Australian Government
Department of Defence
Defence Science and
Technology Organisation

Area of Common Overlap of Three Circles

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DSTO-TN-0722

ABSTRACT

This Note presents the solution to an apparently hitherto unsolved geometrical problem: the derivation of a closed-form algebraic expression of the area of common overlap of three circles, such as can occur in a three-circle Venn diagram. The results presented here have general significance in the corpus of mensuration formulae, and could be of specific use in any quantitative application of the three-circle Venn diagram such as, for example, in search and screening problems.

Published by

DSTO Defence Science and Technology Organisation
PO Box 1500
Edinburgh South Australia 5111   Australia

Telephone: (08) 8259 5555
Fax: (08) 8259 6567

AR-013-763
October 2006

APPROVED FOR PUBLIC RELEASE

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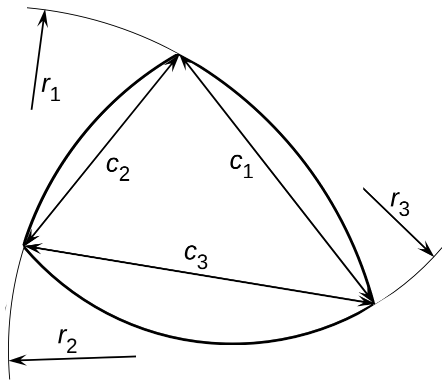
Executive Summary

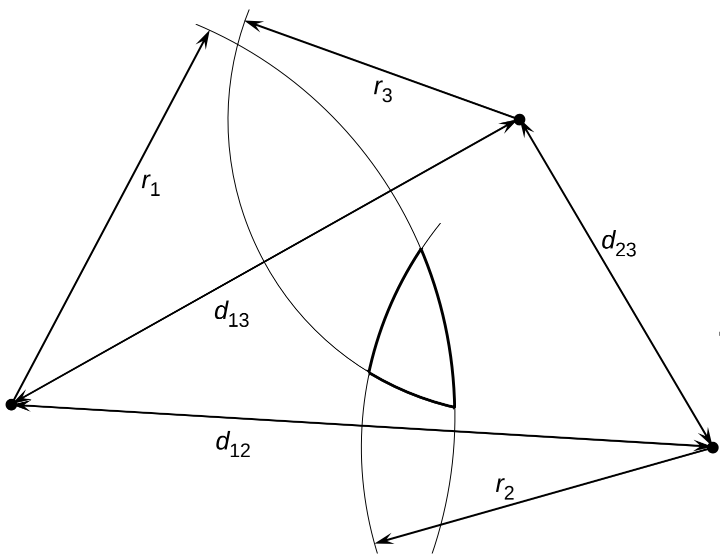
This Note presents the solution to an apparently hitherto unsolved geometric problem: the derivation of a closed-form algebraic expression of the area of common overlap of three circles, such as often occurs in a three-circle Venn diagram.

The area of interest is a type of circular triangle. It is most simply specified by three radii and three chord lengths, as shown in the diagram on the left. Rather than chord lengths, however, it is much more usual for a circular triangle to be specified by the relative locations of the circles of which the sides of the triangle are parts. This is shown in the figure on the right. Formulae are derived for the area and perimeter of the circular triangle in terms of both sets of parameters. The formulae include tests to ensure that the given radii and centre separations actually result in a circular triangle. Expressions for vertex angles, both between arcs and between chords, are also presented. The formulae are general enough to cover all possible cases of the overlap of three circles for which the area of common overlap is a circular triangle. Four special cases, involving substantial simplifications in the formulae, are examined.

The results presented here have general significance in the corpus of mensuration formulae, and could be of specific use in any quantitative application of the three-circle Venn diagram such as, for example, in search and screening problems.



Geometry of a circular triangle, showing the chords and radii involved.



The common method of specifying a circular triangle, in terms of the properties of the three circles of which the sides of the triangle are parts.

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Matthew Fewell joined DSTO in 2001, coming from an academic physics background. He has worked and published in the fields of experimental nuclear structure physics, gaseous electronics, atom–photon interactions including coherent effects, laser physics, the plasma processing of materials, the theory of network-centric warfare and its modelling at the operational level including cognitive issues, human-in-the-loop experimentation, and weapon–target allocation in ship air defence. He is at present working on issues in anti-submarine warfare from surface-ship and maritime-patrol perspectives.

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1. Introduction

The three-circle Venn diagram (Fig. 1) is so familiar and widely used a figure that it is scarcely an overstatement to describe it as an icon of our culture. Usually, applications of this diagram focus on the area of common overlap of the three circles. It is therefore interesting and surprising to learn that a formula for this area is not known, or at least has not previously been published.(a) [1,2,3,4,5,6,7,8,9,10,11]

The area of interest has a shape known as a 'circular triangle': a triangle with circular arcs for sides.(b) Circular triangles, and the Venn diagrams of which they are a part, need not have the high degree of regularity displayed in Figure 1. For example, Figure 2 shows a typical geometry of a screening operation, in which search assets with limited and relatively small detection ranges have the task of securing the perimeter of a large exclusion zone. The problem to be solved might be to determine how the search assets should position themselves for best effect, and the solution may well benefit from an analytic expression for the area of a circular triangle. Whether or not this is so, it is clearly desirable for a formula for the magnitude of the common overlap area to be general enough to include all cases.

The purpose of this Technical Note is to report what seems to be the first published closed-form expression for the area of a common overlap of three circles. It seems odd that it should be the first, for one can straightforwardly construct an expression from well known mensuration formulae, as §2 shows. However, this expression requires values of quantities that usually are not given as part of the problem statement, so that

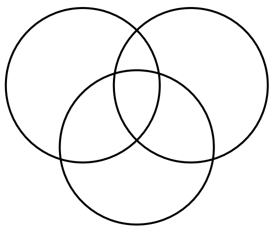

Figure 1. Example of a three-circle Venn diagram. The figure of interest for this Note is the 'circular triangle' where all three circles overlap.

(a) This statement reflects common belief, as I understand it. It is supported by a search of several compilations of mensuration formulae, both print [1–6] and on-line [7–9], and of the web (May 2006) using the key words 'mensuration formulae', 'mensuration formulas' and 'circular triangle'. A web page seeking an expression for the common overlap area was posted on *Math Forum @ Drexel* in December 2002 [10] and so far has attracted no replies.

(b) The term 'circular triangle' seems to have currency among philosophers to mean something than can be conceived of, yet whose existence is impossible. However, the term has been used by geometricians in the meaning given here for well over a century; e.g. Lachlan, writing in 1893, employs it without comment or even explicit definition [11].

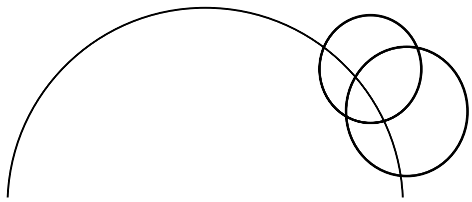

Figure 2: A variation of the 3-circle Venn diagram with circle sizes typical of some screening problems. The large circle represents an exclusion zone and the two smaller circles the detection ranges of search assets tasked with securing the perimeter of the exclusion zone.

it is useless in almost all situations. It is nevertheless presented here because it provides the route towards the more useful result, as shown in §4. Before that, §3 looks in general terms at figures formed by three arbitrarily placed circles, so as to identify situations where an area of common overlap exists and, where it does exist, when it is a circular triangle. Section 5 contains a summary and other comments, including some special cases, and §6 is a brief conclusion. The Appendix indicates how the analysis can be extended to cover other types of circular triangle.

2. Area of a Circular Triangle

2.1 A Straightforward Closed-Form Expression

A circular triangle can be defined by three radii and three chord lengths, as shown in Figure 3. In terms of these quantities, the area A of the figure is[(c)]

$$A = \tfrac{1}{4}\sqrt{(c_1 + c_2 + c_3)(c_2 + c_3 - c_1)(c_1 + c_3 - c_2)(c_1 + c_2 - c_3)} \\ + \sum_{k=1}^{3}\left(r_k{}^2 \arcsin\frac{c_k}{2r_k} - \frac{c_k}{4}\sqrt{4r_k{}^2 - c_k{}^2}\right). \tag{1}$$

Derivation: A circular triangle can be viewed as an ordinary (straight-line) triangle plus three circular segments. The first line of Equation (1) is Heron's well-known formula for the area of a (straight-line) triangle in terms of its three side lengths [1(p.139)]. The second line is the sum of the areas of the three circular segments. The usual expression for the area a of a circular segment is [1(p.143)]

$$a = \frac{r^2}{2}(\phi - \sin\phi), \tag{2}$$

where r is the radius of the circle and ϕ is the angle subtended by the segment at the centre. It is a matter of simple geometry and trigonometry to express ϕ in terms of r and the chord length c, leading to the terms in the second line of Equation (1).[(d)]

It is remarkable how simple Equation (1) is, considering the common belief that there is no known formula for the area of a circular triangle.

