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High-Order Higdon Non-Reflecting Boundary Conditions for the Linearized Euler Equations

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Abstract

In this report we document the implementation of high-order Higdon nonreflecting boundary conditions. We suggest a way to choose the parameters and demonstrate numerically the efficiency of our choice. The model we used is the linearized 2-D Euler equations with zero advection. These equations are solved by the finite difference method. We close with a list of topics for research.

1. Statement of the Problem

Consider the linearized Euler equations in an infinite domain. For simplicity we assume that the domain has a flat bottom and that there is no advection and no Coriolis effect, although these assumptions may be removed in future studies. A Cartesian coordinate system (x, y) is introduced, as shown in the figure.

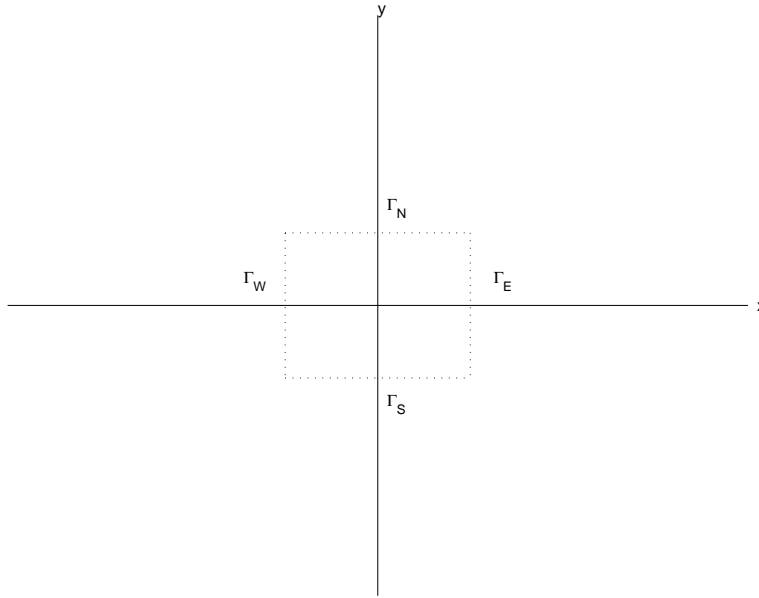


Figure 1: An infinite domain

The linearized Euler equations are (see, e.g., [32] or [33]):

$$\partial_t \rho + \rho_0 (\partial_x u + \partial_y v) = 0 , \quad (1)$$

$$\partial_t u + \frac{1}{\rho_0} \partial_x p = 0 , \quad (2)$$

$$\partial_t v + \frac{1}{\rho_0} \partial_y p = 0 , \quad (3)$$

$$\partial_t p + \gamma p_0 (\partial_x u + \partial_y v) = 0 . \quad (4)$$

Here t is time, $u(x, y, t)$ and $v(x, y, t)$ are the unknown velocities in the x and y directions, $\rho(x, y, t)$ is the density, $p(x, y, t)$ is the pressure and γ is the gas constant. The

linearization was done about mean zero velocities, mean density ρ_0 and mean pressure p_0 . We use the following shorthand for partial derivatives

$$\partial_a^i = \frac{\partial^i}{\partial a^i}$$

The nonlinear Euler equations are

$$\partial_t \rho + \partial_x(\rho u) + \partial_y(\rho v) = 0 , \quad (5)$$

$$\partial_t u + u \partial_x u + v \partial_y u + \frac{1}{\rho} \partial_x p = f v , \quad (6)$$

$$\partial_t v + u \partial_x v + v \partial_y v + \frac{1}{\rho} \partial_y p = -f u , \quad (7)$$

$$\partial_t p + u \partial_x p + v \partial_y p + \gamma p (\partial_x u + \partial_y v) = 0 . \quad (8)$$

It can be shown that *a single* boundary condition must be imposed along the entire boundary to obtain a well-posed problem. At $\vec{x} \rightarrow \infty$ the solution is known to be bounded and not to include any incoming waves. To complete the statement of the problem, initial values for u , v , p and ρ are given at time $t = 0$ in the entire domain.

We now truncate the infinite domain by introducing an artificial east boundary Γ_E , located at $x = x_E$, and similarly at the the other three sides (see Figure 1). To obtain a well-posed problem in the finite domain Ω we need a single boundary condition on each of the artificial boundaries Γ_E , Γ_W , Γ_N , and Γ_S . This should be a Non-Reflecting Boundary Condition (NRBC). We shall apply a high-order NRBC for the variables. A discussion on this NRBC follows.

