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Wei-Min Lu and John Doyle

Control and Dynamical Systems
California Institute of Technology
Pasadena, CA 91125

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Attenuation of Persistent $\mathcal{L}_\infty$-Bounded Disturbances for Nonlinear Systems

Wei-Min Lu* and John C. Doyle†

Abstract

A version of nonlinear generalization of the $\mathcal{L}^1$-control problem, which deals with the attenuation of persistent bounded disturbances in $\mathcal{L}_\infty$-sense, is investigated in this paper. The methods used in this paper are motivated by [23]. The main idea in the $\mathcal{L}^1$-performance analysis and synthesis is to construct a certain invariant subset of the state-space such that achieving disturbance rejection is equivalent to restricting the state-dynamics to this set. The concepts from viability theory, nonsmooth analysis, and set-valued analysis play important roles. In addition, the relation between the $\mathcal{L}^1$-control of a continuous-time system and the ℓ^1-control of its Euler approximated discrete-time systems is established.

Key Words: Controlled Invariance, Disturbance Rejection, $\mathcal{L}^1$-Optimal Control, Nonlinear Systems, Robust Control.

1 Introduction

The problem of **optimal rejection of persistent bounded disturbance** for a linear system was posed by Vidyasagar in [26]. It is a minimax optimization problem, i.e., the problem of minimization of the worst possible impact of a class of persistent bounded disturbances on the system. If the disturbance is denoted by w and the signal measuring the impact by z, and both signals are measured in $\mathcal{L}_\infty$ (in continuous time case), then the performance to be minimized is

$$J := \sup_{w \in \mathcal{L}_\infty[0,\infty), \|w\|_\infty \leq 1} \|z\|_\infty \tag{1}$$

This problem is known as $\mathcal{L}^1$-**optimal control** problem [11, 12, 10], because the minimization (1) amounts to the minimization of the $\mathcal{L}_\infty$-induced norm, i.e., the $\mathcal{L}^1$-norm, of the linear system. The linear $\mathcal{L}^1$(or ℓ^1 in discrete-time case)-optimal control problem was extensively investigated in an input/output setting by using Youla-parameterization [11, 12, 10]. The relation between the $\mathcal{L}^1$-control of a linear continuous-time system and the ℓ^1-control of its Euler approximated discrete-time systems was established in [6]. Recently, for a linear discrete-time system, the corresponding ℓ^1-optimal control problem was solved in a state space setting in terms of dynamical state-feedback [14, 10], continuous nonlinear static state-feedback [22, 23], and piece-wise linear static state-feedback [7]. Furthermore, Shamma [23] showed that if the linear ℓ^1-optimal control has any kind of solution, then there must exist a continuous (nonlinear) static state-feedback ℓ^1-controller. It is possible

*Electrical Engineering 116-81, California Institute of Technology, Pasadena, CA 91125; wmlu@hot.caltech.edu.

†Electrical Engineering/Control and Dynamical Systems 116-81, California Institute of Technology, Pasadena, CA 91125.

that by allowing the class of continuous nonlinear controllers, one can make the closed-loop $\mathcal{L}^1$-performance (1) strictly smaller than one can do using only linear controllers [13, 24]. Therefore, it is natural to consider such an optimal disturbance rejection problem in the nonlinear domain.

In this paper, we will consider the problem of optimal rejection of $\mathcal{L}_\infty$-bounded disturbance for continuous-time nonlinear systems. However, in the nonlinear setting, the minimax optimization problem (1) is not equivalent to the minimization of the $\mathcal{L}_\infty$-induced gains of the corresponding nonlinear operators, while we will borrow the terminology **nonlinear $\mathcal{L}^1$-control** to refer to the corresponding nonlinear minimax optimization problem for convenience. The methods used in this paper are greatly motivated by Shamma [22, 23], in which the ℓ_1-control problem for a linear discrete-time system is constructively solved in terms of continuous nonlinear static state-feedback. The main idea in the $\mathcal{L}^1(\ell^1)$-performance analysis and synthesis is to construct a certain invariant subset of the state-space such that achieving disturbance rejection is equivalent to restricting the state dynamics to this set. The techniques from viability theory, nonsmooth analysis, and set-valued analysis [3, 4, 9, 1] are extensively used; and the notion of **(controlled) invariance** [28, 1, 23] plays a central role. This treatment provides some geometrical insights into the robust ($\mathcal{L}^1$) control problem. It is remarked that the invariance notion has also been employed in other nonlinear contexts, such as the control synthesis with state and control constraints (see [16, 19, 5, 15] and references therein) and the zero dynamics [18, 2, 21].

The remainder of this paper is organized as follows. Some mathematical preliminaries are provided in Appendix A, in which some concepts from set-valued analysis and nonsmooth analysis are reviewed, and the emphasis is on set-valued maps and contingent cones. In section 2, the $\mathcal{L}^1$-performance for a nonlinear system is analyzed. The $\mathcal{L}^1$-performance of a nonlinear system is characterized in terms of $\mathcal{L}^1$-performance domains. In section 3, the nonlinear $\mathcal{L}^1$-control synthesis problem is considered. The $\mathcal{L}^1$-control problem is characterized in terms of controlled $\mathcal{L}^1$-performance domains; a continuous static state-feedback $\mathcal{L}^1$-controller is constructed. In section 4, the (controlled) $\mathcal{L}^1$-performance domains are characterized in terms of the (controlled) invariance domains of some (controlled) differential inclusions. Some algorithms for computing the (controlled) invariance domains are provided. In section 5, the computation issues are considered, and some approximation methods are suggested. In particular, the relation between the $\mathcal{L}^1$-performance analysis and synthesis of a continuous-time system and and the ℓ^1-performance analysis and synthesis of its Euler approximated discrete-time systems is established. Those proofs which are relatively technical and less related are put in Appendix B.

Conventions

The following conventions are made in this paper. $\mathbf{Z}^+$ is the set of non-negative integers. $\mathbb{R}^+ := [0,\infty) \subset \mathbb{R}$. $\mathbb{R}^n$ is n-dimensional real Euclidean space. For $x \in \mathbb{R}^n$, $\|x\| := \max\{|x_i| : i = 1, 2, \cdots, n\}$. $\|u\|_\infty :=$ ess-sup$\{\|u(t)\| : t \in \mathbb{R}^+\}$ for vector-valued measurable function $u : \mathbb{R}^+ \to \mathbb{R}^p$. $\mathcal{L}_\infty[0,\infty)$ is the space of vector-valued functions which are measurable and essentially bounded. The prefix B for a normed space denotes the closed unit ball in this space, e.g., $\mathbf{B}\mathbb{R}^n := \{v \in \mathbb{R}^n | \, \|v\| \leq 1\}$ and $\mathcal{B}\mathcal{L}_\infty[0,\infty) := \{w \in \mathcal{L}_\infty[0,\infty) | \, \|w\|_\infty \leq 1\}$.

2 $\mathcal{L}^1$-Performance Analysis of Nonlinear Systems

In this section, we will give some characterizations of the $\mathcal{L}^1$-performance for a nonlinear system. In the next section the synthesis problem is considered based on the analysis results in this section.

Consider a system with external disturbances as follows,

$$\begin{cases} \dot{x} = f(x, w) \\ z = h(x, w) \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbf{BR}^p := \{v \in \mathbb{R}^p |\, \|v\| \leq 1\}$ and $z \in \mathbb{R}^q$ are the external disturbance input and the regulated output, respectively. Suppose that if $w(t) = 0$ and $x(0) = 0$, then $x(t) = 0$ and $z(t) = 0$. The performance which measures the property of disturbance rejection in $\mathcal{L}_\infty$ for system (2) was formulated by Vidyasagar in [26] as follows,

$$J = \sup_{w \in \mathcal{BL}_\infty[0,\infty)} \|z\|_\infty \tag{3}$$

Note that, in the linear case, this performance J is $\|G\|_1$, i.e., the $\mathcal{L}_\infty$-induced norm of the input/output map $G : w \mapsto z$. We say that the system has a **disturbance attenuation** property if $J \leq 1$. This motivates the following definition.

Definition 2.1 *Consider the given system (2) with $x(0) = 0$. It has* $\mathcal{L}^1$**-performance** *if for all $w(t) \in \mathcal{BL}_\infty[0, \infty)$, $\|x(t)\|_\infty < \infty$ and $\|z(t)\|_\infty \leq 1$.*

Therefore, system (2) has $\mathcal{L}^1$-performance, if and only if it is **bounded-input-bounded-state** (BIBS) stable and $J \leq 1$. The above definition is a natural generalization of the $\mathcal{L}^1$-performance for a linear system. In the next few subsections, we will characterize the $\mathcal{L}^1$-performances.

2.1 $\mathcal{L}^1$-Performances and Reachable Sets

Consider system (2). We will assume f and h are continuous, and $f(0, 0) = 0, h(0, 0) = 0$. Therefore, $0 \in \mathbb{R}^n$ is an equilibrium of the system with $w = 0$. Moreover, we assume the admissible disturbance set is

$$\mathcal{W} := \{w : [0, \infty) \to \mathbf{BR}^p |\ w \text{ is measurable}\} = \mathcal{BL}_\infty[0, \infty) \tag{4}$$

We also define a subset $\mathcal{W}_c \subset \mathcal{W}$ as follows,

$$\mathcal{W}_c := \{w \in \mathcal{W} |\ w \text{ is continuous}\} \tag{5}$$

It is assumed that system (2) has the BIBS property. Therefore, all possible solutions with the admissible inputs are in the space $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^n)$. We assume that system (2) is **complete** in the sense that for each $w(t) \in \mathcal{W}$ and $x_0 \in \mathbb{R}^n$, the solution $x(t)$ to (2) starting at $x(0) = x_0$ is uniquely defined for almost every $t \in [0, \infty)$, and the solution continuously depends on the initial conditions with respect to the compact convergence topology in $\mathcal{C}(\mathbb{R}^+, \mathbb{R}^n)$. The **state transition function** $\phi : \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^n$ is so defined that $x(T) = \phi(T, x_0, w^*)$ means that system (2) evolves from initial state x_0 to state x in time T under the input action w^*. Note that ϕ is well-defined and is continuous with respect to initial state because of completeness of system (2). We define the **reachable state maps** of system (2) with the admissible input set $\mathcal{W}$ and $\mathcal{W}_c$ in (4) as set-valued maps $\mathcal{R} : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ and $\mathcal{R}_c : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ with

$$\mathcal{R}(x) := \{\phi(t, x, w) | \forall w \in \mathcal{W}, t \in \mathbb{R}^+\} \tag{6}$$

$$\mathcal{R}_c(x) := \{\phi(t, x, w) | \forall w \in \mathcal{W}_c, t \in \mathbb{R}^+\} \tag{7}$$

Both $\mathcal{R}(x)$ and $\mathcal{R}_c(x)$ are bounded sets since system (2) is BIBS.

Definition 2.2 *A set $K \in \mathbb{R}^n$ is a* **weak invariant set** *for system (2) with respect to an admissible input set $\mathcal{W}$ if for all $x \in K$, and $w \in \mathcal{W}$, $\phi(t, x, w) \in K$ for almost all $t \geq 0$.*

The sets $\mathcal{R}(0)$ and $\mathcal{R}_c(0)$ have the weak invariance property. The case for $\mathcal{R}(0)$ is stated as in the following proposition.

Proposition 2.3 *For all $x \in \mathcal{R}(0)$, and $w \in \mathcal{W}$, $\phi(t, x, w) \in \mathcal{R}(0)$ for all $t \geq 0$. The closure $\overline{\mathcal{R}}(0)$ of $\mathcal{R}(0)$ has this weak invariance property.*

Proof If $x \in \mathcal{R}(0)$, then by the definition of map $\mathcal{R}(0)$, there exist $w_1 \in \mathcal{W}$ and $T \in \mathbb{R}^+$ such that $x = \phi(T, 0, w_1)$. Now take $w_2 \in \mathcal{W}$, define $w : \mathbb{R}^+ \to \mathbf{BR}^p$ as

$$w(t) = \begin{cases} w_1(t) & \text{if } t \in [0, T] \\ w_2(t) & \text{if } t > T \end{cases}$$

and $w \in \mathcal{W}$. Therefore,

$$\phi(t, 0, w) \in \mathcal{R}(0), \quad \forall t \in \mathbb{R}^+.$$

In particular, if $t \geq T$,

$$\phi(t - T, x, w_2) = \phi(t - T, \phi(T, 0, w_1), w_2) = \phi(t, 0, w) \in \mathcal{R}(0)$$

To show $\overline{\mathcal{R}}(0)$ is invariant, we need to show that given $x \in \overline{\mathcal{R}}(0)$ and $w \in \mathcal{W}$, $\phi(t, x, w) \in \overline{\mathcal{R}}(0)$ for all $t \in \mathbb{R}^+$. In fact, suppose there exists a sequence $\{x_n\} \subset \mathcal{R}(0)$, such that $x_n \to x$ as $n \to \infty$. Therefore, $\phi(t, x_n, w) \in \mathcal{R}(0) \subset \overline{\mathcal{R}}(0)$ for all $t \in \mathbb{R}^+$. $\overline{\mathcal{R}}(0)$ is bounded since $\mathcal{R}(0)$ is bounded, therefore, by the completeness assumption, $\phi(t, x, w) = \lim_{n \to \infty} \phi(t, x_n, w) \in \overline{\mathcal{R}}(0)$ for all $t \in \mathbb{R}^+$. □

Next, we will characterize the $\mathcal{L}^1$-performance for system (2) in terms of the reachable set. We first give a weaker definition as follows,

Definition 2.4 *Consider the given system (2) with $x(0) = 0$. It has* **weak $\mathcal{L}^1$-performance** *if for all $w(t) \in \mathcal{W}_c$, $\|x(t)\|_\infty < \infty$ and $\|z(t)\|_\infty \leq 1$.*

Therefore, if system (2) has weak $\mathcal{L}^1$-performance, then

$$J_W = \sup_{w \in \mathcal{W}_c} \|z\|_\infty \leq 1. \tag{8}$$

As system (2) is BIBS, then there exists a compact set $\mathbf{X} \subset \mathbb{R}^n$, such that $\phi(t, 0, w) \in \mathbf{X}$ for all $w(t) \in \mathcal{W}$. Define a closed set as follows,

$$\Omega := \{x \in \mathbf{X} \mid \|h(x, w)\| \leq 1, \forall w \in \mathbf{BR}^p\} \tag{9}$$

Then Ω is **bounded**. We immediately have the following assertion.