(c) This formula applies to the type of circular triangle shown in Figure 3. Section 4.4.2 and the Appendix describe other types and indicate changes needed to accommodate them.

(d) As elementary as it may be, this step contains a hidden assumption that makes Equation (1) less general than Equation (2). This has an impact on the formula for the area of common overlap of three circles, as detailed in §4.4.2.

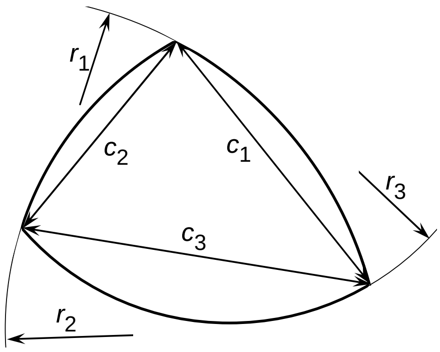

Figure 3: Geometry of a circular triangle, showing the chords and radii involved

2.2 The Difficulty with Equation (1)

The problem with Equation (1) is that usually the chord lengths c_k are not known. Although it is possible to define a circular triangle using these chord lengths, as Figure 3 illustrates, it is much more common to specify the triangle in the manner of Figure 4, using the relative positions of the centres of the three circles. The additional analysis required to handle this very common case is presented in the rest of this Note.

If the vertices of the circular triangle are labelled as in Figure 4 and the chords as in Figure 3, then the length c_k of chord k can be written

$$c_k{}^2 = \left(x_{ik} - x_{jk}\right)^2 + \left(y_{ik} - y_{jk}\right)^2 . \tag{3}$$

To make this equation work, i, j, k must be all distinct from each other and the order of the subscripts must be disregarded; that is, $x_{21} \equiv x_{12}$ etc. Equation (3) reduces the problem to one of determining the coordinates (x_{ij}, y_{ij}) of the vertices.

Before tackling this problem, one must first recognise a difficulty with the schema of Figure 4: it is clearly not the case that all possible sets of radii and separations give

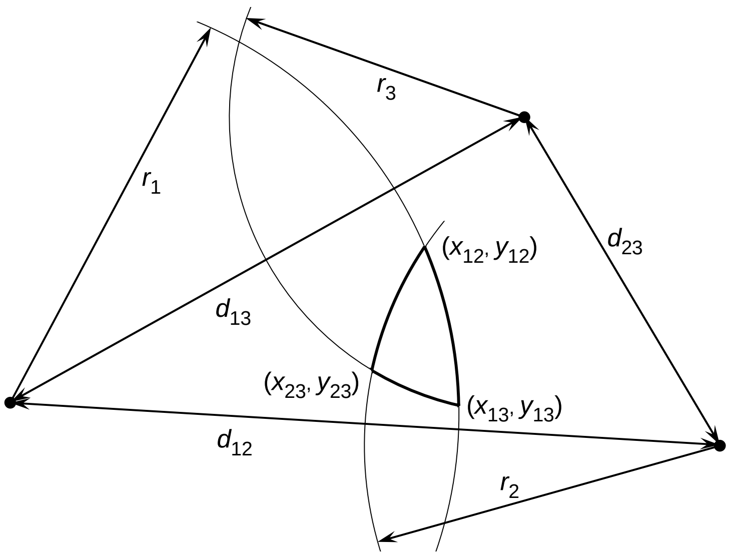

Figure 4: The common method of specifying a circular triangle, in terms of the properties of the three circles of which the sides of the triangle are parts. The vertices of the circular triangle are labelled (x_{ij}, y_{ij}) *in a manner useful for Equation (3).*

circular triangles; for example, there is no *a priori* requirement for the circles even to overlap. Hence it is first necessary to explore the variety of possible cases; this is the task of the next Section. The exercise also helps to build intuition for the range of behaviour inherent in the description of Figure 4, a range that the specification method of Figure 3 so conveniently avoids.

3. A Geometric Taxonomy for Three Circles in a Plane

Figure 5 shows all the configurations of three circles in a plane that involve points of simple intersection between two circles. Cases ⑧ and ⑨ have too few intersection points to produce a circular triangle. Six are required — case ① — but that condition is insufficient, as cases ②, ③ and ⑦ demonstrate. In case ②, the area of common overlap may be termed a circular quadrilateral.[e] Cases ④–⑥ have areas where all three circles overlap, but fewer than six intersection points.

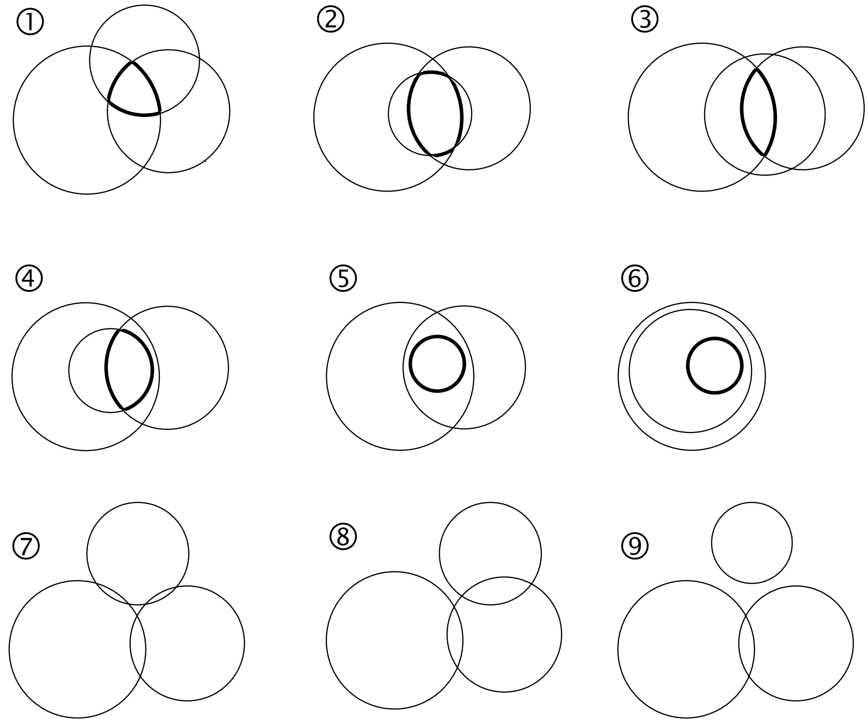

Figure 5: Variety of behaviour in the intersection of three circles in a plane. Areas of overlap of all there circles are delineated by thick lines. Cases ①–③ all have six points of intersection between pairs of circles, but only case ① gives a circular triangle. The overlap area is a circular quadrilateral in case ②, a lens shape in cases ③ and ④, and a circle in cases ⑤ and ⑥. Cases ⑦–⑨ have no area of overlap of all three circles, despite case ⑦ having six intersection points. Degenerate cases, involving point(s) of tangency between two circles or point(s) where all three circles intersect, are omitted from this figure, as is the case where no circle intersects any other.

[e] It is actually a *cyclic* circular quadrilateral, since all four vertices lie on the smallest circle, a fact that slightly simplifies the expression for its area [1(p.141)].

Further cases, not shown in Figure 5, involve different types of intersection point, either points where all three circles intersect or points of tangency between two circles. All such cases can be generated from cases in Figure 5 by suitable displacements of the circles. Many are transitional between the cases in Figure 5.

The logical next step may seem to be examination of the relevant transitional cases to obtain bounds on values of radii and separations that give case ①. However, it turns out to be possible, and easier, to derive expressions for the coordinates of the vertices of the circular triangle in a manner that ensures the existence of six intersections, and then use the resulting expressions to determine bounds on the input parameters that distinguish case ① from cases ②, ③ and ⑦. The next Section sets this out.

4. Vertex Coordinates in Circular Triangles

This Section gives the derivation of expressions for the vertex coordinates, for substitution into Equation (3). It includes a derivation of bounds on the input radii and circle separations so as to ensure that the parameters describe a circular triangle.