2. Higdon's NRBCs

On the artificial boundary Γ_E we use one of the *Higdon NRBCs* [1]. Similarly for the other three sides. These NRBCs were presented and analyzed in a sequence of papers [2]–[6] for non-dispersive acoustic and elastic waves, and were extended in [1] for dispersive waves. Their main advantages are as follows:

1. The Higdon NRBCs are very *general*, namely they apply to a variety of wave problems, in one, two, and three dimensions and in various configurations.
2. They form a *sequence* of NRBCs of increasing order. This enables one, in principle (leaving implementational issues aside for the moment), to obtain solutions with unlimited accuracy.
3. The Higdon NRBCs can be used, without any difficulty, for *dispersive* wave problems and for problems with layers. Most other available NRBCs are either designed for non-dispersive media (as in acoustics and electromagnetics) or are of low order (as in meteorology and oceanography).
4. For certain choices of the parameters, the Higdon NRBCs are equivalent to NRBCs that are derived from rational approximation of the dispersion relation (the Engquist-Majda conditions being the most well-known example). This has been

proved by Higdon in [1] and in earlier papers. Thus, the Higdon NRBCs can be viewed as generalization of rational-approximation NRBCs.

The scheme developed here is different than the original Higdon scheme [1] in the following ways:

1. The discrete Higdon conditions were developed in the literature up to third order only, because of their algebraic complexity which increases rapidly with the order. Here we show how to easily implement these conditions to an *arbitrarily high order*. The scheme is coded once and for all for any order; the order of the scheme is simply an input parameter.
2. The original Higdon conditions were applied to the Klein-Gordon linear wave equation and to the elastic equations. Here we show how to apply them to the *linearized Euler equations* (1)–(4).
3. The Higdon NRBCs involve some parameters which must be chosen. Higdon [1] discusses some general guidelines for their manual a-priori choice by the user. We shall show how a simple choice for these parameters can dramatically simplify the calculations and enable implementation of NRBCs of much higher order with less computational overhead.

The Higdon NRBC of order J is

$$H_J : \quad \left[\prod_{j=1}^J (\partial_t + C_j \partial_x) \right] \eta = 0 \quad \text{on } \Gamma_E . \quad (9)$$

Here, the C_j are parameters which have to be chosen and which signify phase speeds in the x -direction. The boundary condition (9) is exact for all waves that propagate with an x -direction phase speed equal to any of C_1, \dots, C_J . This is easy to see from the reflection coefficient (see below). For the other sides we replace ∂_x by the normal derivative to the boundary.

We make a few observations:

- In their sequence of papers [22]–[31], Givoli, Neta, and van Joolen showed that one should always take $C_j \geq C_0$ because, in general, the solution consists of an infinite number of waves with different phase speeds. For this problem, however, there is only one wave speed C_0 impinging on the boundary at all possible angles; hence, one should instead take $C_j \leq C_0$ (i.e., $C_j = C_0 \cos(\phi_j)$, where $\phi_j \in 0 \dots \frac{\pi}{2}$).
- The first-order condition H_1 is a Sommerfeld-like boundary condition. If we set $C_1 = C_0$ we get the classical Sommerfeld-like NRBC. A lot of work in the meteorological literature is based on using H_1 with a specially chosen C_1 . Pearson [7] used a special but constant value of C_1 , while in the scheme devised by Orlanski [8] and in later improved schemes [9]–[12] the C_1 changes dynamically and locally in each time-step based on the solution from the previous time-step. Some of the limited-area weather prediction codes used today are based on such schemes, e.g., COAMPS [13]. See also the recent papers [14]–[16] where several such schemes are compared. In a series of papers [22]–[31], Givoli, Neta and van Joolen have

demonstrated the use of high order Higdon NRBC to solve the shallow water equations with advection and stratification.

- The condition H_J involves up to J th-order normal and temporal derivatives. In fact, it has the form

$$\sum_{j=0}^J A_j \partial_x^j \partial_t^{J-j} \eta = 0, \quad (10)$$

which is obtained by expanding (9).