Theorem 2.5 *The system (2) has $\mathcal{L}^1$-performance $J \leq 1$ if and only if $\mathcal{R}(0) \subset \Omega$. It has weak $\mathcal{L}^1$-performance $J_W \leq 1$ if and only if $\mathcal{R}_c(0) \subset \Omega$.*

Furthermore, the optimal performance J is given by

$$J = \sup\{\|h(x, w)\| \mid x \in \overline{\mathcal{R}}(0), w \in \mathbf{BR}^p\}.$$

It is noted that in general the reachable set $\mathcal{R}(0)$ and $\mathcal{R}_c(0)$ are not easily computable by the definitions. In the next subsection, we will give some alternative characterizations in terms of the notion of invariance for differential inclusions.

2.2 $\mathcal{L}^1$-Performance Domains

The nonlinear systems with $\mathcal{L}^1$-performances can be described with the aid of a differential inclusions. Indeed, let's consider system (2), define a set-valued map $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ as

$$F(x) := \{f(x,w) | w \in \mathbf{BR}^p\} \tag{10}$$

with the domain $\text{Dom}(F) = \Omega$. It is noted that all solutions to the differential equation $\dot{x} = f(x,w)$ with $w(t) \in \mathcal{W}$ are the solutions of the differential inclusion

$$\dot{x} \in F(x) \tag{11}$$

However, in general, not all solutions of differential inclusion $\dot{x} \in F(x)$ are the admissible solutions for the system (2) for some $w \in \mathcal{W}$; therefore, these two descriptions are not equivalent. We first have the following definition. The contingent cone of a set is defined in Section 7.2.

Definition 2.6 *Consider system (2); the bounded set Ω is defined as in (9). A closed set $K \in \Omega$ is an $\mathcal{L}^1$-**performance domain** for system (2) if $0 \in K$ and for all $x \in K$ and $w \in \mathbf{BR}^p$,*

$$f(x,w) \in T_K(x). \tag{12}$$

where $T_K(x)$ is the contingent cone of set K at x.

It will be seen that an $\mathcal{L}^1$-performance domain is a nonempty invariance domain of its corresponding differential inclusion (10)-(11) (see Section 4.1).

We first have the following theorem about the weak $\mathcal{L}^1$-performance.

Theorem 2.7 *Consider system (2). It has the weak $\mathcal{L}^1$-performance if and only if there exists an $\mathcal{L}^1$-performance domain for system (2).*

The proof of this theorem is given in the next subsection. As for the $\mathcal{L}^1$-performance, we first have the following assertion for a class of nonlinear systems.

Theorem 2.8 *Consider system (2).*

(i) It has $\mathcal{L}^1$-performance, then there exists an $\mathcal{L}^1$-performance domain.

(ii) If $f(x,w)$ is locally Lipschitz in $x \in \mathbb{R}^n$, then system (2) has the $\mathcal{L}^1$-performance if and only if there exists an $\mathcal{L}^1$-performance domain.

The proof of this theorem is given in the next subsection. In the following, we will mainly consider the case of interest in the sequel, where system (2) is affine in w. As a result, the Lipschitz property in Theorem 2.8 (ii) is not required. More concretely, we consider the following system,

$$\begin{cases} \dot{x} = f(x) + g(x)w \\ z = h(x,w) \end{cases} \tag{13}$$

i.e., the function $f(x,w)$ in (2) is replaced by $f(x) + g(x)w$; the other assumptions on $f(x,w)$ are also imposed on $f(x) + g(x)w$.

Theorem 2.9 *Consider system (13) which is affine in w. The following statements are equivalent.*

(i) There exists an $\mathcal{L}^1$-performance domain for system (13).

(ii) System (13) has weak $\mathcal{L}^1$-performance.

(iii) System (13) has $\mathcal{L}^1$-performance.

The proof will be given in the next subsection. Theorems 2.5 and 2.9 imply that if $\overline{\mathcal{R}}(0) \subset \Omega$, then it is an $\mathcal{L}^1$-performance domain for system (13). In fact, it is the smallest $\mathcal{L}^1$-performance domain (Section 4.1). We next give an algorithm to compute the optimal performance J in (3) by using the bisection method (which is used in computing $\mathcal{H}_\infty$-performance [8]).

Given $\epsilon > 0$, one needs to find a $\gamma^* > 0$ such that $\gamma^* - \epsilon \leq J \leq \gamma^* + \epsilon$. Let $\gamma > 0$, define

$$D_\gamma := \{x \in \mathbb{R}^n | \, \|h(x,w)\| \leq \gamma, \forall w \in \mathbf{BR}^p\} \tag{14}$$

and let $\mathrm{D{\small INV}}(D_\gamma)$ be the largest invariance domain of the differential inclusion (10)-(11) in D_γ (see Section 4.1).

Algorithm 2.10 *Give $\gamma_M > \gamma_m \geq 0$ such that $\gamma_m \leq J \leq \gamma_M$.*

Step 1: *If $\gamma_M - \gamma_m \leq 2\epsilon$, let $\gamma^* = (\gamma_M + \gamma_m)/2$, then stop; otherwise go to step 2.*

Step 2: *Let $\gamma = (\gamma_M + \gamma_m)/2$ and compute $K_\gamma := \mathrm{D{\small INV}}(D_\gamma)$.*

Step 3: *If $0 \in K_\gamma$, then redefine $\gamma_M := \gamma$; otherwise let $\gamma_m := \gamma$. Go to step 1.*

The above algorithm can be used to get an approximation of optimal $\mathcal{L}^1$-performance for system (13).

2.3 Proofs of the Main Theorems

Next, we will prove Theorems 2.7, 2.8, and 2.9. The techniques used in the proofs are basically from [1, 25, 29].

PROOF OF THEOREM 2.7

The following lemma from [3, 29] will be used in the following discussion.

Lemma 2.11 *Consider a differential equation $\dot{x} = \psi(x,t)$ with $\psi : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ being continuous. Suppose a set $K \subset \mathbb{R}^n$ is closed. If $\psi(x,t) \in T_K(x)$ for all $x \in K$ and $t \in \mathbb{R}^+$, then for any $x_0 \in K$, there exists a solution $x(t)$ to the differential equation starting at x_0 which is viable in K, i.e. $x(t) \in K$ for almost all $t \in [0, \infty)$.*

Proof [**Theorem 2.7**]

[**Necessity**] Let $K := \overline{\mathcal{R}}_c(0) \subset \Omega$, we now show it is an $\mathcal{L}^1$-performance domain. We need to show that for all $w_0 \in \mathbf{BR}^p$ and $x_0 \in K$, $f(x_0, w_0) \in T_K(x)$. In fact, given $T > 0$, one has $x(t) = \phi(t, x_0, w(t)) \in K$ for all $t \in [0, T]$ where $w(t) \in \mathcal{W}_c$ with $w(0) = w_0$ because of the weak invariance of the set $\overline{\mathcal{R}}_c(0)$. Therefore,

$$x(t) = x_0 + \int_0^t \dot{x}(s)ds = x_0 + t(\frac{1}{t}\int_0^t f(x(s), w(s))ds)$$

Notice that $f(x(s), w(s))$ is bounded in $[0, T]$, then by Lebesgue's differentiation theorem, one has

$$\lim_{t\to 0} \frac{1}{t}\int_0^t f(x(s), w(s))ds = f(x_0, w_0)$$

Therefore, one can find two sequences $\{t_n\}$ and $\{v_n\}$ with $t_n \to 0^+$ and $v_n \to f(x_0, w_0)$ as $n \to \infty$, such that $x_0 + t_n v_n \in K$ for all $n \in \mathbf{Z}^+$. Hence, $f(x_0, w_0) \in T_K(x_0)$, the conclusion then follows by Lemma 7.5.

[**Sufficiency**] Suppose $K \subset \Omega$ is an $\mathcal{L}^1$-performance domain. Given $w(t) \in \mathcal{W}_c$, consider the time-varying differential equation

$$\dot{x} = f(x, w(t)) =: f_w(x, t)$$

Note that the function $f_w : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is continuous; and by assumption, $f_w(x, t) \in T_K(x)$ for all $x \in K$ and $t \in \mathbb{R}^+$, then by Lemma 2.11, for all $x \in K$, $\phi(t, x, w) \in K$ for all $t \geq 0$. In particular, $\mathcal{R}_c(0) \subset K \subset \Omega$. $\square$

PROOF OF THEOREM 2.8

Theorem 2.8 basically follows from [29]. We first restate a result from [29] which is used in the proof. The weak invariance is defined in Definition 2.3.

Lemma 2.12 ([29, Theorem 3.9]) *Consider system (2)*

(i) If K is a closed weak invariant set with respect to $\mathcal{W}$, then $f(x, w) \in T_K(x)$ for all $x \in K$ and $w \in \mathbf{BR^p}$.

(ii) If function $f(x, w)$ is locally Lipschitz in x, K is a closed set, and $f(x, w) \in T_K(x)$ for all $x \in K$ and $w \in \mathbf{BR^p}$, then both K and its interior are weak invariant sets.

Proof [**Theorem 2.8**] (i) Suppose system (2) has $\mathcal{L}^1$-performance $J \leq 1$, by Theorem 2.5, one has that the closure $K := \bar{\mathcal{R}}(0)$ of $\mathcal{R}(0)$ belongs to Ω, since Ω is closed. By Proposition 2.3 and Lemma 2.12 (i), one has that for all $x \in K$,

$$f(x, w) \in T_K(x), \quad \forall w \in \mathbf{BR}^p.$$

or $F(x) \subset T_K(x)$. Thus, K is an invariance domain for F, which is closed, and $0 \in K$. Therefore $0 \in K \subset \textsc{Dinv}(\Omega)$.

(ii) The necessity is proved in (i), only the sufficiency is proved here. Suppose $K := \textsc{Dinv}(\Omega) \ni 0$, then $F(x) \subset T_K(x)$, or

$$f(x, w) \in T_K(x), \quad \forall w \in \mathbf{BR}^p.$$

Then by Lemma 2.12 (ii), one has that for all $x \in K$, and $w \in \mathcal{W}$, $\phi(t, x, w^*) \in K$ for all $t \geq 0$; in particular, $\mathcal{R}(0) \subset K \subset \Omega$. Then the assertion follows from Theorem 2.5. $\square$

PROOF OF THEOREM 2.9

The following lemma is needed in the proof of Theorem 2.9.

Lemma 2.13 *Let continuous functions $f(x)$ and $g(x)$ be defined in (13). Given $\gamma > 0$ and $T > 0$ and $w(t) \in \mathcal{W}$, define*

$$\delta(x, w(t), \gamma) := \sup\{\|f(x) + g(x)w(t) - f(y) - g(y)w(t)\| \,|\forall x, y \in \Omega, \|x - y\| \leq \gamma\} \tag{15}$$

for almost every $t \in [0, T]$. Then map $t \mapsto \delta(x(t), w(t), \gamma)$ is measurable on $[0, T]$. Moreover, let $\{\gamma_n\}$ be a positive decreasing sequence converging to zero, and $\{\tau_n\}$ be a positive sequence converging to zero. Then

$$\lim_{n\to\infty} \frac{1}{\tau_n}\int_t^{t+\tau_n} \delta(x(s), w(s), \gamma_n)ds = 0 \tag{16}$$

almost everywhere in $[0, T]$.