4.1 Coordinate Systems

Figure 6 shows a convenient coordinate system for the derivation. One can assume without loss of generality that the circles are numbered in order of decreasing radius; that is, $r_1 \geq r_2 \geq r_3$. As Figure 6 shows, the origin of coordinates is placed at the centre of circle 1 and the x axis passes through the centre of circle 2. The direction of the y axis is chosen so that the y coordinate of the centre of circle 3 is positive.

The (x, y) system is appropriate for analysing the intersections of circles 1 and 2, as detailed in §4.2. For treating the third circle, it is convenient to work with two other coordinate systems, labelled (x', y') and (x'', y'') in Figure 6, and obtained by rotating and translating the (x, y) coordinates:

- The origin of the (x', y') system is located at the centre of circle 1 and the x' axis passes through the centre of circle 3.
- The origin of the (x'', y'') system is located at the centre of circle 2 and the x'' axis passes through the centre of circle 3.

Figure 6 also shows the angles θ' and θ'' between the x axis and the respective abscissas of the two additional systems.

4.2 Intersection of Two Circles

Consider circles 1 and 2 in Figure 6. It is clear that, for them to intersect, the separation d_{12} between their centres must satisfy

$$r_1 - r_2 < d_{12} < r_1 + r_2. \tag{4}$$

(Recall that we have numbered the circles so that $r_1 \geq r_2$.) If Equation (4) is satisfied, then the circles intersect in two places. Each intersection point satisfies the equations of both circles:

$$x_{12}{}^2 + y_{12}{}^2 = r_1{}^2, \qquad (x_{12} - d_{12})^2 + y_{12}{}^2 = r_2{}^2, \tag{5}$$

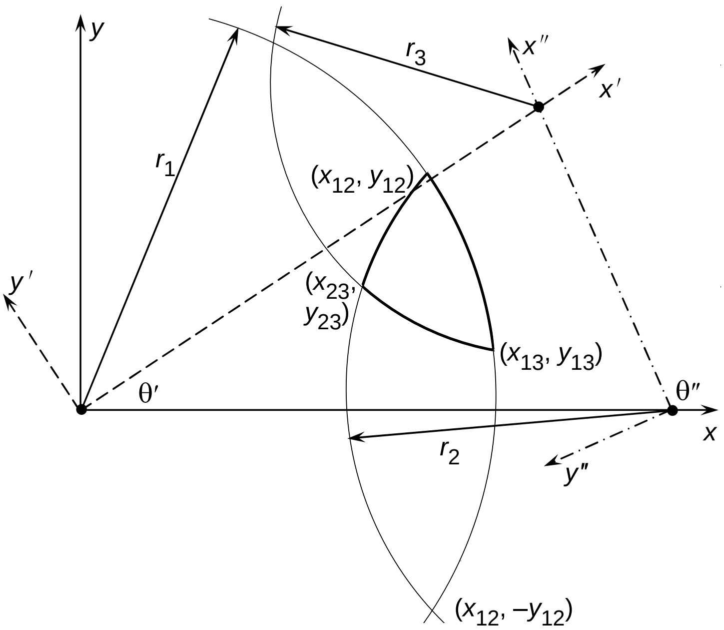

Figure 6: Coordinate systems for the derivation of expressions for the coordinates of the vertices of the circular triangle outlined by thick lines. The (x, y) *system is the primary system, in which all final results are given. The other two systems facilitate the treatment of circle 3.*

where (x_{12}, y_{12}) are the coordinates of the points of intersection. A result from elementary geometry shows that both intersection points have the same x_{12} value and their y_{12} values are the negative of each other, as indicated in Figure 6. Figure 6 also shows that the desired intersection point is the one with positive y coordinate. Hence:

$$x_{12} = \frac{r_1^2 - r_2^2 + d_{12}^2}{2d_{12}}, \qquad y_{12} = \frac{1}{2d_{12}} \sqrt{2d_{12}^2\left(r_1^2 + r_2^2\right) - \left(r_1^2 - r_2^2\right)^2 - d_{12}^4}\,. \tag{6}$$

Equation (6) has long been known (e.g. [12]), but it is interesting to examine the behaviour of x_{12} and y_{12} as d_{12} is varied within the limits given by Equation (4). For this purpose, it is useful to scale all quantities against r_1. That is, the quantities of interest are x_{12}/r_1, y_{12}/r_1 and d_{12}/r_1. The results are shown in Figure 7. In general, $x_{12} \to r_1$ and $y_{12} \to 0$ as either limit in Equation (4) is approached. The exception is the case of $r_1 = r_2$, for which the limiting values are reversed as $d_{12} \to 0$: $x_{12} \to 0$ and $y_{12} \to r_1$.

4.3 Inclusion of the Third Circle

The simplest way of obtaining the coordinates of the intersection points with the third circle would seem to be by making maximum use of the results in §4.2 for circles 1 and 2. That is, use of the (x', y') coordinate system (Fig. 6) allows one to write down expressions for the coordinates of the intersection between circles 1 and 3 by inspection of Equation (6). Further, it is plain from Figure 6 that the desired intersection point is the one with y'_{13} negative. Hence,

$$x'_{13} = \frac{r_1^2 - r_3^2 + d_{13}^2}{2d_{13}}, \qquad y'_{13} = \frac{-1}{2d_{13}} \sqrt{2d_{13}^2\left(r_1^2 + r_3^2\right) - \left(r_1^2 - r_3^2\right)^2 - d_{13}^4}\,. \tag{7}$$

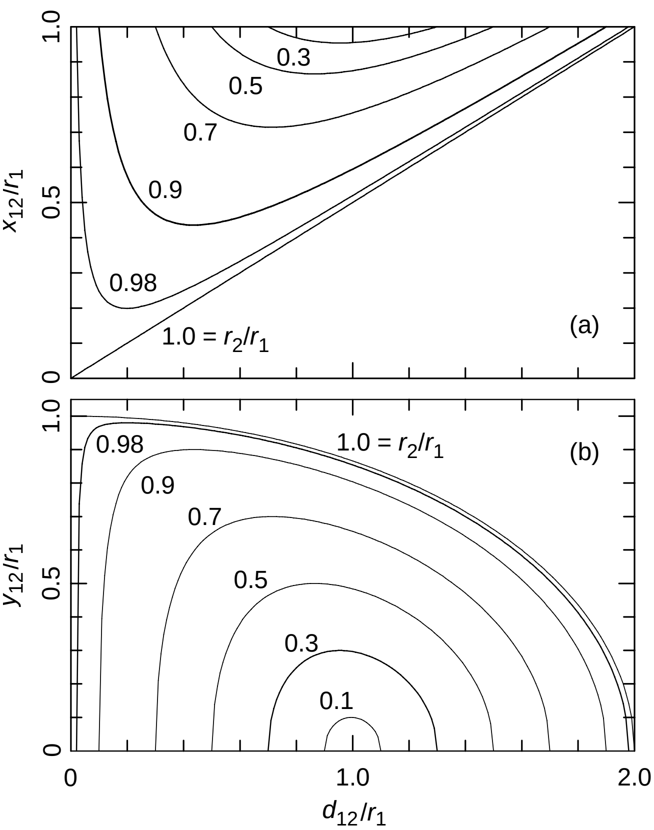

Figure 7: Behaviour of the coordinates (a) x_{12} *and (b)* y_{12} *of a point of intersection of two circles with radii* r_1 *and* r_2 *as the separation* d_{12} *of the centres of the circles is varied, for a range of values of* r_2/r_1*. The extent to which* $r_2/r_1 = 1$ *is a special case is clear.*

Coordinates of the intersection points are to be inserted in Equation (3), but this requires the use of a single set of axes. Hence Equation (7) must be transformed to the (x, y) system:

$$x_{13} = x'_{13}\cos\theta' - y'_{13}\sin\theta', \qquad y_{13} = x'_{13}\sin\theta' + y'_{13}\cos\theta', \tag{8}$$

where

$$\cos\theta' = \frac{d_{12}{}^2 + d_{13}{}^2 - d_{23}{}^2}{2d_{12}\,d_{13}}, \qquad \sin\theta' = \sqrt{1-\cos^2\theta'}\,. \tag{9}$$

The second expression in Equation (9) applies because the choice of coordinate system guarantees that $0 \le \theta' \le \pi$.[f] All this is straightforward and leads to unambiguous, though algebraically messy, expressions.

A similar approach, but using the (x'', y'') coordinate system, gives the coordinates of the intersection between circles 2 and 3. The differences are:

- The desired intersection point has y''_{23} positive.
- The transformation of coordinates involves a translation as well as a reflection.
- The angle θ'' is an exterior angle of the triangle to which the cosine rule is applied, entailing a change of sign.