- It is easy to show (see Higdon [1] for a similar setting) that when a wave of the form $\eta = \Phi(y)e^{ikx-i\omega t}$ impinges on the boundary Γ_E where the NRBC H_J is imposed, the resulting *reflection coefficient* is

$$R = \prod_{j=1}^J \left| \frac{C_j - C_x}{C_j + C_x} \right|. \quad (11)$$

Immediately we see that if $C_j = C_x$ for one of the j 's then $R = 0$, namely there is no reflection and the NRBC is exact. Moreover, we see that the reflection coefficient is a product of J factors, *each of which is smaller than 1*. This implies that the reflection coefficient becomes smaller as the order J increases regardless of the choice made for the parameters C_j . Of course, a good choice for the C_j would lead to better accuracy with a lower order J , but even if we miss the correct C_j 's considerably (say, if we make the simplest choice $C_j = C_0$ for $j = 1, \dots, J$), we are still guaranteed to reduce the spurious reflection as we increase the order J . This is an important property of the Higdon's NRBCs and is the reason for their robustness.

- In [4], Higdon points to the possibility of a long-time instability that might occur when one uses a NRBC with high-order derivatives. If the interior governing equations and the NRBC both admit solutions at zero wave number and frequency, and if the data in the problem include such "zero modes," then a slowly-growing smooth instability is possible. Whether this shows up in practice depends on the order of the derivatives in the NRBC and the number of spatial dimensions. However, these difficulties do not arise in the presence of dispersion, or if the data are confined to nontrivial modes.

3. Discretization of Higdon's NRBCs

The Higdon condition H_J is a product of J operators of the form $\partial_t + C_j \partial_x$. Consider the following Finite Difference (FD) approximations (see e.g. [17]):

$$\partial_t \simeq \frac{I - S_t^-}{\Delta t}, \quad \partial_x \simeq \frac{I - S_x^-}{\Delta x}. \quad (12)$$

In (12), Δt and Δx are, respectively, the time-step size and grid spacing in the x direction, I is the identity operator, and S_t^- and S_x^- are shift operators defined by

$$S_t^- \eta_{pq}^n = \eta_{pq}^{n-1}, \quad S_x^- \eta_{pq}^n = \eta_{p-1,q}^n. \quad (13)$$

Here and elsewhere, η_{pq}^n is the FD approximation of $\eta(x, y, t)$ at grid point (x_p, y_q) and at time t_n . We use (12) in (9) to obtain:

$$\left[\prod_{j=1}^J \left(\frac{I - S_t^-}{\Delta t} + C_j \frac{I - S_x^-}{\Delta x} \right) \right] \eta_{Eq}^n = 0. \quad (14)$$

Here, the index E corresponds to a grid point on the boundary Γ_E . Higdon has solved this difference equation (and also a slightly more involved equation that is based on time- and space-averaging approximations for ∂_x and ∂_t) for $J \leq 3$ to obtain an explicit formula for η_{Eq}^n . This formula is used to find the current values on the boundary Γ_E *after* the solution in the interior points and on the other boundaries has been updated. The formula for $J = 2$ is found in [6], and the one for $J = 3$ appears in the appendix of [5]. The algebraic complexity of these formulas increases rapidly with the order J . It is thus not surprising that we have not found in the literature any report on the implementation of the Higdon NRBCs beyond $J = 3$.

Now we show how to implement the Higdon NRBCs *to any order* using a simple algorithm. To this end, we first multiply (14) by Δt and rearrange to obtain

$$Z \equiv \left[\prod_{j=1}^J \left(a_j I + d_j S_t^- + e_j S_x^- \right) \right] \eta_{Eq}^n = 0, \quad (15)$$

where

$$a_j = 1 + \frac{C_j \Delta t}{\Delta x}, \quad (16)$$

$$d_j = -1, \quad (17)$$

$$e_j = -\frac{C_j \Delta t}{\Delta x}. \quad (18)$$

The coefficient d_j actually does not depend on j , but we keep this notation to allow easy extensions to the scheme (see, e.g., [22]). Now, this formula for Z requires the summation of 3^J terms. If we make the simplification

$$C_j \equiv C_0 \forall j \in 1 \dots J, \quad (19)$$

then our expansion for Z becomes

$$\begin{aligned} Z &\equiv \left(aI + dS_t^- + eS_x^- \right)^J \eta_{Eq}^n \\ &= \left(\sum_{\beta=0}^J \sum_{\gamma=0}^{J-\beta} \frac{J!}{\alpha! \beta! \gamma!} a^\alpha d^\beta e^\gamma \right) \eta_{Eq}^n = 0, \end{aligned} \quad (20)$$

where $\alpha = J - \beta - \gamma$, and this summation consists of only $\frac{(J+1)(J+2)}{2}$ terms, reducing the computational time considerably.