Proof It is obvious that the map $t \mapsto \delta(x(t), w(t), \gamma)$ is measurable on $[0, T]$. It is noted that for almost $t \in [0, T]$, and $\gamma_1 \geq \gamma_2 > 0$ $\delta(x(t), w(t), \gamma_1) \geq \delta(x(t), w(t), \gamma_2)$, and

$$\lim_{\gamma\to 0} \delta(x(t), w(t), \gamma) = 0$$

Now Let $\{\gamma_n\}$ is a decreasing sequence converging to zero. We will show (16) holds almost everywhere in $[0, T]$. In fact, by Lebesgue's differentiation theorem, one has

$$\lim_{\tau\to o^+} \frac{1}{\tau}\int_t^{t+\tau} \delta(x(s), w(s), \gamma_n)ds = \delta(x(t), w(t), \gamma_n) \tag{17}$$

almost everywhere in $[0, T]$ for each $n \in \mathbf{Z}^+$. Therefore, (17) holds almost everywhere in $[0, T]$ for all $n \in \mathbf{Z}^+$, since $\mathbf{Z}^+$ is countable.

Now take $t \in [0, T]$ such that (17) holds for all $n \in \mathbf{Z}^+$. Let $\epsilon > 0$ be given. Then we can find $n_1, n_2 \in \mathbf{Z}^+$, such that $\delta(x(t), w(t), \gamma_{n_1}) < \epsilon/2$, and

$$\frac{1}{\tau_n}\int_t^{t+\tau_n} \delta(x(s), w(s), \gamma_{n_1})ds < \delta(x(t), w(t), \gamma_{n_1}) + \epsilon/2$$

for $n \geq n_2$. Therefore, if $n \geq \max\{n_1, n_2\}$, we have

$$\frac{1}{\tau_n}\int_t^{t+\tau_n} \delta(x(s), w(s), \gamma_n)ds < \frac{1}{\tau_n}\int_t^{t+\tau_n} \delta(x(s), w(s), \gamma_{n_1})ds < \delta(x(t), w(t), \gamma_{n_1}) + \epsilon/2 < \epsilon$$

$\square$

Proof [Theorem 2.9]

It is noted that the implication $(iii) \Rightarrow (ii)$ is obvious, and $(ii) \Rightarrow (i)$ follows from theorem 2.7. We only need to show $(i) \Rightarrow (iii)$. Suppose $K \subset \Omega$ is a compact $\mathcal{L}^1$-performance domain. It is sufficient to show that if $x_0 \in K$, then for all $w \in \mathcal{W}$, $\phi(t, x_0, w) \in K$ for all $t \in [0, \infty)$. The proof is divided into two steps.

Claim 1: Given $T > 0$, for the given $w \in \mathcal{W}$, $\phi(t, x, w) \in K$ for all $t \in [0, T]$.

Indeed, take $\tau > 0$, define a function $f_\tau : K \times [0, T] \rightarrow \mathbb{R}^n$ as follows,

$$f_\tau(x, t) = \frac{1}{\tau} \int_t^{t+\tau} (f(x) + g(x)w(s))ds = f(x) + g(x)w_\tau(t) \tag{18}$$

where

$$w_\tau(t) := \frac{1}{\tau} \int_t^{t+\tau} w(s)ds$$

which is continuous on $[0, T]$. In fact, for all $t_1, t_2 \in [0, T]$,

$$\|w_\tau(t_2) - w_\tau(t_1)\| = \frac{1}{\tau} \left\| \int_{t_1+\tau}^{t_2+\tau} w(s)ds - \int_{t_1}^{t_2} w(s)ds \right\| \leq \frac{2}{\tau} \|t_2 - t_1\|$$

since $w(s) \in \mathbf{BR}^p$ for almost all $s \in [0, T]$. Therefore, f_τ is continuous on the compact set $K \times [0, T]$. because of the continuity of f and g, we can assume $\|f(x) + g(x)w\| \leq \beta$ with some $\beta > 0$ for all $(x, w) \in K \times \mathbf{BR}^p$. Therefore, by (18), it follows that $\|f_\tau(x, t)\| \leq \beta$. Note that

$$\|w_\tau(t)\| \leq \frac{1}{\tau} \int_t^{t+\tau} \|w(s)\| \, ds \leq 1$$

for all $t \in [0, T]$. Therefore, $w_\tau \in \mathcal{W}_c \subset \mathcal{W}$. Then one has

$$f_\tau(x, t) = f(x) + g(x)w_\tau(t) \in T_K(x)$$

for all $x \in K$ $t \in \mathbb{R}^+$. By Lemma 2.11, the solution $x_\tau(t)$ to $\dot{x}_\tau = f_\tau(x_\tau, t)$ for $t \in [0, T]$ is viable in the compact set K, i.e., $x_\tau(t) \in K$ for $t \in [0, T]$.

On the other hand, one has that $\|\dot{x}_\tau\| = \|f_\tau(x, t)\| \leq \beta$. Take a sequence $\{x_n(\cdot)\} := \{x_{\tau_n}(\cdot)\}$, where $\tau_n \rightarrow 0$ as $n \rightarrow \infty$, then the sequence is equicontinuous. Then by Ascoli's Theorem, the sequence remains in a compact subset of the Banach space $\mathcal{C}([0, T], \mathbb{R}^n)$; therefore, there exists a subsequence, denoted as $\{x_n\}$ without loss of generality, which converges uniformly on [0,T] to an absolutely continuous function x which is viable in K since K is closed. Furthermore, the sequence $\{\dot{x}_n\}$ converges to $\dot{x}$ because $\dot{x}_n = f_\tau(x_n(t), t)$ and f is uniformly continuous on the compact set $K \times [0, T]$.

A sequence $\{\gamma_n\}$ is chosen as follows,

$$\gamma_n := \|x_n(t) - x(t)\| + \beta\tau_n$$

such that $\{\gamma_n\}$ is decreasingly converges to zero (otherwise, we can choose a decreasingly subsequence instead), and for all $s \in [t, t + \tau_n]$,

$$\|x_n(t) - x(s)\| \leq \|x_n(t) - x(t)\| + \beta\tau_n = \gamma_n$$

Given $\epsilon > 0$, by Lemma 2.13, there exists an $n_0 \in \mathbf{Z}^+$, such that if $n \geq n_0$,

$$\left\| \dot{x}_n(t) - \frac{1}{\tau_n} \int_t^{t+\tau_n} (f(x(s)) + g(x(s))w(s))ds \right\|$$

$$= \frac{1}{\tau_n} \left\| \int_t^{t+\tau_n} (f(x_n(s)) + g(x_n(s))w(s))ds - \int_t^{t+\tau_n} (f(x(s)) + g(x(s))w(s))ds \right\|$$

$$\leq \frac{1}{\tau_n} \int_t^{t+\tau_n} \delta(x(s), w(s), \gamma_n)ds < \epsilon$$

By Lebesgue's differentiation theorem, one has

$$\lim_{n\to\infty} \frac{1}{\tau_n} \int_t^{t+\tau_n} (f(x(s)) + g(x(s))w(s))ds = f(x(t)) + g(x(t))w(t),$$

since $f(x(s)) + g(x(s))w(s)$ is bounded, is thus integrable on [0,T]. Therefore,

$$\dot{x}(t) = f(x(t)) + g(x(t))w(t), \quad x(0) = x_0,$$

almost everywhere in $[0, T]$. By the completeness of system (13), $\phi(t, x_0, w) = x(t) \in K$ for all $t \in [0, T]$.

Claim 2: The viable solution $x(t)$ in K can be extended to $[0, \infty)$.

In fact, by Zorn's Lemma, one can extend the viable solution $x(t)$ in K to the interval $[0, T_{max})$ for some $T_{max} \geq T$ [29], i.e.,

$$x(t) \in K, \quad \forall t \in [0, T_{max})$$

and T_{max} is such a maximal number. Now, we show $T_{max} = \infty$. In fact, if not so, define

$$C := \limsup_{t\to T_{max}^-} \|x(t)\| < \infty$$

as K is closed and bounded. Thence, $\|x(t)\| \leq C+1$ for $t \in [T_{max} - \tau, T_{max}]$ with some $\tau > 0$. Since $\|w(t)\| \leq 1$, then $\|\dot{x}(t)\| = \|f(x(t)) + g(x(t))w(t)\| \leq \beta$. Therefore, for all $t_1, t_2 \in [T_{max} - \tau, T_{max}]$ with $t_1 \leq t_2$,

$$\|x(t_1) - x(t_2)\| \leq \int_{t_1}^{t_2} \|\dot{x}(s)\| \, ds \leq \beta(t_2 - t_1)$$

Therefore, the Cauchy criterion implies that $\lim_{t\to T_{max}^-} x(t)$ exists. Let $x(T_{max}) := \lim_{t\to T_{max}^-} x(t) \in K$. Since

$$x(t) = x_0 + \int_0^t \dot{x}(s)ds,$$

let $t \to T_{max}^-$, one has

$$x(T_{max}) = x_0 + \int_0^{T_{max}} \dot{x}(s)ds$$

Then the solution can be extended to $[0, T_{max}]$. Now $x(T_{max}) \in K$, the same argument as in **Step 1** shows that there exists $T_0 > 0$ such that $x(t)$ can be extend to $[0, T_{max} + T_0)$. This leads to a contradiction about the maximality of T_{max}. Therefore, $T_{max} = \infty$. □

3 $\mathcal{L}^1$-Control of Nonlinear Systems

In this section, we will consider the nonlinear $\mathcal{L}^1$-control synthesis problem based on the characterizations of $\mathcal{L}^1$-performance in the previous section. A static state-feedback $\mathcal{L}^1$-controller is constructed for a nonlinear system.

Consider the following input-affine system.

$$\begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ z = h(x) + k_1(x)w + k_2(x)u \end{cases} \tag{19}$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbf{BR}^p := \{v \in \mathbb{R}^p | \, \|v\| \leq 1\}$, $u \in \mathbb{R}^m$, and $z \in \mathbb{R}^q$ are the external disturbance input, the control input, and the regulated output, respectively. We will assume f, g_1, g_2, h, k_1, and k_2 are continuous on $\mathbb{R}^n$, $\mathrm{RANK}(g_1(x)) = n$ and $\mathrm{RANK}(g_2(x)) = m$ for all $x \in \mathbb{R}^n$, and $f(0) = 0$ and $h(0) = 0$. Therefore, $0 \in \mathbb{R}^n$ is an equilibrium of the system with $w = 0$ and $u = 0$. Moreover, we assume the admissible disturbance set for system (19) is

$$\mathcal{W} := \{w \in \mathcal{L}_\infty[0,\infty) | \, \|w\|_\infty \leq 1\} = \mathcal{B}\mathcal{L}_\infty[0,\infty) \tag{20}$$

The $\mathcal{L}^1$-control problem for system (19) is defined as follows,

Definition 3.1 *The state-feedback* $\mathcal{L}^1$**-control** *synthesis problem is to find a continuous state-feedback $u = \psi(x)$ for system (19) such that the resulting closed-loop system has the $\mathcal{L}^1$-performance.*

Define a set-valued map $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ as follows,

$$U(x) := \{u \in \mathbb{R}^m | \, \|z\| = \|h(x) + k_1(x)w + k_2(x)u\| \leq 1, \forall w \in \mathbf{BR}^p\} \tag{21}$$

with domain

$$\mathrm{DOM}(U) := \{x \in \mathbb{R}^n | U(x) \neq \emptyset\}.$$

Therefore, if $u = \psi(x)$ is an admissible $\mathcal{L}^1$-controller, then it necessarily satisfies $\psi(x) \in U(x)$ for all $x \in \mathrm{DOM}(U)$. We thus define the set of admissible (state-feedback) controllers for the system G as follows,

$$\mathcal{K} := \{\psi : \mathbb{R}^n \to \mathbb{R}^m | \ \psi \text{ is continuous on } \mathrm{DOM}(U) \text{ and}$$
$$\psi(x) \in U(x) \text{ for all } x \in \mathrm{DOM}(U) \text{ with } \psi(0) = 0\} \tag{22}$$

Let $F : \mathrm{GRAPH}(U) \rightsquigarrow \mathbb{R}^n$ be another set-valued map defined as follows,

$$F(x, u) := \{f(x) + g_1(x)w + g_2(x)u | w \in \mathbf{BR}^p\} \tag{23}$$

One immediately has the following observations. The upper semi-continuous (USC), lower semi-continuous (LSC), and Marchaud maps are defined in Appendix A.

Lemma 3.2 *(i) The set-valued map $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ defined in (21) is locally bounded, USC, as well as LSC with closed values; and* $\mathrm{DOM}(U)$ *is closed.*

(ii) The set-valued map F : $\mathrm{GRAPH}(U) \rightsquigarrow \mathbb{R}^n$ defined in (23) is USC; if in addition $\mathrm{DOM}(U)$ *is bounded, then it is Marchaud.*

Define $F_c : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ by

$$F_c(x) := \bigcup_{u \in U(x)} F(x, u), \tag{24}$$

then the differential inclusion $\dot{x} \in F_c(x)$, which is derived from system (19), is a controlled differential inclusion (F, U) defined by (21) and (23). We first have the following definition.