[f] The choice of the direction of the y axis is crucial here. In fact, the limits on θ' in any given case are rather tighter than this because of the requirement that the third circle be placed so as to form a circular triangle.

The results are:

$$x''_{23} = \frac{r_2{}^2 - r_3{}^2 + d_{23}{}^2}{2d_{23}}, \qquad y''_{23} = \frac{1}{2d_{23}}\sqrt{2d_{23}{}^2\left(r_2{}^2 + r_3{}^2\right) - \left(r_2{}^2 - r_3{}^2\right)^2 - d_{23}{}^4}\,, \tag{10}$$

$$x_{23} = x''_{23}\cos\theta'' - y''_{23}\sin\theta'' + d_{12}\,, \qquad y_{23} = x''_{23}\sin\theta'' + y''_{23}\cos\theta''\,, \tag{11}$$

$$\cos\theta'' = -\frac{d_{12}{}^2 + d_{23}{}^2 - d_{13}{}^2}{2d_{12}\,d_{23}}, \qquad \sin\theta'' = \sqrt{1 - \cos^2\theta''}\,. \tag{12}$$

4.4 Bounds on the Input Parameters

4.4.1 Ensuring the Existence of the Circular Triangle

The equations of §4.3 implicitly assume that the given radii r_k and separations d_{ij} are such that a circular triangle results. This section examines ways of ensuring the fulfilment of this assumption through appropriate bounds on the values of d_{ij}.

The first requirement is obviously that each circle must intersect with both of the others. This leads one to propose a generalisation of Equation (4):

$$r_i - r_j < d_{ij} < r_i + r_j, \tag{13}$$

with a view to applying it to each of d_{12}, d_{13} and d_{23}. (Recall that the circles are numbered so that $r_1 \geq r_2 \geq r_3$.) However, these conditions are insufficient—cases ②, ③ and ⑦ in Figure 5 all satisfy Equation (13), and so are not distinguished from case ①. Inspection of the differences between these cases suggests the another approach.

Case ① differs from the other cases in two properties: point (x_{12}, y_{12}) lies inside circle 3 and point $(x_{12}, -y_{12})$ does *not* lie inside circle 3. Hence, having first used Equation (4) to test that (x_{12}, y_{12}) is a real point, one then computes values of x_{12} and y_{12} and requires that

$$\begin{aligned} \left(x_{12} - d_{13}\cos\theta'\right)^2 + \left(y_{12} - d_{13}\sin\theta'\right)^2 &< r_3{}^2\,, \\ \left(x_{12} - d_{13}\cos\theta'\right)^2 + \left(y_{12} + d_{13}\sin\theta'\right)^2 &> r_3{}^2\,. \end{aligned} \tag{14}$$

If these conditions are satisfied, then one is guaranteed to have case ① of Figure 5 and no other.(g)

4.4.2 When More than Half a Circle is Involved

Passage from Equation (2) to Equation (1) uses the relation $\sin(\phi/2) = c/(2r)$. This is incorrect if ϕ is a reflex angle, which can occur in a circular triangle. Figure 8 illustrates. Figure 8(a) shows the usual picture for the area of a circular segment, which is outlined by heavy lines. This leads to the expression for $\sin(\phi/2)$ quoted above. Figure 8(b) shows the situation when the segment encompasses more than half of the circle. In this case, Equation (2) remains correct, because the sine of a reflex angle is negative. That is, now $\sin(\phi/2) = -c/(2r)$. Hence, a sign change is required in Equation (1). Figure 8(c) gives an example of a circular triangle in which this effect occurs; so demonstrating that the effect is an issue in computing the area of common overlap of three circles.

(g) One can, of course, substitute Equations (6) and (9) into Equation (14) to obtain expressions in which only the input parameters r_k, d_{ij} appear, but the result cannot be rearranged into anything useful.

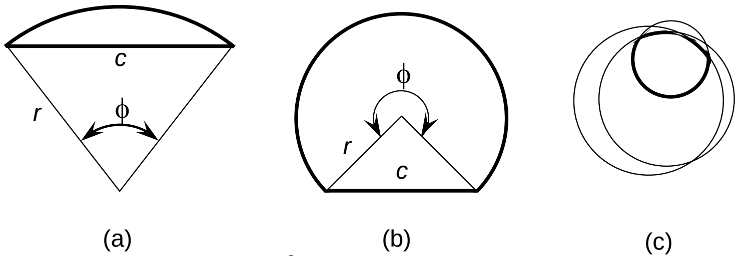

Figure 8: (a) The picture usually employed to illustrate the formula for the area of a circular segment. (b) The situation when the angle subtended at the centre is reflex. (c) An example of a circular triangle involving a reflex segment.

Examination of Figure 8(c) and variations of it leads to the conjecture that the situation can arise only when d_{12} and r_3 are both substantially less than either of r_1 and r_2, and that only the smallest circle—circle 3—can ever contribute more than half of its area to the circular triangle. Hence, one must test whether the centre of circle 3 lies below the line joining the points (x_{13}, y_{13}) and (x_{23}, y_{23}). That is, one asks: is it the case that

$$d_{13}\sin\theta' < y_{13} + \frac{y_{23}-y_{13}}{x_{23}-x_{13}}\left(d_{13}\cos\theta' - x_{13}\right)? \qquad (15)$$

If Equation (15) is true, then the circular triangle includes more than half of circle 3 and Equation (1) must be modified to read

$$A = \tfrac{1}{4}\sqrt{(c_1+c_2+c_3)(c_2+c_3-c_1)(c_1+c_3-c_2)(c_1+c_2-c_3)} + \sum_{k=1}^{3} r_k{}^2 \arcsin\frac{c_k}{2r_k}$$
$$-\frac{c_1}{4}\sqrt{4r_1{}^2 - c_1{}^2} - \frac{c_2}{4}\sqrt{4r_2{}^2 - c_2{}^2} + \frac{c_3}{4}\sqrt{4r_3{}^2 - c_3{}^2}\,. \qquad (16)$$

5. Summary, Other Properties, Special Cases

5.1 Putting It All Together

The expressions needed to evaluate the area of a given circular triangle are scattered through §§2 and 4. For convenience, they are collected together here and set out as an algorithm. Original equation numbers are retained.

The input parameters are the three radii, ordered so that $r_1 \geq r_2 \geq r_3$, and the three separations of circle centres d_{12}, d_{13}, d_{23}, as illustrated in Figure 4. The steps required to compute the area of the circular triangle are:

- *Step 1*. Check whether circles 1 and 2 intersect by testing d_{12}. If

$$r_1 - r_2 < d_{12} < r_1 + r_2 \qquad (4)$$

  is not satisfied, then there is no circular triangle and the algorithm terminates.
- *Step 2*. Calculate the coordinates of the relevant intersection point of circles 1 and 2:

$$x_{12} = \frac{r_1{}^2 - r_2{}^2 + d_{12}{}^2}{2d_{12}}, \qquad y_{12} = \frac{1}{2d_{12}}\sqrt{2d_{12}{}^2\left(r_1{}^2 + r_2{}^2\right) - \left(r_1{}^2 - r_2{}^2\right)^2 - d_{12}{}^4}\,. \qquad (6)$$

- *Step 3*. Calculate the values of the sines and cosines of the angles θ′ and θ″:

$$\cos\theta' = \frac{d_{12}^2 + d_{13}^2 - d_{23}^2}{2d_{12}\,d_{13}}, \qquad \sin\theta' = \sqrt{1-\cos^2\theta'}, \tag{9}$$

$$\cos\theta'' = -\frac{d_{12}^2 + d_{23}^2 - d_{13}^2}{2d_{12}\,d_{23}}, \qquad \sin\theta'' = \sqrt{1-\cos^2\theta''}. \tag{12}$$