Note that we need to store $\eta_{iq}^{\hat{n}}$ values for $\hat{i} = E, E - 1, \dots, E - J$ and $\hat{n} = n, n - 1, \dots, n - J$. In other words, we have to store the history of the values of η for a layer

of thickness $J + 1$ points near the boundary Γ_E and for $J + 1$ time levels (including the current one). If there are N_y grid points in the y direction, then the amount of storage needed in a simple storage scheme is $(J + 1)^2 N_y$. However, one can save in storage by exploiting the fact that not all values $\eta_{i,q}^{\hat{n}}$ are needed, but only those for which $(E - \hat{i}) + (n - \hat{n}) \leq J$. This is clear from (10) and also from (15). For example, the solution at time t_{n-J} should be stored only for points on the boundary Γ_E itself.

4. The Interior Scheme

We consider explicit FD interior discretization schemes for the linearized Euler equations (1)–(4) to be used in conjunction with the H_J condition. The interaction between the H_J condition and the interior scheme is a source of concern, since simple choices for an explicit interior scheme turn out to give rise to long-time instabilities. We have tried the usual second-order centered difference scheme

$$\eta'(x) \approx \frac{\eta(x+h) - \eta(x-h)}{2h}$$

using both Euler's method in time and the second-order centered difference in time. They are stable for a sufficiently small time step when used with the boundary condition H_1 (which is a Sommerfeld-like condition as previously mentioned), but they become unstable for $J \geq 2$. The instability appears earlier in time when J becomes larger.

Higdon [1] has proved, in the context of the scalar Klein-Gordon equation,

$$\partial_t^2 \eta - C_0^2 \nabla^2 \eta + f^2 \eta = 0, \quad (21)$$

that the discrete NRBCs (14) are stable if the interior scheme is the standard *second-order centered* difference scheme

$$\begin{aligned} \eta_{pq}^{n+1} &= 2\eta_{pq}^n - \eta_{pq}^{n-1} + \left(\frac{C_0 \Delta t}{\Delta x}\right)^2 (\eta_{p+1,q}^n - 2\eta_{p,q}^n + \eta_{p-1,q}^n) \\ &+ \left(\frac{C_0 \Delta t}{\Delta y}\right)^2 (\eta_{p,q+1}^n - 2\eta_{p,q}^n + \eta_{p,q-1}^n) - (f \Delta t)^2 \eta_{p,q}^n. \end{aligned} \quad (22)$$

Now we shall show how the linearized Euler equations (1)–(4) can be discretized in such a way as to mimic (22) and to lead to a stable scheme.

Let Δ_t denote a forward difference approximation of $\frac{\partial}{\partial t}$, let ∇_t denote a backward difference approximation to the same, and use similar notation for forward and backward difference approximations in x and y . We can then write this wave equation discretization as

$$\Delta_t \nabla_t u_{i,j}^n = c_0^2 (\Delta_x \nabla_x u_{i,j}^n + \Delta_y \nabla_y u_{i,j}^n) \quad (23)$$

or

$$\nabla_t \Delta_t u_{i,j}^n = c_0^2 (\nabla_x \Delta_x u_{i,j}^n + \nabla_y \Delta_y u_{i,j}^n) \quad (24)$$

where we use $f = 0$ to reduce the Klein-Gordon equation to the standard wave equation.

Try the following discretization for (1)–(4):

$$\begin{aligned}
\nabla_t \rho &= -\rho_0 (\nabla_x u + \nabla_y v) \\
\Delta_t u &= -\frac{\Delta_x p}{\rho_0} \\
\Delta_t v &= -\frac{\Delta_y p}{\rho_0} \\
\nabla_t p &= -\gamma p_0 (\nabla_x u + \nabla_y v)
\end{aligned} \tag{25}$$

Apply ∇_x to the second equation of (25), ∇_y to the third, Δ_t to the fourth, and then make the appropriate substitution. This gives us

$$\Delta_t \nabla_t p = \frac{\gamma p_0}{\rho_0} (\Delta_x \nabla_x p + \Delta_y \nabla_y p), \tag{26}$$

exactly as desired. Alternatively, we can switch the direction of every spatial difference and get the following discretization:

$$\begin{aligned}
\nabla_t \rho &= -\rho_0 (\Delta_x u + \Delta_y v) \\
\Delta_t u &= -\frac{\nabla_x p}{\rho_0} \\
\Delta_t v &= -\frac{\nabla_y p}{\rho_0} \\
\nabla_t p &= -\gamma p_0 (\Delta_x u + \Delta_y v),
\end{aligned} \tag{27}$$

which is then equivalent to

$$\Delta_t \nabla_t p = \frac{\gamma p_0}{\rho_0} (\nabla_x \Delta_x p + \nabla_y \Delta_y p) \tag{28}$$

Which discretization to use is purely an esthetic decision. We have tried both, and both bring stable, albeit asymmetric, results. However, care must be taken in programming this scheme to ensure that the semi-implicit computations of p and ρ use the correct values for u and v . In the numerical example below, we use (25).