Definition 3.3 *Consider system (19) and its corresponding controlled differential inclusion (F, U) defined by (21) and (23). Suppose $K \subset \mathrm{DOM}(U)$ is closed, then K is a* **controlled $\mathcal{L}^1$-performance domain** *if $0 \in K$ and for each $x \in K$ there exists $u \in U(x)$ such that*

$$F(x, u) \subset T_K(x) \tag{25}$$

It will be shown in the next section that a controlled $\mathcal{L}^1$-performance domain for system (19) is a closed controlled invariance domain of (F, U). Next, we will characterize the solvability of $\mathcal{L}^1$-control synthesis problem in terms of the controlled $\mathcal{L}^1$-performance domains. It is assumed that system (19) is **complete** in the sense that for each $w(t) \in \mathcal{W}$, $u(t) \in \mathcal{L}_\infty[0, \infty)$ or $u = \psi(x) \in \mathcal{K}$, and $x_0 \in \mathbb{R}^n$, the solution $x(t)$ to (19) starting at $x(0) = x_0$ is uniquely defined for almost every $t \in [0, \infty)$, and the solutions are continuously dependent on the initial states. We first have the following assertion.

Theorem 3.4 (Necessary Conditions) *Consider system (19). If the $\mathcal{L}^1$-control problem has a static state-feedback solution, then there exists a controlled $\mathcal{L}^1$-performance domain.*

Proof If the $\mathcal{L}^1$-control problem has a state-feedback solution, then there exists a state feedback $\psi \in \mathcal{K}$ such that the following closed-loop system

$$\begin{cases} \dot{x} = f(x) + g_2(x)\psi(x) + g_1(x)w \\ z = h(x) + k_2(x)\psi(x) + k_1(x)w \end{cases} \tag{26}$$

has the $\mathcal{L}^1$-performance. Define a set

$$\Omega_c := \{x \in \mathbb{R}^n | \, \|h(x) + k_2(x)\psi(x) + k_1(x)w\| \leq 1, \forall w \in \mathbf{BR}^p\}.$$

Let $\mathrm{DINV}(\Omega_c)$ be the largest $\mathcal{L}^1$-performance domain of the closed-loop system (26) contained in Ω. By Theorem 2.8, $0 \in \mathrm{DINV}(\Omega_c) \neq \emptyset$; and moreover, $\mathrm{DINV}(\Omega_c)$ is a controlled $\mathcal{L}^1$-performance domain for the original system by Definition 3.3. $\square$

It is noted that, the above theorem holds for more general class of nonlinear systems in addition to the class of input-affine systems. Next, we will give a sufficient condition for a modified $\mathcal{L}^1$-control problem to have a solution. Consider system (19). For $\epsilon \in [0, 1)$, define the following performance.

$$J^\epsilon = \sup_{w \in \mathcal{L}_\infty[0,\infty), \|w\|_\infty \leq 1-\epsilon} \|z\|_\infty \tag{27}$$

We will construct a state-feedback $u(\cdot) \in \mathcal{K}$ such that the closed-loop system satisfies $J^\epsilon \leq 1$ for any $\epsilon \in (0, 1)$. We have the following theorem. The sleek sets are defined in Section 7.2.

Theorem 3.5 (Sufficient Conditions) *If there exists a sleek compact controlled $\mathcal{L}^1$-performance domain for system (19), then for all $\epsilon \in (0, 1)$, there exists a continuous static state feedback such that the closed-loop system satisfies $J^\epsilon \leq 1$.*

Proof Suppose $K \neq \emptyset$ is a sleek compact controlled $\mathcal{L}^1$-performance domain for system (19). It is sufficient to construct a state-feedback $\mathcal{L}^1$-controller $u = \phi \in \mathcal{K}$ such that the following modified system

$$\begin{cases} \dot{x} = f(x) + (1-\epsilon)g_1(x)w + g_2(x)u \\ z = h(x) + (1-\epsilon)k_1(x)w + k_2(x)u \end{cases} \tag{28}$$

with the constructed controller achieves $\mathcal{L}^1$-performance.

As $K \in \textsc{Dom}(U)$ is sleek, $T_K : K \rightsquigarrow \mathbb{R}^n$ is LSC with closed convex values. Define a set valued map $T^\epsilon : K \rightsquigarrow \mathbb{R}^n$ as

$$T^\epsilon(x) := \{\xi | \xi + (1-\epsilon)g_1(x)w \in T_K(x), \forall w \in \mathbf{BR}^p\}. \tag{29}$$

It is easy to see that set-valued map T^ϵ is LSC with closed convex values on K. Define the (allowable control) set-valued map $C^\epsilon : K \rightsquigarrow \mathbb{R}^n$ as

$$C^\epsilon(x) := \{u \in U(x) | f(x) + g_2(x)u \in T^\epsilon(x)\}$$

It can also be seen that the set-valued map C^ϵ has closed convex values on K.

On the other hand, we claim that there exists an $\alpha > 0$ such that for all $x \in K$, there exists a $u \in C^\epsilon(x)$ such that

$$f(x) + g_2(x)u + r \in T^\epsilon(x)$$

for all $r \in \mathbb{R}^n$ such that $\|r\| \leq \alpha$. In fact, since $T^0(x) \subset T^\epsilon(x)$ for $0 < \epsilon$, we have $C^0(x) \subset C^\epsilon(x)$. Also $K \neq \emptyset$ implies $C^0(x) \neq \emptyset$ for all $x \in K$. Therefore, there exists $u \in C^0(x) \subset C^\epsilon(x)$, such that

$$f(x) + g_2(x)u \in T^0(x)$$

or

$$f(x) + g_2(x)u + \epsilon g_1(x)w \in T^\epsilon(x), \quad \forall w \in \mathbf{BR}^p$$

Then the claim is justified since $g_1(x)$ has rank n on the compact set K.

By employing Theorem 7.3, we can immediately deduce that the set-valued map C^ϵ is LSC. Furthermore, it can be verified that $0 \in C^\epsilon(0)$. We now use Michael's selection theorem (Proposition 7.4) to conclude that there exists a continuous selection $\psi : K \to \mathbb{R}^m$ of set-valued map $C^\epsilon : K \rightsquigarrow \mathbb{R}^n$ with $\psi(0) = 0$, then $\psi \in \mathcal{K}$.

Now we claim that the state feedback $u = \psi(x)$ is the desired controller. Indeed, the closed-loop system is

$$\begin{cases} \dot{x} = f(x) + g_2(x)\psi(x) + (1-\epsilon)g_1(x)w \\ z = h(x) + k_2(x)\psi(x) + (1-\epsilon)k_1(x)w \end{cases} \tag{30}$$

From the construction, we know that for all $x \in K$ and $w \in \mathbf{BR}^p$,

$$f(x) + g_2(x)\psi(x) + (1-\epsilon)g_1(x)w \in T_K(x).$$

Thus the sleek set $K \neq \emptyset$ is an $\mathcal{L}^1$-performance domain for the above closed-loop system (30). Thus, Theorem 2.9 shows that the closed-loop system (30) has $\mathcal{L}^1$-performance. Therefore, the resulting controller for system (19) yields $J^\epsilon \leq 1$. □

4 $\mathcal{L}^1$-Control and (Controlled) Invariance

In this section, we will characterize the (controlled) $\mathcal{L}^1$-performance domains in terms of corresponding (controlled) differential inclusions. The notions of (controlled) invariance play a central role.

4.1 Differential Inclusions and Invariance Domains

Given a set-valued map $F : X \rightsquigarrow X$, we mainly consider, in this subsection, the following differential inclusion

$$\dot{x}(t) \in F(x(t)), \text{ for almost all } t \in [0, \infty) \tag{31}$$

A function $x : \mathbb{R}^+ \rightarrow X$ is said to be **viable** in a subset $K \subset X$ if $x(t) \in K$ for all $t \in \mathbb{R}^+$ [1]. We have the following definition.

Definition 4.1 *Consider differential inclusion (31). The subset $K \subset X$ is said to be* **invariant** *under F if for all $x_0 \in K$, any solution to (31) starting at x_o is viable in K. Given any closed subset $\Omega \subset \mathrm{Dom}(F)$. The largest closed subset of Ω which is invariant under F, denoted by $\mathrm{Inv}_F(\Omega)$[1], is called the* **invariance kernel** *(IK) of Ω. The smallest closed subset of X invariant under F containing Ω is called* **invariant envelope** *(IE) of Ω, denoted as $\mathrm{Env}_F(\Omega)$[1].*

We first have the following lemma.

Lemma 4.2 *The class of invariance subsets under F is closed under the operation of subset union.*

The above lemma implies that the invariance kernel, if exists, is unique; The invariant envelope, which always exists by Zorn's lemma, is also unique. It is known that if F is Lipschitz, then there exists an invariance kernel for closed subset $\Omega \subset \mathrm{Dom}(F)$ [1, Theorem 5.4.2]. The following theorem gives another class of such nonlinear systems of interest in this paper.

Theorem 4.3 *Suppose the set-valued map $F : \mathrm{Dom}(F) \rightsquigarrow X$ is Marchaud. Then, for any closed subset $\Omega \subset \mathrm{Dom}(F)$, there exists an IK (possibly empty) of Ω. It is the subset of initial points such that all solutions starting from them are viable in Ω.*

The proof of this theorem is given in Appendix B. Definition 4.1 can hardly be conveniently implemented for checking the invariance sets and computing the invariance kernels. We next give an alternative notion.

Definition 4.4 *Let $F : X \rightsquigarrow X$ be given. $K \subset \mathrm{Dom}(F)$ is an* **invariance domain** *(ID) of F if for all $x \in K$, $F(x) \subset T_K(x)$. Given any closed subset $\Omega \subset \mathrm{Dom}(F)$. We denote by $\mathrm{DInv}_F(\Omega)$ the largest closed invariance domain under F in Ω, and by $\mathrm{DEnv}_F(\Omega)$ the smallest closed invariance domain of F containing Ω.*

Recall the definition of the $\mathcal{L}^1$-performance domains in Section 2, it is known from the above definition that any $\mathcal{L}^1$-performance domain is an invariance domain of the corresponding differential inclusion (10) and (11). Therefore, $\mathrm{DInv}_F(\Omega)$ exists for a class of parameterized set-valued maps $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$. From Theorem 2.9, we immediately have the following theorem which characterizes the $\mathcal{L}^1$-performance of system (2) in terms of the invariance domains.

[1] If clear from context, the subscription F will be dropped.

Theorem 4.5 *Consider system (2) and its corresponding differential inclusion (10) and (11); suppose the system is affine in w; the compact set Ω is defined in (9). Then the following statements are equivalent.*

(i) The system has weak $\mathcal{L}^1$-performance.

(ii) $0 \in \mathrm{DINV}_F(\Omega)$.

(iii) $\mathrm{DENV}_F(\{0\}) \subset \Omega$.

Furthermore, if any of the above statements holds, then for any $\mathcal{L}^1$-performance domain K, which is well-defined, one has

$$0 \in \mathrm{DENV}_F(\{0\}) \subset K \subset \mathrm{DINV}_F(\Omega) \subset \Omega.$$

Therefore, if the system has the $\mathcal{L}^1$-performance, then $\mathrm{DENV}_F(\{0\})$ and $\mathrm{DINV}_F(\Omega)$ are the smallest and the largest $\mathcal{L}^1$-performance domains, respectively. In the following, we will give some algorithms for computing the (closed) invariance domains in a given closed subset $\Omega \subset \mathrm{DOM}(F)$. By modifying **viability kernel** algorithms in [1, pp.147–153], one has the following algorithms.

Algorithm 4.6 *Let $F : X \rightsquigarrow X$ and a closed subset $\Omega \subset \mathrm{DOM}(F)$ be given. Define recursively the subsets K_n by*

$$K_0 := \Omega,$$

$$K_{n+1} := \{x \in K_n | F(x) \subset T_{K_n}(x)\}$$

where if K_n is empty in some step n stop there; otherwise define

$$K_\infty := \bigcap_{n=0}^{\infty} K_n \tag{32}$$

It is observed that if $K_n = \emptyset$ for some $n \in \mathbf{Z}^+$, then $\mathrm{DINV}_F(\Omega) = \emptyset$; otherwise, $\mathrm{DINV}_F(\Omega) \subset K_\infty$ if exists, since $\mathrm{DINV}_F(\Omega) \subset K_n$ for all n. However, in general, the inclusion can not be replaced by equality, i.e., the above algorithm does not yield the maximal invariance domain contained in Ω. Because the algorithm does not guarantee the subsets K_n to be closed; also in general the upper limit of the contingent cones $T_{K_n}(x)$ is not necessarily contained in the contingent cone to the upper limit of the subsets K_n [1]. In the following, an alternative algorithm yielding a closed invariance domain which is a subset of $\mathrm{DINV}_F(\Omega)$ is provided. This algorithm is a modification of viability domain algorithm [1, p.151]; the set-valued map $T_K^c : X \rightsquigarrow X$ defined in Definition 7.8 is used.

Algorithm 4.7 *Let $F : X \rightsquigarrow X$ and a compact subset $\Omega \subset \mathrm{DOM}(F)$ be given. Given a constant $c > 0$, define recursively the subsets K_n^c by*

$$K_0^c := \Omega,$$

$$K_{n+1}^c := \{x \in K_n^c | F(x) \subset T_{K_n^c}^c(x)\}.$$

If K_n^c is empty in some step n, then stop there; otherwise define

$$K_\infty^c := \bigcap_{n=0}^{\infty} K_n^c \tag{33}$$

Theorem 4.8 *Let $F : X \rightsquigarrow X$ be LSC and $\Omega \subset \mathrm{DOM}(F)$ be a compact subset. In the above algorithm, if $K_n^c \neq \emptyset$ for all $n \in \mathbf{Z}^+$, then K_∞^c is a nonempty closed invariance domain of F.*

The proof is given in Appendix B.

