVERT THE GAME INTO LINEAR PROGRAM

For more complicated games, such as 3x3 and more, the best way is to convert the game and use linear programming (LP) solvers or the toolkit provided with this thesis. Otherwise, one would have to try all possible intersections of the expected values functions. This is a polynomial problem, and as such, the number of possible solutions increases quickly.

$$\begin{array}{cccc} & & \text{Colin} & \\ & & \begin{array}{cc} \text{C} & \text{D} \end{array} & \\ \text{Rose} & \begin{array}{c} \text{A} \\ \text{B} \end{array} & \begin{pmatrix} AC & AD \\ BC & BD \end{pmatrix} & \begin{array}{c} p_1 \\ p_2 \end{array} \\ & & \begin{array}{cc} q_1 & q_2 \end{array} & \end{array}$$

In the linear program, there is the objective function and a set of constraints that limit our solution. Objective function in game theory is the value of the game, which is necessary to either maximize or minimize. For Rose (maximizing) it is:

Objective function:

Value of the game (v) ➔ **maximize**

Subject to constraints:

1. $p_1 + p_2 = 1$ Probabilities has to be equal to 1
2. $EV(Colin - C) \geq v$
3. $EV(Colin - D) \geq v$

For $v \geq 0$

$AC * p_1 + BC * p_2 - v \geq 0$

$AD * p_1 + BD * p_2 - v \geq 0$

By changing: p_1, p_2, v

Constraints 2 and 3 are derived from the minimax theorem. All values of the game (possible solutions) have to be less than or equal to expected value of all of Colin's pure strategies. The feasible region is restricted from the above by Colin pure strategies. The value of the game determined by mixed strategy p_1 and p_2 has to be less or equal.

For Colin (minimizing) the LP is:

Objective function:

Value of the game (v) ➔ **minimize**

Subject to constraints:

1. $q_1 + q_2 = 1$
2. $EV(Rose\text{-}A) \le v$
3. $EV(Rose\text{-}B) \le v$

For $v \ge 0$

$$AC * q_1 + AD * q_2 - v \le 0$$
$$BC * q_1 + BD * q_2 - v \le 0$$

By changing: p_1, p_2, v

This time the candidate solutions have to be higher than or equal to the expected value of all of Rose's pure strategies. The feasible region is restricted from below.

We have created the LP in general; the exact procedure for solver set up differs from solver to solver. For example in the Excel solver or the LP solve solver used in the toolkit, it is not possible to define the objective function just as one variable. This limitation can be overcome by adjusting the objective function.

Example: $p_1 + p_2 + v$ ➔ maximize

By adjusting the objective function this way, the solver readily solves the problem. We only have to keep in mind that the result has to be subtracted by $p_1 + p_2 = 1$ in order to get the value of the game.

The Excel solver also uses the approximation method, so be aware of possible errors in the results.

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