- *Step 4*. Check that circle 3 is placed so as to form a circular triangle. The conditions

$$\begin{aligned}(x_{12} - d_{13}\cos\theta')^2 + (y_{12} - d_{13}\sin\theta')^2 &< r_3^2,\\ (x_{12} - d_{13}\cos\theta')^2 + (y_{12} + d_{13}\sin\theta')^2 &> r_3^2\end{aligned} \tag{14}$$

  must both be satisfied. Otherwise, there is no circular triangle and the algorithm terminates.
- *Step 5*. Calculate the values of the coordinates of the relevant intersection points involving circle 3:

$$x'_{13} = \frac{r_1^2 - r_3^2 + d_{13}^2}{2d_{13}}, \qquad y'_{13} = \frac{-1}{2d_{13}}\sqrt{2d_{13}^2\left(r_1^2 + r_3^2\right) - \left(r_1^2 - r_3^2\right)^2 - d_{13}^4}, \tag{7}$$

$$x_{13} = x'_{13}\cos\theta' - y'_{13}\sin\theta', \qquad y_{13} = x'_{13}\sin\theta' + y'_{13}\cos\theta', \tag{8}$$

$$x''_{23} = \frac{r_2^2 - r_3^2 + d_{23}^2}{2d_{23}}, \qquad y''_{23} = \frac{1}{2d_{23}}\sqrt{2d_{23}^2\left(r_2^2 + r_3^2\right) - \left(r_2^2 - r_3^2\right)^2 - d_{23}^4}, \tag{10}$$

$$x_{23} = x''_{23}\cos\theta'' - y''_{23}\sin\theta'' + d_{12}, \qquad y_{23} = x''_{23}\sin\theta'' + y''_{23}\cos\theta''. \tag{11}$$

- *Step 6*. Use the coordinates of the intersection points to calculate the chord lengths c_1, c_2, c_3:

$$c_k^2 = \left(x_{ik} - x_{jk}\right)^2 + \left(y_{ik} - y_{jk}\right)^2. \tag{3}$$

- *Step 7*. Check whether more than half of circle 3 is included in the circular triangle, so as to choose the correct expression for the area. That is, determine whether

$$d_{13}\sin\theta' < y_{13} + \frac{y_{23} - y_{13}}{x_{23} - x_{13}}\left(d_{13}\cos\theta' - x_{13}\right) \tag{15}$$

  is true or false. The area is given by

$$\begin{aligned}A = {}&\tfrac{1}{4}\sqrt{(c_1+c_2+c_3)(c_2+c_3-c_1)(c_1+c_3-c_2)(c_1+c_2-c_3)} + \sum_{k=1}^{3}\left(r_k^2 \arcsin\frac{c_k}{2r_k}\right)\\ &-\sum_{k=1}^{2}\frac{c_k}{4}\sqrt{4r_k^2 - c_k^2} + \begin{cases}\dfrac{c_3}{4}\sqrt{4r_3^2 - c_3^2} & \text{(Eq. 15 true)}\\[2ex] \dfrac{-c_3}{4}\sqrt{4r_3^2 - c_3^2} & \text{(Eq. 15 false).}\end{cases}\end{aligned} \tag{1),(16}$$

The form of the equations makes it clear that Equations (1) and (16) could be written out as single closed-form expressions involving only the six input parameters, but the result would be highly repetitive and not very illuminating. Some special cases are examined in §§5.3–5 below in a search for cases with simple expressions for the area.

5.2 Perimeter and Angles

The chord lengths are the key to obtaining an expression for the perimeter of a circular triangle. As Figure 8 shows, the arc length of a circular segment is simply related to the angle ϕ that the segment subtends at the centre. The only complication arises when the segment includes more than half of the circle (Fig. 8b). Using results from §4.4.2, we find that the perimeter length l is

$$l = \begin{cases} \sum_{k=1}^{3} 2r_k \arcsin\dfrac{c_k}{2r_k} & \text{(Eq. 15 false)} \\ 2\pi r_3 - 2r_3 \arcsin\dfrac{c_3}{2r_3} + \sum_{k=1}^{2} 2r_k \arcsin\dfrac{c_k}{2r_k} & \text{(Eq. 15 true).} \end{cases} \tag{17}$$

Circular triangles have two sets of related angles: those between the chords and those between the arcs. Since the chord lengths are known, the angles between them can be obtained by the cosine rule:

$$\gamma_{ij} = \arccos\frac{c_i^2 + c_j^2 - c_k^2}{2c_i c_j}, \tag{18}$$

where γ_{ij} is the angle between chords i and j, with k being the label of the third chord. The derivation of an expression for the angles between the arcs is sketched in the next paragraph.

Figure 9 shows the geometry of the intersection of two circular arcs, where circles i and j have centres O_i and O_j separated by d_{ij}. The broken lines are tangents at the point of intersection X_{ij}; the angle χ_{ij} between them is the required angle. The angles between the line joining the circle centres and the radii through the point of intersection are denoted α_{ij} and α_{ji}.[h] These are

$$\alpha_{ij} = \arccos\frac{d_{ij}^2 + r_i^2 - r_j^2}{2d_{ij} r_i}. \tag{19}$$

A simple geometrical argument, using the fact that a tangent is perpendicular to the radius through the point of contact (e.g. $AX_{ij} \perp O_j X_{ij}$ in Fig. 9) and the angle properties of triangles, gives

$$\chi_{ij} = \alpha_{ij} + \alpha_{ji}. \tag{20}$$

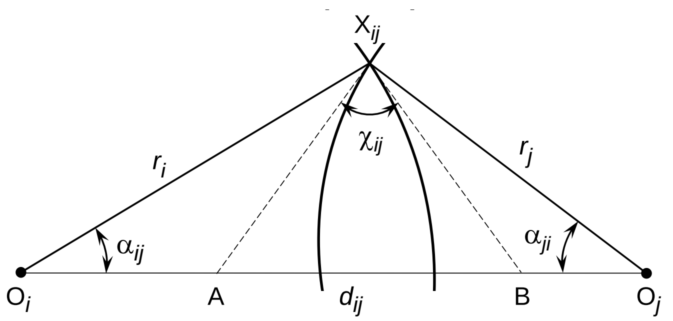

Figure 9: Geometry of the intersection of two circular arcs

[h]Of all the doubly subscripted quantities in this Note, the α_{ij} are the only ones where the order of the subscripts matters, in the manner indicated in Equation (19).

5.3 All Radii Equal

When $r_1 = r_2 = r_3 = r$, Equations (6), (7) and (10) simplify considerably:

$$x_{12} = d_{12}/2\,, \qquad x'_{13} = d_{13}/2\,, \qquad x''_{23} = d_{23}/2\,,$$
$$y_{12} = \sqrt{r^2 - d_{12}{}^2/4}\,, \qquad y'_{13} = -\sqrt{r^2 - d_{13}{}^2/4}\,, \qquad y''_{23} = \sqrt{r^2 - d_{23}{}^2/4}\,. \tag{21}$$

However, there is no simplification of the expressions for θ′ or θ″, and so there is little to be gained by proceeding further with substitutions.

5.4 Equilateral Circular Triangle

Setting $d_{12} = d_{13} = d_{23} = d$ gives θ′ = 60°, θ″ = 120°. This taken together with equal radii means that all sides of the triangle are of equal length, so suggesting the term ‘equilateral circular triangle’ for this figure. The chord lengths c, which are also all equal, are

$$c^2 = 3r^2 - d^2/2 - d\sqrt{3r^2 - 3d^2/4}\,. \tag{22}$$

The fact that the chord lengths are all equal simplifies the expressions for the area (Eq. 1) and perimeter (Eq. 17):[i]

$$A = \frac{\sqrt{3}}{4}c^2 + 3\left(r^2 \arcsin\frac{c}{2r} - \frac{c}{4}\sqrt{4r^2 - c^2}\right), \qquad l = 6r\arcsin\frac{c}{2r}\,. \tag{23}$$

Although there is little point in substituting for c in these equations, the expressions are straightforward to plot, as shown in Figure 10(a). As $d \to 0$, the area approaches πr^2 and the perimeter $2\pi r$;[j] the chord length approaches $r\sqrt{3}$. The area, chord length and perimeter all go to zero at the transition point between cases ① and ⑦ (Fig. 5), which corresponds to $d = r\sqrt{3}$. These and other cases where the arcsine function evaluates to simple fractions of π are listed in Table 1; the case with $d = r$ is shown in Figure 10(b).