4.2 Controlled Differential Inclusions and Controlled Invariance

Let X, Y, and Z be metric spaces. Given two set-valued maps, $U : X \rightsquigarrow Z$ and $F : \text{GRAPH}(U) \rightsquigarrow Y$, Define a parameterized set-valued map $F_c : X \rightsquigarrow Y$ as follows,

$$F_c(x) := \bigcup_{u \in U(x)} F(x, u)$$

with $\text{DOM}(F_c) = \text{DOM}(U)$. Then the differential inclusion $\dot{x} \in F_c(x)$ is called a **controlled differential inclusion** (CDI), denoted as (F, U).

Definition 4.9 *Consider a CDI defined by (F, U). A subset $K \in \text{DOM}(U)$ is* **controlled invariant** *under (F, U) if there exists a measurable function $u : \mathbb{R}^+ \rightarrow \mathbb{R}^m$ such that for all $x_o \in K$, the differential inclusion $\dot{x} \in F(x, u(t))$ has all solutions starting at x_o and viable in K, and $u(t) \in U(x(t))$ for all $t \in \mathbb{R}^+$. Given any closed subset $\Omega \subset \text{DOM}(U)$, The largest closed subset of Ω which is controlled invariant under (F, U), denoted by $\text{CINV}(\Omega)$, is called the* **controlled invariance kernel** *(CIK) of Ω, and the* **controlled invariance envelope** *(CIE) $\text{CENV}(\Omega)$ of Ω is defined as the smallest closed controlled invariant subset containing Ω under (F, U).*

The notion of controlled invariance (or (A, B)-invariance for linear systems) was defined for linear systems to deal with disturbance decoupling in [28], and for nonlinear systems in the context of zero dynamics [18, 2]. The controlled invariance envelope exists and is unique. The controlled invariance kernel, if exists, is also unique, because of the following observation.

Lemma 4.10 *The class of controlled invariance subsets under (F, U) is closed under the operation of subset union.*

The following theorem characterizes the existence of CIK in a given closed set.

Theorem 4.11 *Consider the CDI defined by (F, U). Suppose $\Omega \subset \text{DOM}(U) \subset X$ is compact, the set-valued maps $U : \text{DOM(U)} \rightsquigarrow Z$ is LSC with closed convex values, and $F : \text{GRAPH}(U) \rightsquigarrow X$ is Marchaud. Then there exists a CIK (possibly empty) of Ω.*

Proof Since $U : \text{DOM(U)} \rightsquigarrow Z$ is LSC, by Michael's selection theorem (Proposition 7.4) there exists a continuous selection $u(x) \in U(x)$. Define a new set-valued map $F_u : \Omega \rightsquigarrow X$ as $F_u(x) := F(x, u(x))$. Since $F : \text{GRAPH}(U) \rightsquigarrow X$ is Marchaud, then there exists $C > 0$ such that

$$\|F_u(x)\| = \|F(x, u(x))\| \leq C(\|x\| + \|u(x)\| + 1) \leq C(C_u + 1)(\|x\| + 1)$$

with $C_u > 0$ being such that $\|u(x)\| \leq C_u$ as u is continuous on the compact set Ω, so F_u is also Marchaud. Therefore, by Theorem 4.3, there exists a maximal invariance kernel $\text{INV}_{F_u}(\Omega)$, and it is controlled invariant under (F, U) by the definition. Zorn's Lemma implies that there exists a maximal controlled invariance subset, which is the CIK, of Ω. □

From the above theorem and Lemma 3.2, the controlled differential inclusion (F, U) defined by (21) and (23) has the CIK in $\text{DOM}(U)$ if $\text{DOM}(U)$ is compact.

Definition 4.12 *Consider the CDI defined by (F, U). A subset $K \in \text{DOM}(U)$ is a* **controlled invariance domain** *(CID) of (F, U) if for all $x \in K$, there exists a $u(x) \in U(x)$ such that $F(x, u(x)) \subset T_K(x)$. Given any closed subset $\Omega \subset \text{DOM}(U)$, $\text{DCI}(\Omega)$ is the largest closed controlled invariance domain in Ω under (F, U), and $\text{DCE}(\Omega)$ is the smallest closed controlled invariance domain containing Ω for (F, U).*

Recall the definition of the controlled $\mathcal{L}^1$-performance domains in the last section, it is known from the above definition that any controlled $\mathcal{L}^1$-performance domain of system (19) is a controlled invariance domain of the controlled differential inclusion (F, U) defined by (21) and (23). Therefore, $\textsc{Dci}(\Omega)$ exists for a class of parameterized controlled differential inclusions. We immediately have the following theorem which characterizes the controlled $\mathcal{L}^1$-performance of system (19) in terms of the controlled invariance domains.

Theorem 4.13 *Consider system (19) and its corresponding controlled differential inclusion* (F, U) *defined by (21) and (23). Suppose* $\textsc{Dom}(U)$ *is compact, and* $K \in \textsc{Dom}(U)$ *is a controlled* $\mathcal{L}^1$-*performance domain. Then* $0 \in \textsc{Dce}(\{0\}) \subset K \subset \textsc{Dinv}_F(\textsc{Dom}(U)) \subset \textsc{Dom}(U)$.

Therefore, if the system has controlled $\mathcal{L}^1$-performance, then $\textsc{Dce}(\{0\})$ and $\textsc{Dci}(\textsc{Dom}(U))$ are the smallest and the largest controlled $\mathcal{L}^1$-performance domains, respectively. In the following, we give some algorithms to compute the controlled invariance domains in some given closed set. Those algorithms are modifications of the (A, B)-invariance algorithm for linear systems [28], the **controlled invariance kernel** algorithm for controlled difference inclusions [23], and the **zero dynamics** algorithm [18, 21].

Algorithm 4.14 *Let* $F : X \rightsquigarrow X$ *and a closed subset* $\Omega \subset \textsc{Dom}(F)$ *be given. Define recursively the subsets* K_n *by*

$$K_0 := \Omega,$$

$$K_{n+1} := \{x \in K_n | F(x, u) \in T_{K_n}(x), \text{ for some } u \in U(x).\}$$

Define

$$K_\infty := \bigcap_{n=0}^{\infty} K_n \tag{34}$$

It is observed that $\textsc{Dci}(\Omega) \subset K_n$ for all n, then $\textsc{Dci}(\Omega) \subset K_\infty$. However, in general, the inclusion can not be replaced by equality. In the following, we give a remedy to this problem as in Algorithm 4.7, however, instead of $\textsc{Dci}(\Omega)$ itself, only a closed invariance domain, which is a subset of $\textsc{Dci}(\Omega)$, is obtained.

Algorithm 4.15 *Let* $F : X \times Z \rightsquigarrow X$ *and a compact subset* $\Omega \subset \textsc{Dom}(F)$ *be given. Given a constant* $c > 0$, *define recursively the subsets* K_n^c *by*

$$K_0^c := \Omega,$$

$$K_{n+1}^c := \{x \in K_n^c | F(x, u) \in T^c_{K_n^c}(x), \text{ for some } u \in U(x)\}.$$

Then either K_n^c *is empty in some step* n, *or*

$$K_\infty^c := \bigcap_{n=0}^{\infty} K_n^c \tag{35}$$

is not empty.

Theorem 4.16 *Suppose* Ω *is compact and* $F : X \times Z \rightsquigarrow X$ *is LSC,* $U : Z \rightsquigarrow X$ *is locally bounded USC with closed values. In the above algorithm, either* K_n^c *is empty in some step* n, *or* K_∞^c *is a nonempty closed controlled invariance domain of* F.

The proof is given in the Appendix B.

5 Approximation Methods for $\mathcal{L}^1$-Performance Analysis and Synthesis

The $\mathcal{L}^1$-performance analysis and synthesis for nonlinear systems are reduced to the computations of (controlled) invariance domains for some (controlled) differential inclusions. However, unlike the discrete time systems, the algorithms given in the last section are not easy to implemented. In this section, we will try to give some alternatives characterization for the (controlled) invariance domains and approximate them in terms of the (controlled) invariance domains of the corresponding Euler approximated discrete-time systems. To this end, we give the following definition [4, p.17].

Definition 5.1 *Let $\{K_n\}|_{n\in \mathbf{Z}^+}$ be a sequence of subsets of a metric space X. The* **upper limit** *of the sequence is a closed subset of X defined as*

$$\limsup_{n\to\infty} K_n := \{x \in X \,|\, \liminf_{n\to\infty} d(x, K_n) = 0\}$$

Therefore, $\limsup_{n\to\infty} K_n$ is the set of cluster points of sequence $x_n \in K_n$, i.e., of limits of subsequence $x_{n_i} \in K_{n_i}$.

5.1 ℓ^1-Performance of Discrete-Time Nonlinear Systems

The material in this subsection is just the reformulation of some results from [23].

ℓ^1-PERFORMANCES

Consider the following discrete-time nonlinear system

$$\begin{cases} x(k+1) = f_d(x(k), w(k)) \\ z(k) = h_d(x(k), w(k)) \end{cases} \tag{36}$$

where f_d and h_d are continuous. The ℓ^1-performance for system (36) is defined similarly to that in the continuous times case (see Definition 2.1). Let a set-valued map $F_d : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be defined as

$$F_d(x) := \{f_d(x, w) | w \in \mathbf{BR}^p\}$$

with the domain $\mathrm{DOM}(F_d) = \Omega$, where

$$\Omega := \{x \in \mathbb{R}^n | \, \|h_d(x, w)\| \leq 1, \forall w \in \mathbf{BR}^p\}$$

is assumed bounded. We can also similarly define the invariance and the invariance kernel of a closed set for the corresponding difference inclusion. We have the following result [23, Proposition 4.1].

Proposition 5.2 *The invariant kernel $\mathrm{INV}_{F_d}(\Omega)$ in Ω for difference inclusion $x(k+1) \in F_d(x(k))$ exists, and*

$$\mathrm{INV}_{F_d}(\Omega) = \bigcap_{j=1}^{\infty} K_j$$

where $K_0 = \Omega$, $K_{j+1} = \{x \in K_j : F_d(x) \subset K_j\}$. Moreover, system (36) has ℓ^1-performance

$$J := \sup_{w\in\ell_\infty, \|w\|_\infty \leq 1} \|z\|_\infty \leq 1$$

if and only if $0 \in \mathrm{INV}_{F_d}(\Omega) \neq \emptyset$.

CONTROLLED INVARIANCE AND ℓ^1-CONTROL

Let X and Y be metric spaces. Given two set-valued map $U_d : X \rightsquigarrow Y$, $F_d : \text{GRAPH}(U_d) \rightsquigarrow X$, then the difference inclusion

$$x(k+1) \in \bigcup_{u \in U_d(x(k))} F_d(x(k), u)$$

defines a controlled difference inclusion, denoted as (F_d, U_d), we can similarly define such concepts as controlled invariance and controlled invariance kernel of a closed set under (F_d, U_d) [23, Definitions 4.3 and 4.4]. The following result is due to Shamma [23, Proposition 4.2].

Proposition 5.3 *Consider a controlled difference inclusion defined by (F_d, U_d). Suppose $\Omega \subset \text{DOM}(U_d)$ is compact, $U_d : X \rightsquigarrow Y$ is locally bounded USC with closed-values, and $F_d : \text{GRAPH}(U_d) \rightsquigarrow X$ is LSC. Then the controlled invariance kernel $\text{CINV}(\Omega)$ of Ω for (F_d, U_d) exists (possibly empty). And*

$$\text{CINV}(\Omega) = \bigcap_{j=0}^{\infty} K_j$$

where K_j is recursively defined: $K_0 := \Omega$ and $K_{j+1} = \{x \in K_j | F_d(x, u) \subset K_j \text{ for some } u \in U_d(x)\}$.

Next, consider the following discrete-time control system

$$\left\{ \begin{array}{l} x(k+1) = f_d(x(k)) + g_{d1}(x(k))w(k) + g_{d1}(x(k))u(k) \\ z(k) = h_d(x(k)) + k_{d1}(x(k)w(k) + k_{d2}(x(k))u(k) \end{array} \right. \tag{37}$$

with $f_d, g_{d1}, g_{d2}, h_d, k_{d1}$, and k_{d2} being continuous, and $\text{RANK}(g_1(x)) = n$ for all $x \in \mathbb{R}^n$. Similarly, the ℓ^1-control problem for system (37) can be defined as did for the continuous time case (see Definition 3.1). Define $U_d : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$

$$U_d(x) := \{u \in \mathbb{R}^m | \ \|h_d(x) + k_{d1}(x)w + k_{d2}(x)u\| \leq 1, \forall w \in \mathbf{BR}^p\}$$

Suppose $\Omega := \text{DOM}(U_d)$ is bounded. Note that U_d is locally bounded USC with closed values. Let a set-valued map $F_d : \mathbb{R}^n \times \mathbb{R}^m \rightsquigarrow \mathbb{R}^n$ be defined as

$$F_d(x, u) := \{f_d(x) + g_{d1}(x)w + g_{d2}(x)u) | w \in \mathbf{BR}^p\}$$

with domain $\text{DOM}(F_d) = \Omega$. We have the following results about ℓ^1-control problem slightly generalizing Shamma's theorems [23, Theorem 5.1] and [22, Theorem 3.1].