5. A Numerical example

Let us consider a simple numerical example. Using the discretization scheme in (25), we look at a square domain 100 km on each side, subdividing it into a 50×50 computational domain with the Higdon-like NRBCs on all four sides. Using a mean atmospheric density of $1.2 \frac{\text{kg}}{\text{m}^3}$ and pressure of $1.01 \times 10^5 \frac{\text{N}}{\text{m}^2}$ [34], and zero advection, our initial condition is a biquadratic pressure and density bulge in the center of the domain:

$$\begin{aligned}
p_{i,j}^0 &= \begin{cases} p_0 \left(1 + \frac{(i-21)(30-i)(j-21)(30-j)}{4000} \right) & : 21 \leq i, j \leq 30 \\ p_0 & : \text{otherwise} \end{cases} \\
\rho_{i,j}^0 &= \begin{cases} \rho_0 \left(1 + \frac{(i-21)(30-i)(j-21)(30-j)}{4000} \right) & : 21 \leq i, j \leq 30 \\ \rho_0 & : \text{otherwise} \end{cases}
\end{aligned} \tag{29}$$

For comparison, our reference solution domain is 300 km on each side, with the domain of interest in the center. We define the error norm for each state variable η as

$$E_\eta = \frac{\sqrt{\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} (\eta_J(i, j) - \eta_0(i, j))^2}}{N_x N_y}, \quad (30)$$

where N_x, N_y are the number of grid points in the x and y directions, respectively, η_J is a solution state variable using the J -order NRBC, and η_0 is the reference solution. Our time step is computed by

$$dt = \frac{\sqrt{dx^2 + dy^2}}{4C_0} \quad (31)$$

which is half the CFL limit, thus guaranteeing stability. We run the simulation up to $t = 216$, long enough for the primary wave to exit the computational domain with the wave trough just passing through the corners. The following figures show the four state variables at the end of the run for $J = 6$. In each figure, the top left shows the computed solution using the NRBCs, the top right shows the reference solution, the bottom left shows the reference solution domain truncated to the size of the computed solution's domain, and the bottom right plots the delta between the two solutions and computes the error norm as defined above.