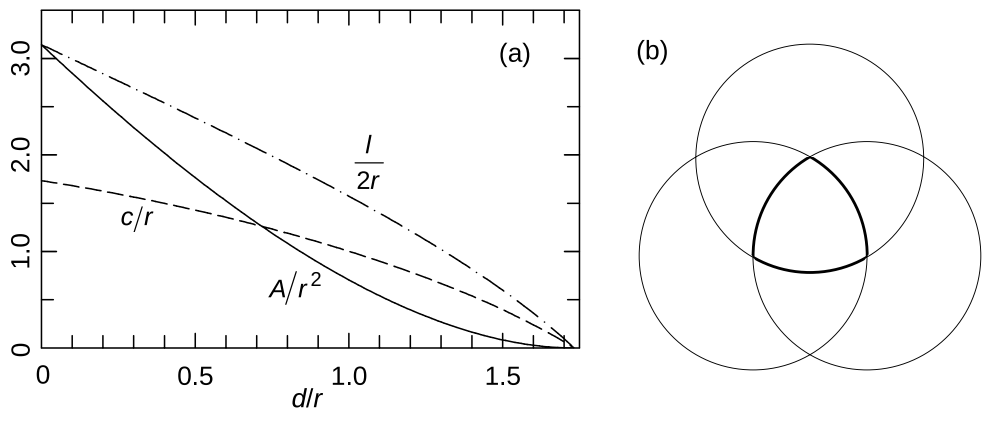

Figure 10: (a) Area A*, chord lengths* c *and perimeter* l *of an equilateral circular triangle (equal radii and equal separations of circle centres) as a function of the ratio* d/r *of separation to radius. (b) An equilateral circular triangle (thick lines) with* d = r.

(i) It is easy to show that Equation (15) cannot be true for equilateral circular triangles, and so Equation (1) is always the correct expression for the area.

(j) As $d \to 0$, the circular triangle approaches a circle. Might this limit represent what philosophers have in mind when they use the term ‘circular triangle’ (footnote b, p. 1)?

Table 1: Area A, *chord length* c *and perimeter* l *of equilateral circular triangles with circle radii* r *and separations* d *between circle centres*

d/r	c/r	A/r^2	l/r
0	$\sqrt{3} = 1.7321$	$\pi = 3.1416$	2π
$\frac{1}{2}\sqrt{9-\sqrt{5}-\sqrt{6}\sqrt{5+\sqrt{5}}} = 0.2091$	$\frac{\sqrt{5}+1}{2} = 1.6180$	$\frac{9\pi}{10}+\frac{\sqrt{3}\left(3+\sqrt{5}\right)}{8}-\frac{3\sqrt{5+\sqrt{5}}}{4\sqrt{2}} = 2.5345$	$\frac{9\pi}{5}$
$\sqrt{2-\sqrt{3}} = 0.5176$	$\sqrt{2} = 1.1412$	$\frac{3\pi}{4}-\frac{3-\sqrt{3}}{2} = 1.7222$	$\frac{3\pi}{2}$
$\frac{1}{2}\sqrt{7+\sqrt{5}-\sqrt{6}\sqrt{5+\sqrt{5}}} = 0.8135$	$\frac{\sqrt{5-\sqrt{5}}}{\sqrt{2}} = 1.1756$	$\frac{3\pi}{5}+\frac{\sqrt{3}\left(5-\sqrt{5}\right)}{8}-\frac{3\sqrt{5+\sqrt{5}}}{4\sqrt{2}} = 1.0558$	$\frac{6\pi}{5}$
1.0	1.0	$\frac{\pi-\sqrt{3}}{2} = 0.7048$	π
$\sqrt{2-\frac{\sqrt{3}-1}{\sqrt{2}}} = 1.2175$	$\sqrt{2-\sqrt{2}} = 0.7652$	$\frac{3\pi}{8}-\frac{1}{2}\left(\frac{3}{\sqrt{2}}+\sqrt{\frac{3}{2}}-\sqrt{3}\right) = 0.3177$	$\frac{3\pi}{4}$
$\frac{1}{2}\sqrt{9+\sqrt{5}-\sqrt{6}\sqrt{5-\sqrt{5}}} = 1.3383$	$\frac{\sqrt{5}-1}{2} = 0.6180$	$\frac{3\pi}{10}+\frac{\sqrt{3}\left(3-\sqrt{5}\right)}{8}-\frac{3\sqrt{5-\sqrt{5}}}{4\sqrt{2}} = 0.2262$	$\frac{3\pi}{5}$
$\sqrt{2} = 1.4142$	$\frac{\sqrt{3}-1}{\sqrt{2}} = 0.5176$	$\frac{\pi}{4}-\frac{3-\sqrt{3}}{2} = 0.1514$	$\frac{\pi}{2}$
$\sqrt{3} = 1.7321$	0	0	0

The angles γ between the chords are $\pi/3$ for all values of d/r, since the three chords are always equal in length. The angles between arcs at each vertex of the circular triangle are all equal to

$$\chi = 2\arccos\frac{d}{2r}. \tag{24}$$

Values of the angles are shown in Figure 11. As $d \to 0$, χ approaches π. As the triangle area approaches zero, $\chi \to \gamma$. Since χ involves d/r rather than c/r, it has a simple form for only a few of the cases listed in Table 1; these are listed in Table 2.

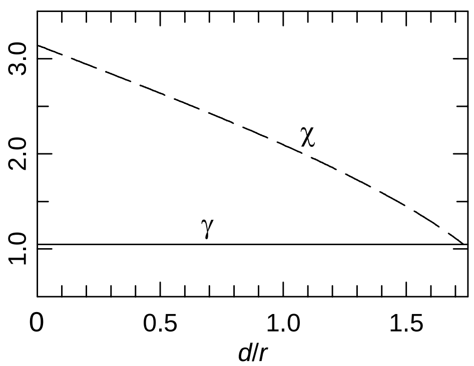

Figure 11: Interarc angles χ and interchord angles γ of an equilateral circular triangle as a function of the ratio d/r *of circle separation to radius*

Table 2: Interarc angles χ of equilateral circular triangles with circle radii r *and separations* d *between circle centres*

d/r	χ
0	$\pi = 180°$
1.0	$\frac{2\pi}{3} = 120°$
$\sqrt{2} = 1.4142$	$\frac{\pi}{2} = 90°$
$\sqrt{3} = 1.7321$	$\frac{\pi}{3} = 60°$

5.5 Two Isosceles Circular Triangles

The next more complicated figure after the equilateral triangle is the isosceles circular triangle, with two sides equal. The most general definition of an isosceles circular triangle would require no more than that two arc lengths be equal. However this is not enough to give simple expressions,[k] so here we explore two definitions that are more restrictive than this.

5.5.1 With Equal Radii

First, we retain the equal radii of the equilateral triangle ($r_1 = r_2 = r_3 = r$) but now set just two of the circle-centre separations equal, choosing to set $d_{13} = d_{23} = d$. Relatively simple results can be obtained by taking the third separation as $d_{12} = d\sqrt{2}$. That is, the circle centres form a right isosceles triangle, with $\theta' = 45°$ and $\theta'' = 135°$.

This choice of parameters gives $c_1 = c_2$. The expressions for chord lengths are more complicated than for the equilateral triangle, but nevertheless simple enough to written in collected form:

$$c_1{}^2 = 2r^2 - d^2/2 + \sqrt{2r^4 - 3r^2 d^2/2 + d^4/4} - d\sqrt{r^2 - d^2/4} - d\sqrt{r^2/2 - d^2/4}\,,$$
$$c_3{}^2 = 2r^2 - d\sqrt{r^2 - d^2/4}\,. \tag{25}$$

Note that the ratio c_1/c_3 does not necessarily equal $\sqrt{2}$. In fact, this value is obtained at one value of d/r only.

As with the equilateral triangle, Equation (15) is never true for this figure, so Equation (1) is always the correct expression for the area. Figures 12(a) and 13 show values of the area, chord lengths ($c_1 = c_2$), perimeter and angles. All these quantities except the angles χ_{13} (= χ_{23}) and γ_{13} (= γ_{23}) go to zero as $d \to r\sqrt{2}$; angles χ_{13} and γ_{13} approach $\pi/2$

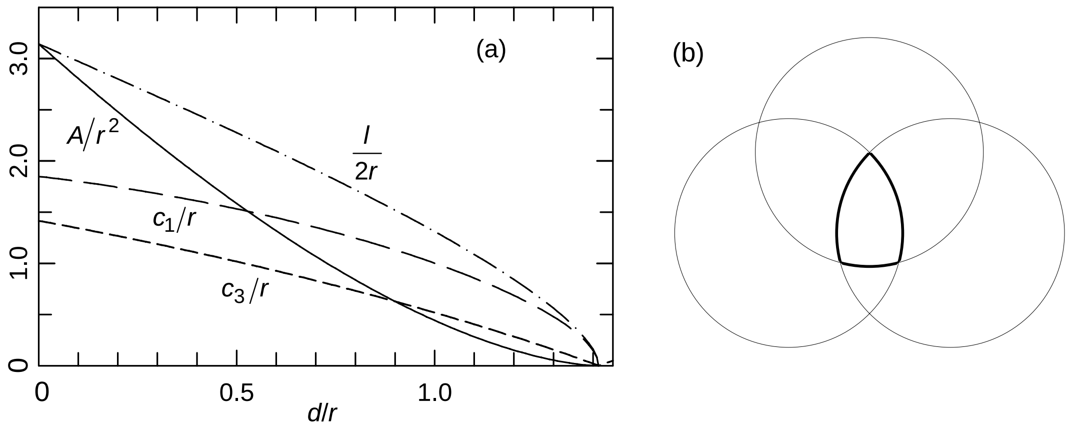

Figure 12: (a) Area A*, chord lengths* c_k *(*$c_1 = c_2$*) and perimeter* l *of an isosceles circular triangle with equal radii as a function of the ratio* d/r*, where* r *is the radius and* d *is the separation of the centre of circle 3 from the centres of each of circles 1 and 2. (The centres of circles 1 and 2 are separated by* $d\sqrt{2}$*.) (b) An isosceles circular triangle (thick lines) with the above parameters and, in addition,* d = r.