Proposition 5.4 *Consider system (37). Then the following statements are true.*

(i) $\text{CINV}(\Omega)$ exists.

(ii) If the system has ℓ^1-control solution such that $J \leq 1$, then $0 \in \text{CINV}(\Omega) \neq \emptyset$.

(iii) If $0 \in \text{CINV}(\Omega) \neq \emptyset$ and $\text{CINV}(\Omega)$ is convex, then for all $\epsilon \in (0,1)$, there exists a continuous static state feedback such that the closed-loop system satisfies:

$$J^\epsilon := \sup_{w \in \ell_\infty, \|w\|_\infty \leq 1-\epsilon} \|z\|_\infty \leq 1. \tag{38}$$

It is remarked that the results about ℓ^1-performance analysis and synthesis can also be characterized in terms of (controlled) invariance envelope. For example, we have the following version of Proposition 5.4.

Proposition 5.5 *Consider system (37). Then the following statements are true.*

(i) If the system has ℓ^1-control solution, then $\mathrm{CENV}(\{0\}) \subset \Omega$.

(iii) If $\mathrm{CENV}(\{0\}) \subset \Omega$ *and* $\mathrm{CENV}(\{0\})$ *is convex, then for all* $\epsilon \in (0,1)$, *there exists a continuous static state feedback such that the closed-loop system satisfies (38).*

5.2 Approximation of $\mathcal{L}^1$-Performance Domains

Consider system (13) which is rewritten as follows,

$$\begin{cases} \dot{x} = f(x,w) \\ z = h(x,w) \end{cases}$$

where $w \in \mathcal{W}$. Given $\tau > 0$, define a corresponding difference equation as

$$\begin{cases} x_\tau(k+1) = f_\tau(x_\tau(k), w_\tau(k)) \\ z_\tau(k) = h(x_\tau(k), w_\tau(k)) \end{cases} \tag{39}$$

where $w_\tau(k) := w(\tau k)$, and

$$f_\tau(x,w) := x + \tau f(x,w). \tag{40}$$

It is noted that the discrete-time system (39) is a Euler approximation of system (13). Let a set-valued map $F_\tau : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be defined as

$$F_\tau(x) := \{f_\tau(x,w) | w \in \mathbf{BR}^p\}$$

with the domain $\mathrm{DOM}(F_\tau) = \Omega$, where

$$\Omega := \{x \in \mathbb{R}^n | \, \|h(x,w)\| \leq 1, \forall w \in \mathbf{BR}^p\}$$

is assumed bounded. Note that the map F_τ is LSC because of the continuity assumption on f for system (13). Consider the difference inclusion $x_\tau(k+1) \in F_\tau(x_\tau(k))$. Then by Proposition 5.2, we know that invariant kernel $\mathrm{INV}_{F_\tau}(\Omega)$ in Ω for difference inclusion $x_\tau(k+1) \in F_\tau(x_\tau(k))$ exists; and the discrete-time system (39) has ℓ^1-performance if and only if $\mathrm{INV}_{F_\tau}(\Omega) \neq \emptyset$.

We have the following result about the approximations of the $\mathcal{L}^1$-performance domains for system (13).

Theorem 5.6 *Consider system (13). Let $\{\tau_n\}$ be a decreasing sequence such that $\tau_n \to 0$ as $n \to \infty$, and $V_{\tau_n} \subset \Omega$ be closed and invariant under F_{τ_n} for each τ_n with $0 \in V_{\tau_n}$. Then $V_\infty := \limsup_{n\to\infty} V_{\tau_n}$ is a $\mathcal{L}^1$-performance domain for system (13).*

It is noted that in Theorem 5.6, for each τ_n, the corresponding Euler approximated discrete-time system has ℓ^1-performance $J \leq 1$. Possible choices for V_{τ_n} are $\mathrm{INV}_{F_{\tau_n}}(\Omega)$ and $\mathrm{ENV}_{F_{\tau_n}}(\{0\})$.

Proof Choose $w \in \mathcal{W}_c$ and $x_0 \in V_\infty$, then the solution $\phi(t, x_0, w)$ to the differential equation $\dot{x} = f(x, w(t))$ with $x(0) = x_0$ is bounded. We first show that, for all $T > 0$, $\phi(t, x_0, w) \in V_\infty$ for all $t \in [0, T)$.

Consider the function $f_w(x, t) := f(x, w(t))$, which is continuous on compact set $\Omega \times [0, T]$; therefore, $\|f(x, w(t))\| < \beta$ for some $\beta > 0$, and it is uniformly continuous on $\Omega \times [0, T]$. Given $\epsilon > 0$, there thus exists $\delta > 0$ such that for all $\tau \in (0, \delta]$,

$$\|f(x_1, w(t_1)) - f(x_2, w(t_2))\| < \epsilon \tag{41}$$

for all $(x_i, t_i) \in \Omega \times [0, T]$ $(i = 1, 2)$ with $\|x_1 - x_2\| \leq \delta$ and $|t_1 - t_2| < \delta$.

Since $x_0 \in V_\infty$, there exists $x_n^0 \in V_{\tau_n}$ such that x_0 is a cluster point of the sequence $\{x_n^0\}$; we assume $x_n^0 \to x_0$ as $n \to \infty$ without loss of generality. On the other hand, $\tau_n \to 0$ as $n \to \infty$, there exists $N > 0$ such that $\tau_n \in [0, \min\{\delta, \delta/\beta\})$ for all $n \geq N$. Take $n \geq N$, we consider a solution $x_{\tau_n}(k)$ for the difference equation defined in (39):

$$\dot{x}_{\tau_n}(k+1) = f_{\tau_n}(x_{\tau_n}(k), w_{\tau_n}(k))$$

with $x_{\tau_n}(0) = x_n^0$. Then $x_{\tau_n}(k) \in V_{\tau_n}$ for all $k \in \mathbf{Z}^+$ by the definition of V_{τ_n}. Now we associate with the solution a function $x_n \in \mathcal{C}([0, T), \mathbb{R}^n)$ as

$$x_n(t) := x_{\tau_n}(k) + \frac{x_{\tau_n}(k+1) - x_{\tau_n}(k)}{\tau_n}(t - k\tau_n)$$

for all $k \geq 0$ and $t \in [k\tau_n, (k+1)\tau_n)$ such that $t \in [0, T)$. Note that $x_{\tau_n}(k+1) - x_{\tau_n}(k) = \tau_n f(x_{\tau_n}(k), w_{\tau_n}(k)$; thus,

$$\|x_n(t) - x_{\tau_n}(k)\| \leq \left\| \frac{x_{\tau_n}(k+1) - x_{\tau_n}(k)}{\tau_n} \right\| \tau_n < \beta\tau_n \leq \delta$$

and $\dot{x}_n(t) = f(x_{\tau_n}(k), w_{\tau_n}(k))$ for $t \in [k\tau_n, (k+1)\tau_n)$ (hence $\|\dot{x}_n(t)\| < \beta$). From (41), we thus have

$$\|\dot{x}_n(t) - f(x_n(t), w(t))\| < \epsilon \tag{42}$$

for all $t \in [0, T)$. On the other hand, $\dot{x}_n(t)$ is bounded, then $x_n(t)$ is equicontinuous. Similar argument in terms of Ascoli's Theorem in the proof of Theorem 2.9 (i) yields that a subsequence of $\{x_n(t)\}$, still denoted as $\{x_n(t)\}$ without loss of generality, converges to an absolutely continuous function $x(t)$, and their derivatives $\dot{x}_n(t) \to \dot{x}(t)$ as $n \to \infty$. (42) implies that

$$\dot{x}(t) = f(x(t), w(t))$$

Since $x_n(0) = x_{\tau_n}(0) \to x_0$ as $n \to \infty$; and each $t \geq 0$ is the limit of nodes $k_t\tau_n$, so $x(t)$ is the limit of $x_{\tau_n}(k_t) \in K_{\tau_n}$. Then $x(t) \in V_\infty$ for all $t \in [0, T)$. By the completeness of the given system, $\phi(t, x_0, w) = x(t) \in V_\infty$ for all $t \in [0, T)$.

Finally, from similar argument in Theorem 2.7, one can conclude that for all $x \in V_\infty$, $f(x, w) \in T_{V_\infty}(x)$ for all $w \in \mathbf{BR}^p$. Therefore, V_∞ is an $\mathcal{L}^1$-performance domain for system (13). □

The following theorem, which generalizes [6, Theorem 2], characterizes the $\mathcal{L}^1$-performance domains for a class of special systems which include the linear systems.

Theorem 5.7 *Consider system (13). Suppose there exists $\tau > 0$ such that $V_\tau \subset \Omega$ is closed, convex, and invariant under F_τ with $0 \in V_\tau$. Then V_τ is an $\mathcal{L}^1$-performance domain for system (13).*

Proof Since $V_\tau \subset \Omega$ is invariant under F_τ, one has that for all $x \in V_\tau$, $F_\tau(x) \in V_\tau$, or given $w \in \mathbf{BR}^p$,

$$x + \tau f(x, w) \in V_\tau$$

By the assumption V_τ is convex, then

$$x + hf(x, w) \in V_\tau$$

for all $h \in [0, \tau]$. Now by Lemma 7.5, it follows that

$$f(x, w) \in T_{V_\tau}(x)$$

for all $w \in \mathbf{BR}^p$. Therefore, V_τ is an $\mathcal{L}^1$-performance domain for system (13). □

5.3 Approximation of Controlled $\mathcal{L}^1$-Performance Domains

Consider system (19), which is rewritten as follows,

$$\begin{cases} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ z = h(x) + k_1(x)w + k_2(x)u \end{cases}$$

where $w \in \mathcal{W}_c$. Given $\tau > 0$, define a corresponding difference equation, which is a Euler approximation of system (19), as follows,

$$\begin{cases} \dot{x}_\tau(k+1) = f_\tau(x_\tau(k), w_\tau(k), u_\tau(k)) \\ z_\tau(k) = h(x_\tau(k)) + k_1(x_\tau(k)w_\tau(k) + k_2(x_\tau(k))u_\tau(k) \end{cases} \tag{43}$$

where $w_\tau(k) := w(\tau k)$, and

$$f_\tau(x, w, u) := x + \tau(f(x) + g_1(x)w + g_2(x)u). \tag{44}$$

Define $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ as

$$U(x) := \{u \in \mathbb{R}^m | \, \|h(x) + k_1(x)w + k_2(x)u\| \le 1, \forall w \in \mathbf{BR}^p\}$$

Let $\Omega := \textsc{Dom}(U)$ be bounded. Since U is locally bounded USC with closed values by Lemma 3.2. Then there exists a compact set $\mathcal{U} \subset \mathbb{R}^m$, such that

$$\bigcup_{x \in \Omega} U(x) \subset \mathcal{U}$$

Let a set-valued map $F_\tau : \mathbb{R}^n \times \mathbb{R}^m \rightsquigarrow \mathbb{R}^n$ be defined as

$$F_\tau(x, u) := \{f_\tau(x, w, u) | w \in \mathbf{BR}^p\}$$

with the domain $\textsc{Dom}(F_\tau) = \Omega$. Consider the controlled difference inclusion defined by (F_τ, U). By Proposition 5.3, the controlled invariance kernel $\textsc{Cinv}(\Omega)$ of (F_τ, U) exists in Ω; and under some mild conditions, the discrete-time system (43) has ℓ^1-control solution if and only $\textsc{Cinv}(\Omega) \neq \emptyset$ (see Theorem 5.4).

We have the following result on the approximations of controlled $\mathcal{L}^1$-performance domains.

Theorem 5.8 *Consider system (19). Let $\{\tau_n\}$ be a decreasing sequence such that $\tau_n \to 0$ as $n \to \infty$, $V_{\tau_n} \subset \Omega$ be controlled invariant under (F_{τ_n}, U) for each τ_n with $0 \in V_{\tau_n}$. Then $V_\infty := \limsup_{n\to\infty} V_{\tau_n}$ is a controlled $\mathcal{L}^1$-performance domain for system (19).*

It is noted that, in Theorem 5.8, for each τ_n, the corresponding Euler approximated discrete-time system has a nonempty controlled ℓ^1-performance domain. Possible choices for V_{τ_n} are $\textsc{Cinv}(\Omega)$ and $\textsc{Cenv}(\{0\})$.

Proof One needs to show that there exists $u \in U(x)$ such that $f(x, w, u) \in T_{V_\infty}(x)$ for all $x \in V_\infty$ and $w \in \mathbf{BR}^n$.