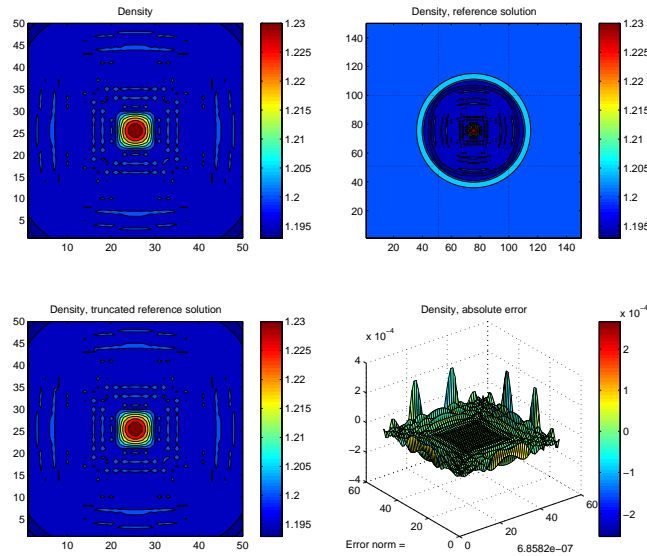


Figure 2: The solution for the density ρ using $J = 6$

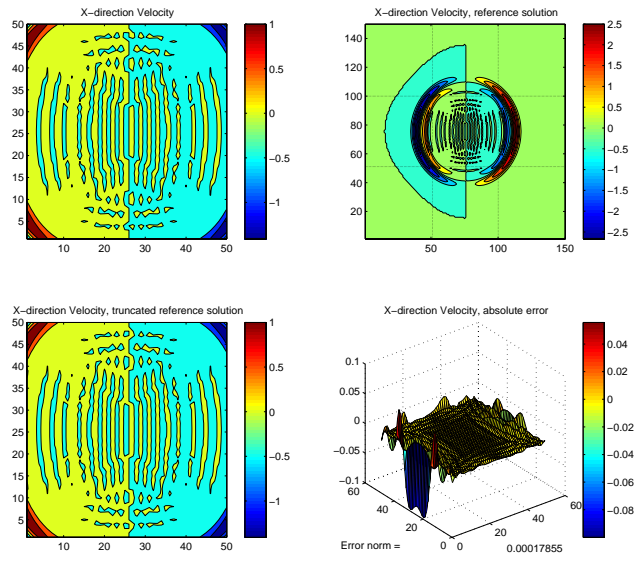


Figure 3: The solution for u using $J = 6$

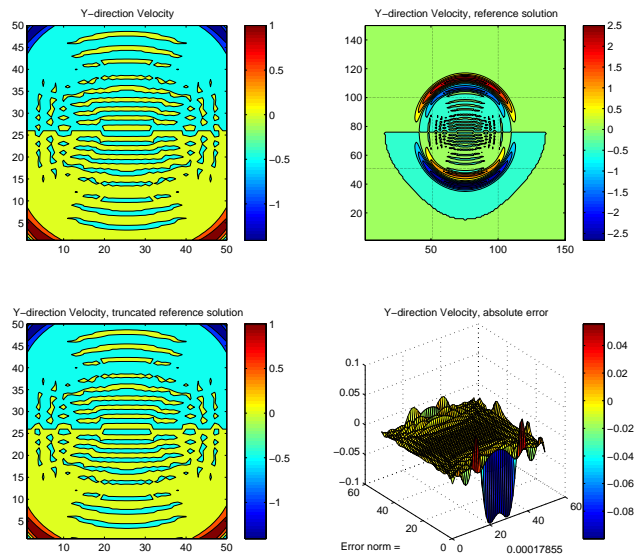


Figure 4: The solution for v using $J = 6$

J	$\rho \times 10^{-5}$	$u \times 10^{-3}$	$v \times 10^{-3}$	p
1	1.7677	3.7601	3.7601	2.0829
2	0.43811	0.94068	0.94068	0.51624
3	0.18223	0.38785	0.38785	0.21473
4	0.10341	0.22628	0.22628	0.12185
5	0.083147	0.16743	0.16743	0.097975
6	0.068582	0.17855	0.17855	0.080812

Table 1: Error Norms for $J \in 1, 6$ with Discretization Scheme (25)

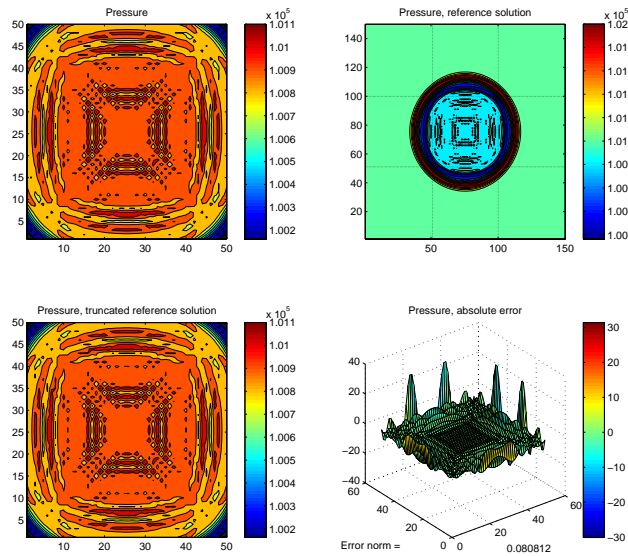


Figure 5: The solution for the pressure p using $J = 6$

Table 1 shows the improvements as J goes from 1 to 6 (graphs omitted) This example demonstrates, albeit in a simplified setting, that the linearized Euler equations are compatible with high-order Higdon-like NRBCs.

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Appendix: Future Research

Here is a list of subjects for further investigation (in random order):

1. Thorough investigation of the numerical properties of the scheme: measuring the error as a function of the location of the artificial boundary; computing time and operation count as a function of the various parameters (such as J and the number of grid points on the boundary); stability with various interior schemes; etc.
2. Implementing the scheme with auxiliary variables, using FDs.
3. Implementing the scheme with auxiliary variables using FEs.
4. Experimenting with the use of the Higdon conditions with the Nonlinear Euler equations in the computational domain. (Need to find a stable interior scheme-NRBC combination.)
5. Applying the scheme in the 3D case.
6. Extending the scheme to the case of the linearized Euler equations with a nonzero mean flow (advection).

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