[k] In fact, the corresponding condition is not enough in the equilateral case, either. That is, it is not enough to require equal arc lengths; in order to obtain simple equations, one must require both equal radii and equal circle-centre separations.

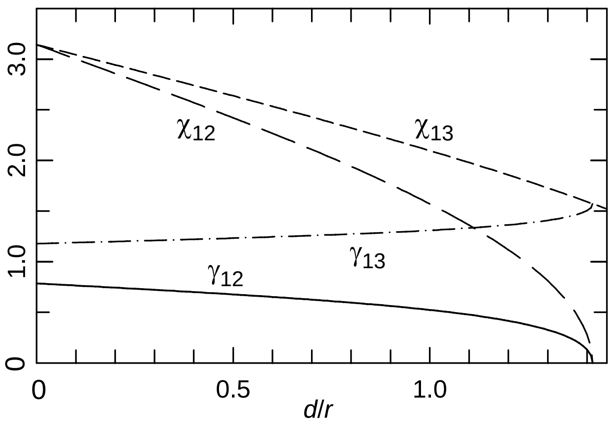

Figure 13: As for Figure 12, but showing values of interarc angles χ_{ij} and interchord angles γ_{ij}

in this limit. As d approaches zero, $\chi_{12} \to \pi/4$, $\chi_{13} = \chi_{23} \to 3\pi/8$, all the interarc angles approach π, c_3 approaches $\sqrt{2}$ and $c_1 \to \sqrt{(2 + \sqrt{2})} = 1.8478$.

Figure 12(b) shows an isosceles circular triangle with the properties specified in the first paragraph of this Subsection and, in addition, $d = r$. It did not prove possible to find a range of simple results, like those listed in Table 1 for the equilateral circular triangle, owing to the complexity of the expression for c_1 (Eq. 25).

5.5.2 With Circle 3 Centred on an Intersection of Circles 1 and 2

A simpler case than that of §5.5.1 is obtained by choosing $r_1 = r_2 = d_{13} = d_{23} = r$ and $r_3 = d_{12} = \rho$. This places the centre of circle 3 at an intersection of circles 1 and 2. In addition, the radius of circle 3 is set equal to the separation between the centres of circles 1 and 2. The angles θ' and θ'' are not constant, as in the two previous examples, but their cosines have simple expressions: $\cos\theta' = -\cos\theta'' = \rho/(2r)$. Equation (4) suggests that, for a given value of r, ρ can vary in the range $0 < \rho < 2r$, but the second condition of Equation (14) impacts here, restricting the upper limit to $r\sqrt{2}$. Values of ρ larger than $r\sqrt{2}$ give case ③ in Figure 5 (p. 4). It should also be noted that the circles are misnumbered when $\rho > 1$, for then circle 3 is bigger than the other two, so that numbers 1 and 3 should be interchanged. However, circle numbering is important only in Equations (4) and (14); the expressions for area and perimeter are symmetric in the circle numbers.

The expressions for the chord lengths of this figure are particularly simple:

$$c_1 = c_2 = \rho, \qquad c_3 = \rho\left(2 - \rho^2/r^2\right), \tag{26}$$

leading to an expression for the area that is the simplest discovered in this work:

$$A = 2r^2 \arcsin\frac{\rho}{2r} + \rho^2 \arcsin\left(1 - \frac{\rho^2}{2r^2}\right) - \frac{\rho}{2}\sqrt{4r^2 - \rho^2}\,. \tag{27}$$

The perimeter and all the angles also have simple expressions. Plots of these quantities are shown in Figure 14. Because $r_3 \to 0$ as $d_{12} \to 0$, the area, chord lengths and perimeter all vanish in this limit (i.e. as $\rho \to 0$). In the other limit, $\rho/r \to \sqrt{2}$, c_3 is the only chord length that goes to zero as the circular triangle degenerates to a lens shape. Hence the area and perimeter remain nonzero.

Tables 3 and 4 give results in cases where the arcsine functions evaluate to simple fractions of π. Figure 15 shows some examples of this type of isosceles circular triangle.

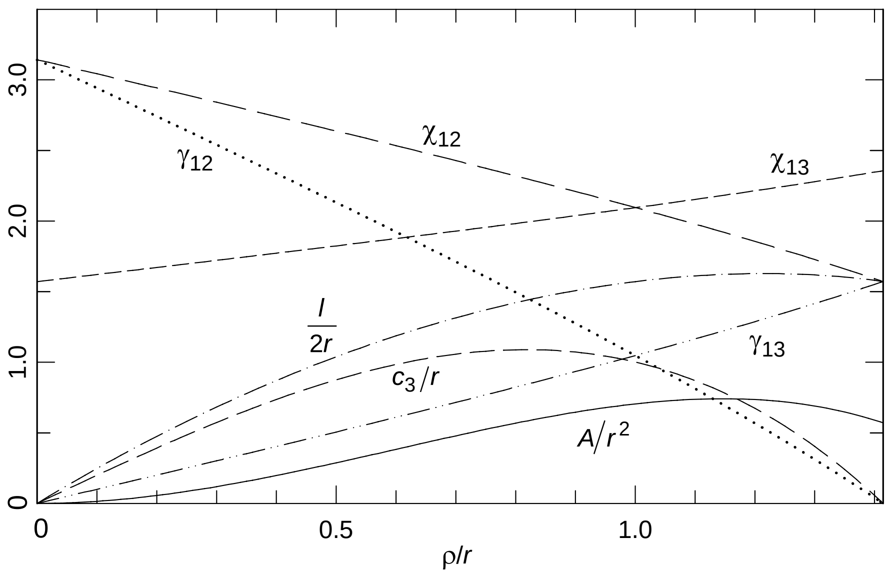

Figure 14: Values of area A*, chord length* c_3*, perimeter* l*, interarc angles* χ_{ij} *and interchord angles* γ_{ij} *of an isosceles circular triangle with circle 3 located at an intersection of circles 1 and 2 as a function of the ratio* ρ/r*, where* $r = r_1 = r_2 = d_{13} = d_{23}$ *and* $\rho = r_3 = d_{12}$

Table 3: Area A*, chord length* c_3 *and perimeter* l *of circular triangles described in the caption above*

ρ/r	c_3/r	A/r^2	l/r
0	0	0	0
$\frac{\sqrt{3}-1}{\sqrt{2}} = 0.5176$	$\frac{3-\sqrt{3}}{\sqrt{2}} = 0.8966$	$\frac{5\pi}{6}-\frac{\pi}{\sqrt{3}}-\frac{1}{2} = 0.3042$	$\frac{1+\sqrt{6}-\sqrt{2}}{3}\pi = 2.1313$
$\frac{\sqrt{5}-1}{2} = 0.6180$	1.0	$\frac{13\pi}{20}-\frac{3\pi}{4\sqrt{5}}-\sqrt{\frac{5-\sqrt{5}}{8}} = 0.4005$	$\frac{1+3\sqrt{5}}{10}\pi = 2.4216$
$\sqrt{2-\sqrt{2}} = 0.7654$	$2\sqrt{\sqrt{2}-1} = 1.2872$	$\frac{3\pi}{4}-\frac{\pi}{2\sqrt{2}}-\frac{1}{\sqrt{2}} = 0.5384$	$\frac{1+\sqrt{2-\sqrt{2}}}{2}\pi = 2.7730$
1.0	1.0	$\frac{\pi}{2}-\frac{\sqrt{3}}{2} = 0.7048$	$\pi = 3.1416$
$\sqrt{\frac{5-\sqrt{5}}{2}} = 1.1756$	$\sqrt{5-2\sqrt{5}} = 0.7256$	$\frac{13\pi}{20}+\frac{\pi}{4\sqrt{5}}-\sqrt{\frac{5+\sqrt{2}}{8}} = 0.7397$	$\frac{\pi}{5}\left(4+\sqrt{\frac{5-\sqrt{5}}{2}}\right) = 3.2519$
$\sqrt{2} = 1.4142$	0	$\frac{\pi}{2}-1 = 0.5708$	$\pi = 3.1416$