Suppose $w \in \mathcal{W}_c$ and $x_0 \in V_\infty$. We first show that, given $T > 0$, there exists a measurable function $u(t)$ such that the solution $\xi(t)$ to the differential equation $\dot{x} = f(x, w(t), u(t))$ with $x(0) = x_0$ is in V_∞ for all $t \in [0, T)$ and $u(t) \in U(\xi(t))$.

Consider the function $f_w(x, t, u) := f(x, w(t), u)$, which is continuous on compact set $\Omega \times [0, T] \times \mathcal{U}$; therefore, $\|f(x, w(t), u)\| < \beta$ for some $\beta > 0$, and it is uniformly continuous on $\Omega \times [0, T]$. Now given $\epsilon > 0$, there exists $\delta > 0$ such that for all $\tau \in (0, \delta]$,

$$\|f(x_1, w(t_1), u) - f(x_2, w(t_2), u)\| < \epsilon \tag{45}$$

for all $(x_i, t_i, u) \in \Omega \times [0, T] \times \mathcal{U}$ with $\|x_1 - x_2\| < \delta$ and $|t_1 - t_2| < \delta$.

Since $x_0 \in V_\infty$, therefore there exists $x_n^0 \in V_{\tau_n}$ such that x_0 is a cluster point of the sequence $\{x_n^0\}$; we assume $x_n^0 \to x_0$ as $n \to \infty$ without loss of generality. On the other hand, $\tau_n \to 0$ as $n \to \infty$, there exists $N > 0$ such that $\tau_n \in [0, \min\{\delta, \delta/\beta\})$ for all $n \geq N$. We consider a solution $x_{\tau_n}(k)$ for the difference equation defined in (39) for some $n \geq N$,

$$\dot{x}_{\tau_n}(k+1) = f_{\tau_n}(x_{\tau_n}(k), w_{\tau_n}(k), u_{\tau_n}(k))$$

with $x_{\tau_n}(0) = x_n^0$. Then by the definition of V_{τ_n}, there exists $u_{\tau_n}(k) \in U(x_{\tau_n}(k))$ such that $x_{\tau_n}(k) \in V_{\tau_n}$ for all $k \in \mathbf{Z}^+$. We define a function $x_n \in \mathcal{C}([0, T), \mathbb{R}^n)$ as

$$x_n(t) := x_{\tau_n}(k) + \frac{x_{\tau_n}(k+1) - x_{\tau_n}(k)}{\tau_n}(t - k\tau_n)$$

for all $k \geq 0$ and $t \in [k\tau_n, (k+1)\tau_n)$ such that $t \in [0, T)$. Note that $x_{\tau_n}(k+1) - x_{\tau_n}(k) = \tau_n f(x_{\tau_n}(k), w_{\tau_n}(k), u_{\tau_n}(k))$; thus,

$$\|x_n(t) - x_{\tau_n}(k)\| \leq \left\| \frac{x_{\tau_n}(k+1) - x_{\tau_n}(k)}{\tau_n} \right\| \tau_n < \beta \tau_n \leq \delta$$

and $\dot{x}_n(t) = f(x_{\tau_n}(k), w_{\tau_n}(k), u_{\tau_n}(k))$ for $t \in [k\tau_n, (k+1)\tau_n)$. Let the function $u_n : [0, T) \to \mathbb{R}^m$ be defined as

$$u_n(t) := u_{\tau_n}(k),$$

for $t \in [k\tau_n, (k+1)\tau_n) \subset [0, T)$. Therefore, from (45), we have

$$\|\dot{x}_n(t) - f(x_n(t), w(t), u_n(t))\| \leq \epsilon \tag{46}$$

for all $t \in [0, T)$.

It is noted that, $\dot{x}_n(t)$ is bounded, hence $x_n(t)$ is equicontinuous. Then the similar argument in terms of Ascoli's Theorem in the proof of Theorem 2.9 (i) yields that a subsequence of $\{x_n(t)\}$, still denoted as $\{x_n(t)\}$ without loss of generality, converges to an absolute continuous function $x(t)$, and their derivatives $\dot{x}_n(t) \to \dot{x}(t)$ as $n \to \infty$.

On the other hand, given $t \in [0, T)$, then $x(t)$ is a limit point of some $x_{\tau_n}(k)$; since for $t \in [k\tau_n, (k+1)\tau_n)$, $u_n(t) = u_{\tau_n}(k) \in U(x_{\tau_n}(k)) \in \mathcal{U}$, there exists a subsequence of $\{u_n(t))\}$, still denoted it as $\{u_n(t)\}$, converges to some $u(t) \in U(x(t))$ since U is USC with closed values. Note that $u : [0, T) \to \mathbb{R}^m$ can be chosen to be measurable by the construction[2].

[2]For example, $u(t) = (\limsup_{n\to\infty} u_n^1(t), \cdots, \limsup_{n\to\infty} u_n^m(t))^T$ where $u_n := (u_n^1, \cdots, u_n^m)^T$; $u(t)$ is measurable since its components are upper limits of simple functions.

Therefore, (46) implies that

$$\dot{x}(t) = f(x(t), w(t), u(t))$$

Since $x_n(0) = x_{\tau_n}(0) \to x_0$ as $n \to \infty$; for each $t \geq 0$ is the limit of nodes $k_t \tau_n$, $x(t)$ is the limit of $x_{\tau_n}(k_t) \in K_{\tau_n}$. Then $x(t) \in V_\infty$ for all $t \in [0, T)$. The completeness of the given system implies $\xi(t) = x(t) \in V_\infty$ for all $t \in [0, T)$.

Finally, we will show that V_∞ is a controlled $\mathcal{L}^1$-performance domain for system (19). Take $x_0 \in V_\infty$. For given $w \in \mathcal{U}_c$, there exists a essentially bounded function $u : \mathbb{R}^+ \to \mathbb{R}^m$, such that the unique solution $x(t)$ for $\dot{x} = f(x) + g_1(x)w(t) + g_2(x)u(t)$ with $x(0) = x_0$ is viable in V_∞ and $u(t) \in U(x(t))$. One only needs to check that $\dot{x}(0) = f(x_0) + g_1(x_0)w(0) + g_2(x_0)u(0) \in T_{V_\infty}(x_0)$. In fact, for all $t \in [0, T]$, $x(t) \in V_\infty$

$$x(t) = x_0 + \int_0^t \dot{x}(s)ds = x_0 + t(\frac{1}{t}\int_0^t (f(x(s)) + g_1(x(s))w(s) + g_2(x(s))u(s))ds)$$

Notice that $f(x(s)) + g_1(x(s))w(s) + g_2(x(s))u(s)$ is essentially bounded in $[0, T]$, then by Lebesgue's differentiation theorem, one has

$$\lim_{t \to 0} \frac{1}{t}\int_0^t (f(x(s)) + g_1(x(s))w(s) + g_2(x(s))u(s))ds = f(x_0) + g_1(x_0)w(0) + g_2(x_0)u(0)$$

Therefore, one can find two sequences $\{t_n\}$ and $\{v_n\}$ with $t_n \to 0^+$ and $v_n \to f(x_0) + g_1(x_0)w(0) + g_2(x_0)u(0)$ as $n \to \infty$, such that $x_0 + t_n v_n \in V_\infty$ for all $n \in \mathbf{Z}^+$. Hence, $f(x_0) + g_1(x_0)w(0) + g_2(x_0)u(0) \in T_{V_\infty}(x_0)$, the conclusion then follows by Lemma 7.5. □

The following theorem characterizes the controlled $\mathcal{L}^1$-performance domains for a class of special systems which include the linear systems.

Theorem 5.9 *Consider system (19). Suppose there exists $\tau > 0$ such that $V_\tau \subset \Omega$ is closed, convex, and controlled invariant under (F_τ, U) with $0 \in V_\tau$. Then V_τ is a controlled $\mathcal{L}^1$-performance domain for system (19).*

Proof The argument is similar to that in the proof of Theorem 5.7. □

6 Conclusions

In this paper, the $\mathcal{L}^1$-control problems for nonlinear systems were investigated. The $\mathcal{L}^1$-performance analysis and the $\mathcal{L}^1$-control synthesis problems were characterized in terms of the $\mathcal{L}^1$-performance domains and the controlled $\mathcal{L}^1$-performance domains, which are the invariance domains and the controlled invariance domains of the corresponding differential inclusions, respectively. This treatment therefore provided some geometrical insights into the robust ($\mathcal{L}^1$) control problem. In addition, the relation between the $\mathcal{L}^1$-control of a continuous-time system and the ℓ^1-control of its Euler approximated discrete-time systems was established. Nonetheless, the computational implications of the results for general nonlinear systems in this paper are not clear. The results in this paper can serve for didactic purpose, and can be used to guide the design of nonlinear control systems with disturbance attenuation properties.

Another issue that was not explicitly addressed in this paper is the asymptotic property of the nonlinear $\mathcal{L}^1$-control systems, i.e., when the initial states are not in any of the $\mathcal{L}^1$-performance domains, in which case the systems do not have $\mathcal{L}^1$-performance initially, do the systems eventually have $\mathcal{L}^1$-performance as they evolve? This issue can be investigated in the framework reported in [20].

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7 Appendix A: Set-Valued Maps

In this section, we will review some basic notions from set-valued analysis. We refer to the books [3, 4, 1] for detailed account about the related issues.

7.1 Set-Valued Maps and Their Selections

Let X and Y be two normed spaces. A **set valued map** F from X to Y is a map that associates with any $x \in X$ a subset $F(x)$ of Y. We denote it as

$$F : X \rightsquigarrow Y.$$

The subset $F(x)$ is called the **value** of F at $x \in X$. The **domain** and **graph** of F are defined as

$$\textsc{Dom}(F) := \{x \in X : F(x) \neq \emptyset\}.$$

$$\textsc{Graph}(F) := \{(x, y) \in X \times Y | y \in F(x)\}$$

Definition 7.1 *Consider a set-valued map $F : X \rightsquigarrow Y$.*

(i) It is said to be **lower semi-continuous** *(LSC) if for all $x_0 \in X$, $y_0 \in F(x_0)$, and any sequence of elements $x_n \in \textsc{Dom}(F)$ converging to x_0, there exists a sequence of elements $y_n \in F(x_n)$ converging to y_0.*

(ii) It is said to be **upper semi-continuous** *(USC) if for all $x_0 \in X$, $y \in F(x_0)$, and for any open subset $\mathbf{N}$ of Y containing $F(x_0)$, there exists a neighborhood $\mathbf{N}(x_0)$ of x_0 such that $F(\mathbf{N}(x_0)) \subset \mathbf{N}$.*

Note that if F is USC with closed domain and closed values, then $\textsc{Graph}(F)$ is closed. Two special classes of continuous set-valued maps are defined as follows,

Definition 7.2 *Consider a set-valued map $F : X \rightsquigarrow Y$.*

(i) It is said to be **Marchaud**, *if it is USC, has compact convex images, and has* **linear growth property**, *i.e., there exists $C > 0$ such that for all $x \in \textsc{Dom}(F)$,*

$$\|F(x)\| \leq C(\|x\| + 1) \tag{47}$$

where $\|F(x)\| := \sup_{y \in F(x)} \|y\|$.

(ii) It is said to be **Lipschitz** *around $x \in X$ if there exist a positive constant L and a neighborhood $\mathcal{W} \subset \textsc{Dom}(F)$ of x such that for all $x_1, x_2 \in \mathcal{W}$*

$$F(x_1) \subset F(x_2) + L \|x_1 - x_2\| B_Y$$

Note that, if $F : X \rightsquigarrow Y$ is bounded on $\textsc{Dom}(F)$, then it has the linear growth property (47).

The following technical result will be used [3, p.49].

Proposition 7.3 *Let X be a metric space and Y and Z be Banach spaces. Let $f : X \times Z \to Y$ be a continuous map such that for all $x \in X, u \mapsto f(x,u)$ is affine. Let set-valued maps $T : X \rightsquigarrow Y$ and $U : X \rightsquigarrow Z$ be LSC, and let U be locally bounded. Suppose there exists an $\alpha > 0$ such that for all $x \in X$, there exists a $u \in U(x)$ such that $f(x,u) + r \in T(x)$ for all $r \in Y$ with $\|r\| \leq \alpha$. Then the set-valued map $C : X \rightsquigarrow U$ defined by*

$$C(x) := \{u \in U(x) | f(x,u) \in T(x)\} \tag{48}$$

is LSC.

Given a set-valued map $F : X \rightsquigarrow Y$, there is map $f : X \to Y$ which is a selection of F, i.e. $f(x) \in F(x)$ for each $x \in X$. For a class of set-valued maps, we have the following lemma which is known as **Michael's selection theorem** (cf [4, p.355]).

Proposition 7.4 *Let X be a metric space, Y a Banach space, $F : X \rightsquigarrow Y$ which has the closed convex subsets as its values be LSC. Then there exists a continuous selection $f : X \to Y$ from F. In addition, if $y_0 \in F(x_0)$, then the continuous selection f of F can be chosen such that $f(x_0) = y_0$.*

7.2 Contingent Cones

Let X be a finite dimensional normed space, K be a nonempty subset of X, for each $x \in X$, define the distance of x to K as

$$d_K(x) := d(x,K) := \inf_{y \in K} \|x - y\| . \tag{49}$$

Define a set-valued map $T_K : X \rightsquigarrow X$,

$$T_K(x) := \{v | \liminf_{h \to 0^+} \frac{d_K(x + hv) - d_K(x)}{\tau} \leq 0\} \tag{50}$$

For all $x \in X$, the value $T_K(x)$ is a closed cone, and is called the **contingent cone** to K at x. Note that if $\overline{K}$ denotes the closure of K, then $T_{\overline{K}} = T_K$; if $x \in \overline{K}$, then $T_K(x) = \{v | \liminf_{h \to 0^+} d_K(x + hv)/\tau = 0\}$, and if $x \in \mathrm{INT}(K) \neq \emptyset$, then $T_K(x) = X$. Also if K is a manifold in X, then for any $x \in K$, $T_K(x)$ defines the tangent space of K at x. The following following lemma characterizes contingent cones in terms of sequences [4, p.122].

Lemma 7.5 *Given a set $K \subset X$ and $x \in K$. $v \in T_K(x)$ if and only if there exist a nonincreasing sequence $h_n \to 0$ and a sequence $v_n \to v$, such that $x + h_n v_n \in K$ for all n.*

A subset K of X is said to be **sleek** if the set-valued map $T_K : K \rightsquigarrow X$ is LSC. The following result is from [1, p.161].

Proposition 7.6 *If K is sleek, then for all $x \in \overline{K}$,*

$$T_K(x) = \{v | \lim_{h \to 0^+, y \to_K x} d_K(y + hv)/h = 0\}$$

and $T_K(x)$ is a closed convex cone for all $x \in K$.

Convex sets are sleek. We next state a result about the computing the contingent cone of a set which is defined by some inequalities [4, p.123].

Proposition 7.7 *Given a C^1 vector-valued function $g = (g_1, g_2, \cdots, g_p) : X \to \mathbb{R}^p$. Define a set*

$$K := \{x \in X | g_i(x) \geq 0, i = 1, 2, \cdots, p\}$$

Given $x \in K$, define $I(x) := \{i = 1, 2, \cdots, p | g_i(x) = 0\}$, then under the regularity condition that there exists $v_0 \in X$ such that for all $i \in I(x)$, $< g_i'(x), v_0 >> 0$. Then one has that for all $x \in K$ such that if $I(x) = \emptyset$, $T_K(x) = X$, otherwise

$$T_K(x) = \{v \in X | < g_i'(x), v > \geq 0, \forall i \in I(x)\}$$

Some alternatives to $T_K : X \rightsquigarrow X$ have some nice properties, one is the Clarke cone which has convex closed values [9]. In the following, we give another set-valued map $T_K^c : X \rightsquigarrow X$, whose values belong to the values of T_K [1, p.148].

Definition 7.8 *Let $K \subset X$ be closed, $c > 0$, and $x \in K$. The* **global contingent set**, *denoted by $T_K^c(x)$, is the subset of all $v \in T_K(x)$ such that there exists a measurable function $\gamma(\cdot)$ bounded by c and satisfying*

$$x + tv + \int_0^t (t - \tau)\gamma(\tau) d\tau \in K$$

It is noted that if $c_1 \geq c_2 > 0$, then $T_K^{c_1} \supset T_K^{c_2}$. And if $v \in T_K(x)$, then there exists $c > 0$ such that $v \in T_K^c(x)$. One of the nice properties about the global contingent set is that its graph is closed (see Lemma 8.2).

8 Appendix B: Proofs

PROOF OF THEOREM 4.3

Given $F : X \rightsquigarrow X$, consider the following differential inclusion

$$\dot{x}(t) \in F(x(t)), \text{ for almost all } t \in [0, \infty) \tag{51}$$

Define a set-valued map

$$S_F : \mathrm{Dom}(F) \rightsquigarrow \mathcal{C}([0, \infty); X) \tag{52}$$

such that $S_F(x)$ is the set of all solutions to the differential inclusion (51) starting at $x(0) = x \in \mathrm{Dom}(F)$. S_F is called the **solution map** of differential inclusion (31) (see [1]). We have the following result about the solution map which follows from [1, Theorem 3.5.2].

Lemma 8.1 *Suppose $F : X \rightsquigarrow X$ is Marchaud, then the set valued map S_F defined in (52) is USC with compact values supplied with the compact convergence topology. Moreover, the graph of the restriction $S_F|_K$ of S_F to any compact subset K of $\mathrm{Dom}(F)$ is compact in $X \times \mathcal{C}([0, \infty); X)$.*

The proof of Theorem 4.3 follows the similar ideas in the proof of existence of viability kernel [1, Theorem 4.1.2].

Proof **[Theorem 4.3]** Let $\mathcal{V}(\Omega) \subset \mathcal{C}([0,\infty); X)$ denote the subset of functions viable in $\Omega \subset \text{DOM}(F)$. Define

$$\text{INV}_F(\Omega) := \{x \in \Omega | S_F(x) \subset \mathcal{V}(\Omega)\}$$

We first show the set $\text{INV}_F(\Omega)$ is closed. In fact, given $x \in \text{INV}_F(\Omega)$, and let $\{x_n\} \subset \text{INV}_F(\Omega)$ be a sequence such that $x_n \to x$ as $n \to \infty$. Therefore, there exists a compact set $K \subset X$ such that x and $x_n \in K$. Take a sequence $\{\xi_n\} \subset S_F(x_n)$, then the sequence $\{(x_n, \xi_n)\}$ belongs to $\text{GRAPH}(S_F)|_K$, which is compact by Lemma 8.1. Therefore, there exists a subsequence of $\{(x_n, \xi_n)\}$ converging to some $(x, \xi) \in \text{GRAPH}(S_F)|_K$. Therefore, $\xi \in S_F(x) \subset \mathcal{V}(\Omega)$. Hence, $x \in \text{INV}_F(\Omega)$.

Next, we show $\text{INV}_F(\Omega)$ is invariant under F. Indeed, take $x \in \text{INV}_F(\Omega)$, we need to show that any $\xi \in S_F(x)$ is viable in $\text{INV}_F(\Omega)$, i.e., $\xi(T) \in \text{INV}_F(\Omega)$ for all $T > 0$. In fact, let $\xi_T \in S_F(\xi(T))$, define a function ξ_0 as follows,

$$\xi_0(t) := \begin{cases} \xi(t) & \text{if } t \in [0, T] \\ \xi_T(t - T) & \text{if } t > T \end{cases}$$

Then ξ_0 is a solution to the differential inclusion starting at x at time 0, and thus, is viable in Ω by the definition of $\text{INV}_F(\Omega)$. Hence for all $t > T$, $\xi_T(t - T) \in \Omega$, therefore $S_F(\xi(T) \subset \mathcal{V}(\Omega)$, i.e., $\xi(T) \in \text{INV}_F(\Omega)$.

Finally, we show $\text{INV}_F(\Omega)$ is the largest invariance set contained in Ω. Indeed, let $K \subset \Omega$ is a closed invariance set of F, then for all $x \in K$, there exists a solution ξ to the differential inclusion starting at x which is viable in K, thus in Ω. Therefore, $x \in \text{INV}_F(\Omega)$. □

PROOFS OF THEOREMS 4.8 AND 4.16

The set-valued map, $T_K^c : X \rightsquigarrow X$, is defined in Definition 7.8. The proofs of the two theorems make use of the following properties of T_K^c [1, Proposition 4.4.2]. The upper limit of a set sequence is defined in Definition 5.1.

Lemma 8.2 *The graph of the set-valued map $T_K^c : K \rightsquigarrow X$ is closed. In addition, let $K_\infty := \limsup_{n\to\infty} K_n$ denote the upper limit of a sequence of closed subsets K_n. Then*

$$\limsup_{n\to\infty} \text{GRAPH}(T_{K_n}^c) \subset \text{GRAPH}(T_{K_\infty}^c)$$

Proof **[Theorem 4.8]** We first show that K_n^c defined in the Algorithm 4.7 is closed for each $n \in \mathbf{Z}^+$. In fact, K_0 is closed by definition. Suppose K_n^c is closed, it is sufficient to show K_{n+1}^c is closed. To this end, take a sequence $\{x_i\} \subset K_{n+1}^c$, such that $x_i \to x \in K_n^c$ as $n \to \infty$. Note that

$$F(x_i) \subset T_{K_n^c}^c(x_i) \tag{53}$$

We need to show $F(x) \subset T_{K_n^c}^c(x)$.

In fact, take any $y \in F(x)$, since F is LSC, then there exists a sequence $y_i \in F(x_i)$, such that $y_i \to y$ as $i \to \infty$. Note that from (53),

$$(x_i, y_i) \in \text{GRAPH}(T_{K_n^c}^c)$$

And $\text{GRAPH}(T_{K_n^c}^c)$ is closed by Lemma 8.2, therefore

$$\lim_{i\to\infty} (x_i, y_i) = (x, y) \in \text{GRAPH}(T_{K_n^c}^c)$$

Therefore $y \in T^c_{K^c_n}(x)$ as required. Therefore $x_i \rightarrow x \in K^c_{n+1}$.

To show K^c_∞ is invariant under F, take $x \in K^c_\infty$, we need to verify $F(x) \subset T^c_{K^c_\infty}(x)$.

Note that $x \in K^c_n$ for $n \in \mathbf{Z}^+$, then we have $F(x) \subset T^c_{K^c_n}(x)$. Now for all $y \in F(x)$, then $y \in T^c_{K^c_n}(x)$. Therefore, $(x, y) \subset \text{GRAPH}(T^c_{K^c_n})$, or

$$(x, y) \subset \bigcap_{n=1}^{\infty} \text{GRAPH}(T^c_{K^c_n}) \subset \text{GRAPH}(T^c_{K^c_\infty})$$

where the last inclusion is from Lemma 8.2. Thus, $y \in T^c_{K^c_\infty}(x)$. □

Next, we prove Theorem 4.16.

Proof **[Theorem 4.16]** We first show that K^c_n defined in the Algorithm 4.15 is closed for each $n \in \mathbf{Z}^+$. In fact, K_0 is closed by definition. Suppose K^c_n is closed, it is sufficient to show K^c_{n+1} is closed. To this end, take a sequence $\{x_i\} \subset K^c_{n+1}$, such that $x_i \rightarrow x \in K^c_n$ as $n \rightarrow \infty$. Note that there exists $u_i \in U(x_i)$ such that

$$F(x_i, u_i) \subset T^c_{K^c_n}(x_i) \tag{54}$$

for each $i \in \mathbf{Z}^+$. Note that U is locally bounded and USC with closed value. Then there exists a subsequence, still denoted as $\{u_i\}$ without loss of generality, converging to some $u \in U(x)$. Now we show that $F(x, u) \subset T^c_{K^c_n}(x)$.

In fact, take any $y \in F(x, u)$, since F is LSC, then there exists a sequence $y_i \in F(x_i, u_i)$, such that $y_i \rightarrow y$ as $i \rightarrow \infty$. Note that from (54),

$$(x_i, y_i) \in \text{GRAPH}(T^c_{K^c_n})$$

Since $\text{GRAPH}(T^c_{K^c_n})$ is closed by Lemma 8.2, therefore

$$\lim_{i \rightarrow \infty} (x_i, y_i) = (x, y) \in \text{GRAPH}(T^c_{K^c_n})$$

Therefore $y \in T^c_{K^c_n}(x)$ as required. Therefore $x_i \rightarrow x \in K^c_{n+1}$.

We next show K^c_∞ is controlled invariant under (F, U). To this end, take $x \in K^c_\infty$, then $x \in K^c_n$ for $n \in \mathbf{Z}^+$, therefore there exists $u_n \in U(x)$ for each $n \in \mathbf{Z}^+$, $F(x, u_n) \subset T^c_{K^c_n}(x)$. Since U is locally bounded and USC with closed value, there exists a subsequence, $\{u_{n_i}\} \subset \{u_n\}$, converging to some $u \in U(x)$. We now show $F(x, u) \subset T^c_{K^c_\infty}(x)$.

In fact, for all $y \in F(x, u)$, there exists a sequence $\{y_{n_i}\}$, such that $y_{n_i} \in F(x, u_{n_i}) \subset T^c_{K^c_{n_i}}(x)$, and $y_{n_i} \rightarrow y \in F(x, u)$ as $i \rightarrow \infty$. On the other hand,

$$(x, y) = \lim_{i \rightarrow \infty} (x, y_{n_i}) \in \text{GRAPH}(T^c_{K^c_{n_i}})$$

Then $y \in T^c_{K^c_{n_i}}(x)$. Note that

$$K^c_\infty = \bigcap_{n=1}^{\infty} K^c_n = \bigcap_{i=1}^{\infty} K^c_{n_i}$$

Then,

$$(x, y) \subset \bigcap_{i=1}^{\infty} \text{GRAPH}(T^c_{K^c_{n_i}}) \subset \text{GRAPH}(T^c_{K^c_\infty})$$

where the last inclusion is from Lemma 8.2. Therefore, $y \in T^c_{K^c_\infty}(x)$. □

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