6. Conclusion

This Note presents closed-form formulae for the area of common overlap of three circles, apparently the first such to be published. The area in question has a shape known as a 'circular triangle'. The formulae are expressed both in terms of the three chord lengths of the circular triangle and, more usefully, in terms of the radii and separations

Table 4: Interchord angles γ_{ij} and interarc angles χ_{ij} of circular triangles described in the caption to Figure 14

ρ/r	γ_{12}	γ_{13}	χ_{12}	χ_{13}
0	$\pi = 180°$	0	$\pi = 180°$	$\frac{\pi}{2} = 90°$
$\frac{\sqrt{3}-1}{\sqrt{2}} = 0.5176$	$\frac{2\pi}{3} = 120°$	$\frac{\pi}{6} = 30°$	$\frac{5\pi}{6} = 150°$	$\frac{7\pi}{12} = 105°$
$\frac{\sqrt{5}-1}{2} = 0.6180$	$\frac{7\pi}{12} = 105°$	$\frac{\pi}{5} = 36°$	$\frac{4\pi}{5} = 144°$	$\frac{3\pi}{5} = 108°$
$\sqrt{2-\sqrt{2}} = 0.7654$	$\frac{\pi}{2} = 90°$	$\frac{\pi}{4} = 45°$	$\frac{3\pi}{4} = 135°$	$\frac{5\pi}{8} = 112.5°$
1.0	$\frac{\pi}{3} = 60°$	$\frac{\pi}{3} = 60°$	$\frac{2\pi}{3} = 120°$	$\frac{2\pi}{3} = 120°$
$\sqrt{\frac{5-\sqrt{5}}{2}} = 1.1756$	$\frac{\pi}{5} = 36°$	$\frac{2\pi}{5} = 72°$	$\frac{3\pi}{5} = 108°$	$\frac{7\pi}{10} = 126°$
$\sqrt{2} = 1.4142$	0	$\frac{\pi}{2} = 90°$	$\frac{\pi}{2} = 90°$	$\frac{3\pi}{4} = 135°$

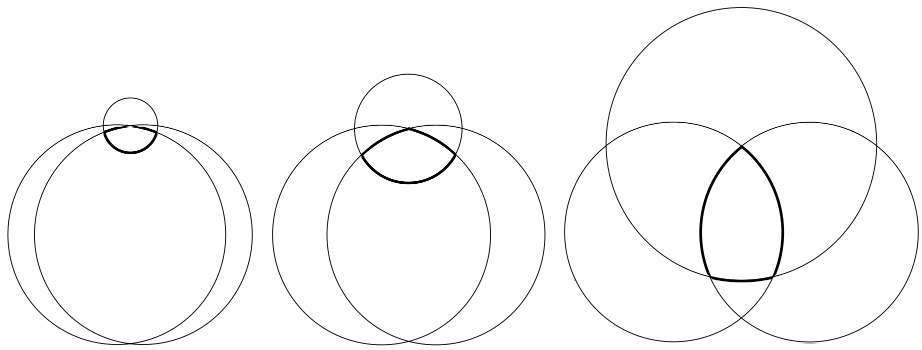

Figure 15: Examples of circular triangles with $r_1 = r_2 = d_{13} = d_{23} = r$ *and* $r_3 = d_{12} = \rho$ *for the cases (a)* $\rho/r = 0.25$, *(b)* $\rho/r = 0.5$, *(c)* $\rho/r = 1.25$. *The case* $\rho/r = 1$ *is identical to the equilateral circular triangle with* $d = r$ *(Fig. 10b).*

of centres of the three circles, parts of which form the sides of the circular triangle. The formulae include tests to ensure that the given radii and centre separations do indeed result in a circular triangle. Formulae for the perimeter and angles of a circular triangle are also presented. The formulae are general enough to cover all possible cases of the overlap of three circles for which the area of common overlap is a circular triangle.

Four special cases are examined, three of which—the equilateral circular triangle and two types of isosceles circular triangle—involve sufficient simplification to give substantially reduced equations, with one being so simple that the expression for the area can be usefully written as a collected formula.

The results presented here have general significance in the corpus of mensuration formulae, and could be of specific use in any quantitative application of the three-circle Venn diagram.

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Appendix A: Concave and Mixed Circular Triangles

The circular segments of the object shown in Figure 3 (p. 3) all add to the area of the straight-line triangle formed by the chords. Clearly, the opposite is possible; Figure 16 contrasts some examples. Taking a cue from the terminology of lenses, the object in Figure 3 and Figure 16(a) may be called a 'convex circular triangle' and that in Figure 16(b) a 'concave circular triangle'. Figure 16(c) shows a mixed case in which one segment subtracts from the area of the chordal triangle and the other two add, and Figure 16(d) shows a mixed case with two negative segments.

Examination of a range of cases indicates that, whenever the area of common overlap of three circles is a circular triangle, it must always be convex. However, a glance at Figure 5 (p. 4) shows that all types of circular triangle occur among the various cases displayed. Case ⑦ has a concave circular triangle at its centre: the area that is outside all three circles but completely surrounded by them. Case ① exhibits three mixed circular triangles like Figure 16(c) and three like Figure 16(d).

It is clear that the methods of §§2 and 4 can be equally well applied to the other types of circular triangle to give expressions for area and perimeter. In many—perhaps all—cases, it is simply a matter of the correct placement of negative signs. The main point to watch for are cases where a segment contains more than half of a circle (as in §4.4.2), which Figure 5 suggests is more common in mixed circular triangles than in the other types, particularly in mixed circular triangles with two negative segments.

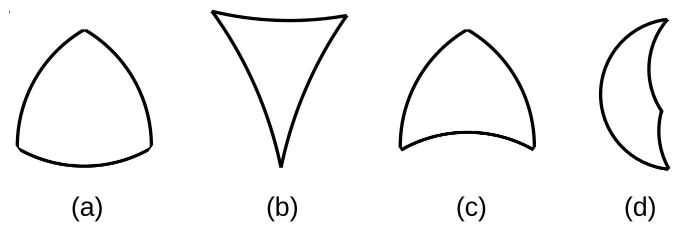

Figure 16: Examples of (a) convex, (b) concave and (c, d) mixed circular triangles

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Area of Common Overlap of Three Circles

M.P. Fewell

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2. TITLE Area of Common Overlap of Three Circles			3. SECURITY CLASSIFICATION (FOR UNCLASSIFIED REPORTS THAT ARE LIMITED RELEASE USE (L) NEXT TO DOCUMENT CLASSIFICATION) Document (U) Title (U) Abstract (U)			
4. AUTHOR(S) M.P. Fewell			5. CORPORATE AUTHOR Defence Science and Technology Organisation, P.O. Box 1500, Edinburgh, SA 5111, Australia			
6a. DSTO NUMBER DSTO-TN-0722	6b. AR NUMBER AR- 013-763		6c. TYPE OF REPORT Technical Note		7. DOCUMENT DATE October 2006	
8. FILE NUMBER 2006/1106831/1	9. TASK NUMBER NAV 04-126	10. TASK SPONSOR COMAUSNAVSUBGP		11. NO. OF PAGES 20	12. NO. OF REFERENCES 12	
13. URL on the World Wide Web http://www.dsto.defence.gov.au/corporate/reports/DSTO-TN-0722.pdf				14. RELEASE AUTHORITY Chief, Maritime Operations Division		

15. SECONDARY RELEASE STATEMENT OF THIS DOCUMENT

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Plane geometry, Mensuration, Circular triangle, Intersecting circles, Venn diagram

19. ABSTRACT
This Note presents the solution to an apparently hitherto unsolved geometrical problem: the derivation of a closed-form algebraic expression of the area of common overlap of three circles, such as can occur in a three-circle Venn diagram. The results presented here have general significance in the corpus of mensuration formulae, and could be of specific use in any quantitative application of the three-circle Venn diagram such as, for example, in search and screening problems.